ACTA ARITHMETICA
142.3 (2010)

Extreme binary forms

by

Andrzej Białynicki-Birula and Andrzej Schinzel (Warszawa)

1. Introduction. Let K be an algebraically closed field of characteristic 0 or $> d$, where d is a positive integer. Let f be a binary form in $K[x,y]$ of degree d. By a *presentation* of f we mean a decomposition

$$f = l_1^d + \cdots + l_u^d$$

of f into a sum of powers of linear forms $l_1, \ldots, l_u \in K[x,y]$. Let $r = r(f)$ be the minimal length of a presentation of f. Then f is said to be *extreme* if for every linear form $l \in K[x,y]$, the sum $f + l^d$ admits a presentation of length $r(f)$. In other words, a form is extreme if and only if it is maximal with respect to the partial order denoted by $\prec$ and defined by: a form f is greater than every form obtained as a sum of some, but not all, summands appearing in a presentation of f of minimal length.

A form is said to be *exotic* if the minimal length of presentation of f is greater than the minimal length of presentation of a generic form of the same degree. Hence exotic forms are defined as those that have the minimal length of presentation greater than the length predicted by an answer to the Waring Problem. In the case of binary forms, the minimal length r_0 of presentation of a general binary form of degree d is

$$(d+1)/2 \text{ in case } d \text{ is odd} \quad \text{and} \quad (d+2)/2 \text{ in case } d \text{ is even.}$$

Hence a binary form of degree d is exotic if its minimal length of presentation is greater than $(d+2)/2$. A form of degree d is said to be *regular* if its minimal length of presentation is equal to the length of presentation of a general form of degree d. A form of degree d is said to be *plain* if its minimal length of presentation is less than the length of presentation of a general form of degree d.

When we consider presentations of forms as sums of forms, then usually we are not able to identify projectively equal forms and simultaneously

2010 *Mathematics Subject Classification*: 12E05, 11D85.

Key words and phrases: sums of powers of linear forms, binary forms, Waring problem.

DOI: 10.4064/aa142-3-2

identify their projectively equal summands. Hence, in the following, we are going to treat forms as elements of the vector space $K[x,y]$, rather than of its projectivization. The properties of forms which we are going to study are invariant under invertible linear substitutions, i.e. under the natural action of $GL(2)$ on $K[x,y]$. Moreover, the studied notions have some interpretation, easy to find but not helpful in our considerations, in the geometric theory of secant varieties obtained as the images of the projective line under Veronese embedding into a projective space.

Let $K[X,Y]$ be the dual *ring of differential operators*, where

$$X = \frac{\partial}{\partial x}, \qquad Y = \frac{\partial}{\partial y}.$$

Let $K[X,Y]_t$ denote the space of homogeneous operators of degree t, $t = 0,1,\ldots$. For $f \in K[x,y]$, let $A(f) \subset K[X,Y]$ be the *annihilating ideal of* f and $A_t(f)$ its homogeneous part of degree t, i.e.

$$A_t(f) = K[X,Y]_t \cap A(f).$$

We shall say that a form $D \in K[X,Y]$ is *squareful* if, for a linear form L, whenever $L \mid D$, then $L^2 \mid D$.

One of the main aims of the paper is to characterize properties of forms by means of properties of annihilating differential operators. Let $j = \lfloor (d+2)/2 \rfloor$. We shall see that a form $f \in K[x,y]_d$ is

- regular if and only if there exist projectively distinct linear differential operators $D_1,\ldots,D_j$ such that $D_1\ldots D_j(f) = 0$ (Theorem 1),
- plain if and only if there exist projectively distinct linear differential operators $D_1,\ldots,D_u$, $u < j$, such that $D_1\ldots D_u(f) = 0$ (Theorem 1),
- exotic if and only if no squarefree differential operator of degree $\le j$ annihilates f (Theorem 1),
- extreme and exotic if and only if it is annihilated by a squareful linear differential operator D of degree $u < j$ (Theorem 1′),
- exotic and regular if and only if (this is the most difficult case) the conditions given in Theorem 5 are satisfied (then d has to be even).

Since there are no extreme plain forms (Lemma 3.4), all cases are covered above.

It follows from these results that, for example, all forms $f = l^m g \in K[x,y]_d$, where l is a linear form, $g \in K[x,y]_{d-m}$ is relatively prime to l and $2m > d+1$, are extreme. Moreover, all extreme forms with minimal length of presentation equal to d are equal to $l_1^{d-1} l_2$, where l_1, l_2 are projectively different linear forms (Corollary 3). This means that they constitute one $GL(2)$-orbit. The set of extreme forms with minimal length of presentation equal to $d-1, d \ge 6$, is composed of two orbits. For exotic forms these are the only cases where the exotic extreme forms with given minimal length of presentation comprise a finite number of $GL(2)$-orbits. However, the extreme regular forms (they exist only if $d = 2m$) comprise a finite number of orbits (Corollary 7). The proof of this fact is the most difficult part of the paper.

In the remaining part of the paper we study, for a binary form f, all presentations of f as a sum of powers $l_1^d + \cdots + l_u^d$ of linear forms of minimal length u. In particular, we would like to know all partial summands $g = l_1^d + \cdots + l_t^d$, $t < u$, that appear in such presentations. This leads to studying the $\prec$ relation. We start with the following simple observation. Plain forms and regular forms in case $d+1$ is even and only those forms have a unique presentation of minimal length. It follows that if, for a form $f = l_1^d + \cdots + l_u^d$, we have already fixed some part $g = l_1^d + \cdots + l_t^d, t < u, u - t < (d+1)/2$, of its minimal decomposition, then the remaining part of the decomposition is already uniquely determined. If f, g are as above and g is regular or exotic, then the conditions are satisfied, hence, then, $f - g$ has a unique minimal presentation. Next we prove that, for f a regular form of even degree or an exotic form, and only for such forms, there exist only finitely many extreme forms h such that $f \prec h$, hence there are only finitely many forms greater than f (Proposition 3).

Remark. The map r which assigns to a form f of degree d its minimal length of presentation $r(f)$ satisfies the triangle inequality $r(f+g) \le r(f) + r(g)$ and defines a metric ρ (by $\rho(f,g) = r(f-g)$) on $K[x,y]_d$. Then $f \prec g$ if and only if $r(f) + r(g-f) = r(g)$. A form h is extreme if and only if r admits at h its local maximum, more exactly, $r(h)$ is maximal among the values of r taken in the closed ball of radius 1. In other words, the distance from an extreme form to 0 is locally maximal.

For every form f we can consider the set of all linear forms $l \in K[x,y]_1$ such that $l^d \prec f$, i.e. l^d appears in some minimal presentation of f. This set is finite for any plain form and any regular form in case $d+1$ is even, and infinite with non-empty complement in $K[x,y]_1$ composed of a finite number of lines in the remaining cases (Proposition 1). Studying these cases leads finally to the result (Corollary 5) that all forms can be divided into two disjoint classes: those that dominate only finitely many other forms and those that are dominated only by finitely many forms.

There are still some open problems connected with the results of the paper.

Problem. What is the number $n(m)$ of linear equivalence classes of regular extreme forms of degree $2m$?

2. Statements of the results

Theorem 1. *Let f be a binary form of degree $d > 1$ with minimal length of presentation $r(f) = r$. Then f is extreme if and only if f is regular or* *exotic and*

- *whenever f is exotic, then every element of $A_{d+2-r}(f)$ is squareful,*
- *whenever f is regular, then d is even and every element of $A_{d+2-r}(f) = A_{(d+2)/2}(f)$ is either squarefree or squareful.*

We shall see that in the case where f is exotic, $\dim A_{d+2-r}(f) = 1$. Thus every element of $A_{d+2-r}(f)$ is squareful if and only if $A_{d+2-r}(f)$ contains a non-zero squareful form.

In order to formulate further assertions we need to quote some auxiliary results.

LEMMA 2.1. *Let $l_1, \ldots, l_u$ be projectively distinct non-zero linear forms and let $L_1, \ldots, L_u$ be non-zero linear operators such that $L_i l_i = 0$ for $i = 1, \ldots, u$. Let $e_1 + \cdots + e_u + u \le d$. A binary form f of degree d admits a decomposition*

$$f = g_1 l_1^{d-e_1} + \cdots + g_u l_u^{d-e_u},$$

where, for $i = 1, \ldots, u$, $g_i \in K[x, y]$ is a form of degree e_i or 0, if and only if

$$L_1^{e_1+1} \ldots L_u^{e_u+1} f = 0.$$

In particular, f admits a presentation

$$f = a_1 l_1^d + \cdots + a_u l_u^d,$$

where $a_1, \ldots, a_u \in K$, if and only if

$$L_1 \ldots L_u f = 0.$$

Proof. See [2, Lemma 1.31].

LEMMA 2.2. *Let f be a binary form of degree d.*

(i) *The sequence $(\dim K[X, Y]_u / A_u(f))_{u=0}^{d}$ has the following shape:*

$$(1, 2, \ldots, s-1, s, \ldots, s, s-1, \ldots, 2, 1),$$

where $s = s(f) := \max_{0 \le u \le d} \dim(K[X, Y]_u / A_u(f)) \le (d+2)/2$.

(ii) *The ideal $A(f)$ is generated by two relatively prime homogeneous forms $\mathcal{D}_f$, $\mathcal{E}_f$ of degree s and $d-s+2$, respectively. For $s \le (d+1)/2$ the form $\mathcal{D}_f$ is uniquely determined up to projective equality and for $u = s, \ldots, d-s+1$ we have*

$$A_u(f) = \mathcal{D}_f K[X, Y]_{u-s}.$$

Proof. For the proof of (i), see [2, Theorems 1.43 and 1.44(i)]. For the proof of (ii), see [2, Theorem 1.44(ii) and (iii) and Claim on p. 31].

LEMMA 2.3. *Let $F_1, \ldots, F_u$ be binary forms of the same degree such that the set of all forms $f = a_1 F_1 + \cdots + a_u F_u$, $a_1, \ldots, a_u \in K$, which are not squarefree is Zariski dense in the linear span of $F_1, \ldots, F_u$. Then there exists a linear form $L \in K[X, Y]$ such that L^2 divides all $F_1, \ldots, F_u$.*

Proof. See the proof of Lemma 1.1 in [3].

With the notation established in Lemma 2.2, we can formulate Theorem 1 in a slightly weaker, but more transparent form:

THEOREM 1′. *A binary exotic form f is extreme if and only if $\mathcal{D}_f$ is squareful.*

COROLLARY 1. *If a binary form f of degree d has a linear factor of multiplicity m, where $(d+1)/2 \le m < d$, then f is extreme.*

COROLLARY 2. *If a binary form f of degree $d \ge 5$ is extreme and $r(f) \ge d-1$, then f has a linear factor of multiplicity $r(f)-1$. For $d \ge 6$ there exists a squarefree extreme form with $r(f) = d-2$.*

The special case of Corollary 2 for $r(f) = d$ is due to J. Kleppe [3, p. 11].

In view of Corollary 2 it is doubtful that extreme forms have a simple characterization in terms of factorization of f instead of that of $\mathcal{D}_f$.

