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On the zeros of degree one L-functions from the extended Selberg class

by

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1. Introduction. In [13], Selberg introduced the class $\mathcal{S}$ consisting of the functions $F(s)$ satisfying the following conditions.

(1) (Dirichlet series) For $\sigma > 1$, $F(s)$ is an absolutely convergent Dirichlet series

$$F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s} \qquad (s = \sigma + it).$$

(2) (Analytic continuation) For some integer $m \geq 0$, $(s-1)^m F(s)$ is an entire function of finite order.

(3) (Functional equation) $F(s)$ satisfies a functional equation of the form

$$\Phi(s) = \omega \bar{\Phi}(1-s),$$

where

$$\Phi(s) = Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s)$$

with $\bar{\Phi}(s) = \overline{\Phi(\bar{s})}$, $Q > 0$, $\lambda_j > 0$, $\operatorname{Re} \mu_j \geq 0$ and $|\omega| = 1$.

(4) (Ramanujan hypothesis) For every $\epsilon > 0$, $a(n) \ll n^{\epsilon}$.

(5) (Euler product) For σ sufficiently large,

$$\log F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s} \qquad (s = \sigma + it),$$

where $b(n) = 0$ unless n is a positive power of a prime, and $b(n) \ll n^{\theta}$ for some $\theta < 1/2$.

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For a function $F(s)$ in the Selberg class $\mathcal{S}$, we define $d = 2\sum_j \lambda_j$ to be the *degree* of F. We denote by $\mathcal{S}_d$ the subclass of functions of degree d in $\mathcal{S}$. We note that the structure of $\mathcal{S}_d$ has been completely determined for $0 \le d \le 1$. From the work of Conrey and Ghosh [4], we have $\mathcal{S}_0 = \{1\}$ and $\mathcal{S}_d = \emptyset$ for $0 < d < 1$. For $d = 1$, by Kaczorowski and Perelli [9], the functions $F \in \mathcal{S}_1$ are of the forms $F(s) = \zeta(s)$ or $F(s) = L(s + i\theta, \chi)$ with a primitive Dirichlet character χ and $\theta \in \mathbb{R}$. On the other hand, we denote by $\mathcal{S}^\#$ the extended Selberg class of functions satisfying conditions (1)–(3), and we define $\mathcal{S}_d^\#$ similarly to $\mathcal{S}_d$. Theorems 1 and 2 in [9] describe the structure of $\mathcal{S}_d^\#$ for $0 \le d \le 1$.

If $d = 0$, the functional equation is $Q^s F(s) = \omega Q^{1-s} \bar{F}(1-s)$. The proof of [9, Theorem 1] shows that the Dirichlet series $F(s) = \sum_n a(n)/n^s \in \mathcal{S}_0^\#$ is absolutely convergent in the whole complex plane. Thus, we have

$$\sum_{n=1}^{\infty} a(n)\left(\frac{Q^2}{n}\right)^s = \omega Q \sum_{n=1}^{\infty} \frac{\overline{a(n)}}{n} n^s.$$

We let $q = Q^2$; then $a(n) = 0$ for $n \nmid q$. For $n \mid q$, we have

$$a(n) = \frac{\omega n}{\sqrt{q}}\, \overline{a\left(\frac{q}{n}\right)}. \tag{1.1}$$

THEOREM A (Theorem 1 of [9]).

(1) *If* $0 < d < 1$, *then* $\mathcal{S}_d^\# = \emptyset$. *If* $F \in \mathcal{S}_0^\#$, *then* $q \in \mathbb{N}$, *the pair* (q, ω) *is an invariant of* $F(s)$ *and* $\mathcal{S}_0^\#$ *is the disjoint union of the subclasses* $\mathcal{S}_0^\#(q, \omega)$ *with* $q \in \mathbb{N}$ *and* $|\omega| = 1$.

(2) *Every* $F \in \mathcal{S}_0^\#(q, \omega)$ *with* q *and* ω *as above is a Dirichlet polynomial of the form*

$$F(s) = \sum_{n \mid q} \frac{a(n)}{n^s}.$$

For $d = 1$, we use the notation

$$\beta = \prod_{j=1}^{r} \lambda_j^{-2\lambda_j}, \qquad \xi = 2\sum_{j=1}^{r} (\mu_j - 1/2) = \eta + i\theta, \qquad q = \frac{2\pi Q^2}{\beta},$$

$$\omega^* = \omega e^{-i\pi(\eta+1)/2} \left(\frac{Q^2}{\beta}\right)^{i\theta} \prod_{j=1}^{r} \lambda_j^{-2i \operatorname{Im} \mu_j}.$$

If χ is a Dirichlet character modulo q, we denote by f_χ its conductor, and by χ^* the primitive character inducing χ. We denote by ω_{χ^*} and Q_{χ^*} the ω-factor and the Q-factor in the standard functional equation for $L(s, \chi^*)$, i.e., $\omega_{\chi^*} = \tau(\chi^*)/i^{\mathfrak{a}}\sqrt{f_\chi}$, where $\tau(\chi^*)$ is the Gauss sum, $\mathfrak{a} = 0$ if $\chi(-1) = 1$

and $\mathfrak{a}=1$ if $\chi(-1)=-1$, and $Q_{\chi^*}=\sqrt{f_\chi/\pi}$. Moreover, we write

$$\mathfrak{X}(q,\xi)=\begin{cases}\{\chi \bmod q \mid \chi(-1)=1\} & \text{if } \eta=-1,\\ \{\chi \bmod q \mid \chi(-1)=-1\} & \text{if } \eta=0.\end{cases}$$

χ_0 denotes the principal character modulo q.

