

Bernoulli–Dedekind sums

by

MATTHIAS BECK (San Francisco, CA) and
ANASTASIA CHAVEZ (Berkeley, CA)

1. Introduction. While studying the transformation properties of

$$\eta(z) := e^{\pi iz/12} \prod_{n \geq 1} (1 - e^{2\pi inz}),$$

under $\mathrm{SL}_2(\mathbb{Z})$, Dedekind, in the 1880's [9], naturally arrived at the following arithmetic function: Let $((x))$ be the sawtooth function defined by

$$(1.1) \quad ((x)) := \begin{cases} \{x\} - 1/2 & \text{if } x \notin \mathbb{Z}, \\ 0 & \text{if } x \in \mathbb{Z}. \end{cases}$$

Here $\{x\} = x - \lfloor x \rfloor$ denotes the fractional part of x . For $a, b \in \mathbb{N} := \{n \in \mathbb{Z} : n > 0\}$, we define the *Dedekind sum* as

$$(1.2) \quad s(a, b) := \sum_{k \bmod b} \left(\left(\frac{ka}{b} \right) \right) \left(\left(\frac{k}{b} \right) \right).$$

The Dedekind sum and its generalizations have since intrigued mathematicians from various areas such as analytic [1, 9, 10] and algebraic number theory [17, 28], topology [13, 18, 31], algebraic [7, 22, 30] and combinatorial geometry [6, 20], and algorithmic complexity [16].

By means of the discrete Fourier series of the sawtooth function (see, for example, [25, p. 14]), it is not hard to write the Dedekind sum in terms of cotangents:

$$(1.3) \quad s(a, b) = \frac{1}{4b} \sum_{k=1}^{b-1} \cot \frac{\pi ka}{b} \cot \frac{\pi k}{b}.$$

2010 *Mathematics Subject Classification*: 11F20, 11B68.

Key words and phrases: Dedekind sum, Bernoulli polynomial, reciprocity theorem.

Starting with these two representations (1.2) and (1.3) of $s(a, b)$, various generalizations of the Dedekind sum have been introduced. A previous paper [5] by the first author attempted to unify generalizations of the Dedekind sum in its “cotangent representation” (1.3). The present paper complements this by introducing a vast generalization of the “sawtooth representation” (1.2) of the Dedekind sum. The sawtooth function is the first periodized Bernoulli polynomial $\bar{B}_1(u)$, where the *Bernoulli polynomial* $B_k(u)$ is defined, as usual, through

$$\frac{e^{uz}}{e^z - 1} = \sum_{k \geq 0} \frac{B_k(u)}{k!} z^{k-1},$$

and its periodized counterpart $\bar{B}_k(u)$ is defined as the unique function that is periodic with period 1 and coincides with $B_k(u)$ on $[0, 1)$, except that we set $\bar{B}_1(u) = 0$ for $u \in \mathbb{Z}$. To define our generalization, let $p_1, \dots, p_n, a_1, \dots, a_n \in \mathbb{N}$, $x_1, \dots, x_n \in \mathbb{R}$, and

$$\begin{aligned} A_k &:= (a_1, \dots, \widehat{a}_k, \dots, a_n), & X_k &:= (x_1, \dots, \widehat{x}_k, \dots, x_n), \\ P_k &:= (p_1, \dots, \widehat{p}_k, \dots, p_n), \end{aligned}$$

where \widehat{a}_k means we omit the entry a_k . Then we define the *Bernoulli–Dedekind sum* as

$$(1.4) \quad S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} := \sum_{h \bmod a_k} \prod_{\substack{i=1 \\ i \neq k}}^n \bar{B}_{p_i} \left(a_i \frac{h + x_k}{a_k} - x_i \right).$$

The Bernoulli–Dedekind sums include as special cases various previously defined Dedekind-like sums, which we will discuss in detail in Section 2.

The most fundamental and important theorems for any of the generalized Dedekind sums are the *reciprocity laws*: an appropriate sum of generalized Dedekind sums (usually permuting the arguments in a cyclic fashion) gives a simple rational expression. The famous reciprocity law for the classical Dedekind sum is as old as the sum itself:

THEOREM 1.1 (Dedekind). *If $a, b \in \mathbb{N}$ are coprime then*

$$s(a, b) + s(b, a) = -\frac{1}{4} + \frac{1}{12} \left(\frac{a}{b} + \frac{1}{ab} + \frac{b}{a} \right).$$

Our main goal in this paper is to prove a reciprocity theorem for the Bernoulli–Dedekind sums, which is most conveniently stated in terms of generating functions. For nonzero variables y_1, \dots, y_n , let $Y_k := (y_1, \dots, \dots, \widehat{y}_k, \dots, y_n)$ and

$$\Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & \end{pmatrix} := \sum_{(p_1, \dots, \widehat{p_k}, \dots, p_n) \in \mathbb{Z}_{\geq 0}^{n-1}} \frac{1}{p_1! \cdots p_{k-1}! p_{k+1}! \cdots p_n!} S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{i=1 \\ i \neq k}}^n \left(\frac{y_i}{a_i} \right)^{p_i-1}.$$

Our main result is as follows.

THEOREM 1.2. *Let $a_1, \dots, a_n \in \mathbb{N}$ be pairwise coprime, $x_1, \dots, x_n \in \mathbb{R}$, and let y_1, \dots, y_n be nonzero variables such that $y_1 + \cdots + y_n = 0$. If $(x_u - h_u)/a_u - (x_v - h_v)/a_v \notin \mathbb{Z}$ whenever $1 \leq u < v \leq n$ and $h_u, h_v \in \mathbb{Z}$, then*

$$\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & \end{pmatrix} = 0.$$

The next section will illustrate the span from (1.2) to (1.4) (and from Theorem 1.1 to Theorem 1.2). Section 3 is devoted to the proof of Theorem 1.2. As an interlude, we exhibit in Section 4 a Petersson–Knopp identity [15] for the Bernoulli–Dedekind sum. In Section 5, we show that—within limits—our ideas can also be applied to more general reciprocity theorems, namely versions of Theorem 1.2 in which the condition $(x_u - h_u)/a_u - (x_v - h_v)/a_v \notin \mathbb{Z}$ can be omitted.