Then we show the following characterization of exotic extreme forms:

THEOREM 2. *An exotic form f is extreme if and only if it is of the shape*

$$f = g_1 m_1^{d-e_1} + \dots + g_u m_u^{d-e_u},$$

where $m_1, \dots, m_u$ are pairwise projectively distinct linear forms, $g_1, \dots, g_u$ are forms of positive degree $e_1, \dots, e_u$, respectively, $m_i \nmid g_i$ and

$$e_1 + \dots + e_u + u < \frac{d+2}{2}.$$

Moreover, for f as above, if $M_1, \dots, M_u$ are non-zero linear operators such that $M_i m_i = 0$ for $i = 1, \dots, u$, then

$$\mathcal{D}_f = M_1^{e_1+1} \dots M_u^{e_u+1}.$$

The following corollary follows immediately from the above Theorem 2.

COROLLARY 3. *Up to a linear transformation we have only the following extreme forms f_0 with the given lengths of minimal presentations:*

1. *$r(f_0) = d$, $d \ge 2$, only $f_0 = xy^{d-1}$.*
2. *$r(f_0) = d-1$, $d \ge 4$, only $f_0 = x(x+y)y^{d-2}$ and $f_0 = x^2y^{d-2}$. For $d \le 7$ these are all exotic extreme forms and for $d < 6$ these are all extreme forms.*
3. *For $r(f_0) = d-2$, $d \ge 7$, we have two infinite families of extreme exotic forms which contain all extreme forms: $f_0 = x(x+y)(x+ay)y^{d-3}$, $f_0 = xy^{d-1} + ax^{d-1}y$, where $a \in K \setminus \{0\}$.*
4. *For $r(f_0) = d-3$, $d \ge 9$, we have two infinite families of extreme exotic forms which contain all extreme forms: $f_0 = x(x+y)(x+ay) \cdot (x+by)y^{d-4}$ and $f_0 = (x+ay)y^{d-1} + e(bx+y)(cx+y)x^{d-2}$, where $a, b, c \in K$, $e \in K \setminus \{0\}$.*

Moreover, for $d = 2m$ we have a regular extreme form $f_0 = x^m y^m$, and for $d{=}2m$ with m odd, we still have extreme regular forms

$$f_0{=}x^{(3m+1)/2}y^{(m-1)/2} + x^{(m-1)/2}y^{(3m+1)/2}.$$

For $d \le 10$ these are all regular extreme forms.

Summarizing the results for small values of d, there are only finitely many, up to a linear transformation, extreme exotic forms of degree $d < 7$. For $d = 6$ these are xy^5, x^2y^4, $x(x+y)y^4$. For $d = 7$, $x(x+y)(x+ay)y^4$, $a \in K$, is an infinite family of exotic extreme linearly distinct forms.

THEOREM 3. *For a binary exotic form f of degree d let*

$$\mathcal{D}_f = L_1^{e_1+1} \dots L_k^{e_k+1} L_{k+1} \dots L_{k+t},$$

where $k \ge 1$, $e_1, \dots, e_k \ge 1$, and L_i are projectively distinct. Let l_i, for $i = k+1, \dots, k+t$, be a non-zero linear form annihilated by L_i. Then there exist $a_1, \dots, a_t \in K \setminus \{0\}$ such that

$$f + a_1 l_{k+1}^d + \dots + a_t l_{k+t}^d = f_0,$$

where f_0 is an extreme form with

$$\mathcal{D}_{f_0} = L_1^{e_1+1} \dots L_k^{e_k+1}.$$

Moreover, f_0 is the only extreme form such that $f \preceq f_0$ and the summands in the above decomposition of f_0 are uniquely determined by f.

It follows from the above theorem that if for a form f we denote by f^- (by f^+) the partially ordered (by $\preceq$) set composed of all forms $g \preceq f$ ($f \preceq g$, resp.), then f^+ for all exotic forms f, and f^- for all plain forms, are isomorphic to a partially ordered (by inclusion) family of all subsets of a finite set.

THEOREM 4. *Let f be a regular form of degree $d = 2m$. Let*

$$\mathcal{D}_1 = L_1^{e_1+1} \dots L_k^{e_k+1} L_{k+1} \dots L_{k+t}, \qquad e_1 + \dots + e_k + k + t = m + 1,$$

where $k \ge 1$, $e_1, \dots, e_k \ge 1$, and L_i are projectively distinct. Let l_i, for $i = k+1, \dots, k+t$, be a non-zero linear form annihilated by L_i.

If $\mathcal{D}_1 f = 0$, then there exist $a_1, \dots, a_t \in K \setminus \{0\}$ such that

$$f + a_1 l_{k+1}^d + \dots + a_t l_{k+t}^d = f_0,$$

where f_0 is an extreme form with

$$\mathcal{D}_{f_0} = L_1^{e_1+1} \dots L_k^{e_k+1}.$$

The summands in the above decomposition of f_0 are uniquely determined by f. Moreover, there are only finitely many extreme forms f_0 such that $f \preceq f_0$, and every such form is defined in the way described above by some differential operator $\mathcal{D}_1$.

Proposition 1. *For a binary exotic form f of degree d, a linear operator L divides $\mathcal{D}_f$ if and only if al^d, where $Ll = 0$ and $a \in K \setminus \{0\}$, does not appear as a summand in any presentation of f of length $r(f)$.*

The above proposition describes some obstacles for a sum $f = l_1^d + \cdots + l_r^d$ to appear in a minimal presentation of an extreme form f_0 given as

$$f_0 = g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u},$$

where $m_1, \ldots, m_u$ are pairwise projectively distinct linear forms, $g_1, \ldots, g_u$ are forms of positive degree $e_1, \ldots, e_u$, respectively, and $m_i \nmid g_i$. The next proposition shows that if $r \le r(f_0) - \lceil (d+2)/2 \rceil$, this is the only obstacle.

Proposition 2. *Let f be a plain binary form and let $f = l_1^d + \cdots + l_r^d$ be its presentation of minimal length. Let f_0 be an extreme form*

$$f_0 = g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u}, \qquad e_1 + \cdots + e_u + u < (d+2)/2,$$

where $m_1, \ldots, m_u$ are pairwise projectively distinct linear forms, $g_1, \ldots, g_u$ are forms of positive degree $e_1, \ldots, e_u$, respectively, and $m_i \nmid g_i$. Assume that every linear form l_i is projectively distinct from any form m_j, for $i = 1, \ldots, r$, $j = 1, \ldots, u$.

If $r \le r(f_0) - \lceil (d+2)/2 \rceil$, then $f \prec f_0$. Moreover, for every degree $d > 1$ the inequality is best possible.

Corollary 4. *For every form f of degree d with $r(f) < \lfloor d/2 \rfloor$ there exist infinitely many extreme forms f_0 such that $f \prec f_0$.*

However, Corollary 4 is not best possible, as shown by

Proposition 3. *For a given form f of degree d, there exist infinitely many extreme forms $f_0 \succ f$ if and only if $r(f) \le (d+1)/2$.*

Corollary 5. *For every binary form f exactly one of the inequalities $f \prec g, f \succ g$ admits infinitely many solutions g.*

In fact, it follows from the above results that for all plain forms and regular forms of odd degree there exist only finitely many forms less than the given form, and infinitely many greater forms. For all exotic forms and for regular forms of even degree there exist infinitely many forms that are less than the given form, and only finitely many greater forms.

The next proposition describes a method for finding some extreme form greater than a given plain form.

Proposition 4. *Let f be a form with $r = r(f) \le (d+1)/2$. Let G be the maximal squareful factor of $\mathcal{E}_f \in A_{d-r+2}(f)$. If $0 < \deg G < (d+2)/2$, then there exists an extreme form f_0 such that $\mathcal{D}_{f_0} = G$ and $f \prec f_0$. Conversely, if $f \prec f_0$, where f_0 is extreme, then $\mathcal{D}_{f_0}$ is a maximal squareful factor of some $\mathcal{E}_f \in A_{d-r+2}(f)$.*

Regular extreme forms f are mysterious. We already know from Theorem 1 that they are of even degree d and in case $d = 2m$, we have $r(f) = m+1$ and every element in $A_{m+1}(f)$ is either squarefree or squareful. The next theorem implies that up to a linear transformation, unexpectedly, there are only finitely many extreme regular forms of any even degree d, but their number, in general, is not explicitly given.

THEOREM 5. *A regular form f of degree $2m$ is extreme if and only if either it is equivalent via a linear transformation to $x^m y^m$, or in $A_{m+1}(f)$ there are three coprime squareful forms that together have exactly $m + 3$ projectively distinct linear factors.*

COROLLARY 6. *For $m \le 5$ a regular form f of degree $2m$ is extreme if and only if either f is equivalent via a linear transformation to $x^m y^m$, or $m=3$ or 5 and f is equivalent via a linear transformation to $x^{(3m+1)/2}y^{(m-1)/2} - x^{(m-1)/2}y^{(3m+1)/2}$.*

COROLLARY 7. *For every m there exist up to a linear transformation only finitely many extreme regular forms of degree $2m$.*

Corollary 6 does not extend to $m > 5$. Indeed, if $m = 2kl - 1$ $(l > 1)$, there is an extreme regular form f of degree $2m$ with three squareful forms in $A_{m+1}(f)$:

$$((X^k + Y^k)^l + Y^{kl})^2, \quad ((X^k + Y^k)^l - Y^{kl})^2, \quad Y^{kl}(X^k + Y^k)^l.$$

For $k = 1$ we obtain forms f mentioned in Corollary 6, but for $k > 1$ the explicit descriptions of the relevant forms f are much more complicated, though, in any case, such a form is uniquely, up to projective equality, determined by two explicitly given differential equations. For $m = 6$, J. Browkin has found an interesting explicit example given at the end of the paper.

In [1], the current authors considers presentations of a given form f of degree d in n variables as a sum of powers $l_1^d + \cdots + l_u^d$, where $l_1, \ldots, l_u$ are linear forms and $l_1, \ldots, l_u$ do not belong to a given finite family of hyperplanes in the space of linear forms. This has led to the following definition:

DEFINITION. One says that f *has a lot of presentations of length* r if for any finite set of points $\{p_1, \ldots, p_m\}$ in $K^2 \setminus \{0\}$ there is a presentation $f = l_1^d + \cdots + l_r^d$ such that $l_i(p_j) \ne 0$ for all $i \le r$, $j \le m$.

For f essentially depending on n variables, let $\overline{r}(f)$ be the minimum r such that f has a lot of presentations of length r. In the case of binary forms f essentially depending on two variables, it has been noticed in [1] that $\overline{r}(f) \le d$ and $\overline{r}(xy^{d-1}) = d$. The next proposition determines $\overline{r}(f)$ for various forms f.

PROPOSITION 5.