Theorem B (Theorem 2 of [9]).

(1) *If $F\in\mathcal{S}_1^\#$, then $q\in\mathbb{N}$ and $\eta\in\{-1,0\}$. The triple (q,ξ,ω^*) is an invariant of $F(s)$, and $\mathcal{S}_1^\#$ is the disjoint union of the subclasses $\mathcal{S}_1^\#(q,\xi,\omega^*)$ with $q\in\mathbb{N}$, $\eta\in\{-1,0\}$, $\theta\in\mathbb{R}$ and $|\omega^*|=1$. Moreover, $a(n)n^{i\theta}$ is periodic with period q.*

(2) *Every $F\in\mathcal{S}_1^\#(q,\xi,\omega^*)$ with q, ξ and ω^* as above can be uniquely written as*

$$F(s)=\sum_{\chi\in\mathfrak{X}(q,\xi)}P_\chi(s+i\theta)L(s+i\theta,\chi^*),$$

where $P_\chi\in\mathcal{S}_0^\#(q/f_\chi,\omega^*\bar{\omega}_{\chi^*})$. Moreover, $P_{\chi_0}(1)=0$ if $\theta\neq 0$.

Bombieri and Hejhal [2] studied the distribution of zeros of the linear combinations $F(s)=\sum_{j=1}^J b_je^{i\alpha_j}L_j(s)$ of various L-functions with the same gamma factor. Assuming an orthonormality condition on $a_j(p)$ (where $a_j(n)$ are the coefficients of $L_j(s)$), the generalized Riemann hypothesis for $L_j(s)$ and a weak condition on the spacing of zeros of $L_j(s)$, they proved that almost all zeros of $F(s)$ are simple and on the critical line $\operatorname{Re}s=1/2$. Hejhal [6] studied the behavior of zeros of $F(s)$ near the critical line and announced that the true order of the number of zeros of $F(s)$ in $\operatorname{Re}s\geq\sigma$, $T\leq\operatorname{Im}s\leq T+H$ is

$$\frac{H}{(\sigma-1/2)\sqrt{\log\log T}}$$

for $1/2+(\log\log T)^\kappa/\log T\leq\sigma\leq 1/2+(\log T)^{-\delta}$, $c_1T^w\leq H\leq c_2T$, $\kappa>2$ with possibly few exceptional $\{b_j\}_{j=1}^J$. Note that this result for the special case $J=2$ was also justified by the same author in [5].

Recently, the second author [11] investigated the off-line zeros of the Epstein zeta function $E(s,Q)$ associated to the quadratic form $Q(x,y)=ax^2+bxy+cy^2$, $a>0$, $b^2-4ac<0$, $a,b,c\in\mathbb{Z}$. It is a classical example that belongs to the class $\mathcal{S}_2^\#$. We find the number of zeros $N_E(\sigma_1,\sigma_2;0,T)$ in the rectangular region $\sigma_1<\operatorname{Re}s<\sigma_2$, $0<\operatorname{Im}s<T$ to be $c(\sigma_1,\sigma_2)T+o(T)$ for $1/2<\sigma_1<\sigma_2$, which improves Voronin's result $N_E(\sigma_1,\sigma_2;0,T)\gg T$ for $1/2<\sigma_1<\sigma_2<1$ (see [14] or Chapter 7 of [10]) based on the joint distribution for Hecke L-functions. We observe that one can apply our method to degree one objects.

For $F \in \mathcal{S}^{\#}$, Kaczorowski and Kulas [8] defined the *density property* to be $N_F(\sigma, T) = o(T)$ for every fixed $1/2 < \sigma < 1$. This property classifies the elements in $\mathcal{S}_1^{\#}$. If $F \in \mathcal{S}_1^{\#}$ has the density property, then $F(s+i\theta) = P(s)L(s,\chi)$ for certain real θ, a Dirichlet polynomial $P \in \mathcal{S}_0^{\#}$ and a primitive Dirichlet character χ. Otherwise, $F(s+i\theta) = \sum_{j\le J} P_j(s)L(s,\chi_j)$ for $J \ge 2$, $\theta \in \mathbb{R}$, Dirichlet polynomials $P_j \in \mathcal{S}_0^{\#}$ and primitive inequivalent Dirichlet characters χ_j. For $F \in \mathcal{S}_1^{\#}$ violating the density property, they obtain $N_F(\sigma_1, \sigma_2; 0, T) \gg T$ for $1/2 < \sigma_1 < \sigma_2 < 1$. Saias and Weingartner [12] extend their method to the strip $1 < \operatorname{Re} s < 1 + \eta$ for some small $\eta > 0$ and achieve $N_F(\sigma_1, \sigma_2; 0, T) \gg T$ for $1/2 < \sigma_1 < \sigma_2 < 1 + \eta$. Our main purpose is to improve these results by obtaining an asymptotic formula for $N_F(\sigma_1, \sigma_2; 0, T)$.

