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Logarithmic vector-valued modular forms

by

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1. Introduction. The present work is a natural sequel to our earlier articles on "normal" vector-valued modular forms [KM1], [KM2]. The component functions of a normal vector-valued modular form F are q-series with at worst real exponents. Equivalently, the finite-dimensional representation ρ associated with F has the property that $\rho(T)$ is (similar to) a matrix that is unitary and diagonal. Here, $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

In the case of a general representation, $\rho(T)$ is not necessarily diagonal but may always be assumed to be in Jordan canonical form [1]. This circumstance leads to *logarithmic*, or *polynomial* q-expansions for the component functions of a vector-valued modular form associated to ρ (see Subsection 2.2), which take the form

$$f(\tau) = \sum_{j=0}^{t} (\log q)^j h_j(\tau), \tag{1}$$

where the $h_j(\tau)$ are ordinary q-series. There follow naturally the definition of logarithmic vector-valued modular form and the concomitant notions of logarithmic meromorphic, holomorphic (i.e., entire in the sense of Hecke) and cuspidal vector-valued modular forms (Subsection 2.3).

The Poincaré series is an indispensable device in every theory of modular (or automorphic) forms, regardless of the level of abstraction. Naturally, then, we introduce appropriately constructed Poincaré series to establish the existence of nontrivial logarithmic vector-valued modular forms. Of course, in the logarithmic case we treat here, the construction is of necessity more complicated, as compared with the classical (i.e., scalar-valued) theory or the normal vector-valued case. The principal new complexity resides in the

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[1] We actually use a modified Jordan canonical form. See Subsection 2.2 for details.

additional matrix factor B_ρ that must be inserted in the definition (cf. Definition 3.1) in order to achieve the desired formal transformation properties with respect to the representation ρ (Subsection 3.1, following the proof of Lemma 3.2).

It is useful to compare Definition 3.1 with the corresponding definition in the normal case [KM2, display (18), pp. 1352–1353]. Definition 3.1 actually defines a *matrix-valued* Poincaré series, each column of which is a logarithmic vector-valued modular form. In fact, the same is true of our definition in the normal case, except that in the latter case we define the Poincaré series as a *single column* of the matrix-valued Poincaré series. Matrix-valued modular forms are a very natural generalization of vector-valued modular forms. In addition to our Poincaré series, for example, the modular Wronskian [M1] is the determinant of a matrix-valued modular form. The passage from vector-valued modular forms to matrix-valued modular forms is analogous to passing from a modular linear differential equation of order p (loc. cit.) to an associated system of p linear differential equations of order 1.

Subsections 3.2 and 3.3 are devoted, respectively, to the proof of convergence of our matrix-valued Poincaré series and the determination of the general form of their logarithmic q-series expansions. Our proof of convergence requires the assumption that the eigenvalues of $\rho(T)$ have absolute value 1, so that the q-series $h_j(\tau)$ in (1) again have at worst real exponents. This condition will be implicitly assumed in the remainder of the Introduction. We also use a simple new estimate (Proposition 3.7) in the proof of convergence.

The remainder of the paper is devoted to applications. In Subsection 3.4 we give some consequences of an algebraic nature. We show (Theorem 3.13) that if ρ has dimension p, the graded space $\mathcal{H}(\rho)$ of all holomorphic vector-valued modular forms associated to ρ is a *free module of rank* p over the algebra $\mathcal{M}$ of (scalar) holomorphic modular forms on Γ. This generalizes the corresponding Theorem proved in [MM] in the normal case. In fact, the proof in [MM] was organized with just such a generalization in mind. The only additional input that is required is the existence of *some* nonzero holomorphic vector-valued modular form associated with ρ, and this is an easy consequence of the existence of a nonzero meromorphic Poincaré series. A consequence of the free module theorem is Theorem 3.14, which implies that if F is a logarithmic vector-valued modular form F then there is a *canonical* modular linear differential equation whose solution space is spanned by the component functions of F.

The occurrence of q-expansions of the form (1) is well known in rational and logarithmic conformal field theory. Indeed, much of the motivation for the present work originates from a need to develop a systematic theory of vector-valued modular forms wide enough in scope to cover possible applications in such field theories. By results in [DLM] and [M], the eigenvalues of $\rho(T)$ for the representations that arise in rational and logarithmic conformal field theory are indeed of absolute value 1 (in fact, they are roots of unity). Thus this assumption is natural from the perspective of conformal field theory. Our earlier results [KM1] on polynomial estimates for Fourier coefficients of entire vector-valued modular forms in the normal case have found a number of applications to the theory of rational vertex operator algebras, and we expect that the extension to the logarithmic case that we prove here will be useful in the study of C_2-cofinite vertex operator algebras, which constitute the algebraic underpinning of logarithmic field theory.

Other properties of logarithmic vector-valued modular forms are also of interest, from both a foundational and applied perspective. These include polynomial estimates for the Fourier coefficients, a Petersson pairing, generation of the space of cusp-forms by Poincaré series, existence of a natural boundary for the component functions, and explicit formulas (in terms of Bessel functions and Kloosterman sums) for the Fourier coefficients of Poincaré series. This program was carried through in the normal case in [KM2]. We expect that the more general logarithmic case will yield a similarly rich harvest, but one must expect more complications. For example, there are logarithmic vector-valued modular forms with nonconstant component functions that may be extended to the whole of the complex plane, so that the usual natural boundary result is false *per se*. Furthermore, our preliminary calculations indicate that the explicit formulas exhibit genuinely new features. We hope to return to these questions in the future.

2. Logarithmic vector-valued modular forms

2.1. Unrestricted vector-valued modular forms. We start with some notation that will be used throughout. The *modular group* is

$$\Gamma = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, a,b,c,d \in \mathbb{Z},\ ad-bc=1 \right\}.$$

It is generated by the matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

The complex upper half-plane is

$$\mathfrak{H} = \{\tau \in \mathbb{C} \mid \Im(\tau) > 0\}.$$

There is a standard left action $\Gamma \times \mathfrak{H} \to \mathfrak{H}$ given by Möbius transformations:

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \tau\right) \mapsto \frac{a\tau + b}{c\tau + d}.$$

Let $\mathfrak{F}$ be the space of holomorphic functions in $\mathfrak{H}$. There is a standard 1-cocycle $j : \Gamma \to \mathfrak{F}$ defined by

$$j(\gamma, \tau) = j(\gamma)(\tau) = c\tau + d, \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

In what follows, $\rho : \Gamma \to \mathrm{GL}(p, \mathbb{C})$ will always denote a p-dimensional matrix representation of Γ. An *unrestricted vector-valued modular form of weight k with respect to ρ* is a holomorphic function $F : \mathfrak{H} \to \mathbb{C}^p$ satisfying

$$\rho(\gamma) F(\tau) = F|_k \gamma(\tau), \qquad \gamma \in \Gamma,$$

where the right-hand side is the usual stroke operator

$$F|_k \gamma(\tau) = j(\gamma, \tau)^{-k} F(\gamma\tau). \tag{3}$$

We could take $F(\tau)$ to be *meromorphic* in $\mathfrak{H}$, but we will not consider that more general situation here. Choosing coordinates, we can rewrite (3) as

$$\rho(\gamma) \begin{pmatrix} f_1(\tau) \\ \vdots \\ f_p(\tau) \end{pmatrix} = \begin{pmatrix} f_1|_k \gamma(\tau) \\ \vdots \\ f_p|_k \gamma(\tau) \end{pmatrix} \tag{4}$$

with each $f_j(\tau) \in \mathfrak{F}$. We also refer to (F, ρ) as an *unrestricted* vector-valued modular form.

2.2. Logarithmic q-expansions. In this subsection we consider the q-expansions associated to unrestricted vector-valued modular forms. We make use of the polynomials defined for $k \geq 1$ by

$$\binom{x}{k} = \frac{x(x-1)\cdots(x-k+1)}{k!},$$

and with $\binom{x}{0} = 1$ and $\binom{x}{k} = 0$ for $k \leq -1$.