1. *For every exotic form f, $\overline{r}(f) = r(f)$.*
2. *For every plain form f, $\overline{r}(f) = d - r(f) + 2$.*
3. *For every regular form f,*

$$\overline{r}(f) = \begin{cases} d/2+1 & \text{for } d \text{ even,} \\ (d+3)/2 & \text{for } d \text{ odd.} \end{cases}$$

3. Proofs of Theorems 1 and 1′

LEMMA 3.1. *For*

$$f = \sum_{i=0}^{d} \binom{d}{i} a_i x^{d-i} y^i, \qquad d = 2n+1,$$

let

$$C(f) = \begin{vmatrix} x^{n+1} & -x^n y & \ldots & \pm y^{n+1} \\ a_{n+1} & a_n & \ldots & a_0 \\ a_{n+2} & a_{n+1} & \ldots & a_1 \\ \ldots & \ldots & \ldots & \ldots \\ a_{2n+1} & a_{2n} & \ldots & a_n \end{vmatrix}.$$

If $r(f) \le n$, then $C(f) = 0$. If $r(f) = n+1$ and

$$f = \sum_{i=1}^{n+1} l_i^d, \tag{1}$$

where l_i are projectively distinct non-zero linear forms, i.e. (1) *is a presentation of f, then*

$$C(f) = k \prod_{i=1}^{n+1} l_i, \tag{2}$$

where $k \in K$ and the presentation is unique up to the order of summands.

Proof. See [5].

LEMMA 3.2. *If $r(f) > n$ and* $\operatorname{discr} C(f) = 0$, *then $r(f) > n+1$.*

Proof. If we had $r(f) = n+1$, then it would follow from (1) that

$$\operatorname{discr} C(f) \neq 0.$$

LEMMA 3.3. *If $f = \sum_{i=1}^{n+1} l_i^d$ is a presentation of length $n+1 = r(f)$, then*

$$r(f + (ax+by)^d) \le n \tag{3}$$

if and only if $(ax+by)^d = -l_j^d$ for some $j \le n+1$.

Proof. Assume that (3) holds and

$$f + (ax + by)^d = \sum_{j=1}^{s} m_j^d$$

is a presentation of length $s \le n$. Then

$$f = -(ax + by)^d + \sum_{j=1}^{s} m_j^d$$

and $s+1 \ge n+1$, hence $s = n$, $ax+by$ is projectively distinct from m_j, and by uniqueness of presentation (Lemma 3.1)

$$-(ax + by)^d = l_j^d$$

for some $j \le n + 1$. Conversely, if the above equality holds, then

$$f + (ax + by)^d = \sum_{i=1,\, i\neq j}^{n+1} l_i^d$$

and (3) holds.

LEMMA 3.4. *If a form f of degree $d > 1$ is extreme, then $r(f) > (d + 1)/2$, i.e. f is exotic or regular of even degree.*

Proof. Assume that f is extreme, $r = r(f)$ and let

$$f = l_1^d + \cdots + l_r^d$$

be an r-presentation. For every linear form $l \in K[x, y]$, we have

$$f + l^d = m_1^d + \cdots + m_s^d$$

for some linear forms $m_1, \ldots, m_s$ and $s \le r$. Then

$$l_1^d + \cdots + l_r^d + l^d = m_1^d + \cdots + m_s^d$$

and this gives a presentation of zero of length at most $r + 1 + s$. Hence, by [1, Corollary 3],

$$r + 1 + s \ge d + 2 \text{ and } 2r + 1 \ge d + 2, \quad \text{i.e.} \quad r \ge (d + 1)/2.$$

It remains to exclude the case $d = 2n+1$, $r(f) = n+1$. Let $F = f+(ax+by)^d$. For $d > 1$, we have

$$\operatorname{discr} C(F) = P \in K[a, b].$$

If $f = \sum_{i=1}^{n+1}(a_i x + b_i y)^d$ then by Lemmas 3.1 and 3.3,

$$P(-a_i, -b_i) = 0, \qquad P(0, 0) \neq 0,$$

hence $P(a, b) = 0$ defines a curve. This curve contains a point (a_0, b_0) different from

$$(-\zeta_d^j a_i, -\zeta_d^j b_i) \quad (1 \le i \le n + 1,\ 0 < j < d)$$

and the form $F_0 = f + (a_0x + b_0y)^d$ satisfies the assumptions of Lemma 3.2, hence $r(F_0) = n + 1$ and f is not extreme.

LEMMA 3.5. *Let $s(f)$ be defined as in Lemma 2.2. If $s(f) \le (d+1)/2$ and $\mathcal{D}_f$ is not squarefree, then $r(f) = d + 2 - s(f)$ and $\mathcal{E}_f$ can be chosen to be squarefree. If $s(f) = (d+2)/2$, then $r(f) = s(f)$.*

Proof. Let $r = r(f)$ and $s = s(f)$. By Lemma 2.1, $A_r(f)$ contains a squarefree form G. If $s \le (d+1)/2$, then by Lemma 2.2(ii), $G = A\mathcal{D}_f + B\mathcal{E}_f$, where $A, B \in K[X, Y]$. Since $\mathcal{D}_f$ is not squarefree, we have $B \ne 0$, hence

$$r = \deg G \ge \deg \mathcal{E}_f = d + 2 - s(f).$$

However, again by Lemma 2.2, we have $(\mathcal{D}_f, \mathcal{E}_f) = 1$. Let D_3 be any form of degree $d+2-2s(f)$ prime to $\mathcal{E}_f$. By Lemma 2.3 there are $a, b \in K$ such that $G_{a,b} = a\mathcal{D}_f D_3 + b\mathcal{E}_f$ is squarefree. Hence $r = d + 2 - s(f)$ by Lemma 2.1 and since $b \ne 0$ we can replace $\mathcal{D}_f$ as one of the generators of $A(f)$ by $G_{a,b}$.

If $s = (d+2)/2$, then by Lemma 2.2(ii), $A(f)$ is generated by two coprime forms $\mathcal{D}_1$, $\mathcal{D}_2$ both of degree s. By Lemma 2.3 there is a squarefree linear combination $a\mathcal{D}_1 + b\mathcal{D}_2$ and this is a squarefree form of the least degree in $A(f)$. By Lemma 2.1, $r(f) = s$.

LEMMA 3.6. *If f is exotic then $s(f) \le (d+1)/2$, $\mathcal{D}_f$ is not squarefree and*

$$A_{r-1} = \mathcal{D}_f K[X, Y]_{r-s-1}.$$

Proof. If we had $s(f) = (d+2)/2$, then by Lemma 2.2, $\mathcal{D}_f$ and $\mathcal{E}_f$ would be both of degree $s(f)$ and since $(\mathcal{D}_f, \mathcal{E}_f) = 1$, $A(f)$ would contain, by Lemma 2.3, a squarefree operator of degree $(d+2)/2$, hence by Lemma 2.1 we should have $r(f) \le (d+2)/2$, contrary to the assumption. If $\mathcal{D}_f$ were squarefree, we should have $r \le s \le (d+1)/2$, a contradiction again. Finally, $A_{r-1}(f)$ does not contain any element $A\mathcal{D}_f + B\mathcal{E}_f$ with $B \ne 0$.

Proofs of Theorems 1 and 1′. We shall first prove that the condition is necessary. So assume that f is extreme. Then by Lemma 3.4 we have

$$r = r(f) \ge (d+2)/2, \qquad s = s(f) = d + 2 - r.$$

Suppose that $A_s(f)$ contains an element D which is neither squarefree nor squareful. Hence $D = D_0L$, where L is a linear and L does not divide D_0. Choose a linear form $l \ne 0$ such that $Ll = 0$.

Since $LD_0f = 0$ we have

$$D_0f = c_1l^{d-s+1} \quad \text{for some } c_1 \in K.$$

Since L does not divide D_0,

$$D_0l^d = c_2l^{d-s+1} \quad \text{for some } c_2 \in K \setminus \{0\}.$$

It follows that

$$D_0(f - al^d) = 0 \quad \text{for } a = c_1/c_2,$$

and since D_0 of degree $s-1 \le d/2$ is not squarefree,

$$\mathcal{D}_{f-al^d} = D_0, \quad r(f-al^d) = d+2-\deg D_0 = d+3-s = r(f)+1.$$

The contradiction obtained shows that all elements of $A_s(f)$ are either squarefree or squareful. If f is exotic, then, by Lemma 2.1, $A_s(f)$ contains no squarefree elements, hence all elements there are squareful.

In order to prove that the condition is sufficient, assume that

$$r = r(f) \ge (d+2)/2, \quad s = s(f) = d+2-r$$

and that all elements of $A_s(f)$ are either squarefree or squareful. Let $l \in K[x,y]$ be a linear form and choose a linear operator $L \neq 0$ such that $Ll = 0$. Suppose that

$$r(f+l^d) = r+1.$$

Then by Lemma 3.5,

$$s(f+l^d) = d+2-r-1 = d+1-r \le d/2$$

and for $D_1 = \mathcal{D}_{f+l^d}$ we have

$$D_1 f + D_1 l^d = 0.$$

It follows that

$$LD_1 f = -LD_1 l^d = -D_1 L l^d = 0, \quad LD_1 \in A_s(f).$$

By the assumption, LD_1 is either squarefree or squareful. If LD_1 is squarefree then D_1 is squarefree, contrary to Lemma 3.6. Therefore LD_1 is squareful, L divides D_1,

$$D_1 f = -D_1 l^d = 0,$$

and by Lemma 3.5,

$$r(f) = d+2-\deg D_1 = r+1,$$

a contradiction. This completes the proof of Theorem 1. Theorem 1$'$ follows now from the last statement of Lemma 3.6.

Proof of Corollary 1. We can assume without loss of generality that the linear factor in question is y. Then $\mathcal{D}_f \mid X^{d-m+1}$ and, since $\mathcal{D}_f$ does not divide X for $m < d$, $\mathcal{D}_f$ is squareful.

Proof of Corollary 2. If $d \ge 5$, $r(f) \ge d-1$ and f is extreme, then by Lemma 3.5, $s = s(f) \le 3$, and by Theorem 1, $\mathcal{D}_f = L^s$, where L is a linear operator. Assuming without loss of generality that $L = Y$ we obtain

$$f = \sum_{i=0}^{s-1} a_i x^{d-i} y^i, \quad a_{s-1} \neq 0,$$

and f has a factor x with multiplicity $d-s+1 = r+1$.

To prove the second assertion of the corollary put $f = x^{d-1}y + xy^{d-1}$. If $d \geq 7$, we have $\mathcal{D}_f = X^2Y^2$ and Theorem 1 applies. If $d = 6$ every element of $A_4(f)$ is of the form $a(X^4 - Y^4) + bX^2Y^2$ and is either squarefree, if $a(b^2 + 4a^2) \neq 0$, or squareful otherwise, and Theorem 1 applies.

4. Proof of Theorem 2. Assume that f is extreme and exotic. Then it follows from Theorem 1′ that $\mathcal{D}_f$ is squareful and

$$\mathcal{D}_f = M_1^{e_1+1} \dots M_u^{e_u+1},$$

where $e_i \geq 1$ for $i = 1, \dots, u$ and $e_1 + \dots + e_u + u < (d+2)/2$. For $i = 1, \dots, u$, let m_i be a non-zero linear form such that $M_i m_i = 0$. Then it follows from Lemma 2.1 that

$$f = g_1 m_1^{d-e_1} + \dots + g_u m_u^{d-e_u},$$

where g_i, e_i satisfy the above conditions. Thus the conditions given in the propositions for a form to be extreme and exotic are necessary.