By Theorems A and B, we can write the function $E(s+i\theta) \in \mathcal{S}_1^{\#}$ as

$$E(s) = \sum_{j=1}^{J} h_j(p_1^{-s}, \ldots, p_k^{-s}) \prod_{p > p_k} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1} \tag{1.2}$$

for some integer $k > 0$, where

$$h_j(x_1, \ldots, x_k) = \tilde{h}_j(x_1, \ldots, x_k) \prod_{l \le k} (1 - \chi_j(p_l) x_l)^{-1}$$

and $\tilde{h}_j$ is a polynomial of k variables. Let

$$E_n(s) = \sum_{j=1}^{J} h_j(p_1^{-s}, \ldots, p_k^{-s}) \prod_{p_k < p \le p_n} \left(1 - \frac{\chi_j(p)}{p^s}\right)^{-1}$$

for $n > k$. Then, $E_n(s)$ converges in the mean with index 2 towards $E(s)$ in $[1/2, \infty]$ by Parseval's identity for almost periodic functions, i.e.,

$$\limsup_{T\to\infty} \frac{1}{T} \int_1^T \int_\alpha^\beta |E(\sigma+it) - E_n(\sigma+it)|^2 \, d\sigma \, dt \to 0$$

as $n \to \infty$ for any $1/2 < \alpha < \beta$ (for the method of proof, see Proposition 2.3 of [11]). Applying Lemma 2.3 to $E_n(s)$, we get an asymptotic formula for $N_{E_n}(\sigma_1, \sigma_2; 0, T)$. The theory of mean motions partially preserves this property through the convergence in the mean with index $p > 0$ via Lemma 2.4.

If $J = 1$, then we encounter the Riemann hypothesis. Our method does not work in this case, since we are using the Euler product $\log \zeta(s) = \sum_p \sum_{m=1}^{\infty} \frac{1}{mp^{ms}}$ and this cannot give any information about $\zeta(s) = 0$. Concerning this matter, see Borchsenius and Jessen [3]. From now on, we only consider the case $J > 1$.

We consider $\mathcal{S}_1^{\#}(p,\xi,\omega^*)$ for p prime or 1. By (1.1) and Theorem B, we have

$$\tilde{h}_j = a_j(1) \text{ or } a_j(1) + \frac{\omega\overline{a_j(1)}}{p^{s-1/2}},$$

and as a result $\tilde{h}_j \neq 0$ for $\operatorname{Re} s > 1/2$. In this case, the method in [11] works, and we have the following theorem.

THEOREM 1.1. *Let $E(s+i\theta) \in \mathcal{S}_1^{\#}(p,\xi,\omega^*)$ for p prime or $p=1$, and $|\omega|=1$, and let $1/2 < \sigma_1 < \sigma_2$. Suppose $J > 1$ in* (1.2)*. Then*

$$N_E(\sigma_1,\sigma_2;0,T) = c(\sigma_1,\sigma_2)T + o(T)$$

as $T \to \infty$. The constant $c(\sigma_1,\sigma_2)$ can be represented as an integral $\int_{\sigma_1}^{\sigma_2} H_\sigma(0)\,d\sigma$ for the density function $H_\sigma(x)$ of some distribution μ_σ, and $c(\sigma_1,\sigma_2) > 0$ if $1/2 < \sigma_1 \le 1$. In particular, for $\sigma_0 > 1/2$, the number of zeros on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ is $o(T)$.

When q is a prime power, the $\tilde{h}_j$ are polynomials of the same single variable by Theorems A(2) and B(2). If these polynomials have the same factor with $cT + o(T)$ zeros on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ for some $1/2 < \sigma_0 < 1$, then we cannot expect the integral form of the constant $c(\sigma_1,\sigma_2)$ in general. Indeed, we may take $\tilde{h}_j(p^{-s}) = 1 + 2p^{3/4-s} + p^{1-2s}$ by letting $\omega = a(1) = 1$, and $a(p) = 2p^{-3/4}$. Then the function $s \mapsto \tilde{h}_j(p^{-s})$ has $\frac{\log p}{2\pi}T + O(1)$ zeros on $\operatorname{Re} s = \log(p^{3/4} + \sqrt{p^{3/2}-p})/\log p$, $0 < \operatorname{Im} s < T$. We still have the following.

THEOREM 1.2. *Let $E(s+i\theta) \in \mathcal{S}_1^{\#}(q,\xi,\omega^*)$ for q a prime power, and let $1/2 < \sigma_1 < \sigma_2$. Suppose $J > 1$ in* (1.2)*. Then*

$$N_E(\sigma_1,\sigma_2;0,T) = c(\sigma_1,\sigma_2)T + o(T)$$

as $T \to \infty$, and $c(\sigma_1,\sigma_2) > 0$ if $1/2 < \sigma_1 \le 1$. Suppose that the closed interval $[\sigma_1,\sigma_2]$ does not contain the real part of exceptional points satisfying $h_j = 0$. Then the constant $c(\sigma_1,\sigma_2)$ can be represented as an integral $\int_{\sigma_1}^{\sigma_2} H_\sigma(0)\,d\sigma$ for the density function $H_\sigma(x)$ of some distribution μ_σ. In this case for $\sigma_0 \in [\sigma_1,\sigma_2]$, the number of zeros on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ is $o(T)$.

For general q, we could also prove a similar theorem, although it is not easy to classify the common zeros of $\tilde{h}_j$ with multiple variables. We will discuss and prove a general theorem in Section 3.

2. Lemmas. We begin with the work of Jessen and Tornehave [7] that concerns zeros of a Dirichlet series in the region of its absolute convergence. For the basic theory of almost periodic functions, we refer to [1].

Lemma 2.1 (Theorem 8 of [7]). *A function $f(s)$ almost periodic in $[\alpha,\beta]$ and not identically zero has no zeros in the substrip $(\alpha\le)\ \alpha_0<\sigma,\beta_0\ (\le\beta)$, if and only if its Jensen function*

$$\varphi(\sigma)=\lim_{T_2-T_1\to\infty}\frac{1}{T_2-T_1}\int_{T_1}^{T_2}\log|f(\sigma+it)|\,dt$$

is linear in the interval (α_0,β_0).