We consider a finite-dimensional subspace $W \subseteq \mathfrak{F}_k$ that is *invariant* under T, i.e. $f(\tau+1) \in W$ whenever $f(\tau) \in W$. We introduce the $m \times m$ matrix

$$J_{m,\lambda} = \begin{pmatrix} \lambda & & & \\ \lambda & \ddots & & \\ & \ddots & \ddots & \\ & & \lambda & \lambda \end{pmatrix}, \tag{5}$$

i.e. $J_{i,j} = \lambda$ for $i = j$ or $j + 1$ and $J_{i,j} = 0$ otherwise.

Lemma 2.1. *There is a basis of W with respect to which the matrix $\rho(T)$ representing T is in block diagonal form:*

$$\rho(T) = \begin{pmatrix} J_{m_1,\lambda_1} & & \\ & \ddots & \\ & & J_{m_t,\lambda_t} \end{pmatrix}. \tag{6}$$

Proof. The existence of such a representation is basically the theory of the Jordan canonical form. The *usual* Jordan canonical form is similar to the above, except that the subdiagonal of each block then consists of 1's rather than λ's. The λ's that appear in (6) are the eigenvalues of $\rho(T)$, and in particular they are nonzero on account of the invertibility of $\rho(T)$. Then it is easily checked that (6) is indeed similar to the usual Jordan canonical form, and the lemma follows. ∎

We refer to (6) as the *modified Jordan canonical form* of $\rho(T)$, and J_{m_i,λ_i} as a *modified Jordan block*. To a certain extent at least, Lemma 2.1 reduces the study of the functions in W to those associated to one of the Jordan blocks. In this case we have the following basic result.

Theorem 2.2. *Let $W \subseteq \mathfrak{F}_k$ be a T-invariant subspace of dimension m. Suppose that W has an ordered basis $(g_0(\tau), \dots, g_{m-1}(\tau))$ with respect to which the matrix $\rho(T)$ is a single modified Jordan block $J_{m,\lambda}$. Set $\lambda = e^{2\pi i\mu}$. Then there are m convergent q-expansions $h_t(\tau) = \sum_{n\in\mathbb{Z}} a_t(n)q^{n+\mu}, 0 \leq t \leq m-1$, such that*

$$g_j(\tau) = \sum_{t=0}^{j} \binom{\tau}{t} h_{j-t}(\tau), \qquad 0 \leq j \leq m-1. \tag{7}$$

The case $m = 1$ of the theorem is well known. We will need it for the proof of the general case, so we state it as

Lemma 2.3. *Let $\lambda = e^{2\pi i\mu}$, and suppose that $f(\tau) \in \mathfrak{F}$ satisfies $f(\tau+1) = \lambda f(\tau)$. Then $f(\tau)$ is represented by a convergent q-expansion*

$$f(\tau) = \sum_{n\in\mathbb{Z}} a(n)q^{n+\mu}. \ \blacksquare \tag{8}$$

Proof of Theorem 2.2. We have

$$g_j(\tau+1) = \lambda(g_j(\tau) + g_{j-1}(\tau)), \qquad 0 \leq j \leq m-1, \tag{9}$$

where $g_{-1}(\tau) = 0$. Set

$$h_j(\tau) = \sum_{t=0}^{j} (-1)^t \binom{\tau+t-1}{t} g_{j-t}(\tau), \qquad 0 \leq j \leq m-1.$$

These equalities can be displayed as a system of equations. Indeed,

$$B_m(\tau)\begin{pmatrix} g_0(\tau) \\ \vdots \\ g_{m-1}(\tau)\end{pmatrix} = \begin{pmatrix} h_0(\tau) \\ \vdots \\ h_{m-1}(\tau)\end{pmatrix}, \tag{10}$$

where $B_m(x)$ is the $m \times m$ lower triangular matrix with

$$B_m(x)_{ij} = (-1)^{i-j}\binom{x+i-j-1}{i-j}. \tag{11}$$

Then $B_m(x)$ is invertible and

$$B_m(x)^{-1}_{ij} = \binom{x}{i-j}. \tag{12}$$

We will show that each $h_j(\tau)$ has a convergent q-expansion. This being the case, (7) holds and the theorem will be proved. Using (9), we have

$$\begin{aligned} h_j(\tau+1) &= \lambda \sum_{t=0}^{j} (-1)^t \binom{\tau+t}{t}(g_{j-t}(\tau) + g_{j-t-1}(\tau)) \\ &= \lambda\Big\{\sum_{t=0}^{j}(-1)^t\Big(1+\frac{t}{\tau}\Big)\binom{\tau+t-1}{t} g_{j-t}(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t}{t} g_{j-t-1}(\tau)\Big\} \\ &= \lambda\Big\{h_j(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t-1}{t}\frac{t}{\tau} g_{j-t}(\tau) + \sum_{t=0}^{j}(-1)^t\binom{\tau+t}{t} g_{j-t-1}(\tau)\Big\}. \end{aligned}$$

But the sum of the second and third terms in the braces vanishes, being equal to

$$\begin{aligned} &\sum_{t=1}^{j}(-1)^t\binom{\tau+t-1}{t}\frac{t}{\tau}g_{j-t}(\tau) + \sum_{t=1}^{j}(-1)^{t-1}\binom{\tau+t-1}{t-1}g_{j-t}(\tau) \\ &\qquad = \sum_{t=1}^{j}(-1)^{t-1}g_{j-t}(\tau)\left\{\binom{\tau+t-1}{t-1} - \binom{\tau+t-1}{t}\frac{t}{\tau}\right\} = 0. \end{aligned}$$

Thus we have established the identity $h_j(\tau+1) = \lambda h_j(\tau)$. By Lemma 2.3, $h_j(\tau)$ is indeed represented by a q-expansion of the desired shape, and the proof of Theorem 2.2 is complete. ∎

We call (7) a *polynomial* q-expansion. The space of polynomials spanned by $\binom{x}{t}$, $0 \le t \le m-1$, is also spanned by the powers x^t, $0 \le t \le m-1$. Bearing in mind that $(2\pi i\tau)^t = (\log q)^t$, it follows that in Theorem 2.2 we can find a basis $\{g'_j(\tau)\}$ of W such that

$$g'_j(\tau) = \sum_{t=0}^{j} (\log q)^t h'_{j-t}(\tau) \tag{13}$$

with $h'_t(\tau) = \sum_{n\in\mathbb{Z}} a'_t(n) q^{n+\mu}$. We refer to (13) as a *logarithmic* q-expansion.

2.3. Logarithmic vector-valued modular forms. We say that a function $f(\tau)$ with a q-expansion (8) is *meromorphic at infinity* if

$$f(\tau) = \sum_{n+\Re(\mu)\geq n_0} a(n) q^{n+\mu}.$$

That is, the Fourier coefficients $a(n)$ *vanish* for exponents $n+\mu$ whose *real parts* are small enough. A polynomial (or logarithmic) q-expansion (7) is holomorphic at infinity if each of the associated ordinary q-expansions $h_{j-t}(\tau)$ is holomorphic at infinity. Similarly, $f(\tau)$ *vanishes* at ∞ if the Fourier coefficients $a(n)$ vanish for $n+\Re(\mu)\leq 0$; a polynomial q-expansion vanishes at ∞ if the associated ordinary q-expansions vanish at ∞. These conditions are independent of the chosen representations.

Now assume that $F(\tau) = (f_1(\tau),\dots,f_p(\tau))^t$ is an unrestricted vector-valued modular form of weight k with respect to ρ. It follows from (4) that the span W of the functions $f_j(\tau)$ is a right Γ-submodule of $\mathfrak{F}$ satisfying $f_j(\tau+1)\in W$. Choose a basis of W so that $\rho(T)$ is in modified Jordan canonical form. By Theorem 2.2 the basis of W consists of functions $g_j(\tau)$ which have polynomial q-expansions. We call $F(\tau)$, or (F,ρ), a *logarithmic meromorphic, holomorphic, or cuspidal vector-valued modular form* respectively if each of the functions $g_j(\tau)$ is meromorphic, is holomorphic, or vanishes at ∞, respectively.

From now on we generally drop the adjective "logarithmic" from this terminology, and say that $F(\tau)$ is *semisimple* if the component functions have ordinary q-expansions, i.e. they are free of logarithmic terms. This holds if, and only if, $\rho(T)$ is a semisimple operator.