In order to prove that they are also sufficient, assume that f can be decomposed as

$$f = g_1 m_1^{d-e_1} + \dots + g_u m_u^{d-e_u}$$

with g_i, e_i and m_i as above. Let $M_i m_i = 0$ for some non-zero linear operators M_i, where $i = 1, \dots, u$. Then

$$M_i^{e_i+1} g_i m_i^{d-e_i} = m_i^{d-e_i}(M_i^{e_i+1} g_i) = 0,$$

since

$$\deg(M_i^{e_i+1} g_i) = e_i + 1 > e_i = \deg g_i.$$

Hence f is annihilated by $M_1^{e_1+1} \dots M_u^{e_u+1}$. On the other hand,

$$M_i^{e_i} g_i m_i^{d-e_i} = m_i^{d-e_i}(M_i^{e_i} g_i) = a_i m_i^{d-e_i} \neq 0,$$

where $a_i \in K \setminus \{0\}$ (g_i is not divisible by m_i, hence $M_i^{e_i} g_i \neq 0$) and

$$\begin{aligned} M_i^{e_i}\Big(\prod_{j=1,\, j\neq i}^{u} M_j^{e_j+1}\Big) f &= \Big(\prod_{j=1,\, j\neq i}^{u} M_j^{e_j+1} M_i^{e_i}\Big) g_i m_i^{d-e_i} \\ &= \prod_{j=1,\, j\neq i}^{u} M_j^{e_j+1} a_i m_i^{d-e_i} = b_i m_i^{d-(e_1+\dots+e_u+u-1)} \neq 0. \end{aligned}$$

The last inequality follows from $d - (e_1 + \dots + e_u + u - 1) > 0$, $M_j m_i \neq 0$ for $j = 1, \dots, u$, $j \neq i$. Thus f is annihilated by

$$D = \prod_{i=1}^{u} M_i^{e_i+1}$$

but, for $i = 1, \dots, u$, f is not annihilated by D/M_i. Thus $D = \mathcal{D}_f$. Now, it follows from Theorem $1'$ that f is extreme and exotic with $r(f) = d - \deg \mathcal{D}_f + 2 = d - (e_1 + \cdots + e_u + u) + 2 > (d+2)/2$.

5. Proofs of Theorems 3 and 4

LEMMA 5.1. *Let $r = r(f) \geq (d+2)/2$, and let L_i for $1 \leq i \leq k+t$ denote projectively distinct linear operators. If $t \geq 1$ and*

$$\mathcal{D}_f = L_1^{e_1+1} \ldots L_k^{e_k+1} L_{k+1} \ldots L_{k+t},$$

then for some $a_t \neq 0$, $f_1 = f + a_t l_{k+t}^d$ has the minimal length of presentation equal to $r+1$ and

$$\mathcal{D}_{f_1} = L_1^{e_1+1} \ldots L_k^{e_k+1} L_{k+1} \ldots L_{k+t-1}.$$

Proof. Let $L_i l_i = 0$ for $1 \leq i \leq k+t$ and

$$H = L_1^{e_1+1} \ldots L_k^{e_k+1} L_{k+1} \ldots L_{k+t-1}.$$

Then $Hf \neq 0$, but $L_{k+t} Hf = 0$. Hence $Hf = a l_{k+t}^{d-s+1}$ for some $a \in K \setminus \{0\}$ and $s = d + 2 - r$. Thus

$$H(f + a_t l_{k+t}^d) = 0, \tag{4}$$

where a_t is chosen in such a way that $H(a_t l_{k+t}^d) = -a l_{k+t}^{d-s+1}$. Since $\deg H = s - 1$ it follows from Lemma 3.5 that

$$r(f + a_t l_t^d) \geq d - (s-1) + 2 = r + 1.$$

Since obviously $r(f + a_t l_t^d) \leq r+1$ we have $r(f + a_t l_t^d) = r+1$ and $s(f + a_t l_t^d) = s - 1 = \deg H$. In view of (4) we have

$$H = \mathcal{D}_{f + a_t l_{k+t}^d} = \mathcal{D}_{f_1}.$$

Proof of Theorem 3. It follows from Lemma 5.1, by induction on t, that there exist $a_1, \ldots, a_t$ in $K \setminus \{0\}$ such that

$$f + a_1 l_{k+1}^d + \cdots + a_t l_{k+t}^d = f_0,$$

where f_0 is an extreme form with $\mathcal{D}_{f_0} = L_1^{e_1} \ldots L_k^{e_k}$. The minimal length of presentation of f_0 is equal to $d - (s - t) + 2 = r + t$. Thus

$$f_0 - f = a_1 l_{k+1}^d + \cdots + a_t l_{k+t}^d$$

is a presentation of minimal length and hence it is unique for fixed f_0, since its length is $t < s - 1 < d/2$.

It suffices to show that f_0 is unique. Notice that anyway

$$L_1^{e_1+1} \ldots L_k^{e_k+1}(f_0) = 0$$

and

$$\begin{aligned} L_{k+1} \ldots L_{k+t}(f_0) &= L_{k+1} \ldots L_{k+t}(f + (a_1 l_{k+1}^d + \cdots + a_t l_{k+t}^d)) \\ &= L_{k+1} \ldots L_{k+t} f \neq 0. \end{aligned}$$

If f_0, f_0' both satisfy these conditions, then

$$L_{k+1} \dots L_{k+t}(f_0 - f_0') = 0,$$

hence $f_0 - f_0'$, annihilated by $\mathcal{D}_{f_0} = \mathcal{D}_{f_0'}$, can be represented, by Lemma 2.1, as a sum of powers of linear forms projectively equal to $l_{k+1}, \dots, l_{k+t}$. This is not possible unless $f_0 - f_0' = 0$, because both operators annihilating $f_0 - f_0'$, namely $\mathcal{D}_{f_0}$ and $L_{k+1} \dots L_{k+t}$, have degrees smaller than $(d+2)/2$, but are relatively prime.

Proof of Theorem 4. For the proof of the first part of the theorem it suffices to use the same argument as in the proof of Theorem 3. Finiteness of the set of extreme forms f_0 such that $f \preceq f_0$ follows from Lemma 2.3.

6. Proof of Proposition 1. In order to prove that the condition for $L \mid \mathcal{D}_f$ is necessary, let l be a non-zero linear form such that $Ll = 0$. If al^d appears in a presentation of f of length $r = r(f)$, then $A_r(f)$ contains a squarefree operator D of degree r divisible by L. By Lemma 2.2 we have $D = \mathcal{D}_f A + \mathcal{E}_f B$, where A, B are forms from $K[X,Y]$. Since $\deg D = \deg \mathcal{E}_f$, we have $B \in K$, and $(\mathcal{D}_f, \mathcal{E}_f) = 1$ gives $B = 0$. Thus D divisible by $\mathcal{D}_f$ is not squarefree, a contradiction.

In the opposite direction assume that L does not divide $\mathcal{D}_f$ and that $Ll = 0$. Now, if $\deg H < r$ and

$$HLf = H(Lf) = 0,$$

then $\mathcal{D}_f \mid HL$, but since L does not divide $\mathcal{D}_f$, we have $\mathcal{D}_f \mid H$. On the other hand, $\mathcal{D}_f Lf = 0$. Thus if $s(Lf) \le d/2$, then $\mathcal{D}_{Lf} = \mathcal{D}_f$ and by Lemma 3.5,

$$r(Lf) = \deg Lf + 2 - \deg \mathcal{D}_{Lf} = d + 1 - (d + 2 - r(f)) = r - 1.$$

If $s(Lf) = (d+1)/2$, then again

$$r(Lf) = d + 1 - s(Lf) = (d + 2 - r(f)) = (d+1)/2 = r - 1.$$

Let

$$Lf = l_1^{d-1} + \dots + l_{r-1}^{d-1}. \tag{5}$$

If l is not projectively equal to any of l_i, then integrating this presentation with respect to L we obtain a form

$$f_0 = a_1 l_1^d + \dots + a_{r-1} l_{r-1}^d, \qquad a_1, \dots, a_{r-1} \in K,$$

of degree d such that $f - f_0 \neq 0$ is annihilated by L, hence equal to al^d, $a \neq 0$. Thus we have a presentation of length r

$$f = a_1 l_1^d + \dots + a_{r-1} l_{r-1}^d + al^d$$

containing al^d.

If l is projectively equal to l_i, then $A_{r-1}(f)$ has a squarefree element G_0 divisible by L. Let A_0 be an element of $K[X,Y]_{2r-d-3}$ not divisible by L.

Since G_0 is squarefree, the discriminant of $aA_0\mathcal{D}_f + bG_0$ is not identically 0 in a, b. Hence there exist $a_0 \in K \setminus \{0\}$ and $b_0 \in K$ such that

$$G_1 = a_0\mathcal{D}_f + b_0G_0$$

is squarefree. We have $G_1 \in A_{r-1}(Lf)$ and L does not divide G_1, hence by Lemma 2.1, Lf has a presentation (5) such that no l_i is projectively equal to l and this case has been considered earlier.

7. Proofs of Proposition 2 and of Corollary 4. First we prove the following

LEMMA 7.1. *The formula*

$$\sum_{i=1}^{d} \frac{(x+z_iy)^d}{\prod_{j=1,\, j\neq i}^{d}(z_i - z_j)} = dxy^{d-1} \tag{6}$$

for distinct z_i satisfying

$$z_1 + \cdots + z_d = 0 \tag{7}$$

gives all presentations of dxy^{d-1} of the minimal length d.

Proof. Since X^2 annihilates dxy^{d-1}, by Lemma 3.5 the minimal length of a presentation of dxy^{d-1} is d. By Proposition 1 every form in such a presentation depends essentially on x. An identity

$$\sum_{j=1}^{d} a_j(x+z_jy)^d = dxy^{d-1} \tag{8}$$

implies that

$$\sum_{j=1}^{d} a_jz_j^i = 0 \quad (i = 0, \ldots, d-2, d), \qquad \sum_{j=1}^{d} a_jz_j^{d-1} = 1, \tag{9}$$

hence

$$\begin{vmatrix} 1 & \ldots & 1 & 0 \\ z_1 & \ldots & z_d & 0 \\ \multicolumn{4}{c}{\dotfill} \\ z_1^{d-2} & \ldots & z_d^{d-2} & 0 \\ z_1^{d-1} & \ldots & z_d^{d-1} & 1 \\ z_1^d & \ldots & z_d^d & 0 \end{vmatrix} = - \begin{vmatrix} 1 & \ldots & 1 \\ z_1 & \ldots & z_d \\ \multicolumn{3}{c}{\dotfill} \\ z_1^{d-2} & \ldots & z_d^{d-2} \\ z_1^d & \ldots & z_d^d \end{vmatrix} = 0, \tag{10}$$

and since z_j are distinct, we have (7) by a well known formula (see [4, p. 333]) for the last determinant. The coefficients a_j are determined by a system of equations obtained from (9) for $i = 0, \ldots, d-1$. Using Cramer's formulae and the formula for the Vandermonde determinant we obtain

$$(11) \qquad a_j = \prod_{\substack{j=1\\ j\neq i}}^{d} (z_i - z_j)^{-1},$$

which gives (6). Conversely, if (7) and (11) are satisfied for distinct z_i, we obtain (10) and (9), hence (8).