2. Various Dedekind-like sums. In this section we give an overview of previously defined generalizations of the Dedekind sum (rather, of its “sawtooth representation” (1.2)). We do not claim any completeness but hope to give some picture of what has been introduced in the past.

Apostol [2] replaced one of the sawtooth functions in (1.2) by an arbitrary Bernoulli function:

$$(2.1) \quad \sum_{k \bmod b} \left(\left(\frac{k}{b} \right) \right) \bar{B}_n \left(\frac{ka}{b} \right).$$

Apostol’s idea was generalized by Carlitz [8] and Mikolás [19] to

$$\sum_{k \bmod b} \bar{B}_m \left(\frac{kb}{a} \right) \bar{B}_n \left(\frac{kc}{a} \right).$$

Another way of generalizing (1.2) is to shift the argument of the sawtooth functions. This was introduced by Meyer [17] and Dieter [10], and brought to a solid ground by Rademacher [24]: For $a, b \in \mathbb{N}$, $x, y \in \mathbb{R}$, the *Dedekind–*

Rademacher sum is defined by

$$\sum_{k \bmod b} \left(\left(a \frac{k+y}{b} - x \right) \right) \left(\left(\frac{k+y}{b} \right) \right).$$

The ideas of Apostol and Rademacher can also be combined: Takács [29] introduced a shift in Apostol’s sum (2.1):

$$\sum_{k \bmod b} \left(\left(\frac{k+y}{b} \right) \right) \bar{B}_n \left(a \frac{k+y}{b} - x \right).$$

This was further generalized by Halbritter [11] and later by Hall, Wilson, and Zagier [12] to

$$\sum_{h \bmod c} \bar{B}_m \left(a \frac{h+z}{c} - x \right) \bar{B}_n \left(b \frac{h+z}{c} - y \right),$$

where $a, b, c, m, n \in \mathbb{N}$ and $x, y, z \in \mathbb{R}$. The Hall–Wilson–Zagier sum, in turn, is a special case of our Bernoulli–Dedekind sum, namely,

$$S_{m,n} \begin{pmatrix} A_3 & a_3 \\ X_3 & x_3 \end{pmatrix},$$

where $(a_1, a_2, a_3) = (a, b, c)$, $(x_1, x_2, x_3) = (x, y, z)$, and $P_3 = (m, n)$ (and thus $A_3 = (a, b)$ and $X_3 = (x, y)$).

The central theorems of all of the above-cited papers are reciprocity theorems for each generalized Dedekind sum. To give one example, we state the reciprocity theorem of [12].

THEOREM 2.1 (Hall–Wilson–Zagier). *Let $a_1, a_2, a_3 \in \mathbb{N}$ be pairwise coprime, $x_1, x_2, x_3 \in \mathbb{R}$, and let y_1, y_2, y_3 be nonzero variables such that $y_1 + y_2 + y_3 = 0$. Then*

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & \end{pmatrix} \\ = \begin{cases} -1/4 & \text{if } (x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

As noted elsewhere, the statement of this theorem in [12] missed the minus sign in front of 1/4 (the proof is correct nevertheless). We should also remark that it is a somewhat nontrivial (but funny) exercise to derive Dedekind’s Theorem 1.1 from Hall–Wilson–Zagier’s Theorem 2.1.

It is the generic (“otherwise”) case of Theorem 2.1 that our Theorem 1.2 extends. The other case (which essentially deals with Bernoulli–Dedekind sums for which $x_1 = \dots = x_n = 0$) was recently extended by Bayad and

Raouj [4]. For Theorem 2.1, there are essentially only these two “extreme” cases; however, for higher-dimensional Bernoulli–Dedekind sums, there are more intermediate cases (in which we have a partial linear relation such as $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$), and it is not clear to us how one can easily deal with them. We address this issue in Section 6.

It is worth mentioning Hu’s thesis [14], which contains another variant of a “multidimensional” Hall–Wilson–Zagier sum. Hu’s reciprocity theorem is equivalent to Theorem 2.1 for $n = 3$ but Hu’s generalized Dedekind sums and their reciprocity theorems become different from our Bernoulli–Dedekind sums for $n \geq 4$.

Finally, we note that both Theorems 1.2 and 2.1 can be implicitly seen in the work of Sczech [26, 27]. Our main contribution is to give a unifying picture and a simple combinatorial reciprocity proof for Bernoulli–Dedekind sums.

3. Proof of Theorem 1.2. We start our journey towards a proof of Theorem 1.2 with the following lemma on fractional parts, whose easy proof is left to the reader.

LEMMA 3.1. *Given $a, b, c \in \mathbb{R}$,*

$$\begin{aligned} \{a - b\} - \{a - c\} \geq 0 &\Rightarrow \{a - b\} - \{a - c\} = \{c - b\}, \\ \{a - b\} - \{a - c\} \leq 0 &\Rightarrow \{a - b\} - \{a - c\} = -\{b - c\}. \end{aligned}$$

Almost as easy is the proof of the following well-known lemma [23].