Let $\mathcal{H}(k,\rho)$ be the space of holomorphic vector-valued modular forms of weight k with respect to ρ, with $\mathcal{H}(\rho) = \bigoplus_{k\in\mathbb{Z}} \mathcal{H}(k,\rho)$ the $\mathbb{Z}$-graded space of all holomorphic vector-valued modular forms.

2.4. Matrix-valued modular forms. *Matrix-valued modular forms* are a natural generalization of vector-valued modular forms. They arise naturally in several contexts, including (as we shall see) Poincaré series. Let $\rho : \Gamma \to \mathrm{GL}_p(\mathbb{C})$ be a representation, and let $\mathrm{Mat}_{p\times n}(\mathbb{C})$ be the space of $p\times n$ matrices. Let $\underline{k} = (k_1,\dots,k_n)\in\mathbb{Z}^n$. Consider a holomorphic map $A:\mathfrak{H}\to\mathrm{Mat}_{p\times n}(\mathbb{C})$ satisfying

$$\rho(\gamma)A(\tau) = A|_{\underline{k}}\gamma(\tau), \quad \gamma\in\Gamma,$$

where the right-hand side is defined as

$$A|_{\underline{k}}\gamma(\tau) = A(\gamma\tau)J_{\underline{k}}(\gamma,\tau)^{-1}$$

and J is the matrix automorphy factor

$$J_{\underline{k}}(\gamma,\tau) = \begin{pmatrix} j(\gamma,\tau)^{k_1} & \dots & 0 \\ \vdots & \ddots & 0 \\ 0 & 0 & j(\gamma,\tau)^{k_n} \end{pmatrix}. \tag{14}$$

This defines an unrestricted matrix-valued modular form of weight $\underline{k}$ with respect to ρ.

Let $p_j : \mathrm{Mat}_{p\times n}(\mathbb{C}) \to \mathrm{Mat}_{p\times 1}(\mathbb{C})$ be projection onto the jth column. Then $p_j \circ A$ is an unrestricted vector-valued modular form of weight k_j with respect to ρ, and we say that $A(\tau)$ is a meromorphic, holomorphic, or cuspidal vector-valued modular form of weight $\underline{k}$ if each $p_j \circ A$ is meromorphic, holomorphic or cuspidal, respectively. Thus, a matrix-valued modular form associated to ρ consists of n vector-valued modular forms of weight $k_1, \dots, k_n$, each associated to ρ with the component functions organized into the columns of a matrix.

2.5. The nontriviality condition. Let $\rho : \Gamma \to \mathrm{GL}_p(\mathbb{C})$ be a matrix representation. Because $S^2 = -I_2$ has order 2, we can choose a basis of the underlying representation space such that

$$\rho(S^2) = \begin{pmatrix} I_{p_1} & 0 \\ 0 & -I_{p_2} \end{pmatrix}.$$

Since S^2 is in the *center* of Γ, we see that $\rho(\Gamma)$ acts on the two eigenspaces of $\rho(S^2)$, and therefore the matrices

$$\rho(\gamma) = \begin{pmatrix} \rho_{11}(\gamma) & 0 \\ 0 & \rho_{22}(\gamma) \end{pmatrix}, \quad \gamma \in \Gamma, \tag{15}$$

are correspondingly in block diagonal form. It follows that if (F,ρ) is a vector-valued modular form of weight k, and if we write $F(\tau) = (F_1(\tau), F_2(\tau))$ with $F_i(\tau)$ having p_i components, $i = 1, 2$, then $F_i(\tau)$ is a vector-valued modular form of weight k with respect to the representation ρ_{ii}. More is true. The equality $\rho(S^2)F^t(\tau) = F^t|_k S^2(\tau)$ says that

$$(F_1(\tau), -F_2(\tau)) = (-1)^k (F_1(\tau), F_2(\tau)).$$

Assuming that $F \neq 0$, it follows that either $F_2 = 0$ and k is *even*, or else $F_1 = 0$ and k is *odd*. It follows that there are natural identifications

$$\mathcal{H}(k,\rho) = \begin{cases} \mathcal{H}(k,\rho_{11}), & k \text{ even}, \\ \mathcal{H}(k,\rho_{22}), & k \text{ odd}, \end{cases} \qquad \mathcal{H}(\rho) = \mathcal{H}(\rho_{11}) \oplus \mathcal{H}(\rho_{22}). \tag{16}$$

The upshot of this discussion is that for most considerations, we may assume that $\rho(S^2)$ is a *scalar*, i.e.

$$\rho(S^2) = \epsilon I_p, \qquad \epsilon = \pm 1. \tag{17}$$

In this case, if $F(\tau) \in \mathcal{H}(k,\rho)$ is nonzero then

$$\epsilon = (-1)^k. \tag{18}$$

This is the *nontriviality condition* in weight k.

In the case of semisimple vector-valued modular forms, it is proved in [KM1] and [M1] that there is an integer k_0 such that $\mathcal{H}(k,\rho) = 0$ for $k < k_0$. The proof in [M1] applies to the general (logarithmic) case. Thus if ρ satisfies (17) then

$$\mathcal{H}(\rho) = \bigoplus_{k \geq k_0} \mathcal{H}(k_0 + 2k). \tag{19}$$

3. Matrix-valued Poincaré series. We develop a theory of Poincaré series in order to prove *existence* of nontrivial logarithmic vector-valued modular forms.

3.1. Definition and formal properties. Fix a representation $\rho : \Gamma \to \mathrm{GL}(p,\mathbb{C})$. We may, and shall, assume that $\rho(T)$ is in modified Jordan canonical form with t blocks, the rth block being the $m_r \times m_r$ matrix J_{m_r,λ_r} (see (5), (6)) and with $\lambda_r = e^{2\pi i \mu_r}$ the associated eigenvalue of $\rho(T)$.

We will need several more block diagonal matrices. The matrices in question will all have t blocks, the rth block having the same size as the rth block of $\rho(T)$. Set

$$B_\rho(x) = \mathrm{diag}(B_{m_1}(x), \ldots, B_{m_t}(x)) \tag{20}$$

where $B_m(x)$ is given in (11). For $(z_1, \ldots, z_t) \in \mathbb{C}^t$ let

$$\Lambda_\rho(z_1, \ldots, z_t) = \mathrm{diag}(z_1 I_{m_1}, \ldots, z_t I_{m_t}). \tag{21}$$

Definition 3.1. Let $\underline{\nu} = (\nu_1, \ldots, \nu_t) \in \mathbb{Z}^t$, $\underline{k} = (k_1, \ldots, k_p) \in \mathbb{Z}^p$. The *Poincaré series* is defined to be

$$\begin{aligned} &P_{\underline{k}}(\underline{\nu}, \tau) \\ &\quad = \frac{1}{2} \sum_M \rho(M)^{-1} \Lambda_\rho(\ldots, e^{2\pi i(\nu_r + \mu_r) M\tau}, \ldots) B_\rho(M\tau)^{-1} J_{\underline{k}}(M,\tau)^{-1}, \end{aligned} \tag{22}$$

where M ranges over a set of representatives of the coset space $\langle T \rangle \backslash \Gamma$ and $J_{\underline{k}}(M,\tau)$ is the matrix automorphy factor (14).

Notice that $B_\rho(\tau)^{-1}$ should be considered as an additional matrix automorphy factor. At least formally, $P_{\underline{k}}(\underline{\nu}, \tau)$ is a $p \times p$ *matrix-valued* function.

We interpolate a lemma.

LEMMA 3.2. *The matrices $\rho(T)$, $\Lambda_\rho(z_1,\dots,z_t)$ and $B_\rho(\tau)$ ($\tau\in\mathfrak{H}$) commute with each other, and satisfy*

$$\rho(T)B_\rho(\tau)^{-1}=B_\rho(\tau+1)^{-1}\Lambda_\rho(\lambda_1,\dots,\lambda_t).$$

Proof. All of the matrices in question are block diagonal with corresponding blocks of the same size. So it suffices to show that for a given m and λ, the $m\times m$ matrices $J_{m,\lambda}$, zI_m and $B_m(\tau)$ commute and satisfy

$$J_{m,\lambda}B_m(\tau)^{-1}=\lambda B_m(\tau+1)^{-1}. \tag{23}$$

The $m\times m$ matrices all have the following properties: they are lower triangular and the (i,j)-entry *depends only on* $i-j$. It is easy to check that any two such matrices commute.