Proof of Proposition 2. Let L_i be non-zero linear operators such that

$$L_i l_i = 0 \quad \text{ for } i = 1, \ldots, r+1, \ldots, r+u.$$

We shall prove below that

(∗) $f_1 = -f + (g_1 l_{r+1}^{d-e_1} + \cdots + g_u l_{r+u}^{d-e_u})$ is either exotic and

$$\mathcal{D}_{f_1} = L_1 \ldots L_r L_{r+1}^{e_1+1} \ldots L_{r+u}^{e_u+1},$$

or regular of even degree and

$$L_1 \ldots L_r L_{r+1}^{e_r+1} \ldots L_{r+u}^{e_u+1} \in A_{(d+2)/2}(f).$$

Denote $e_1 + \cdots + e_u$ by e. Notice that

$$D = L_1 \ldots L_r L_{r+1}^{e_1+1} \ldots L_{r+u}^{e_u+1}$$

annihilates f_1 and has

$$\begin{aligned} \deg D &= r + e + u = r + d - r(f_0) + 2 \\ &\le r(f_0) - \lfloor (d+2)/2 \rfloor + d - r(f_0) + 2 \\ &= d + 2 - \lfloor (d+2)/2 \rfloor = \lceil (d+2)/2 \rceil. \end{aligned}$$

By Lemma 3.5 and Theorem 2, in order to prove (∗), it remains to show that no proper factor of D annihilates f_1. Hence it remains to show that $L_i^{-1} D f_1 \neq 0$ for $i = 1, \ldots, r+u$. For $i = 1, \ldots, r$, this is clear:

$$L_i^{-1} D f_1 = c_i l_i^{d-e-u-r+1}, \qquad c_i \neq 0,$$

since $L_j l_i \neq 0$ whenever $j \neq i$. For $i = r+1, \ldots, r+u$, we have

$$\begin{aligned} L_i^{-1} D f_1 &= L_i^{-1} D g_{i-r} l_i^{d-e_{i-r}} \\ &= (L_i^{-e_{i-r}-1} D) L_i^{e_{i-r}} g_{i-r} l_i^{d-e_{i-r}} = a_i (L_i^{-e_{i-r}-1} D) l_i^{d-e_{i-r}}, \end{aligned}$$

where $a_i \in K \setminus \{0\}$, since g_{i-r} is of degree e_{i-r} and is not divisible by l_i. Then

$$(L_i^{-e_{i-r}-1} D) l_i^{d-e_{i-r}} = b_i l_i^{d-e-u} \neq 0, \qquad b_i \in K \setminus \{0\},$$

since no L_j with $j \neq i$ annihilates l_i.

Then we consider the decomposition of f_1 given by

$$f_1 = -l_1^d - \cdots - l_r^d + (g_1 l_{r+1}^{d-e_1} + \cdots + g_u l_{r+u}^{d-e_u}).$$

We have $r(f_1) = d - (r + (e+u)) + 2$. Denote $r(f_1)$ by r_1. Let

$$f_1 = m_1^d + \cdots + m_{r_1}^d$$

be a presentation of minimal length. Then

$$g_1 l_{r+1}^{d-e_1} + \dots + g_u l_{r+u}^{d-e_u} = l_1^d + \dots + l_r^d + m_1^d + \dots + m_{r_1}^d$$

is also of minimal length equal to $d-(e+u)+2$. Hence

$$f = l_1^d + \dots + l_r^d \preceq g_1 l_{r+1}^{d-e_1} + \dots + g_u l_{r+u}^{d-e_u},$$

where $g_1 l_{r+1}^{d-e_1} + \dots + g_u l_{r+u}^{d-e_u}$ is extreme.

This completes the proof of the first part of Proposition 2. In order to prove the second part we proceed in the following way. Let ζ_n be a primitive root of unity of order n and consider first $d=2r$. If $r=1$ we have $x^2 \not\preceq xy$, hence the inequality $r(f) \le r(f_0) - 2$ cannot be weakened. If $r > 1$ we shall show that $f \not\preceq f_0$, where

$$f = x^{2r} + \sum_{i=2}^{r} (x + \zeta_{r-1}^i y)^{2r}, \qquad f_0 = 2(1 - 2\lfloor 2/r \rfloor) r x y^{2r-1}.$$

Assuming the contrary we infer from Lemma 7.1 the existence of $z_1, \dots, z_{2r}$ in K such that

$$z_1 + \dots + z_{2r} = 0, \tag{12}$$

$$z_1 = 0, \qquad z_i = \zeta_{r-1}^i \quad (2 \le i \le r), \tag{13}$$

$$\prod_{\substack{j=1 \\ j \ne i}}^{2r} (z_i - z_j) = (1 - 2\lfloor 2/r \rfloor)^{-1} = 1 - 2\lfloor 2/r \rfloor. \tag{14}$$

Now, (12) and (13) give

$$z_{r+1} + \dots + z_{2r} = -\lfloor 2/r \rfloor, \tag{15}$$

while (13) and (14) give

$$\prod_{j=r+1}^{2r} (-z_j) = (1 - 2\lfloor 2/r \rfloor) \prod_{j=2}^{r} (-\zeta_{r-1}^j)^{-1} = 2\lfloor 2/r \rfloor - 1,$$

$$\begin{aligned} \prod_{j=r+1}^{2r} (\zeta_{r-1}^i - z_j) &= (1 - 2\lfloor 2/r \rfloor) \prod_{\substack{j=1 \\ j \ne i}}^{r} (\zeta_{r-1}^i - \zeta_{r-1}^j)^{-1} \\ &= (1 - 2\lfloor 2/r \rfloor) \prod_{k=1}^{r-2} (1 - \zeta_{r-1}^k)^{-1} \\ &= (1 - 2\lfloor 2/r \rfloor)(r-1)^{-1} \qquad (2 \le i \le r). \end{aligned}$$

Denoting by y_k the $(r-k)$th elementary symmetric polynomial of $-z_{r+1}, \dots, -z_{2r}$ and using (11) we can write the above system of equations in the form

$$y_0 = 2\lfloor 2/r\rfloor - 1,$$

$$\sum_{k=0}^{r-2} \zeta_{r-1}^{ik} y_k = (1 - 2\lfloor 2/r\rfloor)(r-1)^{-1} - \zeta_{r-1}^{ir} - \zeta_{r-1}^{i(r-1)} \lfloor 2/r\rfloor.$$

For $r = 2$ we obtain an insoluble system of equations $y_0 = 1$, $y_0 = -3$. For $r > 2$ solvability of the system implies

$$\mathcal{F} = \begin{vmatrix} 1 & 0 & \dots & 0 & -1 \\ 1 & 1 & \dots & 1 & (r-1)^{-1} - 1 \\ \hdotsfor{5} \\ 1 & \zeta_{r-1}^{r-2} & \dots & (\zeta_{r-1}^{r-2})^{r-2} & (r-1)^{-1} - \zeta_{r-1}^{r-2} \end{vmatrix} = 0. \tag{16}$$

Adding the second column to the last one and then developing the determinant according to the first row we obtain

$$\begin{aligned} \mathcal{F} &= (r-1)^{-1} \prod_{i=0}^{r-2} \zeta_{r-1}^{i} \prod_{j>i} (\zeta_{r-1}^{j} - \zeta_{r-1}^{i}) + (-1)^r \prod_{j>i} (\zeta_{r-1}^{j} - \zeta_{r-1}^{i}) \\ &= (-1)^r \frac{r-2}{r} \prod_{j>i} (\zeta_{r-1}^{j} - \zeta_{r-1}^{i}) \neq 0, \end{aligned}$$

contrary to (16). This contradiction shows that $f \not\prec f_0$.

Consider now $d = 2r + 1$. If $r = 1$, we have $(x + 2y)^3 \not\prec 27xy^2$, since $(x+2y)^3 - 27xy^2$ has the factor $x - y$ with multiplicity 2. Thus the inequality $r(f) \le r(f_0) - 3$ cannot be weakened.

If $r > 1$, we shall show that $f \not\prec f_0$, where

$$f = \sum_{i=1}^{r} (x + \zeta_r^i r^{-1/2r} y)^{2r+1} = r x^{2r+1} + r^{1/2} \binom{2r+1}{r} x^{r+1} y^r + \binom{2r+1}{2r} x y^{2r},$$

$$f_0 = (2r+1) x y^{2r}.$$

Assuming the contrary we infer from Lemma 7.1 the existence of distinct $z_1, \dots, z_{2r+1} \in K$ such that

$$z_1 + \dots + z_{2r+1} = 0, \tag{17}$$

$$z_i = \zeta_r^i r^{-1/2r}, \qquad \prod_{\substack{j=1 \\ j \neq i}}^{2r+1} (z_i - z_j) = 1 \quad (1 \le i \le r). \tag{18}$$

Now, (17) and (18) give

$$z_{r+1} + \cdots + z_{2r+1} = 0, \tag{19}$$

$$\prod_{j=r+1}^{2r+1} (\zeta_r^i r^{-1/2r} - z_j) = \prod_{\substack{j=1\\ j\neq i}}^{r} (\zeta_r^i r^{-1/2r} - \zeta_r^j r^{-1/2r})^{-1}$$

$$= r^{(r-1)/2r} (\zeta_r^i)^{1-r} \prod_{k=r-1}^{r-1} (1-\zeta_r^k)^{-1} = r^{-(r+1)/2r} \zeta_r^{i(r+1)},$$

hence denoting by y_k the $(r+1-k)$th elementary symmetric polynomial of $-z_{r+1}, \ldots, -z_{2r+1}$ and using (19) we obtain

$$\zeta_r^{i(r+1)} r^{-(r+1)/2r} + \sum_{k=0}^{r-1} (\zeta_r^{ri} r^{-1/2r})^k y_k = \zeta_r^{i(r+1)} r^{-(r+1)/2r}. \tag{20}$$

It follows from (19) and (20) that $y_k = 0$ $(0 \le k \le r)$, hence $z_{r+1} = \cdots = z_{2r+1} = 0$, contrary to the condition that the z_i are distinct. This contradiction shows that $f \not\prec f_0$ and completes the proof.

Proof of Corollary 4. For $r \le d - \lceil (d+2)/2 \rceil = \lfloor d/2 \rfloor - 1$ we infer from Proposition 2 that the form $f = l_1^d + \cdots + l_r^d$, where l_i are projectively distinct and not divisible by y, satisfies $f \prec xy^{d-1}$. Thus for any two linearly independent linear forms l and m such that $m \nmid l_i$ $(1 \le i \le r)$ we have $f \prec lm^{d-1}$.