LEMMA 3.2 (Raabe’s formula). *For $a \in \mathbb{N}$ and $x \in \mathbb{R}$,*

$$\sum_{h \bmod a} \bar{B}_m \left(x + \frac{h}{a} \right) = a^{1-m} \bar{B}_m(ax).$$

Consequently, we can manipulate a Bernoulli–Dedekind sum as follows:

$$\begin{aligned} (3.1) \quad S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{j=1 \\ j \neq k}}^n a_j^{1-m_j} &= \sum_{h \bmod a_k} \prod_{\substack{j=1 \\ j \neq k}}^n \bar{B}_{p_j} \left(a_j \frac{h + x_k}{a_k} - x_j \right) a_j^{1-m_j} \\ &= \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \prod_{\substack{j=1 \\ j \neq k}}^n \bar{B}_{p_j} \left(\frac{x_k + h_k}{a_k} - \frac{x_j + h_j}{a_j} \right), \end{aligned}$$

where the sum includes the original summand h we now call h_k . With the short-hand $r_j := (x_j + h_j)/a_j$, (3.1) gives

$$\begin{aligned}
(3.2) \quad & \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & \end{pmatrix} \\
&= \sum_{(p_1, \dots, \widehat{p}_k, \dots, p_n) \in \mathbb{Z}_{\geq 0}^{n-1}} \frac{1}{p_1! \cdots p_{k-1}! p_{k+1}! \cdots p_n!} S_{P_k} \begin{pmatrix} A_k & a_k \\ X_k & x_k \end{pmatrix} \prod_{\substack{j=1 \\ j \neq k}}^n \left(\frac{y_j}{a_j} \right)^{p_j-1} \\
&= \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \sum_{(p_1, \dots, \widehat{p}_k, \dots, p_n) \in \mathbb{Z}_{\geq 0}^{n-1}} \frac{1}{p_1! \cdots p_{k-1}! p_{k+1}! \cdots p_n!} \prod_{\substack{j=1 \\ j \neq k}}^n \bar{B}_{p_j}(r_k - r_j) y_i^{p_j-1} \\
&= \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \prod_{\substack{j=1 \\ j \neq k}}^n \beta(r_k - r_j, y_j),
\end{aligned}$$

where

$$\beta(u, z) := \sum_{k \geq 0} \frac{\bar{B}_k(u)}{k!} z^{k-1}.$$

Note that

$$(3.3) \quad \beta(u, z) = \begin{cases} \frac{1}{2} \frac{e^z + 1}{e^z - 1} & \text{if } u \in \mathbb{Z}, \\ \frac{e^{\{u\}z}}{e^z - 1} & \text{if } u \notin \mathbb{Z}, \end{cases}$$

and so it is clear that (3.2) depends on the differences $r_k - r_j$, and the $\beta(r_k - r_j, y_j)$ depend on whether or not these differences are integers. This is the reason for our crucial assumption that $(x_u - h_u)/a_u - (x_v - h_v)/a_v \notin \mathbb{Z}$ whenever $1 \leq u < v \leq n$ and $h_u, h_v \in \mathbb{Z}$ in Theorem 1.2: it allows us to use the second case of (3.3) throughout. For the rest of this section, we assume the differences $r_k - r_j$ are not integers. Then

$$\begin{aligned}
\sum_{k=1}^n \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & \end{pmatrix} &= \sum_{k=1}^n \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \prod_{\substack{j=1 \\ j \neq k}}^n \beta(r_k - r_j, y_j) \\
&= \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n \frac{e^{\{r_k - r_j\}y_j}}{e^{y_j} - 1} \frac{e^{y_k} - 1}{e^{y_k} - 1} \\
&= \sum_{\substack{h_1 \bmod a_1 \\ \dots \\ h_n \bmod a_n}} \frac{\sum_{k=1}^n e^{y_k} \prod_{j=1, j \neq k}^n e^{\{r_k - r_j\}y_j} - \sum_{k=1}^n \prod_{j=1, j \neq k}^n e^{\{r_k - r_j\}y_j}}{\prod_{j=1}^n (e^{y_j} - 1)}.
\end{aligned}$$

Theorem 1.2 will be proved once we can show that the numerator

$$N := \sum_{k=1}^n e^{y_k} \prod_{\substack{j=1 \\ j \neq k}}^n e^{\{r_k - r_j\}y_j} - \sum_{k=1}^n \prod_{\substack{j=1 \\ j \neq k}}^n e^{\{r_k - r_j\}y_j}$$

in this expression vanishes. We separate the $k = n$ terms and use the assumption $y_1 + \cdots + y_n = 0$:

$$\begin{aligned} N &= \sum_{k=1}^{n-1} e^{\{r_k - r_n\}y_n + y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{\{r_k - r_j\}y_j} - \sum_{k=1}^{n-1} e^{\{r_k - r_n\}y_n} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{\{r_k - r_j\}y_j} \\ &\quad + e^{y_n} \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j} - \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j} \\ &= \sum_{k=1}^{n-1} e^{\{r_k - r_n\}(-y_1 - \cdots - y_{n-1}) + y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{\{r_k - r_j\}y_j} \\ &\quad - \sum_{k=1}^{n-1} e^{\{r_k - r_n\}(-y_1 - \cdots - y_{n-1})} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{\{r_k - r_j\}y_j} \\ &\quad + e^{-y_1 - \cdots - y_{n-1}} \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j} - \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j}. \end{aligned}$$

Note that $1 - \{r_k - r_n\} = \{r_n - r_k\}$ since $r_k - r_n \notin \mathbb{Z}$. Thus

$$\begin{aligned} (3.4) \quad N &= \sum_{k=1}^{n-1} e^{\{r_n - r_k\}y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{(\{r_k - r_j\} - \{r_k - r_n\})y_j} \\ &\quad - \sum_{k=1}^{n-1} e^{-\{r_k - r_n\}y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{(\{r_k - r_j\} - \{r_k - r_n\})y_j} \\ &\quad + \prod_{j=1}^{n-1} e^{-\{r_j - r_n\}y_j} - \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j}. \end{aligned}$$

We will show that we can find identical pairs of exponents in terms with opposite signs in this expression, and so the sum vanishes. Only three types of exponents appear in (3.4):

$$\{r_n - r_k\}y_k, \quad -\{r_k - r_n\}y_k, \quad \text{and} \quad (\{r_k - r_j\} - \{r_k - r_n\})y_j.$$