As for (23), let $G(\tau)$ and $H(\tau)$ denote the column vectors of functions that occur in (10), so that we can write the equation as

$$B_m(\tau)G(\tau)=H(\tau).$$

By definition of $G(\tau)$ and $H(\tau)$ (cf. Theorem 2.2) we have

$$J_{m,\lambda}G(\tau)=G(\tau+1),\qquad H(\tau+1)=\lambda H(\tau).$$

Therefore,

$$\begin{aligned}J_{m,\lambda}B_m(\tau)^{-1}H(\tau)&=J_{m,\lambda}G(\tau)=G(\tau+1)\\&=B_m(\tau+1)^{-1}H(\tau+1)=\lambda B_m(\tau+1)^{-1}H(\tau).\end{aligned}$$

Since the components of $H(\tau)$ are linearly independent, (23) follows. ∎

Now make the replacement $M\mapsto TM$ in a summand of (22). Using Lemma 3.2 we calculate that the summand maps to

$$\begin{aligned}&\rho(TM)^{-1}\Lambda_\rho(\dots,e^{2\pi i(\nu_r+\mu_r)TM\tau},\dots)B_\rho(TM\tau)^{-1}J_{\underline{k}}(TM,\tau)^{-1}\\&\quad=\rho(M)^{-1}\rho(T)^{-1}\Lambda_\rho(\dots,e^{2\pi i(\nu_r+\mu_r)M\tau},\dots)\Lambda_\rho(\lambda_1,\dots,\lambda_t)\\&\qquad\times B_\rho(M\tau+1)^{-1}J_{\underline{k}}(M,\tau)^{-1}\\&\quad=\rho(M)^{-1}\Lambda_\rho(\dots,e^{2\pi i(\nu_r+\mu_r)M\tau},\dots)B_\rho(M\tau)^{-1}J_{\underline{k}}(M,\tau)^{-1}.\end{aligned}$$

This calculation confirms that the sum defining $P_{\underline{k}}(\underline{\nu},\tau)$ is *independent* of the choice of coset representatives. We also note that

$$\begin{aligned}P_{\underline{k}}|_{\underline{k}}\gamma(\tau)&=\frac12\sum_M\rho(M)^{-1}\Lambda_\rho(\dots,e^{2\pi i(\nu_r+\mu_r)M\gamma\tau},\dots)\\&\qquad\times B_\rho(M\gamma\tau)^{-1}J_{\underline{k}}(M,\gamma\tau)^{-1}J_{\underline{k}}(\gamma,\tau)^{-1}\\&=\frac12\rho(\gamma)\sum_M\rho(M\gamma)^{-1}\Lambda_\rho(\dots,e^{2\pi i(\nu_r+\mu_r)M\gamma\tau},\dots)\\&\qquad\times B_\rho(M\gamma\tau)^{-1}J_{\underline{k}}(M\gamma,\tau)^{-1}\\&=\rho(\gamma)P_{\underline{k}}(\underline{\nu},\tau)\end{aligned}$$

where we used independence of coset representatives for the last equality. This confirms that each $P_{\underline{k}}(\underline{\nu},\tau)$ is, at least formally, a matrix-valued modular form of weight $\underline{k}$ with respect to ρ.

3.2. Convergence of $P_{\underline{k}}(\underline{\nu},\tau)$. From now on we assume that the constants μ_r are *real*, i.e. the eigenvalues λ of $\rho(T)$ satisfy $|\lambda|=1$. With this assumption, we show in this subsection that the Poincaré series $P_{\underline{k}}(\underline{\nu},\tau)$ is an unrestricted matrix-valued modular form for $\underline{k}\gg 0$. After the results of the previous subsection, this amounts to the fact that $P_{\underline{k}}(\underline{\nu},\tau)$ is holomorphic in $\mathfrak{H}$ as long as the component weights k_j of $\underline{k}$ are large enough.

Define the vertical strip

$$\mathcal{S}=\{\tau\in\mathfrak{H}\mid|\Re(\tau)|\le 1/2,\ \Im(\tau)\ge\sqrt{3}/2\}.$$

Notice that $\mathcal{S}$ contains the closure of the standard fundamental region for Γ. We will prove

THEOREM 3.3. *$P_{\underline{k}}(\underline{\nu},\tau)$ converges absolutely-uniformly in $\mathcal{S}$ for $\underline{k}\gg 0$.*

It is a consequence of Theorem 3.3 and the formal transformation law for $P_{\underline{k}}(\underline{\nu},\tau)$ (cf. Subsection 3.1) that $P_{\underline{k}}(\underline{\nu},\tau)$ is holomorphic throughout $\mathfrak{H}$.

We split off the two terms of the Poincaré series corresponding to $\pm I$, so that

$$\begin{aligned}P_{\underline{k}}(\underline{\nu},\tau)&=\Lambda_\rho(\ldots,e^{2\pi i(\nu_r+\mu_r)\tau},\ldots)B_\rho(\tau)^{-1}\\&\quad+\frac{1}{2}\sum_{M\in\mathcal{M}^*}\rho(M)^{-1}\Lambda_\rho(\ldots,e^{2\pi i(\nu_r+\mu_r)M\tau},\ldots)\\&\quad\times B_\rho(M\tau)^{-1}J_{\underline{k}}(M,\tau)^{-1}.\end{aligned}\tag{24}$$

Here, $\mathcal{M}^*$ is a set of representatives of the cosets $\langle T\rangle\backslash\Gamma$ distinct from $\pm\langle T\rangle$, and $\pm I$ are the representative of $\pm\langle T\rangle$. The matrices $M\in\mathcal{M}^*$ have bottom row (c,d) with $c\ne 0$. The entries of $B_\rho(\tau)^{-1}$ are polynomials in τ (cf. (12), (20)), so the first term in (24) is holomorphic.

LEMMA 3.4. *We can choose coset representatives $M\in\mathcal{M}^*$ so that $|M\tau|$ is* uniformly bounded *in $\mathcal{S}$. That is, there is a constant K such that $|M\tau|\le K$ for all $\tau\in\mathcal{S}$ and all $M\in\mathcal{M}^*$.*

Proof. Suppose that $\gamma=\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in\Gamma$ with $c\ne 0$, and consider the γ-image $\gamma(\mathcal{S})$ of the strip. Apart from two exceptional cases (when $c=\pm1, d=\mp1$), the γ-image of the circle $|\tau|=1$ is a circle with center b/d and radius at most 1. Moreover $\gamma(\infty)=a/c$ lies on or inside the boundary of this circle. From this it is easy to see that

$$\gamma(\mathcal{S})\subseteq\{\tau\in\mathfrak{H}\mid|\Re(\tau)-a/c|\le 1\}.$$

Replacing γ by $T^l\gamma$ for suitable l, the corresponding value of $|a/c|$ can be made less than 1, so that

$$\gamma(\mathcal{S}) \subseteq \{\tau \in \mathfrak{H} \mid |\Re(\tau)| \le 2\}.$$

This also holds in the exceptional cases. Therefore we may, and shall, choose a set of coset representatives $\mathcal{M}^*$ so that $|\Re(M\tau)|$ is uniformly bounded in $\mathcal{S}$ for $M \in \mathcal{M}^*$.

On the other hand, it is easy to see that we always have $|c\tau + d|^2 \ge c^2\Im(\tau)^2$. Since $|\Im(\tau)| \ge \sqrt{3}/2$ for $\tau \in \mathcal{S}$, it follows that

$$\Im(\gamma\tau) = \frac{\Im(\tau)}{|c\tau + d|^2} \le \frac{\Im(\tau)}{c^2\Im(\tau)^2} = \frac{1}{c^2\Im(\tau)} \le \frac{1}{\Im(\tau)} \le \frac{2}{\sqrt{3}},$$

so that $|\Im(\gamma\tau)|$ is uniformly bounded in $\mathcal{S}$ for $c \ne 0$. Therefore, with our earlier choice of $\mathcal{M}^*$, it follows that $|M\tau|$ is also uniformly bounded in $\mathcal{S}$. This completes the proof of the lemma. ∎

Until further notice, we assume that $\mathcal{M}^*$ satisfies the conclusion of Lemma 3.4.