8. Proofs of Propositions 3 and 4

LEMMA 8.1. *Let d be odd, $u = (d+1)/2$, and $f = \sum_{i=1}^{u} l_i^d$. Let l^d be projectively different from all l_i^d, $i = 1, \ldots, u$. For some $a \in K^*$, $f + al^d$ admits a minimal presentation of length $u+1$ if and only if $A_{u+1}(f)$ contains an operator $D = D_1 L D_2$, where $Ll = 0$, $D_1, D_2 \in K[X,Y], D_1$ is squareful,* $\deg D_1 > 0$ *and $D_1 D_2$ is not divisible by L.*

Proof. Assume that $f + al^d$ has a minimal presentation of length $u+1$. It is sufficient to consider the case $a = 1$. Then $\mathcal{D}_{f+l^d}$ is of degree u and $\mathcal{D}_{f+l^d} = D_1 D_2$, where D_1 is squareful, $\deg D_1 > 0$ and

$$\mathcal{D}_{f+l^d} f = \mathcal{D}_{f+l^d}(f + l^d) - \mathcal{D}_{f+l^d} l^d = 0 - cl^{d-u} = -cl^{d-u}.$$

Thus

$$D_1 L D_2 f = 0$$

and $D_1 L D_2 \in A_{u+1}(f)$. Moreover, since, by Proposition 1, no linear factor of $\mathcal{D}_{f+l^d}$ can annihilate a summand in the minimal presentation of $f + l^d$, it follows that $D_1 D_2$ is not divisible by L.

Now, assume that $D_1 L D_2 \in A_{u+1}(f)$, where D_1, D_2, L satisfy the assumptions of the lemma. Then $D_1 D_2(f) \neq 0$ is annihilated by L and hence $D_1 D_2 f = bl^{d-u}$ for some $b \in K^*$. Since $D_1 D_2$ is not divisible by L, we have

$D_1D_2(l^d) \neq 0$. Thus, for some $a \in K^*$, we have $D_1D_2(f + al^d) = 0$. Notice now that $f + al^d$ admits a presentation of length $u + 1$,

$$f + al^d = l_1^d + \cdots + l_u^d + al^d$$

and this presentation is of minimal length. In fact, $f + al^d$ cannot have presentations of length less than u, since otherwise f would admit two different presentations of length u (only one of them containing al^d). Hence the minimal length of presentation of $f + al^d$ is at least u. If it were u, then $A(f + al^d)_u$ would contain both D_1D_2 and a squarefree operator $\mathcal{D}_{f+al^d}$. This is not possible, since $u = (d + 1)/2$.

Thus the minimal length of presentation of $f+al^d$ is $u+1$. This completes the proof of the lemma.

Proof of Proposition 3. Necessity of the condition follows from Theorems 3 and 4. In order to prove sufficiency, consider first the case where f is plain. Let $f = l_1^d + \cdots + l_u^d$ be its presentation of minimal length. Since f is plain, $2u < d + 1$. Let l be a non-zero linear form projectively different from all $l_1, \ldots, l_u$ and let $g = f + l^d$. Then $g = l_1^d + \cdots + l_u^d + l^d$ is a presentation of minimal length. In fact, otherwise $g = l_1^d + \cdots + l_u^d + l^d = r_1^d + \cdots + r_u^d$ for some linear forms $r_1, \ldots, r_u$ and we would have

$$l_1^d + \cdots + l_u^d + l^d - r_1^d - \cdots - r_u^d = 0.$$

Since $l_1, \ldots, l_u, l$ are projectively different and $2u + 1 \leq d + 1$, this is not possible by Lemma 5.3 of [1]. It follows that for any form l as above, there exists an extreme form $h \succ f$ such that l^d appears in a presentation of $h - f$ of minimal length. In fact, every extreme form $h \succ g \succ f$ has this property.

Now, if there exists a plain form which is dominated by only finitely many extreme forms, then there also exists such a plain form with minimal length of presentation equal to $u = \lfloor d/2 \rfloor$. Let us consider this case. If an extreme form h dominates f, then $h - f$ is plain, since $h - f$ admits a presentation of length at most $r(h) - r(f) \leq d - u \leq d/2 < (d + 1)/2$. Thus the presentation of $h - f$ is unique and if the number of extreme forms dominating f is finite, only a finite number of powers of linear forms can appear in all presentations of differences $h - f$, where $h \succ f$ and h is extreme. This contradicts the result proved above saying that every form l projectively different from $l_1, \ldots, l_u$ appears in a minimal presentation of some difference $h - f$, where h is a properly chosen extreme form dominating f.

Consider the case where f is regular of odd degree d. Then f can be presented as $l_1^d + \cdots + l_u^d$, where $u = (d + 1)/2$, and the presentation is unique. Consider the space $A_{u+1}(f)$, of differential operators of degree $u + 1$ annihilating f. Then

$$A_{u+1}(f) = \{a\mathcal{D}_f X + b\mathcal{D}_f Y + c\mathcal{E}_f : a, b, c \in K\},$$

hence is of dimension 3. Moreover $\mathcal{D}_f = L_1 \ldots L_u$, where $L_1, \ldots, L_u$ are non-zero differential operators such that $L_i l_i = 0$ for $i = 1, \ldots, u$. Let W be the subset of $A_{u+1}(f)$ corresponding to a, b, c such that $\operatorname{discr}(a\mathcal{D}_f X + b\mathcal{D}_f Y + c\mathcal{E}_f) = 0$. Then $\dim W = 2$. Take the component W_1 of W which contains $L_1^2 L_2 \ldots L_u$. Then W_1 contains an open non-empty, hence two-dimensional, subset consisting of the operators $M_1^2 M_2 \ldots M_u$, where $M_1, \ldots, M_u$ are projectively different linear operators. If among $M_2, \ldots, M_u$ there are infinitely many projectively different linear operators, then it follows from Lemma 8.1 that for infinitely many linear forms l_{u+1}, $f + l_{u+1}^d$ is of minimal length $u+1$. Hence $f \prec g$ for infinitely many extreme forms g.

Otherwise, every operator from W_1 can be presented as $M^2 L_2 \ldots L_u$, where M is linear and can be chosen to be projectively different from L_1. Then

$$M^2 L_2 \ldots L_u f = M^2 L_2 \ldots L_u l_1^d = e l_1^{d-u-1}$$

for some $e \in K^*$. This contradicts the assumption that $M^2 L_2 \ldots L_u \in A_{u+1}(f)$.

This proves that there are infinitely many extreme forms g such that $f \prec g$.

Remark. It follows from the proof that, in fact, each f as above is dominated by infinitely many extreme forms g whose minimal length of presentation is maximal (equal to d). The number of remaining extreme forms dominating f is finite.

Proof of Corollary 5. For $r(f) \le (d+1)/2$ the inequality $g \prec f$ has finitely many solutions by Lemma 5.2 of [1], and the reverse inequality has infinitely many solutions by Proposition 3. For $r(f) > (d+1)/2$ the inequality $g \prec f$ has infinitely many solutions by Theorem 5 of [1] and the reverse inequality has finitely many solutions by Theorems 3 and 4.

Proof of Proposition 4. Let $f = l_1^d + \cdots + l_r^d$ be a form and let $r = r(f) \le (d+1)/2$. Then $\mathcal{D}_f = L_1 \ldots L_r$, where L_i are non-zero linear operators such that $L_i l_i = 0$ for $i = 1, \ldots, r$. We already know that always $(\mathcal{D}_f, \mathcal{E}_f) = 1$. Let

$$\mathcal{E}_f = M_1^{e_1+1} \ldots M_u^{e_u+1} M_{u+1} \ldots M_{u+t},$$

where $M_1, \ldots, M_{u+t}$ are projectively distinct linear operators and

$$0 < e_1 + \cdots + e_u + u < (d+2)/2.$$

Let $m_1, \ldots, m_{u+t}$ be non-zero linear forms such that $M_i m_i = 0$, $i = 1, \ldots, u+t$. It follows from Lemma 2.1 that

$$f = g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u} + a_1 m_{u+1}^d + \cdots + a_t m_{u+t}^d,$$

where g_i are forms of degree e_i and $a_j \in K$, for $i = 1, \ldots, u$, $j = 1, \ldots, t$. Notice that $g_i \ne 0$, $a_j \ne 0$, since otherwise f would be annihilated by a proper factor of $\mathcal{E}_f$. Then Theorem 2 implies that

$$f_0 = g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u}$$

is extreme. Moreover, $f \preceq f_0$. Indeed,

$$f_0 = f - (a_1 m_{u+1}^d + \cdots + a_{u+t} m_{u+t}^d) = l_1^d + \cdots + l_r^d - (a_1 m_{u+1}^d + \cdots + a_t m_{u+t}^d)$$

and this is a presentation of f_0 of minimal length, since

$$\begin{aligned} r(f_0) &= d - (e_1 + \cdots + e_u + u) + 2 \\ &= d - (e_1 + \cdots + e_u + u + t) + t + 2 = d - \deg \mathcal{E}_f + t + 2 \\ &= r(f) + t = r + t, \end{aligned}$$

and this is exactly the length of the above presentation of f_0. Thus $f \preceq f_0$. Also $r(f_0) > (d+1)/2 \geq r(f)$. Hence $f \prec f_0$.

Conversely, let $f \prec f_0$, where f_0 is extreme. Then it follows from our assumptions and Theorem 2 that

$$\begin{aligned} f_0 &= g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u} = l_1^d + \cdots + l_t^d, \\ f &= l_1^d + \cdots + l_r^d, \qquad r < t, \end{aligned}$$

for some linear forms m_i, l_j with $e_1 + \cdots + e_u + u < (d+2)/2$ and $t = d - (e_1 + \cdots + e_u + u) + 2$. Thus

$$f = g_1 m_1^{d-e_1} + \cdots + g_u m_u^{d-e_u} - (l_{r+1}^d + \cdots + l_t^d)$$

and an operator

$$L_{r+1} \ldots L_t M_1^{e_1+1} \ldots M_u^{e_u+1}$$

of degree

$$t - r + e_1 + \cdots + e_u + u = d - (e_1 + \cdots + e_u + u) + 2 - r + e_1 + \cdots + e_u + u = d - r + 2$$

annihilates f. This completes the proof.

9. Proof of Theorem 5

Lemma 9.1. *If $A(f) = A(g)$, then $f/g \in K \setminus \{0\}$.*

Proof. See [2, Lemma 2.12].

