A new kind of Diophantine equations

by

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1. Introduction. Among various kinds of Diophantine equations, a famous one is

\[(1.1) \quad \binom{n}{k} = m^l, \quad 2 \leq k \leq n-2, \ l \geq 2.\]

The complete solution of (1.1) was given by Erdős [3] for \(4 \leq k \leq n-4\), and by Győry [5] for \(k \leq 3\) and \(k \geq n-3\). In 1975, Erdős and Selfridge [4] proved that the product of consecutive positive integers is never a perfect power. Actually the Diophantine equation

\[(1.2) \quad n(n-1) \cdots (n-k+1) = bm^l, \quad 2 \leq k \leq n, \ l \geq 2,\]

under the assumption \(P(b) < k\) was solved in [4], where \(P(b)\) denotes the greatest prime divisor of \(b\), with \(P(1) = 1\). As a common generalization of the above two results, (1.2) was resolved under the assumption \(P(b) \leq k\) by Saradha [11] and Győry [6] for \(k \geq 4\) and \(k \leq 3\) respectively. When \(\gcd(n,d) = 1\), the Diophantine equation

\[n(n-d) \cdots (n-(k-1)d) = bm^l, \quad k \geq 2, \ (k-1)d < n, \ l \geq 2,\]

under the assumption \(P(b) \leq k\) has also been considered. For related results, we refer to [1], [2], [7], [9], [12].

For \(t \geq 1\), let \((2t-1)!! = 1 \cdot 3 \cdots (2t-1)\), and define an analogue of the binomial coefficient

\[\binom{n}{k}!! = \frac{(2n-1)(2n-3) \cdots (2n-2k+1)}{(2k-1)(2k-3) \cdots 1} = \frac{(2n-1)!!}{(2k-1)!!(2(n-k)-1)!!}\]

for \(1 \leq k \leq n-1\). In this paper, we consider in which case \((\frac{n}{k})!!\) is a power of a rational number. We completely solve the equation

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\[(1.3) \quad \binom{n}{k}!! = \binom{m}{M}^l\]
in integers \(n \geq 4, 2 \leq k \leq n - 2, m \geq 1, M \geq 1, \gcd(m, M) = 1, l \geq 2.\)

**Theorem 1.1.** All the solutions of (1.3) are \(l = k = 2, M = 1, 2n + m\sqrt{3} = (2 + \sqrt{3})^{2t+1} + 2\) \((t \in \mathbb{N}^*)\) and \(l = 2, k = n - 2, M = 1, 2n + m\sqrt{3} = (2 + \sqrt{3})^{2t+1} + 2\) \((t \in \mathbb{N}^*)\).

2. **Preliminaries.** Due to the observation \(\binom{n}{k}!! = \binom{n}{n-k}!!\), we assume \(n \geq 2k\) in the following. Define \(\Delta = \Delta(n, k) = (2n - 1)(2n - 3) \cdots (2n - 2k + 1)\).

We have

**Lemma 2.1.** Let \(k \geq 9\). Then \(\Delta\) is divisible by a prime exceeding \(2k\).

*Proof.* Write \(W(\Delta)\) for the number of terms in \(\Delta\) divisible by a prime exceeding \(k\) and \(\pi(x)\) for the number of primes not exceeding \(x\). It is shown in [8, Theorem 1] that
\[(2.1) \quad W(\Delta) \geq \pi(2k) - \pi(k) + 1, \quad k \geq 9.\]
On the other hand, we note that every prime exceeding \(k\) divides at most one term of \(\Delta\), and every odd prime less than \(k\) divides \(\Delta\). Hence
\[(2.2) \quad \omega(\Delta) - \pi(k) + 1 \geq W(\Delta),\]
where \(\omega(\Delta)\) denotes the number of distinct prime divisors of \(\Delta\). Combining (2.1) and (2.2), we have \(W(\Delta) \geq \pi(2k)\), which implies Lemma 2.1 as \(2 \nmid \Delta\). □

The next lemma is a consequence of [10, Theorem 1].

**Lemma 2.2.** Both of the equations
\[9z_1^4 - 5z_2^2 = 4 \quad \text{and} \quad 25z_1^4 - 21z_2^2 = 4\]
have the unique positive integer solution \((z_1, z_2) = (1, 1)\).

**Lemma 2.3.** Let \(l \geq 3\) be an integer, \(t \in \{1, 2, 3\}\). If the equation
\[(2.3) \quad |z_1^l - 3z_2^l| = 2t,\]
where \(\mid \mid \) denotes the absolute value, has a positive integer solution \((z_1, z_2)\), then \(z_1 = z_2 = t = 1\).