Corollary 3.5. *The entries of the matrices $\Lambda_\rho(\ldots, e^{2\pi i(\nu_r+\mu_r)M\tau}, \ldots)$ and $B_\rho(M\tau)^{-1}$ are uniformly bounded in $\mathcal{S}$ for $M \in \mathcal{M}^*$.*

Proof. For $\Lambda_\rho(\ldots, e^{2\pi i(\nu_r+\mu_r)M\tau}, \ldots)$ the assertion is an immediate consequence of Lemma 3.4. As for $B_\rho(M\tau)^{-1}$, we have already pointed out that it has polynomial entries, indeed the (i,j)-entry is $\binom{M\tau}{i-j}$. Uniform boundedness in this case is then also a consequence of Lemma 3.4. ∎

Next we state a modification of [E, p. 169, display (4)] which we call *Eichler's canonical form* for elements of Γ.

Lemma 3.6. *Let $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$. Then the following hold:*

(i) *γ has a unique representation*

$$\gamma = \pm(ST^{l_{\nu+1}})(ST^{l_\nu}) \cdots (ST^{l_1})(ST^{l_0}) \tag{25}$$

such that $(-1)^{j-1}l_j > 0$ for $1 \le j \le \nu$ and $(-1)^\nu l_{\nu+1} \ge 0$. Thus l_1 is positive, the l_j alternate in sign for $j \ge 1$, $l_{\nu+1}$ may be 0, and there is no condition on l_0.

(ii) *If γ is normalized so that $|a/c| < 1$ (as in Lemma 3.4), then $l_{\nu+1} \ne 0$. With γ fixed for now, we further set*

$$P_0 = ST^{l_0},$$
$$P_{j+1} = (ST^{l_j})P_j, \qquad 0 \le j \le \nu,$$
$$P_j = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}, \quad 0 \le j \le \nu + 1, \tag{26}$$

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{27}$$

PROPOSITION 3.7.

(i) *If* $l_{\nu+1} \neq 0$ *in* (25), *then*

$$\begin{aligned} |l_0 l_1 \cdots l_{\nu+1}| &\leq |d| && \text{if } l_0 < 0, \\ |l_1 \cdots l_{\nu+1}| &\leq |d-c| && \text{if } l_0 = 0, \\ |l_0 l_1 \cdots l_{\nu+1}| &\leq |c| + |d| && \text{if } l_0 > 0. \end{aligned} \tag{28}$$

(ii) *If* $l_{\nu+1} = 0$, *then*

$$\begin{aligned} |l_0 l_1 \cdots l_{\nu-1}| &\leq |d| && \text{if } l_0 < 0, \\ |l_1 \cdots l_{\nu-1}| &\leq |d-c| && \text{if } l_0 = 0, \\ |l_0 l_1 \cdots l_{\nu-1}| &\leq |c| + |d| && \text{if } l_0 > 0. \end{aligned} \tag{29}$$

Proof.

CASE A: $l_0 < 0$. We will prove by induction on $j \geq 0$ that

$$\begin{aligned} &\text{(i)} \quad |l_0 l_1 \cdots l_j| \leq |d_j|, \\ &\text{(ii)} \quad (-1)^j b_j d_j \geq 0. \end{aligned} \tag{30}$$

Once this is established, the case $j = \nu + 1$ of (30)(i) proves (28) in Case A. Now

$$P_0 = \begin{pmatrix} 0 & -1 \\ 1 & l_0 \end{pmatrix},$$

and the case $j = 0$ is clear. For the inductive step, we have

$$P_{j+1} = \begin{pmatrix} 0 & -1 \\ 1 & l_{j+1} \end{pmatrix} \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} = \begin{pmatrix} -c_j & -d_j \\ a_j + l_{j+1} c_j & b_j + l_{j+1} d_j \end{pmatrix}. \tag{31}$$

Thus $(-1)^{j+1} b_{j+1} d_{j+1} = (-1)^j b_j d_j + (-1)^j l_{j+1} d_j^2 \geq 0$ where the last inequality uses induction and the inequality stated in Lemma 3.6. So (30)(ii) holds.

Back to (30)(i), note that because $(-1)^j b_j d_j$ and $(-1)^j l_{j+1} d_j^2$ are both nonnegative, b_j and $l_{j+1} d_j$ have *the same sign*. Therefore, using induction again, we have $|l_0 l_1 \cdots l_{j+1}| \leq |d_j l_{j+1}| \leq |b_j| + |l_{j+1} d_j| = |b_j + l_{j+1} d_j| = |d_{j+1}|$. This completes the proof of Case A.

CASE B: $l_0 = 0$. Notice in this case that $\gamma T^{-1} = (ST^{l_v}) \cdots (ST^{l_1})(ST^{-1})$, which falls into Case A with $l_0 = -1$. Since

$$\gamma T^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b - a \\ c & d - c \end{pmatrix}$$

it follows from Case A that $|l_1 \cdots l_v| \leq |d - c|$, as was to be proved.

Case C: $l_0 > 0$. We will prove by induction on j that

$$(32)\qquad \begin{aligned} &\text{(i)} \quad |l_0 l_1 \cdots l_j| \le |c_j| + |d_j|,\ j \ge 0, \\ &\text{(ii)} \quad (-1)^j b_j d_j, (-1)^j a_j c_j \ge 0,\ j \ge 1. \end{aligned}$$

Once again, the case $j = v$ of (32)(i) proves (28) in Case C, and this will complete the proof of the proposition. Now

$$P_0 = \begin{pmatrix} 0 & -1 \\ 1 & l_0 \end{pmatrix}, \qquad P_1 = \begin{pmatrix} -1 & -l_0 \\ l_1 & l_0 l_1 - 1 \end{pmatrix}.$$

So when $j = 0$, (32)(i) is clearly true, and because $l_0, l_1 > 0$ we also have

$$-a_1 c_1 = l_1 > 0, \qquad -b_1 d_1 = l_0(l_0 l_1 - 1) \ge 0.$$

So (32)(ii) holds for $j = 1$. As for the inductive step, P_{j+1} is as in (31), and the proof that $(-1)^j b_j d_j \ge 0$ is the same as in Case A. Similarly $(-1)^{j+1} a_{j+1} c_{j+1} = (-1)^j c_j a_j + (-1)^j l_{j+1} c_j^2 \ge 0$ is the sum of two nonnegative terms and hence is itself nonnegative, so (32)(ii) holds. Finally, by an argument similar to that used in Case A, we have $|l_0 \cdots l_{j+1}| \le |c_j + d_j|\,|l_{j+1}| < |l_{j+1} c_j| + |l_{j+1} d_j| + |a_j| + |b_j| = |a_j + l_{j+1} c_j| + |b_j + l_{j+1} d_j| = |c_{j+1}| + |d_{j+1}|$. Part (i) of the proposition is proved. The proof goes through for part (ii) without modification. ∎

The *Eichler length* of γ is given by

$$(33)\qquad L(\gamma) = \begin{cases} 2\nu + 4, & l_0 \ne 0, \\ 2\nu + 3, & l_0 = 0, \end{cases}$$

provided $l_{\nu+1} \ne 0$, and by

$$(34)\qquad L(\gamma) = \begin{cases} 2\nu + 1, & l_0 \ne 0, \\ 2\nu, & l_0 = 0, \end{cases}$$

if $l_{\nu+1} = 0$. By Lamé's theorem we have the estimate

$$(35)\qquad L(\gamma) \le K(\log |c| + 1)$$

with a positive constant K independent of γ.

