A note on global units and local units of function fields

by

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1. Introduction. Let $K$ be any Galois extension of $\mathbb{Q}$, and $U_K$ be the unit group of $K$. For any place $v$ of $K$, let $U_v$ be the group of local units of $K_v$. Recently, the second author and Xianke Zhang [6] considered the problem of whether there exists an odd prime $p$ such that the map

$$U_K/U_K^2 \to \prod_{v|p} U_v/U_v^2$$

is injective. In fact, they proved that the existence of such primes is equivalent to $\text{Hom}(U_K/U_K^2, \{\pm 1\})$ is a cyclic $\mathbb{F}_2[\text{Gal}(K/\mathbb{Q})]$-module. Moreover, they also proved that if the class number $h_{\mathbb{Q}(\zeta_p^r)^+}$ is odd, then such primes exist for $\mathbb{Q}(\zeta_p^r)^+$ and $\mathbb{Q}(\zeta_p^r)$, where $p$ is an odd prime and $\mathbb{Q}(\zeta_p^r)^+$ is the maximal real subfield of $\mathbb{Q}(\zeta_p^r)$.

Let $K$ be a geometric Galois extension of the rational function field $k = \mathbb{F}_q(t)$. Let $O_K$ be the integral closure of $\mathbb{F}_q[t]$ in $K$. Let $U_K$ be the group of units of $O_K$ and $U_v$ be the group of local units of $K_v$. In this note, we will generalize the second author and Zhang’s methods to consider the question whether there exists a finite place $P$ of $\mathbb{F}_q(t)$ such that the map

$$U_K/U_K^d \to \prod_{v|P} U_v/U_v^d$$

is injective, where $d > 1$ is a factor of $q - 1$. Let $\mu_d$ be the group of $d$th roots of unity. We will prove that there exist such places $P$ if and only if $\text{Hom}(U_K/U_K^d, \mu_d)$ is a cyclic $\mathbb{Z}/d\mathbb{Z}[\text{Gal}(K/k)]$-module. When $K$ is a quadratic function field, we will prove in Section 4 that such places exist if and only if either $K$ is imaginary, or $K$ is real and $d$ is odd, or $K$ is real, $d$ is even and there exists a fundamental unit $\epsilon_0$ of $O_K$ such that $N(\epsilon_0)$ is a generator of $\mathbb{F}_q^*$. Let $A$ be a monic irreducible polynomial. Suppose that $K = k(A)$ is the $A$th cyclotomic function field and $K^+$ is the maximal real subfield of $K$. In

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Section 5, we will prove that such places exist for $K$ and $K^+$ if the class number of $O_{K^+}$ is relatively prime to $d$. It should be noted that the proof heavily relies on Galovich and Rosen’s work on Sinnott’s circular units in cyclotomic function fields [3].

2. Preliminaries. For each $d | q - 1$, define $L = K(\sqrt[d]{U_K})$. Since $\mathbb{F}_q$ contains $d$th roots of unity, $L$ is an abelian extension of $K$ of exponent $d$. Set $\text{Gal}(L/K) = H$ and $\text{Gal}(K/\mathbb{F}_q(t)) = G$. Define an action of $G$ on $H$ by $gh = \tilde{g}h\tilde{g}^{-1}$, where $g \in G$, $h \in H$ and $\tilde{g}$ is a lift of $g$ in $\text{Gal}(L/\mathbb{F}_q(t))$. By Kummer theory (e.g. Theorem 8.1 of [5]), there is a non-degenerate $G$-equivariant bilinear pairing $H \times U_K/U_K^d \rightarrow \mu_d$, $(h, \bar{u}) = h(\sqrt[d]{u})\sqrt[d]{u}$. Therefore we have $H \cong \text{Hom}(U_K/U_K^d, \mu_d)$ as $G$-modules. The action of $G$ on $\text{Hom}(U_K/U_K^d, \mu_d)$ is defined by $gf(\bar{u}) = f(g^{-1}\bar{u})$ for $g \in \text{Gal}(K/\mathbb{F}_q(t))$, $f \in \text{Hom}(U_K/U_K^d, \mu_d)$, $\bar{u} \in U_K/U_K^d$.