*Proof.* Let \(x = \min(z_1^l, 3z_2^l), y = z_1z_2\). Then
\[(2.4) \quad x(x + 2t) = 3y^l.\]
If \(t = 1, 2\), as \(x\) is odd, we have \(x = 1\) according to [1, Theorem 1.1], which happens only when \(z_1 = 1\) and consequently \(z_2 = t = 1\).
If \( t = 3 \), we have \( x = 3X \) for some positive odd integer \( X \) by virtue of (2.4). Then we deduce from

\[
3X(X + 2) = y^l
\]

that \( y = 3Y \) for some positive integer \( Y \). Thus (2.4) changes to

\[
X(X + 2) = 3^{l-1}Y^l.
\]

According to [1, Theorem 1.1], \( X = 1 \), but this gives no solution of (2.3) for odd \( z_1 \) as 
\[
3 = x \leq z_1^l \leq x + 6 = 9.
\]

3. Proof of Theorem 1.1. Suppose (1.3) has solutions. If \( k \geq 9 \), then according to Lemma 2.1, there exists a prime \( p > 2k \) such that \( p \mid \Delta \). Noting that \( p^l \mid \Delta \) from (1.3) and the fact that \( p \) divides only one term of \( \Delta \), we deduce that \( 2n > p^l > (2k)^l \geq (2k)^2 \). If \( 2 \leq k \leq 8 \) and \( 2k \leq n < k^2 \), one can easily check that (1.3) has no solution. Thus in the following we assume

\[
(3.1) \quad n \geq k^2.
\]

Write \( 2n - 2i - 1 = a_im_i^l \) for \( i = 0, 1, \ldots, k - 1 \), where \( a_i \) is \( l \)th power free. We claim that \( a_0, a_1, \ldots, a_{k-1} \) are distinct. Otherwise there exist integers 
\( 0 \leq i < j \leq k - 1 \) such that \( a_i = a_j \), from which we deduce \( m_i > m_j \). Then it follows that

\[
2k > 2(j - i) = a_j(m_j^l - m_j^l) \geq a_j((m_j + 1)^l - m_j^l) \geq la_jm_j^{l-1}
\]

\[
= la_j^{1/l}(a_jm_j^{l/(l-1)}) \geq 2(a_jm_j^{l})^{1/2} \geq 2(2n - 2k + 1)^{1/2} > 2n^{1/2},
\]

contradicting (3.1).

Now rewrite (1.3) as

\[
(3.2) \quad a_0a_1 \cdots a_{k-1}(m_0m_1 \cdots m_{k-1}M)^l = (2k - 1)!!m^l.
\]

Let

\[
u = \frac{m_0m_1 \cdots m_{k-1}M}{\gcd(m_0m_1 \cdots m_{k-1}M, m)}, \quad v = \frac{m}{\gcd(m_0m_1 \cdots m_{k-1}M, m)}.
\]

Then (3.2) can be written as

\[
(3.3) \quad a_0a_1 \cdots a_{k-1}u^l = (2k - 1)!!v^l.
\]

Suppose \( v \) has a prime divisor \( p \). Obviously, \( p \) is odd and \( p \nmid u \). Therefore from (3.3) we infer that
\( (3.4) \quad \text{ord}_p(a_0 a_1 \cdots a_{k-1}) \)
\[
\geq \text{ord}_p((2k - 1)!!) + l \\
= \text{ord}_p((2k - 1)! - \text{ord}_p((2k - 2)(2k - 4) \cdots 2) + l \\
= \text{ord}_p((2k - 1)! - \text{ord}_p((k - 1)!) + l
\]
\[
= \sum_{i=1}^{l-1} \left( \left\lfloor \frac{2k - 1}{p^i} \right\rfloor - \left\lfloor \frac{k - 1}{p^i} \right\rfloor \right) + l
\]
\[
\geq \sum_{i=1}^{l-1} \left( \left\lfloor \frac{2k - 1}{p^i} \right\rfloor - \left\lfloor \frac{k - 1}{p^i} \right\rfloor \right) + l.
\]

On the other hand, \( \text{ord}_p(a_0 a_1 \cdots a_{k-1}) \) can be evaluated in the following way:

\[
\text{ord}_p(a_0 a_1 \cdots a_{k-1})
\]
\[
= \sum_{i=1}^{l-1} \#\{j : p^i \mid a_j, 0 \leq j \leq k - 1\}
\]
\[
\leq \sum_{i=1}^{l-1} \#\{j : p^i \mid (2n - 2j - 1), 0 \leq j \leq k - 1\}
\]
\[
= \sum_{i=1}^{l-1} (\#\{j : p^i \mid j, 1 \leq j \leq 2n - 1\} - \#\{j : p^i \mid 2j, 1 \leq j \leq n - 1\}
\]
\[
- \#\{j : p^i \mid j, 1 \leq j \leq 2n - 2k - 1\} + \#\{j : p^i \mid 2j, 1 \leq j \leq n - k - 1\})
\]
\[
= \sum_{i=1}^{l-1} \left( \left\lfloor \frac{2n - 1}{p^i} \right\rfloor - \left\lfloor \frac{n - 1}{p^i} \right\rfloor - \left\lfloor \frac{2n - 2k - 1}{p^i} \right\rfloor + \left\lfloor \frac{n - k - 1}{p^i} \right\rfloor \right).
\]