Lemma 2.2 (Theorem 31 of [7]). *For an ordinary Dirichlet series*

$$f(s)=\sum_{n=n_0}^{\infty}\frac{a_n}{n^s},\qquad a_{n_0}\neq 0,$$

with the uniform convergence abscissa α, the Jensen function $\varphi(\sigma)$ has on every half-line $\sigma>\alpha_1\ (>\alpha)$ only a finite number of linearity intervals and a finite number of points of non-differentiability. The values of $\varphi'(\sigma)$ in the linearity intervals belong to the set of numbers $-\log n$, $n\ge n_0$. For $\sigma>$ (some) σ_0, we have

$$\varphi(\sigma)=-(\log n_0)\sigma+\log|a_{n_0}|.$$

For an arbitrary strip (σ_1,σ_2), where $\alpha<\sigma_1<\sigma_2<\infty$, the relative frequency $H(\sigma_1,\sigma_2)$ of zeros exists and is determined by

$$H(\sigma_1,\sigma_2)=\frac{1}{2\pi}(\varphi'(\sigma_2-)-\varphi'(\sigma_1+)).$$

The following lemma guarantees the existence of the second derivative of Jensen functions for almost periodic functions and gives another representation by a certain distribution. The proof can be found in §9 of [3].

Lemma 2.3 (Proposition 2.1 of [11]). *Let $f(s)$ be almost periodic in the strip $[\alpha,\beta]$ and not identically zero. Let ν_σ be the asymptotic distribution function of $f(\sigma+it)$ with respect to $|f'(\sigma+it)|^2$. Suppose ν_σ is absolutely continuous for every σ and its density $G_\sigma(x)$ is a continuous function of x and σ. Then the Jensen function $\varphi_{f-x}(\sigma)$ is twice differentiable with $\varphi''_{f-x}(\sigma)=2\pi G_\sigma(x)$.*

The next lemma is an extension of Lemma 2.3 which is applicable inside the critical strip and which plays the main role in this method.

Lemma 2.4 (Theorem 1 of [3]). *Let $-\infty\le\alpha<\alpha_0<\beta_0<\beta\le\infty$ and let $f_1(s),f_2(s),\dots$ be a sequence of functions almost periodic in $[\alpha,\beta]$ converging uniformly in $[\alpha_0,\beta_0]$ towards a function $f(s)$. Suppose that none of the functions is identically zero and $f(s)$ may be continued as a regular function in the half-strip $\alpha<\sigma<\beta$, $t>\gamma_0$, and that $f_n(s)$ converges in* *mean with an index* $p>0$ *towards* $f(s)$ *in* $[\alpha,\beta]$*. Then the Jensen function*

$$\varphi_f(\sigma)=\lim_{T\to\infty}\frac{1}{T}\int_{\gamma}^{T}\log|f(\sigma+it)|\,dt$$

exists uniformly in $[\alpha,\beta]$ *for some* $\gamma>\gamma_0$*, and* $\varphi_{f_n}(\sigma)$ *converges uniformly in* $[\alpha,\beta]$ *towards* $\varphi_f(\sigma)$ *as* $n\to\infty$*. The function* $\varphi_f(\sigma)$ *is convex in* (α,β)*, and for every strip* (σ_1,σ_2) *where* $\alpha<\sigma_1<\sigma_2<\beta$*, the two relative frequencies of zeros defined by*

$$\underline{H}_f(\sigma_1,\sigma_2)=\liminf_{T\to\infty}\frac{1}{T}N_f(\sigma_1,\sigma_2;\gamma,T),$$
$$\overline{H}_f(\sigma_1,\sigma_2)=\limsup_{T\to\infty}\frac{1}{T}N_f(\sigma_1,\sigma_2;\gamma,T),$$

satisfy the inequalities

$$\begin{aligned}\frac{1}{2\pi}(\varphi_f'(\sigma_2-)-\varphi_f'(\sigma_1+))\le\underline{H}_f(\sigma_1,\sigma_2)&\le\overline{H}_f(\sigma_1,\sigma_2)\\&\le\frac{1}{2\pi}(\varphi_f'(\sigma_2+)-\varphi_f'(\sigma_1-)).\end{aligned}$$

Suppose further that $\varphi_f(\sigma)$ *is twice differentiable. Then*

$$N_f(\sigma_1,\sigma_2;0,T)=\frac{T}{2\pi}\int_{\sigma_1}^{\sigma_2}\varphi_f''(\sigma)\,d\sigma+o(T)$$

for $\alpha<\sigma_1<\sigma_2<\beta$ *as* $T\to\infty$*.*

Together with the above lemmas, we investigate the Fourier transforms of certain distributions. We need two more lemmas, in which we use the following notation:

$$\begin{aligned}\mathfrak{L}_n(\sigma,\Theta;\chi_j)&=L_k(\sigma,\theta;\chi_j)L_{k,n}(\sigma,\vartheta;\chi_j),\\L_k(\sigma,\theta;\chi_j)&=h_j(p_1^{-\sigma}e^{2\pi i\theta_1},\dots,p_k^{-\sigma}e^{2\pi i\theta_k}),\\L_{k,n}(\sigma,\vartheta;\chi_j)&=\prod_{k<l\le n}\left(1-\frac{\chi_j(p_l)e^{2\pi i\vartheta_l}}{p_l^{\sigma}}\right)^{-1},\\M_{n,\sigma}(\vartheta)&=(\log L_{k,n}(\sigma,\vartheta;\chi_1),\dots,\log L_{k,n}(\sigma,\vartheta;\chi_J)),\\E_{n,\sigma}(\Theta)&=\sum_{j=1}^{J}\mathfrak{L}_n(\sigma,\Theta;\chi_j)\end{aligned}$$

for $n>k$, $\Theta=(\theta,\vartheta)\in[0,1]^n$, $\theta=(\theta_1,\dots,\theta_k)\in[0,1]^k$ and $\vartheta=(\vartheta_{k+1},\dots,\vartheta_n)\in[0,1]^{n-k}$. Let $\mu_{n,\sigma}$ be the distribution function of $E_{n,\sigma}$ with respect to $\left|\frac{\partial}{\partial\sigma}E_{n,\sigma}\right|^2$. Its Fourier transform is

$$\hat{\mu}_{n,\sigma}(y)=\int_{[0,1]^n}e^{i\sum_j\mathfrak{L}_n(\sigma,\Theta;\chi_j)\cdot y}\Big|\sum_j\mathfrak{L}_n'(\sigma,\Theta;\chi_j)\Big|^2d\Theta.$$