Lemma 9.2. *A regular form of degree $2m$ is extreme if and only if either it is equivalent via a linear transformation to $x^m y^m$, or in $A_{m+1}(f)$ there are exactly three projectively distinct non-zero forms with a square factor and each of them is squareful.*

Proof. By Lemma 2.2, $A_{m+1}(f)$ is generated by two coprime forms $\mathcal{D}_1$ and $\mathcal{D}_2$ of degree $m+1$. Let

$$B_1(X) = \mathcal{D}_1(X, 1), \qquad B_2(X) = \mathcal{D}_2(X, 1).$$

Since $\mathcal{D}_1$, $\mathcal{D}_2$ are not both divisible by Y we shall assume without loss of generality that $\deg B_1 = m+1$ and that B_1, B_2 are monic. Let

$$C = \begin{cases} B_2 & \text{if } \deg B_2 < \deg B_1, \\ B_2 - B_1 & \text{if } \deg B_2 = \deg B_1, \end{cases}$$

so that $n = \deg C \le m$, and let the leading coefficient of C be c. It follows from

$$B_1' B_2 - B_1 B_2' = B_1' C - B_1 C'$$

(prime denotes differentiation) that the coefficient of X^{m+n} in $B_1'B_2 - B_1B_2'$ is $c(m+1-n) \ne 0$, hence $B_1'B_2 - B_1B_2'$ is of degree $m+n > 0$, thus the set Z of its zeros is not empty. Let

$$B_1'B_2 - B_1B_2' = c(m+1-n)\prod_{z\in Z}(X-z)^{e(z)}. \tag{21}$$

It follows by induction on i that for every $z \in Z$ and for every positive integer $i \le e(z)$,

$$B_1^{(i)}(z)B_2(z) - B_1(z)B_2^{(i)}(z) = 0. \tag{22}$$

For every $z_0 \in Z$ the form

$$F = B_2(z_0)\mathcal{D}_1 - B_1(z_0)\mathcal{D}_2 \in A_{m+1}(f)$$

has the square factor $(X - z_0Y)^2$. Moreover $F \ne 0$ since $(\mathcal{D}_1, \mathcal{D}_2) = 1$ and $B_j(z_0) \ne 0$ for some $j = 1, 2$. If F is, up to projective equality, the only non-zero element of $A_{m+1}(f)$ with a square factor, then for every $z \in Z$ we have $B_j(z) \ne 0$ and

$$\frac{B_{3-j}(z)}{B_j(z)} = \frac{B_{3-j}(z_0)}{B_j(z_0)}.$$

Now, by (22),

$$P = B_{3-j}(X) - \frac{B_{3-j}(z_0)}{B_j(z_0)}\,B_j(X)$$

contains the factor $X - z$ with multiplicity $e(z)+1$, hence by (21),

$$m+1 \ge \deg P \ge \deg(B_1'B_2 - B_1B_2') + |Z| = m+n+|Z| > m+n,$$

thus $n = 0$. It follows that $B_2 - B_1 = b \in K \setminus \{0\}$, hence

$$bY^{m+1} = \mathcal{D}_2 - \mathcal{D}_1 \in A_{m+1}(f)$$

and $A_{m+1}(f)$ contains at least two non-zero, projectively distinct elements with a square factor, namely F and Y^{m+1}. This contradiction proves that the number of projectively distinct non-zero elements of $A_{m+1}(f)$ with a square factor is at least two.

We proceed to prove that there are at most three such elements. Let F_1, F_2 be two of them and suppose that

$$a_1F_1 + a_2F_2 = F_3, \qquad b_1F_1 + b_2F_2 = F_4, \tag{23}$$

where $a_i, b_i \in K \setminus \{0\}$, F_3, F_4 have a square factor and are projectively distinct. Since $(\mathcal{D}_1, \mathcal{D}_2) = 1$ the forms F_i $(1 \le i \le 4)$ are coprime, i.e. pairwise relatively prime, and since f is extreme, F_i are squareful. Let

$$\overline{F}_i(X) = F_i(X,1) = \prod_{z\in Z_i} (X-z)^{e_i(z)}. \tag{24}$$

We see that Z_i are disjoint (one Z_i may be empty) and

$$e_i(z) \ge 2 \quad (1 \le i \le 4,\ z \in Z_i). \tag{25}$$

From (20) we obtain

$$\begin{aligned} a_1\overline{F}_1 + a_2\overline{F}_2 &= \overline{F}_3,\\ b_1\overline{F}_1 + b_2\overline{F}_2 &= \overline{F}_4, \end{aligned}$$

and by differentiation

$$\begin{aligned} a_1(\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2') &= \overline{F}_3'\overline{F}_2 - \overline{F}_3\overline{F}_2',\\ b_1(\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2') &= \overline{F}_4'\overline{F}_2 - \overline{F}_4\overline{F}_2', \end{aligned}$$

thus

$$\prod_{i=1}^{4}(\overline{F}_i', \overline{F}_i) \mid (\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2'). \tag{26}$$

By (24) and (25),

$$\deg(\overline{F}_i', \overline{F}_i) \ge \frac{1}{2}\deg \overline{F}_i.$$

If $\deg \overline{F}_1 \ne \deg \overline{F}_2$, then

$$\deg \overline{F}_3 = \deg \overline{F}_4 = \max(\deg \overline{F}_1, \deg \overline{F}_2)$$

and (26) gives

$$\begin{aligned} \frac{3}{2}\max(\deg \overline{F}_1, \deg \overline{F}_2) &+ \frac{1}{2}\min(\deg \overline{F}_1, \deg \overline{F}_2)\\ &\le \max(\deg \overline{F}_1, \deg \overline{F}_2) + \min(\deg \overline{F}_1, \deg \overline{F}_2) - 1, \end{aligned} \tag{27}$$

a contradiction. If $\deg \overline{F}_i$ are equal, then the same argument applies. If $\deg \overline{F}_1 = \deg \overline{F}_2$ but $\deg \overline{F}_j < \deg \overline{F}_1$ for $j = 3$ or 4, say $j = 3$, we have

$$\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2' = \frac{1}{a_2}(\overline{F}_1'\overline{F}_3 - \overline{F}_1\overline{F}_3')$$

and (27) holds with $\overline{F}_2$ replaced by $\overline{F}_3$. This contradiction shows that the number of projectively distinct non-zero elements of $A_{m+1}(f)$ with a square factor is at most three.

It remains to prove that for a form f in question, $A_{m+1}(f)$ contains exactly two projectively distinct non-zero forms with a square factor if and only if f is equivalent to $x^m y^m$. Let the two relevant forms in $A_{m+1}(f)$ be F_1 and F_2, and let $\overline{F}_i(X) = F_i(X,1)$ for $i = 1,2$. We have

$$(\overline{F}_1, \overline{F}_2) = 1 \tag{28}$$

and we assert that

$$\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2' = c_1(\overline{F}_1, \overline{F}_1')(\overline{F}_2, \overline{F}_2'), \qquad c_1 \in K \setminus \{0\}. \tag{29}$$

If it were not so, then

$$H = \frac{\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2'}{(\overline{F}_1, \overline{F}_1')(\overline{F}_2, \overline{F}_2')}$$

would have a zero ζ. If ζ were a zero of $\overline{F}_i$ with multiplicity $m_1 > 0$, then ζ would be a zero of $\overline{F}_i/(\overline{F}_i, \overline{F}_i')$ of multiplicity 1, hence $H(\zeta) = 0$ would imply

$$\frac{\overline{F}_i'\overline{F}_{3-i}}{(\overline{F}_1, \overline{F}_1')(\overline{F}_2, \overline{F}_2')}(\zeta) = 0$$

and since $\frac{\overline{F}_i'}{(\overline{F}_i, \overline{F}_i')}(\zeta) \neq 0$ we should obtain $\overline{F}_{3-i}(\zeta) = 0$, contrary to (28). Therefore $\overline{F}_1(\zeta)\overline{F}_2(\zeta) \neq 0$ and the form $\overline{F}_2(\zeta)F_1 - \overline{F}_1(\zeta)F_2$ projectively distinct from F_1, F_2 would have the square factor $(X - \zeta Y)^2$. This proves (29).

Now, suppose that $\overline{F}_i$ has exactly q_i projectively distinct factors and $q_1 \geq q_2$. Then

$$\deg\,(\overline{F}_1, \overline{F}_1')(\overline{F}_2, \overline{F}_2') = \deg \overline{F}_1 + \deg \overline{F}_2 - q_1 - q_2 \tag{30}$$

and

$$\deg(\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2') = \deg \overline{F}_1 + \deg \overline{F}_2 - 1 \quad \text{ if } \deg \overline{F}_1 \neq \deg \overline{F}_2.$$

This gives, by (29), that $q_1 = 1$, $q_2 = 0$, thus $\overline{F}_1$ is a power of a linear polynomial, $\overline{F}_2 \in K \setminus \{0\}$. It follows that $F_1 = L^{m+1}$, $F_2 = Y^{m+1}$, where L is a linear form and by a linear transformation we can achieve that $F_1 = X^{m+1}$, $F_2 = Y^{m+1}$. Then f annihilated by F_1 and F_2 is projectively equal to $x^m y^m$.

It remains to consider the case where $\deg \overline{F}_1 = \deg \overline{F}_2$. Then $\overline{F}_2 = c_3\overline{F}_1 + R$, where $c_3 \neq 0$, $\deg R < \deg \overline{F}_1$. If $\deg R \leq \deg \overline{F}_1 - 2$, then $F_2 - c_3F_1$ is divisible by Y^2 and projectively distinct from F_1, F_2, contrary to the assumption. Thus $\deg R = \deg \overline{F}_1 - 1$ and

$$\deg(\overline{F}_1'\overline{F}_2 - \overline{F}_1\overline{F}_2') = \deg(\overline{F}_1'R - \overline{F}_1R') = \deg \overline{F}_1 + \deg R - 1 = 2\deg \overline{F}_1 - 2.$$

Hence from (29) and (30) we obtain

$$2\deg \overline{F}_1 - 2 = 2\deg \overline{F}_1 - q_1 - q_2,$$

thus $q_1 + q_2 = 2$, $q_1 = q_2 = 1$, $\overline{F}_i = L_i(X,1)^{m+1}$, where L_i is a linear form, $F_i = L_i^{m+1}$ and the previous argument shows that f annihilated via F_1, F_2 is equivalent via a linear transformation to $x^m y^m$.

Conversely, if $f = x^m y^m$, then $A_{m+1}(f)$ contains X^{m+1}, Y^{m+1} and up to a constant factor no other form with a square factor.

LEMMA 9.3. *For every positive integer s there exist up to a fractional linear transformation only finitely many rational functions $a, b \in K(x) \setminus \{0\}$ with the set S of zeros and poles such that $a + b = 1$, $|S| = s$ and $-\sum_{p\in S} \min\{\operatorname{ord}_p a, \operatorname{ord}_p b\} = s - 2$.*

Proof. See [6, Theorem 3] in which one takes $k \mapsto K$, $K \mapsto K(x)$.

Proof of Theorem 5. We begin by proving that the condition is necessary. By Lemma 9.2 it suffices to show that if P, Q, R are three projectively distinct forms in $A_{m+1}(f)$, then they are coprime and the total number of their projectively distinct linear factors is $m + 3$. The former fact results from Lemma 2.3. Also, replacing if necessary P, Q, R by their scalar multiples, we may assume that $R = P + Q$.