By Lemma 3.1, $(\{r_k - r_j\} - \{r_k - r_n\})y_j$ equals $\{r_n - r_j\}y_j$ or $-\{r_j - r_n\}y_j$, and so the exponents can be condensed to just the first two types. Moreover, the sign of $\{r_k - r_j\} - \{r_k - r_n\}$ determines if it is equal to $\{r_n - r_j\}$ or $-\{r_j - r_n\}$, and so all the information of N can be encoded by a sign matrix: the term $\{r_n - r_k\}y_k$ gets encoded by $+$, the term $-\{r_j - r_n\}y_j$ by $-$, and the term $(\{r_k - r_j\} - \{r_k - r_n\})y_j$ by the sign of $\{r_k - r_j\} - \{r_k - r_n\}$, which we denote by σ_{kj} . For example, the exponent corresponding to $k = 1$ in the first sum

$$(3.5) \quad \sum_{k=1}^{n-1} e^{\{r_n - r_k\}y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{(\{r_k - r_j\} - \{r_k - r_n\})y_j}$$

is

$$\{r_n - r_1\}y_1 + (\{r_1 - r_2\} - \{r_1 - r_n\})y_2 + \cdots + (\{r_1 - r_{n-1}\} - \{r_1 - r_n\})y_{n-1}$$

and is represented by the sign vector $(+, \sigma_{12}, \dots, \sigma_{1, n-1})$. In general, the k th term in (3.5) is represented by the sign vector $(\sigma_{k,1}, \dots, \sigma_{k, k-1}, +, \sigma_{k, k+1}, \dots, \sigma_{k, n-1})$. Similarly, the k th term in the second sum of (3.4),

$$-\sum_{k=1}^{n-1} e^{-\{r_k - r_n\}y_k} \prod_{\substack{j=1 \\ j \neq k}}^{n-1} e^{(\{r_k - r_j\} - \{r_k - r_n\})y_j},$$

is represented by the sign vector $(\sigma_{k,1}, \dots, \sigma_{k, k-1}, -, \sigma_{k, k+1}, \dots, \sigma_{k, n-1})$. The two final terms in (3.4),

$$\prod_{j=1}^{n-1} e^{-\{r_j - r_n\}y_j} - \prod_{j=1}^{n-1} e^{\{r_n - r_j\}y_j},$$

are represented by the respective sign vectors $(-, \dots, -)$ and $(+, \dots, +)$.

Let M_{pos} , resp. M_{neg} , be the matrix of all sign vectors representing the exponents of the positive, resp. negative, terms of N , where we place the sign vector representing, e.g., an exponent from the k th positive term in the k th row of the matrix M_{pos} . Thus we have constructed the matrices

$$M_{\text{pos}} = \begin{pmatrix} + & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1, n-1} \\ \sigma_{21} & + & \sigma_{23} & \cdots & \sigma_{2, n-1} \\ \sigma_{31} & \sigma_{32} & + & \cdots & \sigma_{3, n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1,1} & \sigma_{n-1,2} & \sigma_{n-1,3} & \cdots & + \\ - & - & - & \cdots & - \end{pmatrix}$$

and

$$M_{\text{neg}} = \begin{pmatrix} - & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1,n-1} \\ \sigma_{21} & - & \sigma_{23} & \cdots & \sigma_{2,n-1} \\ \sigma_{31} & \sigma_{32} & - & \cdots & \sigma_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{n-1,1} & \sigma_{n-1,2} & \sigma_{n-1,3} & \cdots & - \\ + & + & + & \cdots & + \end{pmatrix},$$

and our goal $N = 0$ will follow from proving that $M_{\text{pos}} = M_{\text{neg}}$ after row swapping. To show the latter, we first collect some properties of M_{pos} and M_{neg} .

LEMMA 3.3.

- (a) *The matrix M_{pos} has + entries on the diagonal and the last row consists entirely of – entries; M_{neg} has – entries on the diagonal and the last row consists entirely of + entries.*
- (b) *$\sigma_{ij} = +$ if and only if $\sigma_{ji} = -$.*
- (c) *If $\sigma_{ij} = +$ and $\sigma_{ik} = -$ then $\sigma_{jk} = -$.*

Proof. (a) follows by construction.

(b) follows with Lemma 3.1.

(c) Assume $\sigma_{ij} = +$ and $\sigma_{ik} = -$. Then $\{r_i - r_j\} - \{r_i - r_n\} > 0$ and $\{r_i - r_k\} - \{r_i - r_n\} < 0$, and by Lemma 3.1,

$$(3.6) \quad \{r_i - r_j\} - \{r_i - r_n\} = \{r_n - r_j\},$$

$$(3.7) \quad \{r_i - r_k\} - \{r_i - r_n\} = -\{r_k - r_i\}.$$

Then the difference (3.6) – (3.7) is positive and we get

$$\{r_i - r_j\} - \{r_i - r_k\} = \{r_n - r_j\} + \{r_k - r_i\}.$$

The final identity is positive, which means the left-hand side is positive. Then by Lemma 3.1 again,

$$\{r_i - r_j\} - \{r_i - r_k\} = \{r_k - r_j\},$$

and so

$$\{r_k - r_j\} = \{r_n - r_j\} + \{r_k - r_i\}.$$

But then $\sigma_{jk} = -$ follows from

$$\begin{aligned} \{r_j - r_k\} - \{r_j - r_n\} &= \{r_j - r_k\} - 1 + 1 - \{r_j - r_n\} \\ &= -\{r_k - r_j\} + \{r_n - r_j\} = -\{r_k - r_i\}. \quad \blacksquare \end{aligned}$$

Part (b) of this lemma allows us to update the sign matrices:

$$M_{\text{pos}} = \begin{pmatrix} + & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1,n-1} \\ -\sigma_{12} & + & \sigma_{23} & \cdots & \sigma_{2,n-1} \\ -\sigma_{13} & -\sigma_{23} & + & \cdots & \sigma_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{1,n-1} & -\sigma_{2,n-1} & -\sigma_{3,n-1} & \cdots & + \\ - & - & - & \cdots & - \end{pmatrix},$$

$$M_{\text{neg}} = \begin{pmatrix} - & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1,n-1} \\ -\sigma_{12} & - & \sigma_{23} & \cdots & \sigma_{2,n-1} \\ -\sigma_{13} & -\sigma_{23} & - & \cdots & \sigma_{3,n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\sigma_{1,n-1} & -\sigma_{2,n-1} & -\sigma_{3,n-1} & \cdots & - \\ + & + & + & \cdots & + \end{pmatrix}.$$

LEMMA 3.4. *Each of the matrices M_{pos} and M_{neg} has a unique row with k +’s, for each $0 \leq k \leq n - 1$.*

Proof. We will prove this for M_{pos} ; the statement for M_{neg} then follows immediately.