The *norm* $\|\rho(\gamma)\|$, defined to be $\max_{i,j} |\rho(\gamma)_{ij}|$, satisfies

$$(36)\qquad \|\rho(\gamma)\| \le \|\rho(S)\|^{\nu+2} \prod_{j=0}^{\nu+1} \|\rho(T^{l_j})\|.$$

Lemma 3.8. *Let s be the maximum of the sizes m_j of the Jordan blocks J_{m_j,λ_j} of $\rho(T)$ as in* (5), (6). *There is a constant C_s depending only on s such that for $l \ne 0$,*

$$(37)\qquad \|\rho(T^l)\| \le C_s |l|^{s-1}.$$

Proof. We have

$$J_{m,\lambda}^l = \lambda^l J_{m,1}^l = \lambda^l (I_m + N)^l = \lambda^l \sum_{i\geq 0} \binom{l}{i} N^i$$

where N is the nilpotent $m \times m$ matrix with each $(i, i-1)$-entry equal to 1 ($i \geq 2$), and all other entries zero. Note that $N^m = 0$ and the entries of N^i for $1 \leq i < m$ are 1 on the ith subdiagonal and zero elsewhere. Bearing in mind that $|\lambda| = 1$, it follows that $\|J_{m,\lambda}^l\|$ is majorized by the maximum of the binomial coefficients $\binom{l}{i}$ over the range $0 \leq i \leq m-1$. Since $\binom{l}{i}$ is a polynomial in l of degree i, we certainly have $\|J_{m,\lambda}^l\| \leq C_m |l|^{m-1}$ for a universal constant C_m, and since this applies to each Jordan block of $\rho(T^l)$, the lemma follows immediately. ∎

Corollary 3.9. *There are universal constants K_3, K_4 such that*

$$\|\rho(\gamma)\| \leq \begin{cases} K_3(c^2+d^2)^{K_4}, & l_{\nu+1} \neq 0, \\ K_3(c^2+d^2)^{K_4} |l_\nu|^{s-1}, & l_{\nu+1} = 0. \end{cases} \tag{38}$$

Moreover the same *estimates hold for* $\|\rho(\gamma^{-1})\|$.

Proof. First assume that $l_{\nu+1} \neq 0$. From Lemma 3.8 and (36) we obtain

$$\|\rho(\gamma)\| \leq \begin{cases} K_1^{\nu+1} \prod_{j=0}^{\nu+1} |l_j|^{s-1}, & l_0 \neq 0, \\ K_1^{\nu+1} \prod_{j=1}^{\nu+1} |l_j|^{s-1}, & l_0 = 0, \end{cases}$$

for a constant K_1 depending only on ρ. Now use (33), (35) and Proposition 3.7 to see that

$$\|\rho(\gamma)\| \leq e^{(\log K_1) K_2 \log(|c|+1)} (|c| + |d|) \leq K_3 (c^2 + d^2)^{K_4}.$$

Concerning the second assertion of the corollary, since

$$\gamma^{-1} = (T^{-l_0} S)(T^{-l_1} S) \cdots (T^{-l_{\nu+1}} S),$$

we have

$$\|\rho(\gamma^{-1})\| \leq \|\rho(S)\|^{\nu+2} \prod_{j=0}^{\nu+1} \|\rho(T)^{-l_j}\|,$$

and (37) then holds by Lemma 3.8. The rest of the proof is identical to the previous case, so that we indeed obtain estimate (38) for γ^{-1} as well as γ. The second case, in which $l_{\nu+1} = 0$, is entirely analogous. ∎

Proof of Theorem 3.3. Let $P_{\underline{k}}^*(\underline{\nu}, \tau)$ denote the infinite sum in (24). Since $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathcal{M}^*$ is normalized by the condition $|a/c| < 1$ (see the proof of Lemma 3.4), it follows from Lemma 3.6(ii) that $l_{\nu+1} \neq 0$. Thus, we can apply Corollary 3.5 and the first case of Corollary 3.9 to find that

$$\begin{aligned}
&\|P^*_{\underline{k}}(\underline{\nu},\tau)\| \\
&\quad \le \sum_{M\in\mathcal{M}^*} \|\rho(M)^{-1}\|\,\|\Lambda_\rho(\ldots,e^{2\pi i(\nu_r+\mu_r)M\tau},\ldots)\|\,\|B_\rho(M\tau)^{-1}\|\,\|J_{\underline{k}}(M,\tau)^{-1}\| \\
&\quad \le K_5 \sum_{(c,d)=1} (c^2+d^2)^{K_4}\|J_{\underline{k}}(M,\tau)^{-1}\|,
\end{aligned}$$

with constants K_4, K_5 that depend only on ρ. We also know [KM1, display (13)] that

$$c^2+d^2 \le K_6|c\tau+d|^2 \tag{39}$$

for a universal constant K_6. Because of the nature of the matrix automorphy factor $J_{\underline{k}}$ (see (14)), it follows from the previous two displays that if the *minimum* of the weights k_i in $\underline{k}=(k_1,\ldots,k_p)$ is *large enough*, then

$$\|P^*_{\underline{k}}(\underline{\nu},\tau)\| \le K_7 \sum_{(c,d)=1} (c\tau+d)^{-k}$$

with $k>2$. It is well known that this series converges absolutely-uniformly in $\mathcal{S}$, so the same is true for $P^*_{\underline{k}}(\underline{\nu},\tau)$. This completes the proof. ∎

3.3. q-Expansions of the component functions. We now assume (cf. the discussion in Subsection 2.5) that (17) holds. Consider the substitution $M\mapsto -M$ in the expression for $P_{\underline{k}}(\underline{\nu},\tau)$. Because the sum is independent of the order of the terms, (17) implies that

$$P_{\underline{k}}(\underline{\nu},\tau) = P_{\underline{k}}(\underline{\nu},\tau)\Lambda_\rho(\epsilon(-1)^{k_1},\ldots,\epsilon(-1)^{k_t}).$$

If the nontriviality condition (18) holds in weight k_j then $\epsilon(-1)^{k_j}=1$ and the jth column of $P_{\underline{k}}(\underline{\nu},\tau)$ is unchanged. If the nontriviality condition does *not* hold then the jth column is zero and as such it too is unchanged. We conclude that

$$\begin{aligned}
&P_{\underline{k}}(\underline{\nu},\tau) \\
&\quad = \sum_{M} \rho(M)^{-1}\Lambda_\rho(\ldots,e^{2\pi i(\nu_r+\mu_r)M\tau},\ldots)B_\rho(M\tau)^{-1}J_{\underline{k}}(M,\tau)^{-1},
\end{aligned} \tag{40}$$

where the matrices M now range over an arbitrary set of coset representatives of $\pm\langle T\rangle\backslash\Gamma$.

We will show that $P_{\underline{k}}(\underline{\nu},\tau)$ is a meromorphic vector-valued modular form for $\underline{k}\gg 0$. We have already proved that it is an unrestricted vector-valued modular form, so that the component functions that occur in the matrix representation

$$P_{\underline{k}}(\underline{\nu},\tau) = (P_{mn}(\tau))$$

have polynomial q-expansions (7) by Theorem 2.2. It remains to show that these q-expansions are meromorphic at infinity if the weights are large enough. Note that because of our assumption that the constants μ_r are real, the q-expansions in question have only real powers of q.

To describe $P_{mn}(\tau)$, let us assume that the nontriviality condition holds in weight k_n. Let r be such that the mth row of $P(\tau)$ falls into the rth Jordan block. Setting $M_r = m_1 + \cdots + m_r$, this means that

$$M_{r-1} < m \le M_r.$$

Now take I as the coset representative of $\pm\langle T\rangle\backslash\Gamma$ and set $\mathcal{M} = \mathcal{M}^* \cup \{I\}$. From (40) we have

$$\begin{aligned} &P(\tau)_{mn} \\ &\quad = \sum_{l=1}^{p} \sum_{M\in\mathcal{M}} \Lambda_\rho(\ldots, e^{2\pi i(\nu_s+\mu_s)M\tau}, \ldots)_{mm} \rho(M^{-1})_{ml} B_\rho(M\tau)^{-1}_{ln} j(M,\tau)^{-k_n} \\ &\quad = e^{2\pi i(\nu_r+\mu_r)\tau} B_\rho(\tau)^{-1}_{mn} \\ &\qquad + \sum_{l=1}^{p} \Big\{ \sum_{M\in\mathcal{M}^*} e^{2\pi i(\nu_r+\mu_r)M\tau} \rho(M^{-1})_{ml} B_\rho(M\tau)^{-1}_{ln} j(M,\tau)^{-k_n} \Big\}. \end{aligned}$$