Noting that
\[
\left\lfloor \frac{2n - 1}{p^i} \right\rfloor - \left\lfloor \frac{2n - 2k - 1}{p^i} \right\rfloor \leq \left\lfloor \frac{2k}{p^i} \right\rfloor + 1, \quad \left\lfloor \frac{n - 1}{p^i} \right\rfloor - \left\lfloor \frac{n - k - 1}{p^i} \right\rfloor \geq \left\lfloor \frac{k}{p^i} \right\rfloor,
\]
we have

\( (3.5) \quad \text{ord}_p(a_0 a_1 \cdots a_{k-1}) \leq \sum_{i=1}^{l-1} \left( \left\lfloor \frac{2k}{p^i} \right\rfloor + 1 - \left\lfloor \frac{k}{p^i} \right\rfloor \right)
\]
\[
= \sum_{i=1}^{l-1} \left( \left\lfloor \frac{2k}{p^i} \right\rfloor - \left\lfloor \frac{k}{p^i} \right\rfloor \right) + l - 1.
\]

However, in view of
\[
\left\lfloor \frac{2k}{p^i} \right\rfloor - \left\lfloor \frac{2k - 1}{p^i} \right\rfloor = \left\lfloor \frac{k}{p^i} \right\rfloor - \left\lfloor \frac{k - 1}{p^i} \right\rfloor,
\]
we see that (3.5) contradicts (3.4). Therefore \( v = 1 \), whence \( a_0 a_1 \cdots a_{k-1} | (2k-1)! \). This together with the assertion that \( a_0, a_1, \ldots, a_{k-1} \) are distinct odd integers tells us that

\[
(3.6) \quad \{a_0, a_1, \ldots, a_{k-1}\} = \{1, 3, \ldots, 2k-1\}.
\]

**I.** The case \( l \geq 3, \ k \geq 5 \). Let \( k \equiv \sigma \pmod{3} \), where \( \sigma \in \{-1, 0, 1\} \). According to (3.6),

\[
2k - 2\sigma - 3 = a_i, \quad \frac{2k - 2\sigma - 3}{3} = a_j
\]

for some \( 0 \leq i, j \leq k-1 \). Then

\[
0 < |m_j^l - 3m_i^l| = \frac{3|a_j m_j^l - a_i m_i^l|}{2k - 2\sigma - 3} \leq \frac{3(2k-2)}{2k-5} < 5.
\]

As \( m_i, m_j \) are odd, \( |m_j^l - 3m_i^l| = 2, 4 \), which implies \( m_i = m_j = 1 \) by Lemma 2.3. Hence \( 2n - 2k + 1 \leq a_i m_i^l = 2k - 2\sigma - 3 \leq 2k - 1 \), contradicting \( n \geq 2k \).

**II.** The case \( l \geq 3, \ 2 \leq k \leq 4 \). Let \( a_i = 3, a_j = 1 \). Then \( 0 < |m_j^l - 3m_i^l| \leq 2k - 2 \leq 6 \), which means \( |m_j^l - 3m_i^l| = 2, 4, 6 \) and thus \( m_i = m_j = 1 \) by Lemma 2.3. Hence \( 2n - 2k + 1 \leq a_j m_j^l = 1 \), which is impossible.

**III.** The case \( l = 2, \ k \geq 5 \). This is impossible as, by (3.6), there exists some \( i \) with \( a_i = 9 \), but \( a_i \) must be square free.

**IV.** The case \( l = 2, \ 3 \leq k \leq 4 \). As \( 2, -4 \) are quadratic nonresidues modulo 3, we know that \( x^2 - 3y^2 \neq 2, -4 \) for any integers \( x, y \). Similar argument can be applied to \( 3y^2 - 7w^2, x^2 - 5z^2, x^2 - 7w^2 \) modulo \( 3, 5, 7 \), respectively. Then we have

\[
(3.7) \quad \begin{cases} 
  x^2 - 3y^2 \neq 2, -4, \\
  x^2 - 5z^2 \neq 2, -2, -6, \\
  x^2 - 7w^2 \neq -2, -4, 6, \\
  3y^2 - 7w^2 \neq -2, 4.
\end{cases}
\]