By a linear transformation we can achieve that $Y \mid Q$. Now, let

$$\overline{P} = P(X, 1), \quad \overline{Q} = Q(X, 1), \quad \overline{R} = R(X, 1)$$

and let $\omega(\overline{P}), \omega(\overline{Q}), \omega(\overline{R})$ denote the number of zeros of $\overline{P}, \overline{Q}, \overline{R}$, respectively. Since P and Q cannot be both divisible by Y we have

$$\deg \overline{P} = \deg \overline{R} = m + 1 > \deg \overline{Q}, \tag{31}$$

and the *abc*-theorem (see [6]) gives

$$\omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) \ge m + 2.$$

We shall show that equality holds here. Therefore assume

$$\omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) \ge m + 3. \tag{32}$$

Since

$$\deg(\overline{P}, \overline{P}') = \deg \overline{P} - \omega(\overline{P}), \qquad \deg(\overline{Q}, \overline{Q}') = \deg \overline{Q} - \omega(\overline{Q}),$$
$$\deg(\overline{R}, \overline{R}') = \deg \overline{R} - \omega(\overline{R}),$$

inequalities (31) and (32) give

$$\deg(\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}') = \deg \overline{Q} + m > \deg(\overline{P}, \overline{P}') + \deg(\overline{Q}, \overline{Q}') + \deg(\overline{R}, \overline{R}'). \tag{33}$$

The polynomials $(\overline{P}, \overline{P}'), (\overline{Q}, \overline{Q}'), (\overline{R}, \overline{R}')$ are coprime and each of them divides $\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}'$. For the first two this is evident, for the third it follows from the identity

$$2(\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}') = (\overline{P} + \overline{Q})(\overline{P}' - \overline{Q}') - (\overline{P}' + \overline{Q}')(\overline{P} - \overline{Q}). \tag{34}$$

By virtue of (33),

$$\frac{\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}'}{(\overline{P}, \overline{P}')(\overline{Q}, \overline{Q}')(\overline{R}, \overline{R}')}$$

has a zero ζ. If we had $\overline{P}(\zeta) = 0$ it would follow that

$$\frac{\overline{P}'\overline{Q}}{(\overline{P}, \overline{P}')}(\zeta) = 0,$$

contrary to

$$\left(\overline{P}, \frac{\overline{P}'\overline{Q}}{(\overline{P}, \overline{P}')}\right) = 1.$$

This contradiction shows that $\overline{P}(\zeta) \neq 0$ and similarly $\overline{Q}(\zeta) \neq 0$, while identity (34) implies $\overline{R}(\zeta) \neq 0$. Now consider the form

$$F = \overline{Q}(\zeta)P - \overline{P}(\zeta)Q.$$

F has the square factor $(X - \zeta Y)^2$, since $\overline{Q}(\zeta)\overline{P} - \overline{P}(\zeta)\overline{Q}$ has ζ as a multiple zero. However, since $(PQR)(\zeta) \neq 0$, F is projectively different from P, Q, R contrary to Lemma 9.2. This contradiction shows that

$$\omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) = m + 2.$$

Since Q is divisible by Y, the number of projectively distinct linear factors of PQR is $m + 3$.

We proceed to prove that the condition is sufficient. If P, Q, R are three coprime squareful forms in $A_{m+1}(f)$ with the number of projectively distinct linear factors of PQR equal to $m + 3$ we may assume that $P + Q = R$ and Q is divisible by Y. Let again

$$\overline{P} = P(X, 1), \quad \overline{Q} = Q(X, 1), \quad \overline{R} = R(X, 1).$$

The condition on PQR now gives

$$\omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) = m + 2. \tag{35}$$

By Lemma 9.2, it suffices to show that every non-zero form $F = aP + bQ$ $(a, b \in K)$ with a square factor is projectively equal ($\simeq$) to P, Q or R. If $Y^2 \mid F$, then since $Y \mid Q$ and $Y \nmid P$, we have $a = 0$ and $F \simeq Q$. If F has a square factor and $Y^2 \nmid F$, then $a\overline{P} + b\overline{Q}$ has a multiple zero ζ. Thus

$$a\overline{P}(\zeta) + b\overline{Q}(\zeta) = 0, \qquad a\overline{P}'(\zeta) + b\overline{Q}'(\zeta) = 0, \tag{36}$$

hence

$$(\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}')(\zeta) = 0. \tag{37}$$

However, as shown in the first part of the proof,

$$(\overline{P}, \overline{P'})(\overline{Q}, \overline{Q}')(\overline{R}, \overline{R}') \mid (\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}').$$

Now, by (35),

$$\begin{aligned} \deg\,(\overline{P}, \overline{P}')(\overline{Q}, \overline{Q}')(\overline{R}, \overline{R}') &= \deg \overline{P} + \deg \overline{Q} + \deg \overline{R} - \omega(\overline{P}) + \omega(\overline{Q}) - \omega(\overline{R}) \\ &= m + \deg \overline{Q} = \deg(\overline{P}'\overline{Q} - \overline{P}\,\overline{Q}'), \end{aligned}$$

hence

$$\overline{P}'\overline{Q} - \overline{P}\overline{Q}' = c(\overline{P}, \overline{P}')(\overline{Q}, \overline{Q}')(\overline{R}, \overline{R}'), \quad c \in K \setminus \{0\}.$$

Therefore (37) implies $\overline{P}(\zeta) = 0$ or $\overline{Q}(\zeta) = 0$ or $\overline{R}(\zeta) = 0$. If $\overline{P}(\zeta) = 0$, then $\overline{Q}(\zeta) \neq 0$ and (36) gives $b = 0$, thus $F \simeq P$. If $\overline{Q}(\zeta) = 0$, then similarly $a = 0$, thus $F \simeq Q$. Finally, if $\overline{R}(\zeta) = 0$, then $\overline{P}(\zeta) + \overline{Q}(\zeta) = 0$, thus by (36), $a = b$ and $F \simeq R$.

Proof of Corollary 6. We begin with a remark valid for every m. If any of the three squareful forms is a power of a linear form, then, since each of the others has at most $(m+1)/2$ projectively distinct linear factors, the total number of such factors is too small, namely at most $m+2$. Thus none of the three forms is a power of a linear form, which settles the case $m \leq 2$, where every squareful form is of this type. For $m = 3$ the relevant squareful forms are of the type X^2Y^2, thus we have $P = P_1^2$, $Q = Q_1^2$, $R = R_1^2$ where P_1, Q_1, R_1 have one linear factor each and $P_1^2 + Q_1^2 = R_1^2$. Therefore $R_1 + P_1$, $R_1 - P_1$ have one linear factor each, and taking them equal to $2X^2, 2Y^2$ we obtain

$$P = (X^2 - Y^2)^2, \quad Q = 4X^2Y^2, \quad R = (X^2 + Y^2)^2, \quad f \simeq x^5y - xy^5.$$

For $m = 4$ the relevant squareful forms are of type X^3Y^2, thus the total number of projectively distinct linear factors of three such forms is $6 < m+3$. For $m = 5$ the relevant squareful forms are of type X^4Y^2, X^3Y^3, or $X^2Y^2Z^2$. The condition on the total number of projectively distinct linear factors leads to $P = P_1^2$, $R = R_1^2$, where P_1, R_1 have three factors each, and $Q = R - P$ has two such factors. Therefore, $R_1 + P_1$, $R_1 - P_1$ have one linear factor each, and taking them equal to $2X^3, 2Y^3$ we obtain

$$P = (X^3 - Y^3)^2, \quad Q = X^3Y^3, \quad R = (X^3 + Y^3)^2, \quad f \simeq x^8y^2 - x^2y^8.$$

Proof of Corollary 7. For a form f of degree $2m$, coprime polynomials $P, Q \in A_{m+1}(f)$ determine $A(f)$ by virtue of Lemma 2.2, hence by Lemma 9.1, f is determined by them up to a constant factor. Transforming P, Q by a linear substitution results in transforming f by the inverse of that substitution, thus it suffices to prove that up to a linear substitution there are only finitely many triples $P, Q, R = P + Q$ of coprime forms of degree $m+1$, which together have exactly $m+3$ projectively distinct linear factors. Assuming without loss of generality that $Y \mid Q$ and putting

$$\overline{P} = P(X, 1), \quad \overline{Q} = Q(X, 1), \quad \overline{R} = R(X, 1)$$

we obtain

$$\omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) = m + 2.$$

Now, we apply Lemma 9.3 with $a = \overline{P}/\overline{R}$ and $b = \overline{Q}/\overline{R}$. Here

$$-\sum_{p\in S} \min\{\operatorname{ord}_p a, \operatorname{ord}_p b\} = \deg \overline{R} = m+1,$$

$$|S| = \omega(\overline{P}) + \omega(\overline{Q}) + \omega(\overline{R}) = m+3.$$

Thus the assumptions of the lemma are satisfied and so there exist, up to a fractional linear transformation, only finitely many pairs $\overline{P}/\overline{R}$, $\overline{Q}/\overline{R}$ in question. It follows that up to a linear transformation there exist only finitely many relevant triples P, Q, R.

Example (due to J. Browkin). For $m = 6$ the conditions of Theorem 5 are satisfied by the following forms in $K[X,Y]_{m+1}$:

$$P = (X-Y)^3(4X^2 - XY + Y^2)^2, \quad Q = Y^3(7X^2 - 7XY + 4Y^2)^2,$$
$$R = X^3(7XY + 7Y^2)^2.$$

10. Proof of Proposition 5. In all three cases we argue in the same way. First we notice that, in all three cases, the number on the right hand side is equal to $d - s(f) + 2 = \deg \mathcal{E}_f$. Denote that number by $t(f)$. Next notice that $\dim A_{t(f)}(f) \geq 2$, $t(f) \leq \overline{r}(f)$. Moreover, $A_{t(f)}(f)$ contains coprime forms and contains a Zariski dense subset consisting of squarefree forms. Then it follows from Lemma 2.1 that it is sufficient to prove that, for any given finite family $M_1, \ldots, M_t$ of projectively distinct linear operators, we can find linear forms $L_1, \ldots, L_r$ such that all forms $L_1, \ldots, L_r, M_1, \ldots, M_t$ are pairwise projectively distinct and $G = L_1 \ldots L_r \in A_{t(f)}(f)$. However, for $i = 1, \ldots, r$, M_i does not divide all forms from $A_{t(f)}(f)$, since $A_{t(f)}$ contains coprime forms. Thus forms in $A_{t(f)}(f)$ divisible by M_i belong to a hyperplane and hence for a Zariski dense subset $U \subset A_{t(f)}(f)$ the condition $G \in U$ implies that $M_i \nmid G$ for all $i = 1, \ldots, r$. On the other hand, as noticed above, for a Zariski dense subset $V \subset A_{t(f)}(f)$ the condition $G \in V$ implies that G is squarefree. Hence, $U \cap V \neq \emptyset$ and every form $G \in U \cap V$ satisfies the required conditions.

Acknowledgments. The research of A. Białynicki-Birula was partially supported by Polish MNiSW Grant N201 019 32/0805.

References

[1] A. Białynicki-Birula and A. Schinzel, *Representations of multivariate polynomials as sums of univariate polynomials in linear forms*, Colloq. Math. 112 (2008), 201–233.

[2] A. Iarrobino and V. Kanev, *Power Sums, Gorenstein Algebras and Determinantal Loci*, Lecture Notes in Math. 1721, Springer, 1999.

[3] J. Kleppe, *Representing a homogeneous polynomial as a sum of powers of linear forms*, Thesis for the degree of Candidatus Scientiarum, Department of Mathematics, Univ. of Oslo, 1999.
[4] Th. Muir, *A Treatise on the Theory of Determinants*, Dover, 1960.
[5] J. J. Sylvester, *On a remarkable discovery in the theory of canonical forms and of hyperdeterminant*, in: The Collected Mathematical Papers, Cambridge, 1904, 215–263.
[6] U. Zannier, *Some remarks on the S-unit equation in function fields*, Acta Arith. 64 (1993), 87–98.

Andrzej Białynicki-Birula
Department of Mathematics
University of Warsaw
Banacha 2
02-097 Warszawa, Poland
E-mail: bbirula@mimuw.edu.pl

Andrzej Schinzel
Institute of Mathematics
Polish Academy of Sciences
P.O. Box 21
00-956 Warszawa, Poland
E-mail: schinzel@impan.gov.pl

Received on 15.5.2009
and in revised form on 5.10.2009 (6029)