Lemma 2.5. *For $\sigma > 1/2$, $\delta > 0$ and $j \le J$, define*

$$A_{j,\sigma}(\delta) = \{\theta \in [0,1]^k : |\tilde{h}_j(p_1^{-\sigma} e^{2\pi i\theta_1}, \ldots, p_k^{-\sigma} e^{2\pi i\theta_k})| < \delta\}.$$

Then for any integer $K \le J$ we have

$$\hat{\mu}_{n,\sigma}(y) \ll \Big| \bigcap_{r_1 < \cdots < r_K \le J} (A_{r_1,\sigma}(\delta) \cup \cdots \cup A_{r_K,\sigma}(\delta)) \Big| + |\delta y|^{-K}$$

as $|y| \to \infty$, where the corresponding constant does not depend on n.

Proof. We write

$$\hat{\mu}_{n,\sigma}(y) = \sum_{l_1,l_2} \int_{[0,1]^n} e^{i\sum_j \mathfrak{L}_n(\sigma,\Theta;\chi_j)\cdot y} \mathfrak{L}_n'(\sigma,\Theta;\chi_{l_1}) \overline{\mathfrak{L}_n'(\sigma,\Theta;\chi_{l_2})}\, d\Theta.$$

Define set functions

$$\lambda_{n,\sigma;l_1,l_2}(B) = \int_{M_{n,\sigma}^{-1}(B)} \frac{L_{k,n}'}{L_{k,n}}(\sigma,\vartheta;\chi_{l_1}) \overline{\frac{L_{k,n}'}{L_{k,n}}(\sigma,\vartheta;\chi_{l_2})}\, d\vartheta,$$

$$\lambda_{n,\sigma;l}(B) = \int_{M_{n,\sigma}^{-1}(B)} \frac{L_{k,n}'}{L_{k,n}}(\sigma,\vartheta;\chi_l)\, d\vartheta,$$

$$\lambda_{n,\sigma}(B) = |M_{n,\sigma}^{-1}(B)|,$$

for any Borel set $B \subset \mathbb{C}^J$. Applying the identity

$$a\bar{b} = \frac{1}{4}\sum_{m=1}^{4} i^m |a + i^m b|^2, \qquad a, b \in \mathbb{C},$$

one can prove that $\hat{\mu}_{n,\sigma}(y)$ is a linear combination of at most four absolutely continuous distribution functions. (See [11] for details.) We denote by $G_{n,\sigma;l_1,l_2}(x)$, $G_{n,\sigma;l}(x)$, $G_{n,\sigma}(x)$ the densities of $\lambda_{n,\sigma;l_1,l_2}$, $\lambda_{n,\sigma;l}$, $\lambda_{n,\sigma}$, respectively. By Theorem 6 of [3], all these densities have majorants of the form $Ke^{-\lambda|x|^2}$, and their partial derivatives of order $\le q$ have majorants of the form $K_q e^{-\lambda|x|^2}$ for $n \ge n_q$. Thus,

$$\hat{\mu}_{n,\sigma}(y) = \sum_{l_1,l_2} \int_{[0,1]^k} \int_{\mathbb{C}^J} e^{i\sum_j (L_k(\sigma,\theta;\chi_j)e^{x_j})\cdot y + x_{l_1} + \bar{x}_{l_2}} \mathfrak{G}_{n,\sigma;l_1,l_2}(x,\theta)\, dx\, d\theta,$$

where

$$\begin{aligned} &\mathfrak{G}_{n,\sigma;l_1,l_2}(x,\theta) \\ &= L_k'(\sigma,\theta;\chi_{l_1})\overline{L_k'(\sigma,\theta;\chi_{l_2})}G_{n,\sigma}(x) + L_k'(\sigma,\theta;\chi_{l_1})\overline{L_k(\sigma,\theta;\chi_{l_2})G_{n,\sigma;l_2}(x)} \\ &\quad + L_k(\sigma,\theta;\chi_{l_1})\overline{L_k'(\sigma,\theta;\chi_{l_2})}G_{n,\sigma;l_1}(x) + L_k(\sigma,\theta;\chi_{l_1})\overline{L_k(\sigma,\theta;\chi_{l_2})}G_{n,\sigma;l_1,l_2}(x). \end{aligned}$$

We only consider the first term $L_k'(\sigma,\theta;\chi_{l_1})\overline{L_k'(\sigma,\theta;\chi_{l_2})}G_{n,\sigma}(x)$, since the others can be treated similarly. If $\theta \notin A_{j,\sigma}(\delta)$ for K-many j, we will prove

$$\int_{\mathbb{C}^J} e^{i\sum_j (L_k(\sigma,\theta;\chi_j)e^{x_j})\cdot y + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x)\,dx = O(|\delta y|^{-K}). \tag{2.1}$$

For the other θ, we give a trivial upper bound by the measure of the set of those θ:

$$\hat{\mu}_{n,\sigma}(y) \ll \Big| \bigcap_{r_1<\cdots<r_K\le J} (A_{r_1,\sigma}(\delta)\cup\cdots\cup A_{r_K,\sigma}(\delta))\Big| + |\delta y|^{-K},$$

where the corresponding constant does not depend on n as $y\to\infty$.