When \( k = 3 \), noting that \( a_i m_i^2 - a_j m_j^2 = \pm 2, \pm 4 \) for \( 0 \leq i < j \leq 2 \), we deduce from (3.6) and (3.7) that \( (a_0, a_1, a_2) = (5, 3, 1) \). In fact, \( (a_0, a_1, a_2) = (1, 3, 5) \) implies \( m_1^2 - 3m_2^2 = 2 \), which has no integer solution according to (3.7), but since \( x^2 - 5z^2 \neq \pm 2 \), \( (a_0, a_1, a_2) \) can only be \( (1, 3, 5) \) or \( (5, 3, 1) \), so \( (a_0, a_1, a_2) = (5, 3, 1) \). Therefore,

\[
9m_1^4 - 4 = (3m_1^2 + 2)(3m_1^2 - 2) = 5m_0^2 \cdot m_2^2 = 5(m_0 m_2)^2.
\]

By Lemma 2.2, \( m_0 = m_1 = m_2 = 1 \), but this means \( n = 3 \), contradicting \( n \geq 2k \).
When \( k = 4 \), we can deduce similarly that \((a_0, a_1, a_2, a_3) = (7, 5, 3, 1)\) or \((1, 7, 5, 3)\). Let \( i = 1 \) resp. 2. Then we have
\[
25m_i^4 - 4 = (5m_i^2 + 2)(5m_i^2 - 2) = 21(m_{i-1}m_{i+1})^2.
\]
By Lemma 2.2, \( m_{i-1} = m_i = m_{i+1} = 1 \), which implies \( 2n - 1 \leq 7 \), contradicting \( n \geq 2k \).

V. The case \( l = 2, k = 2 \). As \( M^2 \mid (2k - 1)!! \), we have \( M = 1 \), whence what we are going to solve is
\[
(3.8) \quad (2n - 1)(2n - 3) = 3m^2.
\]
Let \( 2n - 2 = x \), with which (3.8) takes the form
\[
(3.9) \quad x^2 - 3m^2 = 1.
\]
All the positive integer solutions of the above Pell equation are given by
\[
x_t + m_t \sqrt{3} = (2 + \sqrt{3})^t \quad (t \in \mathbb{N}^*).
\]
This implies that all the positive integer solutions of (3.9) with \( 2 \mid x \) and \( x \geq 6 \) are given by
\[
x + m \sqrt{3} = (2 + \sqrt{3})^{2t+1} \quad (t \in \mathbb{N}^*).
\]
Hence all the solutions of (3.8) are given by
\[
2n + m \sqrt{3} = (2 + \sqrt{3})^{2t+1} + 2 \quad (t \in \mathbb{N}^*).
\]
This completes the proof of Theorem 1.1 as \( (n, k)!! = (n - k)!! \). □

4. A generalization of equation (1.3). As a generalization of \((n, k)\) and \((n, k)!!\), we define
\[
\left( \begin{array}{c} n \\ k \end{array} \right)_{a,b} = \frac{(an - a + b)(an - 2a + b) \cdots (an - ak + b)}{(ak - a + b)(ak - 2a + b) \cdots b},
\]
\[
1 \leq b \leq a, \gcd(a, b) = 1,
\]
and ask whether \( \left( \begin{array}{c} n \\ k \end{array} \right)_{a,b} \) is a power of a rational number when \( 2 \leq k \leq n - 2 \).

In view of \( \left( \begin{array}{c} n \\ k \end{array} \right)_{a,b} = \left( \begin{array}{c} n \\ n-k \end{array} \right)_{a,b} \), we only need to consider the following Diophantine equation:
\[
(4.1) \quad \left( \begin{array}{c} n \\ k \end{array} \right)_{a,b} = \left( \frac{m}{M} \right)^l,
\]
\[
in integers n \geq 4, 4 \leq 2k \leq n, m \geq 1, M \geq 1, \gcd(m, M) = 1, l \geq 2.
\]
When \((a, b) = (1, 1)\), (4.1) is (1.1), and when \((a, b) = (2, 1)\), (4.1) is (1.3). However, we cannot solve (4.1) using the method of this paper when \( a \geq 3 \).

Furthermore, for \( 1 \leq b \leq a, \gcd(a, b) = 1 \), we can consider the quotient of two products of consecutive \( k \) terms in the arithmetic progression \( b, a + b, \)
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a + 2b, . . ., and ask for the solutions of the Diophantine equation

\[
\frac{(an - a + b)(an - 2a + b) \cdots (an - ak + b)}{(aN - a + b)(aN - 2a + b) \cdots (aN - ak + b)} = \left( \frac{m}{M} \right)^l
\]

in integers |N − n| ≥ k, m ≥ 1, M ≥ 1, gcd(m, M) = 1, l ≥ 2.

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References

[6] —, On the diophantine equation \( n(n+1) \cdots (n-k+1) = bx^l \), ibid. 83 (1998), 87–92.

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