We begin by showing that every row of the matrix M_{pos} is unique. Suppose on the contrary that row m and row l of M_{pos} are equal. Then these rows look as follows:

$$\begin{aligned} \text{row } m: & \quad -\sigma_{1m} \quad -\sigma_{2m} \quad \cdots \quad + \quad \cdots \quad \sigma_{ml} \quad \cdots \quad \sigma_{m,n-1}, \\ \text{row } l: & \quad -\sigma_{1l} \quad -\sigma_{2l} \quad \cdots \quad -\sigma_{ml} \quad \cdots \quad + \quad \cdots \quad \sigma_{l,n-1}. \end{aligned}$$

Then $\sigma_{ml} = +$ and $-\sigma_{ml} = +$, which contradicts Lemma 3.3(b).

Next we will show that no two rows contain the same number of +’s. Suppose on the contrary that row m and row l of M_{pos} contain the same number of +’s (and are not equal).

Assume $\sigma_{ml} = +$. Since the m th row does not entirely consist of +’s, there exists a $-$ in column, say, k . Then by Lemma 3.3(c), the entry σ_{kl} is $-$. So, for every $-$ in row m , Lemma 3.3(c) can be applied to show there is a $-$ in the same column entry of row l . But $-\sigma_{ml} = -$, and so row l contains at least one more $-$ than row m , a contradiction. If, on the other hand, $\sigma_{ml} = -$, then we can repeat the above argument for row l , starting with the entry $-\sigma_{ml} = +$.

We have shown that no two rows contain the same number of +’s and so, for each $0 \leq k \leq n - 1$, there exists a unique row with k +’s. ■

Lemma 3.4 allows us to match up the unique rows of k +’s in M_{pos} and M_{neg} . Thus, after row swapping, $M_{\text{pos}} = M_{\text{neg}}$, which, by our previous argument, finally proves Theorem 1.2. ■

4. Petersson–Knopp identities. Another basic identity on the classical Dedekind sum is the following ([15]):

THEOREM 4.1 (Petersson–Knopp). *Suppose $a, b \in \mathbb{N}$ are coprime. Then*

$$\sum_{d|m} \sum_{k \bmod d} s\left(\frac{m}{d}b + ka, ad\right) = \sigma(m)s(b, a).$$

Here $\sigma(m)$ denotes the sum of the positive divisors of m .

This result has been extended to certain generalized Dedekind sums [3, 21, 32] and takes its most general form [5] for sums

$$S(a; a_1, \dots, a_n) := \sum_{k \bmod a} f_1\left(\frac{ka_1}{a}\right) \cdots f_n\left(\frac{ka_n}{a}\right)$$

of Dedekind type with weight (m_1, \dots, m_n) , i.e., when for all $j = 1, \dots, n$ we have $f_j(x + 1) = f_j(x)$ and, for all $a \in \mathbb{N}$,

$$(4.1) \quad \sum_{k \bmod a} f_j\left(x + \frac{k}{a}\right) = a^{m_j} f_j(ax).$$

Note that the Bernoulli functions $\bar{B}_k(x)$ satisfy (4.1) (with “weight” $1 - k$), due to Lemma 3.2. The following extension of Theorem 4.1 was proved in [5].

THEOREM 4.2. *Let $a, a_1, \dots, a_n \in \mathbb{N}$. If*

$$S(a; a_1, \dots, a_n) := \sum_{k \bmod a} f_1\left(\frac{ka_1}{a}\right) \cdots f_n\left(\frac{ka_n}{a}\right)$$

is of Dedekind type with weight (m_1, \dots, m_n) then

$$\begin{aligned} \sum_{d|m} d^{-m_1 - \dots - m_n} \sum_{r_1, \dots, r_n \bmod d} S\left(ad; \frac{m}{d}a_1 + r_1a, \dots, \frac{m}{d}a_n + r_na\right) \\ = m\sigma_{n-1-m_1-\dots-m_n}(m) S(a; a_1, \dots, a_n). \end{aligned}$$

This theorem together with Lemma 3.2 immediately gives a Petersson–Knopp identity for the Bernoulli–Dedekind sums

$$S_{p_1, \dots, p_n} \begin{pmatrix} (a_1, \dots, a_n) & a_0 \\ (0, \dots, 0) & 0 \end{pmatrix} = \sum_{h \bmod a_0} \prod_{i=1}^n \bar{B}_{p_i} \left(a_i \frac{h}{a_0}\right).$$

COROLLARY 4.3.

$$\begin{aligned} \sum_{d|m} d^{p_1+\dots+p_n-n+1} \sum_{r_1, \dots, r_n \bmod d} S_{p_1, \dots, p_n} \begin{pmatrix} (\frac{m}{d}a_1 + r_1a_0, \dots, \frac{m}{d}a_n + r_na_0) & a_0 \\ (0, \dots, 0) & 0 \end{pmatrix} \\ = m\sigma_{p_1+\dots+p_n}(m) S_{p_1, \dots, p_n} \begin{pmatrix} (a_1, \dots, a_n) & a_0 \\ (0, \dots, 0) & 0 \end{pmatrix}. \end{aligned}$$

5. Hall–Wilson–Zagier revisited. In this section, we show how our ideas can be used to prove Hall–Wilson–Zagier’s Theorem 2.1.