Because of absolute-uniform convergence in the strip $\mathcal{S}$, $\lim_{\tau\to i\infty}$ may be taken inside the summations. By Lemma 3.4 and Corollary 3.5 we find that

$$\lim_{\tau\to i\infty} \{P(\tau)_{mn} - e^{2\pi i(\nu_r+\mu_r)\tau} B_\rho(\tau)^{-1}_{mn}\} = 0$$

$(k_n > 0)$. It follows that the polynomial q-expansion of

$$P(\tau)_{mn} - e^{2\pi i(\nu_r+\mu_r)} B_\rho(\tau)^{-1}$$

can have only *positive* powers of q, so that

$$(41)\qquad \begin{aligned} P(\tau)_{mn} &= d_{mn}\binom{\tau}{m-n} q^{\nu_r+\mu_r} + \sum_{u=0}^{m-M_{r-1}-1} \binom{\tau}{u} \sum_{l+\mu_r>0} \hat{a}_{unr}(l) q^{l+\mu_r}, \\ d_{mn} &= \begin{cases} 1 & \text{if } M_{r-1} < n \le m, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Notice that the diagonal terms have polynomial q-expansions

$$P(\tau)_{mm} = q^{\nu_r+\mu_r} + \text{regular terms}.$$

In particular, if $\nu_r + \mu_r < 0$ then there is a pole at $i\infty$, and if $\nu_r + \mu_r = 0$ then the constant term is 1. So in both cases $P_{mm}(\tau)$ is *nonzero*. We have established the following.

THEOREM 3.10. *Suppose that ρ satisfies $\rho(S^2) = \epsilon I_p$. Then $P_{\underline{k}}(\underline{\nu}, \tau)$ is a meromorphic matrix-valued modular form of weight $\underline{k}$ for all $\underline{k} \gg 0$. If* *the nontriviality condition holds in all weights k_n then one of the following holds:*

(a) $\nu_r+\mu_r>0$ *for all* r *and* $P_{\underline{k}}(\underline{\nu},\tau)$ *is a cuspidal matrix-valued modular form, possibly zero.*
(b) $\nu_r+\mu_r\ge 0$ *for all* $r, \nu_r+\mu_r=0$ *for* some r, *and* $P_{\underline{k}}(\underline{\nu},\tau)$ *is a* nonzero, *holomorphic matrix-valued modular form of weight* $\underline{k}$.
(c) $\nu_r+\mu_r<0$ *for* some r *and* $P_{\underline{k}}(\underline{\nu},\tau)$ *is a* nonzero *meromorphic matrix-valued modular form of weight* $\underline{k}$.

If the nontriviality condition is not *satisfied in weight* k_n, *then the* n*th column of* $P_{\underline{k}}(\underline{\nu},\tau)$ *vanishes identically.*

3.4. Further consequences. We record a consequence of the nature of the q-expansions (41).

THEOREM 3.11. *Suppose that* $\rho(S^2)=\epsilon I_p$. *For large enough weight* k *there is* $F(\tau)\in\mathcal{H}(k,\rho)$ *such that the component functions of* $F(\tau)$ *are* linearly independent.

Proof. Let $\underline{k}=(k,\dots,k)$ have constant weight k, and choose k large enough so that $P(\tau)=P_{\underline{k}}(\underline{\nu},\tau)$ is holomorphic throughout $\mathfrak{H}$. This holds for any choice of $\underline{\nu}$. We may, and shall, also assume that the nontriviality condition in weight k is satisfied.

Now choose $\underline{\nu}$ so that the exponents $\nu_r+\mu_r$ are negative and pairwise distinct for each r in the range $1\le r\le t$. Consider the resulting t vector-valued modular forms $p_{M_r}\circ P(\tau)=P(\tau)_{M_r}$ where p_{M_r} is projection onto the M_rth column (cf. Subsection 2.4). By (41) we see that the component functions of $P(\tau)_{M_r}$ are holomorphic outside of the rth block, and in the rth block they have q-expansions $q^{\nu_r+\mu_r}\binom{\tau}{u}+\cdots$, $0\le u\le m_r-1$. Clearly then, these functions are linearly independent.

Consider the vector-valued modular form

$$F(\tau)=\Delta(\tau)^v\sum_{r=1}^{t}P(\tau)_{M_r},$$

with v an integer satisfying $v+\nu_r+\mu_r\ge 0,\ 1\le r\le t$. Since the $\nu_r+\mu_r$ are pairwise distinct, it follows from the discussion of the preceding paragraph that $\sum_r P(\tau)_{M_r}$ has linearly independent component functions. The choice of v ensures that $F(\tau)$ is holomorphic at $i\infty$ and it also has linearly independent component functions. Since $F(\tau)$ is a vector-valued modular form associated with the same representation ρ, the theorem follows. ∎

As discussed in Subsection 2.5, ρ is equivalent to the direct sum $\rho_1\oplus\rho_{-1}$ of a pair of representations ρ_ϵ of Γ with the property that $\rho_\epsilon(S^2)=\epsilon I$,

$\epsilon = \pm 1$. From (16)–(19) it follows that there is a natural identification

$$\mathcal{H}(\rho) = \mathcal{H}(\rho_1) \oplus \mathcal{H}(\rho_{-1}) \tag{42}$$

with

$$\mathcal{H}(\rho_1) = \bigoplus_{k\,\mathrm{even}} \mathcal{H}(k, \rho_1), \qquad \mathcal{H}(\rho_{-1}) = \bigoplus_{k\,\mathrm{odd}} \mathcal{H}(k, \rho_{-1}).$$

In other words, $\mathcal{H}(\rho_1)$ and $\mathcal{H}(\rho_{-1})$ are the even and odd parts respectively of $\mathcal{H}(\rho)$.

Corollary 3.12. *For any representation $\rho : \Gamma \to \mathrm{GL}_p(\mathbb{C})$, there is a* nonzero *holomorphic vector-valued modular form $F(\tau) \in \mathcal{H}(k, \rho)$ for* large enough *weight k.*

Proof. If $\rho(S^2) = \pm I_p$ then the corollary follows immediately from Theorem 3.11. The general result is then a consequence of the preceding comments. ∎

Let $\mathcal{M} = \bigoplus_{k \geq 0} \mathcal{M}_k = \mathbb{C}[Q, R]$ be the weighted polynomial algebra of holomorphic modular forms of level 1 on Γ, where $Q = E_4(\tau)$, $R = E_6(\tau)$. As in [M1],

$$\mathcal{R} = \mathcal{M}[d]$$

is the ring of differential operators obtained by adjoining to $\mathcal{M}$ an element d satisfying

$$df - fd = D(f), \qquad f \in \mathcal{M},$$

where D is the modular derivative defined by

$$Df = D_k f = (\theta + kP)f \qquad (f \in \mathcal{M}_k). \tag{43}$$

Here, $\theta = q\,d/dq$ and $P = -1/12 + 2\sum_{n\geq 1} \sigma_1(n) q^n$ is the weight 2 quasi-modular Eisenstein series, normalized as indicated.

The ring $\mathcal{R}$ is a $2\mathbb{Z}$-graded algebra (d has degree 2), and $\mathcal{H}(\rho)$ is a $\mathbb{Z}$-graded $\mathcal{R}$-module in which $f \in \mathcal{M}$ acts as a multiplication operator and d acts on $F \in \mathcal{H}(\rho)$ via its action on components of F given by (43). In particular, it follows that $\mathcal{R}$ operates on the even and odd parts of $\mathcal{H}(\rho)$, so that the identification (42) is one of $\mathcal{R}$-modules.

Theorem 3.13 (Free module theorem). *$\mathcal{H}(\rho)$ is a* free *$\mathcal{M}$-module of* rank *p.*

This means that there are p weights $k_1, \ldots, k_p$ and p vector-valued modular forms $F_j(\tau) \in \mathcal{H}(k_j, \rho)$, $1 \leq j \leq p$, such that every $F(\tau) \in \mathcal{H}(k, \rho)$ has a *unique* expression in the form

$$F(\tau) = \sum_{j=1}^{p} f_j(\tau) F_j(\tau), \qquad f_j(\tau) \in \mathcal{M}_{k-k_j}.$$

It is an immediate consequence of this result that the Hilbert–Poincaré series for $\mathcal{H}(\rho)$ is a rational function:

$$\sum_{k \geq k_0} \dim \mathcal{H}(k,\rho)t^k = \frac{\sum_{j=1}^{p} t^{k_j}}{(1-t^4)(1-t^6)}.$$

With Corollary 3.12 available, the remaining details of the proof of Theorem 3.13 are essentially identical to that of the semisimple case given in [MM] and involve mainly arguments from commutative algebra. In the few places where the nature of the component functions of vector-valued modular forms is relevant, the argument is the same whether the q-expansions are ordinary or logarithmic. We forgo further discussion.