So, it is enough to prove (2.1). We decompose

$$\begin{aligned}\int_{\mathbb{C}^J} & e^{i\sum_j (L_k(\sigma,\theta;\chi_j)e^{x_j})\cdot y + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x)\,dx \\ &= \sum_{m\in\mathbb{Z}^J} \int_{(\mathbb{R}\times[0,2\pi])^J} e^{i\sum_j e^{x_j}\cdot(\overline{L_k(\sigma,\theta;\chi_j)}y) + x_{l_1} + \bar{x}_{l_2}} G_{n,\sigma}(x+2\pi m i)\,dx.\end{aligned}$$

Changing variables $e^{x_j} = r_j e^{z_j}$ with Jacobian r_j^{-1} shows that the above equals

$$\begin{aligned}\sum_{m\in\mathbb{Z}^J} \int_{[0,2\pi]^J} \int_{(0,\infty)^J} & e^{i\sum_j r_j e^{z_j}\cdot(\overline{L_k(\sigma,\theta;\chi_j)}y) + z_{l_1} - z_{l_2}} r_{l_1} r_{l_2} \\ & \times \prod_j r_j^{-1} G_{n,\sigma}(\log r + i(z+2\pi m))\,dr\,dz,\end{aligned}$$

where $r=(r_1,\ldots,r_J)$, $z=(z_1,\ldots,z_J)$, and $\log r = (\log r_1,\ldots,\log r_J)$. Consider the integral

$$\begin{aligned}&\int_0^{2\pi}\int_0^{\infty} e^{ir_j e^{z_j}\cdot(\overline{L_k(\sigma,\theta;\chi_j)}y) + z_{l_1} - z_{l_2}} r_{l_1} r_{l_2} r_j^{-1} G_{n,\sigma}(\log r + i(z+2\pi m))\,dr_j\,dz_j \\ &= \int_0^{2\pi}\int_0^{\infty} e^{ir_j|L_k(\sigma,\theta;\chi_j)y|\cos(z_j-\alpha_j) + z_{l_1} - z_{l_2}} r_{l_1} r_{l_2} r_j^{-1} G_{n,\sigma}(\log r + i(z+2\pi m))\,dr_j\,dz_j\end{aligned}$$

for some α_j. For $\theta\notin A_{j,\sigma}(\delta)$, we integrate by parts with respect to z_j for $|\cos(z_j-\alpha_j)|<1/2$, and with respect to r_j for $|\cos(z_j-\alpha_j)|>1/2$. With the uniform upper bound $K_q e^{-\lambda|x|^2}$ of partial derivatives of G of order $\le q$, we obtain (2.1). ■

Lemma 2.6. *$\hat{\mu}_{n,\sigma}(y)$ converges uniformly for every disc $|y|\le a$ and $1/2<\sigma_1\le\sigma\le\sigma_2$.*

Proof. By definition, we have

$$\hat{\mu}_{n+1,\sigma}(y) = \int_{[0,1]^n}\int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y} \Big|\frac{\partial}{\partial\sigma}E_{n+1,\sigma}(\Theta,u)\Big|^2\,du\,d\Theta.$$

We get

$$\int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y}\left|\frac{\partial}{\partial\sigma}E_{n+1,\sigma}(\Theta,u)\right|^2 du$$
$$= \int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y}\,du\left|\frac{\partial}{\partial\sigma}E_{n,\sigma}(\Theta)\right|^2 + \int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y}$$
$$\times 2\,\mathrm{Re}\biggl[\frac{\partial}{\partial\sigma}\overline{E_{n,\sigma}(\Theta)}e^{2\pi iu}\frac{\partial}{\partial\sigma}\sum_{j=1}^{J}h_j(\dots)\prod_{k<j\le n}(\dots)^{-1}\frac{\chi_j(p_{n+1})}{p_{n+1}^{\sigma}}\biggr]\,du$$
$$+ O\biggl(\frac{F_n(\sigma,\Theta)^2}{p_{n+1}^{2\sigma}}\biggr),$$

where

$$F_n(\sigma,\Theta) = \sum_{j=1}^{J}\prod_{k<l\le n}\left|1-\frac{\chi_j(p_l)}{p_l^{\sigma}}e^{2\pi i\vartheta_l}\right|^{-1}.$$

As $e^{ix} = 1 + ix + O(|x|^2)$ $(x\in\mathbb{R})$, we have

$$\int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y}\,du$$
$$= \int_0^1 e^{iE_{n,\sigma}(\Theta)\cdot y}(1 + i(E_{n+1,\sigma}(\Theta,u)-E_{n,\sigma}(\Theta))\cdot y)\,du + O\biggl(\frac{F_n(\sigma,\Theta)^2}{p_{n+1}^{2\sigma}}\biggr)$$
$$= e^{iE_{n,\sigma}(\Theta)\cdot y} + O\biggl(\frac{F_n(\sigma,\Theta)^2}{p_{n+1}^{2\sigma}}\biggr).$$

Since $e^{ix} = 1 + O(|x|)$ $(x\in\mathbb{R})$, we have

$$\int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y\pm 2\pi iu}\,du = \int_0^1 e^{iE_{n,\sigma}(\Theta)\cdot y\pm 2\pi iu}\,du + O\biggl(\frac{F_n(\sigma,\Theta)}{p_{n+1}^{\sigma}}\biggr)$$
$$= O\biggl(\frac{F_n(\sigma,\Theta)}{p_{n+1}^{\sigma}}\biggr).$$