Proof of Theorem 2.1. We want to show that

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & \end{pmatrix} \\ = \begin{cases} -1/4 & \text{if } (x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

By (3.2),

$$\sum_{k=1}^3 \Omega \begin{pmatrix} A_k & a_k \\ X_k & x_k \\ Y_k & \end{pmatrix} = \sum_{k=1}^3 \sum_{\substack{h_1 \bmod a_1 \\ h_2 \bmod a_2 \\ h_3 \bmod a_3}} \prod_{\substack{i=1 \\ i \neq k}}^3 \beta(r_k - r_i, y_i),$$

where $r_i = (x_i + h_i)/a_i$. We have to examine the following cases:

- (i) $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$;
- (ii) $(x_i, x_j) \in (a_i, a_j)\mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 3$ but not (i);
- (iii) none of the above.

Case (iii) is covered by Theorem 1.2.

Case (i). We have $x_i = \lambda a_i + z_i$ for each i , for some $\lambda \in \mathbb{R}$ and $z_i \in \mathbb{Z}$. Thus

$$r_i - r_j = \frac{h_i + \lambda a_i + z_i}{a_i} - \frac{h_j + \lambda a_j + z_j}{a_j} = \frac{h_i + z_i}{a_i} - \frac{h_j + z_j}{a_j}$$

and so the z_i ’s simply permute the indices h_i . But since each h_i gets summed over a complete residue system mod a_i ,

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \sum_{\substack{h_1 \bmod a_1 \\ h_2 \bmod a_2 \\ h_3 \bmod a_3}} \sum_{k=1}^3 \prod_{\substack{i=1 \\ i \neq k}}^3 \beta \left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i \right). \end{aligned}$$

Since $(a_i, a_j) = 1$, $h_i/a_i - h_j/a_j \in \mathbb{Z}$ occurs only when $h_i = h_j = 0$. Thus, we can split up the above sum:

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \beta(0, y_2)\beta(0, y_3) + \beta(0, y_1)\beta(0, y_3) + \beta(0, y_1)\beta(0, y_2) \\ + \sum_{\substack{h_1, h_2, h_3 \\ (h_1, h_2, h_3) \neq (0, 0, 0)}} \sum_{k=1}^3 \prod_{\substack{i=1 \\ i \neq k}}^3 \beta \left(\frac{h_k}{a_k} - \frac{h_i}{a_i}, y_i \right), \end{aligned}$$

where the last sum is over all triples $(h_1 \bmod a_1, h_2 \bmod a_2, h_3 \bmod a_3) \neq (0, 0, 0)$. This term vanishes for the same reasons as in Section 3, since the crucial assumption $h_k/a_k - h_i/a_i \notin \mathbb{Z}$ holds.

For the remaining terms we use the cotangent identity

$$\cot \alpha + \cot \beta = \frac{\cot \alpha \cot \beta - 1}{\cot(\alpha + \beta)}$$

and note that, by the definition of $\cot y$,

$$\beta(0, y) = -i \cot \frac{y}{2i}.$$

Let $y_k^* := y_k/2i$. Then

$$\begin{aligned} \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & y_1 \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & y_2 \end{pmatrix} \\ = \beta(0, y_2)\beta(0, y_3) + \beta(0, y_1)\beta(0, y_3) + \beta(0, y_1)\beta(0, y_2) \\ = -\frac{1}{4}(\cot y_2^* \cot y_3^* + \cot y_1^* \cot y_3^* + \cot y_1^* \cot y_2^*) \\ = -\frac{1}{4} \left(\cot y_2^* \left(\frac{\cot y_1^* \cot y_3^* - 1}{\cot(y_1^* + y_3^*)} \right) + \cot y_1^* \cot y_3^* \right). \end{aligned}$$

By assumption, $y_1 + y_2 + y_3 = 0$, so

$$\cot(y_1^* + y_3^*) = \cot(-y_2^*) = -\cot y_2^*,$$

which yields

$$\Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & \end{pmatrix} = -\frac{1}{4}.$$

Case (ii). Without loss of generality, we assume $(x_1, x_2) \in (a_1, a_2)\mathbb{R} + \mathbb{Z}^2$ (but not $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$). Then, as in case (i),

$$r_1 - r_2 = \frac{h_1 + \lambda a_1 + z_1}{a_1} - \frac{h_2 + \lambda a_2 + z_2}{a_2} = \frac{h_1 + z_1}{a_1} - \frac{h_2 + z_2}{a_2}.$$

Since z_1 and z_2 permute the summands over h_1 and h_2 , we introduce a change of variables and let $\bar{h}_1 := h_1 + z_1$, $\bar{h}_2 := h_2 + z_2$, and $\bar{h}_3 := h_3$. We can rewrite the differences involving r_3 as

$$r_i - r_3 = \frac{\bar{h}_i}{a_i} + \lambda - r_3 = \frac{\bar{h}_i}{a_i} - \tilde{r}_3.$$

Again we will split up our reciprocity sum into two parts:

$$\begin{aligned} (5.1) \quad & \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & \end{pmatrix} \\ &= \sum_{\bar{h}_3 \bmod a_3} (\beta(0, y_2)\beta(-\tilde{r}_3, y_3) + \beta(0, y_1)\beta(-\tilde{r}_3, y_3) + \beta(\tilde{r}_3, y_1)\beta(\tilde{r}_3, y_2)) \\ &+ \sum_{\substack{\bar{h}_1, \bar{h}_2, \bar{h}_3 \\ (\bar{h}_1, \bar{h}_2) \neq (0,0)}} \sum_{k=1}^3 \prod_{\substack{i=1 \\ i \neq k}}^3 \beta\left(\frac{\bar{h}_k}{a_k} - \frac{\bar{h}_i}{a_i}, y_i\right), \end{aligned}$$

where the last sum is over all triples $(\bar{h}_1 \bmod a_1, \bar{h}_2 \bmod a_2, \bar{h}_3 \bmod a_3)$ such that $(\bar{h}_1, \bar{h}_2) \neq (0, 0)$. As before, the last term in (5.1) vanishes for the same reasons as in Section 3. Thus