We give an application of the free module theorem. Let $F(\tau) \in \mathcal{H}(k,\rho)$. If the elements $F, DF, \ldots, D^pF$ are linearly independent over $\mathcal{M}$ then they span a free $\mathcal{M}$-submodule of $\mathcal{H}(\rho)$ of rank $p+1$. Since $\mathcal{H}(\rho)$ has rank p, this is not possible. Therefore, $F(\tau)$ satisfies an equality of the form

$$(g_0(\tau)D_k^p + g_1(\tau)D_k^{p-1} + \cdots + g_p(\tau))F = 0 \tag{44}$$

where $g_0 \in \mathcal{M}_l$ for some weight l and $g_j(\tau) \in \mathcal{M}_{l+2j}$. We may think of (44) as a modular linear differential equation (MLDE) [M1] of order at most p, in which case the component functions of $F(\tau)$ are solutions. Now suppose that the component functions are *linearly independent*. Since they are solutions of any MLDE satisfied by $F(\tau)$, the solution space must have dimension at least p, and therefore the order of the MLDE must itself be at least p. We have therefore shown that if $F(\tau) \in \mathcal{H}(k,\rho)$ has linearly independent component functions, it satisfies an MLDE of order p and none of order less than p.

Continuing with the assumption that the component functions of $F(\tau)$ are linearly independent, let $I \subseteq \mathcal{M}$ be the set of *all* leading coefficients $g_0(\tau)$ that occur in order p MLDE's (44) satisfied by F. Taking account of the trivial case when all coefficients $g_j(\tau)$ vanish, we see easily that I is a graded ideal. Moreover, our previous comments show that $I \neq 0$. We will show that I contains a *unique* nonzero modular form $g(\tau)$ of least weight, normalized so that the leading coefficient of its q-expansion is 1, and that $I = g(\tau)\mathcal{M}$.

For nonzero $h_0(\tau) \in I$ of weight k, we let

$$L_h = h_0(\tau)D^p + h_1(\tau)D^{p-1} + \cdots + h_p(\tau)$$

be the unique order p differential operator in $\mathcal{R}$ with leading coefficient $h_0(\tau)$ and satisfying $L_hF = 0$. Let $g_0(\tau)$ be any nonzero element in I of least weight, say m. Then we have $L_gF = L_hF = 0$, and therefore also

$$(g_0L_h - h_0L_g)F = 0.$$

The differential operator in the last display has order at most $p-1$, and therefore (by our earlier remarks) must vanish identically. It follows that for all indices j we have

$$g_0 h_j = h_0 g_j. \tag{45}$$

Suppose that the order of vanishing of $g_0(\tau)$ at ∞ is *greater* than that of $h_0(\tau)$. By (45) it follows that all $g_j(\tau)$ vanish to order at least 1 at ∞, i.e. each $g_j(\tau)$ is divisible by the discriminant $\Delta(\tau)$ in $\mathcal{M}$. But then $L_g F = \Delta(\tau) L_{g'} F = 0$, whence $L_{g'} F = 0$ for some nonzero $g'(\tau) \in \mathcal{M}_{m-12}$. Then $g'(\tau) \in I$, and this contradicts the minimality of the weight of g. Thus we have shown that the order of vanishing of $g_0(\tau)$ at ∞ is minimal among nonzero elements in I, and that this assertion holds for *any* nonzero element of least weight in I.

If there are two linearly independent elements $a(\tau), b(\tau)$, say, of least weight in I then some linear combination of them vanishes at ∞ to an order that *exceeds* that of at least one of $a(\tau)$ and $b(\tau)$. By the last paragraph this cannot occur, and we conclude that up to scalars, $g_0(\tau)$ is the unique nonzero element in I of least weight.

We use similar arguments to show that $g_0(\tau)$ generates I. If not, choose an element $h_0 \in I$ of least weight n, say, subject to $h_0(\tau) \notin g_0(\tau)\mathcal{M}$. If $h_0(\tau)$ has greater order of vanishing at ∞ than $g_0(\tau)$, then (45) and a previous argument show that every $h_j(\tau)$ is divisible by $\Delta(\tau)$. Then as before, $h_0(\tau) = \Delta(\tau) h_0'(\tau)$ with $h'(\tau) \in I$. By minimality of the weight of $h_0(\tau)$ we get $h'(\tau) \in g_0(\tau)\mathcal{M}$, and therefore also $h_0(\tau) \in g_0(\tau)\mathcal{M}$, contradiction. Therefore, every element of weight n in $I \setminus g_0(\tau)\mathcal{M}$ has the same order of vanishing at ∞ as $g_0(\tau)$. This again implies the unicity of $h_0(\tau)$ up to scalars.

If $n - m \geq 4$ then $h_0(\tau) + E_{n-m}(\tau) g_0(\tau)$ has weight n and lies in $I \setminus g_0(\tau)\mathcal{M}$. (Here, $E_k(\tau)$ is the usual weight k Eisenstein series.) Thus $h_0(\tau)$ is a scalar multiple of $h_0(\tau) + E_{n-m}(\tau) g_0(\tau)$ and therefore lies in $g_0(\tau)\mathcal{M}$, contradiction. Therefore, $n - m = 2$. In this case we consider $h'(\tau) = E_4(\tau) h_0(\tau) - \beta E_6(\tau) g_0(\tau)$ and $h''(\tau) = E_6(\tau) h_0(\tau) - \gamma E_4^2(\tau) g_0(\tau)$ for scalars β, γ chosen in each case so that the order of vanishing at ∞ is greater than that of $g_0(\tau)$. A previous argument shows that $L_{h'} F = \Delta L_{h_1'} F = 0$ for some $h_1'(\tau)$ of weight $n + 4 - 12 = m - 6$. Since $h_1'(\tau) \in I$ has weight less than m, we have $h_1'(\tau) = 0$, so that $E_4(\tau) h_0(\tau) = \beta E_6(\tau) g_0(\tau)$. The same reasoning applied to $h''(\tau)$ also shows that $E_6(\tau) h_0(\tau) = \gamma E_4^2(\tau) g_0(\tau)$. From these equalities we deduce that $g_0(\tau)(\gamma E_4^3(\tau) - \beta E_6^2(\tau)) = 0$. This can only happen if $\beta = \gamma = 0$, whence $E_4(\tau) h_0(\tau) = 0$. This is impossible since $h_0(\tau)$ is nonzero, and we have contradicted the assumed existence of $h_0(\tau)$. To summarize, we have established

THEOREM 3.14. *Suppose that $F(\tau) \in \mathcal{H}(k, \rho)$ has linearly independent component functions. Then the component functions are a basis of the solu-*

tion space of a modular linear differential equation

$$(g_0(\tau)D_k^p + g_1(\tau)D_k^{p-1} + \cdots + g_p(\tau))f = 0 \tag{46}$$

where $g_j(\tau) \in \mathcal{M}_{l+2j}$, $0 \le j \le p$, *for some* $l \ge 0$. *The set of leading coefficients* $g_0(\tau)$ *that can occur in* (46) *is a (nonzero) principal graded ideal* $I \subseteq \mathcal{M}$ *generated by the* unique *normalized modular form* $g(\tau)$ *of least weight in* I.

If the condition that the component functions of $F(\tau)$ are linearly independent is *not* met, one can replace ρ by the representation ρ' of Γ furnished by the span of the component functions. Then the theorem applies to ρ'. In this way, we see that to any logarithmic vector-valued modular form we can associate an MLDE in a canonical way: it is the MLDE of least order and with normalized leading coefficient of least weight whose solution space is spanned by the component functions of $F(\tau)$.

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