Combining the above equalities yields

$$\int_0^1 e^{iE_{n+1,\sigma}(\Theta,u)\cdot y}\left|\frac{\partial}{\partial\sigma}E_{n+1,\sigma}(\Theta,u)\right|^2 du = e^{iE_{n,\sigma}(\Theta)\cdot y}\left|\frac{\partial}{\partial\sigma}E_{n,\sigma}(\Theta)\right|^2$$
$$+ O\biggl(\frac{F_n(\sigma,\Theta)^2+F_n(\sigma,\Theta)^3+F_n(\sigma,\Theta)^4}{p_{n+1}^{2\sigma}}\log p_{n+1}\biggr).$$

Thus, we have

$$\hat{\mu}_{n+1,\sigma}(y) - \hat{\mu}_{n,\sigma}(y) = O(p_{n+1}^{-2\sigma}\log p_{n+1})$$

and

$$\hat{\mu}_{m,\sigma}(y) - \hat{\mu}_{n,\sigma}(y) = O(p_n^{1-2\sigma_1})$$

for $m > n > k$. Hence, Lemma 2.6 follows. ∎

3. Main results. We consider separately the cases $J = 2$ and $J \geq 3$. For $J = 2$, our function is the sum of two spoiled Euler products $f_1(s) + f_2(s)$. We then apply the theory of value distribution for $f_1(s)$ and $\frac{f_2}{f_1}(s)$.

PROPOSITION 3.1. *Let $J = 2$ and $1/2 < \sigma_1 < \sigma_2$. Suppose that $h_j(p_1^{-\sigma} e^{2\pi i\theta_1}, \ldots, p_k^{-\sigma} e^{2\pi i\theta_k}) \neq 0$ for $j = 1, 2$, $\sigma_1 \leq \sigma \leq \sigma_2$, and $\theta \in [0, 1]^k$. Then*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(-1)\, d\sigma + o(T),$$

where $H_\sigma(x)$ is the density of some distribution function μ_σ. Moreover, $H_\sigma(x) > 0$ for $1/2 < \sigma \leq 1$.

Proof. By Lemma 2.4, $\varphi_{E_n}(\sigma)$ converges uniformly to $\varphi_E(\sigma)$ on $[1/2, \infty)$. If $\varphi_E(\sigma)$ is twice differentiable, then

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi_E''(\sigma)\, d\sigma + o(T).$$

By direct calculation,

$$\varphi_{E_n}(\sigma) = \varphi_{h_2}(\sigma) + \varphi_{\tilde{L}_n+1}(\sigma),$$

where

$$\tilde{L}_n(s) = \frac{h_1}{h_2}(p_1^{-s}, \ldots, p_k^{-s}) \prod_{p_k < p \leq p_n} \frac{1 - \chi_2(p)/p^s}{1 - \chi_1(p)/p^s}.$$

By Lemma 2.1, we have $\varphi_{h_2}''(\sigma) = 0$ for $\sigma_1 \leq \sigma \leq \sigma_2$. For $\tilde{L}_n$, the method in Chapter II of [3] works. Define

$$\tilde{L}_{n,\sigma}(\Theta) = \frac{h_1}{h_2}(p_1^{-\sigma} e^{2\pi i\theta_1}, \ldots, p_k^{-\sigma} e^{2\pi i\theta_k}) \prod_{k < l \leq n} \frac{1 - \chi_2(p_l)e^{2\pi i\vartheta_l}/p_l^\sigma}{1 - \chi_1(p_l)e^{2\pi i\vartheta_l}/p_l^\sigma},$$

$$\mu_{n,\sigma}(B) = \int_{\tilde{L}_{n,\sigma}^{-1}(B)} \left| \frac{\partial}{\partial\sigma} \tilde{L}_{n,\sigma}(\Theta) \right|^2 d\Theta$$

for any Borel set $B \subset \mathbb{C}$ and $n > k$, $\Theta = (\theta_1, \ldots, \theta_k, \vartheta_{k+1}, \ldots, \vartheta_n) \in [0, 1]^n$. Applying Theorems 5–10 in [3] with some modifications, we deduce that the absolutely continuous distributions $\mu_{n,\sigma}$ converge to a distribution μ_σ with a density $H_\sigma(x)$ and $\varphi_{\tilde{L}_n+1}''(\sigma) = 2\pi H_\sigma(-1) > 0$ for $1/2 < \sigma \leq 1$. ∎

For the case $J \geq 3$, we cannot do the same thing as for $J = 2$. However, by the method of [11], we obtain the following.

Proposition 3.2. *Let $J \geq 3$ and $1/2 < \sigma_1 < \sigma_2$. Suppose that $h_j(p_1^{-\sigma}e^{2\pi i\theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i\theta_k}) \neq 0$ for $j = l_1, l_2, l_3$, $\sigma_1 \leq \sigma \leq \sigma_2$, and $\theta \in [0,1]^k$. Then*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0)\, d\sigma + o(T),$$

where $H_\sigma(x)$ is the density of some distribution function μ_σ.