$$\begin{aligned} & \Omega \begin{pmatrix} a_1 & a_2 & a_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_1 \\ x_2 & x_3 & x_1 \\ y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_3 & a_1 & a_2 \\ x_3 & x_1 & x_2 \\ y_3 & y_1 & \end{pmatrix} \\ &= \sum_{\bar{h}_3 \bmod a_3} (\beta(0, y_2)\beta(-\tilde{r}_3, y_3) + \beta(0, y_1)\beta(-\tilde{r}_3, y_3) + \beta(\tilde{r}_3, y_1)\beta(\tilde{r}_3, y_2)) \\ &= \sum_{\bar{h}_3 \bmod a_3} \left(\frac{1}{2} \frac{e^{y_2} + 1}{e^{y_2} - 1} \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3} - 1} + \frac{1}{2} \frac{e^{y_1} + 1}{e^{y_1} - 1} \frac{e^{\{-\tilde{r}_3\}y_3}}{e^{y_3} - 1} + \frac{e^{\{\tilde{r}_3\}y_1}}{e^{y_1} - 1} \frac{e^{\{\tilde{r}_3\}y_2}}{e^{y_2} - 1} \right). \end{aligned}$$

After bringing the fractions to a common denominator, we obtain the numerator

$$\begin{aligned}
& e^{y_1+y_2+\{-\tilde{r}_3\}y_3} - e^{y_2+\{-\tilde{r}_3\}y_3} + e^{y_1+\{-\tilde{r}_3\}y_3} - e^{\{-\tilde{r}_3\}y_3} + e^{y_1+y_2+\{-\tilde{r}_3\}y_3} \\
& - e^{y_1+\{-\tilde{r}_3\}y_3} + e^{y_2+\{-\tilde{r}_3\}y_3} - e^{\{-\tilde{r}_3\}y_3} \\
& + 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2+y_3} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} \\
= & e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{-\{-\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} + e^{\{\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} - e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} \\
& + e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{\{\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} + e^{-\{-\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} - e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} \\
& + 2e^{-\{-\tilde{r}_3\}y_1-\{-\tilde{r}_3\}y_2} - 2e^{\{\tilde{r}_3\}y_1+\{\tilde{r}_3\}y_2} = 0. \blacksquare
\end{aligned}$$

6. Final remarks. Trying to extend Hall–Wilson–Zagier’s Theorem 2.1 to the next case of four sets of variables, the task is to study the reciprocity sum

$$\begin{aligned}
\Omega \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_4 & a_1 \\ x_2 & x_3 & x_4 & x_1 \\ y_2 & y_3 & y_4 & \end{pmatrix} \\
+ \Omega \begin{pmatrix} a_3 & a_4 & a_1 & a_2 \\ x_3 & x_4 & x_1 & x_2 \\ y_3 & y_4 & y_1 & \end{pmatrix} + \Omega \begin{pmatrix} a_4 & a_1 & a_2 & a_3 \\ x_4 & x_1 & x_2 & x_3 \\ y_4 & y_1 & y_2 & \end{pmatrix}
\end{aligned}$$

in the four cases

- (i) $(x_1, x_2, x_3, x_4) \in (a_1, a_2, a_3, a_4)\mathbb{R} + \mathbb{Z}^4$;
- (ii) $(x_i, x_j, x_k) \in (a_i, a_j, a_k)\mathbb{R} + \mathbb{Z}^3$ for some $1 \leq i < j < k \leq 4$ but not (i);
- (iii) $(x_i, x_j) \in (a_i, a_j)\mathbb{R} + \mathbb{Z}^2$ for some $1 \leq i < j \leq 4$ but not (i) or (ii);
- (iv) none of the above.

Case (i) is covered by [4] (and can be easily recovered with a calculation similar to that in the last section); in this case the reciprocity sum equals

$$\frac{i}{8} \left(\cot \frac{y_0}{2i} + \cot \frac{y_1}{2i} + \cot \frac{y_2}{2i} + \cot \frac{y_3}{2i} \right).$$

Case (iv) is covered by Theorem 1.2.

For case (iii), a calculation similar to that in the last section reveals that the reciprocity sum vanishes.

For case (ii), similar calculations yield

$$\begin{aligned} & \Omega \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & \end{pmatrix} + \Omega \begin{pmatrix} a_2 & a_3 & a_4 & a_1 \\ x_2 & x_3 & x_4 & x_1 \\ y_2 & y_3 & y_4 & \end{pmatrix} \\ & + \Omega \begin{pmatrix} a_3 & a_4 & a_1 & a_2 \\ x_3 & x_4 & x_1 & x_2 \\ y_3 & y_4 & y_1 & \end{pmatrix} + \Omega \begin{pmatrix} a_4 & a_1 & a_2 & a_3 \\ x_4 & x_1 & x_2 & x_3 \\ y_4 & y_1 & y_2 & \end{pmatrix} \\ & = \sum_{h_4 \bmod a_4} \frac{\begin{pmatrix} e^{\{\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} \\ + e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \\ + 2e^{-\{-\tilde{r}_4\}y_1 + \{\tilde{r}_4\}y_2 + \{\tilde{r}_4\}y_3} - 2e^{\{\tilde{r}_4\}y_1 - \{-\tilde{r}_4\}y_2 - \{-\tilde{r}_4\}y_3} \end{pmatrix}}{4(e^{y_1} - 1)(e^{y_2} - 1)(e^{y_3} - 1)(e^{y_4} - 1)}, \end{aligned}$$

where \tilde{r}_4 is defined analogously to the way we defined \tilde{r}_3 in the previous section. As with all the previous summands, this final sum exhibits an intriguing symmetry, but it is not clear to us if it vanishes or evaluates to a simple expression. As mentioned earlier, for higher-dimensional Bernoulli–Dedekind sums (i.e., for larger n), there are more intermediate cases, in which we have a partial linear relation such as $(x_1, x_2, x_3) \in (a_1, a_2, a_3)\mathbb{R} + \mathbb{Z}^3$, and it is not clear to us how one can easily deal with them.

We conclude with one more open problem, namely, that of the computational complexity of Bernoulli–Dedekind sums. Any Dedekind-like sum that obeys a two-term reciprocity law is instantly computable through the Euclidean algorithm. However, the computational complexity of “higher-dimensional” Dedekind-like sums is more subtle. It was proved in [5] that the cotangent-generalizations of the Dedekind sum are polynomial-time computable (in the input length of the integer parameters). It is not clear to us how the argument in [5] could be modified to say anything about the complexity of Bernoulli–Dedekind sums.

Acknowledgements. Research of M. Beck partially supported by NSF (DMS-0810105). Research of A. Chavez partially supported by NSF (HRD-0331537) and NIH (RISE 5R25-6M059298-10).

We thank Mejid Bayad for stimulating discussions about a previous version of this paper.

References

[1] G. Almkvist, *Asymptotic formulas and generalized Dedekind sums*, Experiment. Math. 7 (1998), 343–359.

- [2] T. M. Apostol, *Generalized Dedekind sums and transformation formulae of certain Lambert series*, Duke Math. J. 17 (1950), 147–157.
- [3] T. M. Apostol and T. H. Vu, *Identities for sums of Dedekind type*, J. Number Theory 14 (1982), 391–396.
- [4] A. Bayad and A. Raouj, *Reciprocity formulae for general multiple Dedekind–Rademacher sums and enumeration of lattice points*, Acta Arith. 145 (2009), 137–154.
- [5] M. Beck, *Dedekind cotangent sums*, ibid. 109 (2003), 109–130.
- [6] M. Beck and S. Robins, *Computing the Continuous Discretely: Integer-Point Enumeration in Polyhedra*, Undergrad. Texts Math., Springer, New York, 2007.
- [7] M. Brion and M. Vergne, *Residue formulae, vector partition functions and lattice points in rational polytopes*, J. Amer. Math. Soc. 10 (1997), 797–833.
- [8] L. Carlitz, *Some theorems on generalized Dedekind sums*, Pacific J. Math. 3 (1953), 513–522.
- [9] R. Dedekind, *Erläuterungen zu den Fragmenten XXVIII*, in: Collected Works of Bernhard Riemann, Dover, New York, 1953, 466–478.
- [10] U. Dieter, *Das Verhalten der Kleinschen Funktionen $\log \sigma_{g,h}(\omega_1, \omega_2)$ gegenüber Modultransformationen und verallgemeinerte Dedekindsche Summen*, J. Reine Angew. Math. 201 (1959), 37–70.
- [11] U. Halbritter, *Some new reciprocity formulas for generalized Dedekind sums*, Results Math. 8 (1985), 21–46.
- [12] R. R. Hall, J. C. Wilson, and D. Zagier, *Reciprocity formulae for general Dedekind–Rademacher sums*, Acta Arith. 73 (1995), 389–396.
- [13] F. Hirzebruch and D. Zagier, *The Atiyah–Singer Theorem and Elementary Number Theory*, Publish or Perish, Boston, MA, 1974.
- [14] S. B. Hu, *Shintani cocycles and generalized Dedekind sums*, Ph.D. thesis, Univ. of Pennsylvania, 1997.
- [15] M. I. Knopp, *Hecke operators and an identity for the Dedekind sums*, J. Number Theory 12 (1980), 2–9.
- [16] D. E. Knuth, *The Art of Computer Programming. Vol. 2*, 2nd ed., Addison-Wesley, Reading, MA, 1981.
- [17] C. Meyer, *Über einige Anwendungen Dedekindscher Summen*, J. Reine Angew. Math. 198 (1957), 143–203.
- [18] W. Meyer und R. Sczech, *Über eine topologische und zahlentheoretische Anwendung von Hirzebruchs Spitzenauflösung*, Math. Ann. 240 (1979), 69–96.
- [19] M. Mikolás, *On certain sums generating the Dedekind sums and their reciprocity laws*, Pacific J. Math. 7 (1957), 1167–1178.
- [20] L. J. Mordell, *Lattice points in a tetrahedron and generalized Dedekind sums*, J. Indian Math. Soc. (N.S.) 15 (1951), 41–46.
- [21] L. A. Parson and K. H. Rosen, *Hecke operators and Lambert series*, Math. Scand. 49 (1981), 5–14.
- [22] J. E. Pommersheim, *Toric varieties, lattice points and Dedekind sums*, Math. Ann. 295 (1993), 1–24.
- [23] J. L. Raabe, *Zurückführung einiger Summen und bestimmten Integrale auf die Jacob-Bernoullische Function*, J. Reine Angew. Math. 42 (1851), 348–376.
- [24] H. Rademacher, *Some remarks on certain generalized Dedekind sums*, Acta Arith. 9 (1964), 97–105.
- [25] H. Rademacher and E. Grosswald, *Dedekind Sums*, Carus Math. Monographs 16, Math. Assoc. America, Washington, DC, 1972.

- [26] R. Szezech, *Dedekindsummen mit elliptischen Funktionen*, Invent. Math. 76 (1984), 523–551.
- [27] —, *Eisenstein group cocycles for GL_n and values of L -functions*, ibid. 113 (1993), 581–616.
- [28] D. Solomon, *Algebraic properties of Shintani's generating functions: Dedekind sums and cocycles on $PGL_2(\mathbb{Q})$* , Compos. Math. 112 (1998), 333–362.
- [29] L. Takács, *On generalized Dedekind sums*, J. Number Theory 11 (1979), 264–272.
- [30] G. Urzúa, *Arrangements of curves and algebraic surfaces*, J. Algebraic Geom. 19 (2010), 335–365.
- [31] D. Zagier, *Higher dimensional Dedekind sums*, Math. Ann. 202 (1973), 149–172.
- [32] Z. Zheng, *The Petersson–Knopp identity for the homogeneous Dedekind sums*, J. Number Theory 57 (1996), 223–230.

Matthias Beck
Department of Mathematics
San Francisco State University
San Francisco, CA 94132, U.S.A.
E-mail: beck@math.sfsu.edu

Anastasia Chavez
Department of Mathematics
University of California
Berkeley, CA 94720, U.S.A.
E-mail: anastasia.chavez@gmail.com

*Received on 18.8.2010
and in revised form on 23.3.2011*

(6467)