Proof. By Lemma 2.4, $\varphi_{E_n}(\sigma)$ converges uniformly to $\varphi_E(\sigma)$ on $[1/2, \infty)$. If $\varphi_E(\sigma)$ is twice differentiable, then

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi} \int_{\sigma_1}^{\sigma_2} \varphi_E''(\sigma)\, d\sigma + o(T).$$

By Lemma 2.5 with

$$\delta = \min\{|\tilde{h}_j(p_1^{-\sigma}e^{2\pi i\theta_1}, \ldots, p_k^{-\sigma}e^{2\pi i\theta_k})| \mid j = l_1, l_2, l_3, \\ \sigma_1 \leq \sigma \leq \sigma_2, \theta \in [0,1]^k\} > 0,$$

we have $\hat{\mu}_{n,\sigma}(y) \ll |y|^{-3}$ and this implies that $\mu_{n,\sigma}$ is absolutely continuous and its density $H_{n,\sigma}(x)$ is continuous. Let $\nu_{n,\sigma}$ be the asymptotic distribution of $E_n(\sigma + it)$ with respect to $|E_n'(\sigma + it)|^2$. Since $\hat{\mu}_{n,\sigma}(y) = \hat{\nu}_{n,\sigma}(y)$ by Kronecker's theorem, $\mu_{n,\sigma} = \nu_{n,\sigma}$ and $H_{n,\sigma}$ is their common density. By Lemma 2.3, $\varphi''_{E_n - x}(\sigma) = 2\pi H_{n,\sigma}(x)$. By Lemma 2.6, $H_{n,\sigma}(x)$ converges to $H_\sigma(x)$ which is the density of some distribution $\mu_\sigma = \lim_{n\to\infty} \mu_{n,\sigma}$. Therefore,

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0)\, d\sigma + o(T). \quad \blacksquare$$

By Lemma 2.2, each Dirichlet polynomial $h_j(p_1^{-s}, \ldots, p_k^{-s})$ has at most a finite number of linearity intervals of its Jensen function $\varphi_{h_j}(\sigma)$ in $[1/2, \infty)$. Let $\mathfrak{I}_j$ be the union of those intervals. By Lemmas 2.3 and 2.4 and almost periodicity, h_j has no zero in $\mathfrak{I}_j$. We let $\varsigma_j = \inf \mathfrak{I}_j \geq 1/2$, and ς_E be the third smallest ς_j, more precisely, $\varsigma_E = \varsigma_{l_3}$ when $\varsigma_{l_1} \leq \varsigma_{l_2} \leq \varsigma_{l_3} \leq \cdots$ is the linear order of $\varsigma_1, \ldots, \varsigma_J$. By combining Lemma 2.4 and Proposition 3.2, we obtain the following theorem.

Theorem 3.3. *Let $J \geq 3$ and $\varsigma_E < \sigma_1 < \sigma_2$. Suppose that $\sigma_1, \sigma_2 \in \mathfrak{I}_j$ for at least three j. Then*

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi}(\varphi_E'(\sigma_2-) - \varphi_E'(\sigma_1+)) + o(T).$$

Suppose further that $[\sigma_1, \sigma_2] \subset \mathfrak{I}_j$ for at least three j. Then

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0)\, d\sigma + o(T),$$

where $H_\sigma(x)$ is the density of some distribution μ_σ. In this case, for $\sigma_1 < \sigma_0 < \sigma_2$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ *is $o(T)$.*

If each $\tilde{h}_j$ is non-vanishing on $\operatorname{Re} s > 1/2$, the conclusion of Theorem 3.3 holds.

THEOREM 3.4. *Let $J \geq 3$ and $1/2 < \sigma_1 < \sigma_2$. Suppose that $\tilde{h}_j \neq 0$ for* $\operatorname{Re} s > 1/2$. *Then*

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0)\, d\sigma + o(T), \tag{3.1}$$

where $H_\sigma(x)$ is the density of a distribution μ_σ. For $\sigma_0 > 1/2$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$ *and* $0 < \operatorname{Im} s < T$ *is $o(T)$.*

As a consequence, we obtain Theorem 1.1.

We now consider the case when $\tilde{h}_j$ is a one-variable polynomial. Then it has only finitely many solutions, say $\beta_1, \ldots, \beta_m \in \mathbb{C}$. So $\tilde{h}_j(p^{-s}) = 0$ if and only if $p^{-s} = \beta_i$ for some i. Thus, each line segment $\operatorname{Re} s = -\log|\beta_j|/\log p$, $0 < \operatorname{Im} s < T$ contains $cT + O(1)$ zeros of $\tilde{h}_j(p^{-s})$. So we may not have the equation (3.1) for $E(s)$. However, if we disregard these exceptional points, we obtain the following theorem.

THEOREM 3.5. *Let $J \geq 3$ and $1/2 < \sigma_1 < \sigma_2$. Let*

$$E(s) = \sum_{j \leq J} \tilde{h}_j(p_1^{-s}, \ldots, p_k^{-s}) L(s, \chi_j),$$

where each $\tilde{h}_j$ is a polynomial of one variable. Then

$$N_E(\sigma_1, \sigma_2; 0, T) = \frac{T}{2\pi}(\varphi'_E(\sigma_2-) - \varphi'_E(\sigma_1+)) + o(T).$$

Suppose $\mathfrak{I} = \bigcup_{l_1 < l_2 < l_3 \leq J}(I_{l_1} \cap I_{l_2} \cap I_{l_3})$ is $(1/2, \infty)$ minus finitely many points. If $[\sigma_1, \sigma_2] \subset \mathfrak{I}$, then

$$N_E(\sigma_1, \sigma_2; 0, T) = T \int_{\sigma_1}^{\sigma_2} H_\sigma(0)\, d\sigma + o(T),$$

where $H_\sigma(x)$ is the density of some distribution μ_σ. For $\sigma_0 \in \mathfrak{I}$, the number of zeros of $E(s)$ on the line segment $\operatorname{Re} s = \sigma_0$, $0 < \operatorname{Im} s < T$ *is $o(T)$.*

As a consequence, we obtain Theorem 1.2.

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