Assume the infinite place $(1/t)$ of $\mathbb{F}_q(t)$ splits into $r$ places of $K$. By Dirichlet’s unit theorem, the rank of $U_K/U_K^d$ as $\mathbb{Z}/d\mathbb{Z}$-module is equal to $r$. Let $\{u_1, \ldots, u_r\} \subset U_K$ be representatives such that $\overline{u}_1, \ldots, \overline{u}_r$ form a $\mathbb{Z}/d\mathbb{Z}$-basis of $U_K/U_K^d$. Then it is easy to show that $H \cong \text{Gal}(K(\sqrt[d]{u_1})/K) \times \cdots \times \text{Gal}(K(\sqrt[d]{u_r})/K)$. The isomorphism is given by restriction to the subfields.

The following is Chebotarev’s density theorem for global function fields (Theorem 9.13A of [7]).

**Theorem 2.1 (Chebotarev).** Let $L/K$ be a Galois extension of global function fields and $\text{Gal}(L/K) = H$. Let $C \subset H$ be a conjugacy class and $S'_K$ be the set of primes of $K$ which are unramified in $L$. Then

$$\delta(\{\mathfrak{p} \in S'_K \mid (\mathfrak{p}, L/K) = C\}) = #C/#H,$$

where $\delta$ means Dirichlet density. In particular, every conjugacy class $C$ is of the form $(\mathfrak{p}, L/K)$ for infinitely many places $\mathfrak{p}$ of $K$.

**Lemma 2.2.** Let $u \in U_K$ and $\mathfrak{p}$ be a place of $K$ which is unramified in $L$. Then $u \in U_K^d$ if and only if $(\mathfrak{p}, L/K)$ fixes $K(\sqrt[d]{u})$, where $L = K(\sqrt[d]{U_K})$ (see the beginning of this section).

**Proof.** $u \in U_K^d$ is equivalent to $\mathfrak{p}$ splitting completely in $K(\sqrt[d]{u})$. Since $\mathfrak{p}$ is unramified in $K(\sqrt[d]{u})$, this is equivalent to $(\mathfrak{p}, K(\sqrt[d]{u})/K) = \text{Id}$. As the Artin symbol satisfies $(\mathfrak{p}, L/K)|_{K(\sqrt[d]{u})} = (\mathfrak{p}, K(\sqrt[d]{u})/K)$, the result follows. ■
3. Proof of the main result

**Proposition 3.1.** The natural map $U_K/U_K^d \to \prod_v U_v/U_v^d$ is injective, where $v$ runs over all finite places of $K$.

**Proof.** Let $u$ belong to the kernel of the map. Then $u \in U_v^d$ for all $v$. By Lemma 2.2, $(v, K(\sqrt[d]{u})/K) = \text{Id}$ for all finite places $v$. Consequently, $\delta(\{p \in S_K' \mid (p, L/K) = \text{Id}\}) = 1$. By Chebotarev’s density theorem, the extension $K(\sqrt[d]{u})/K$ is trivial. Thus $u \in U_K^d$. ■

**Proposition 3.2.** There exist places $p_1, \ldots, p_r$ of $K$ such that the natural map $U_K/U_K^d \to \prod_{1 \leq i \leq r} U_{p_i}/U_{p_i}^d$ is injective.

**Proof.** Let $\sigma_1, \ldots, \sigma_r \in H$ be such that the restriction of $\sigma_i$ to $K(\sqrt[d]{u_j})$ is trivial when $j \neq i$ and is a generator of $\text{Gal}(K(\sqrt[d]{u_j})/K)$ for $j = i$. By Chebotarev’s density theorem, there exist finite places $p_1, \ldots, p_r$ of $K$ such that $(p_i, L/K) = \sigma_i$. If $u$ belongs to the kernel, then $u \in U_{p_i}^d$. By Lemma 2.2, $\sigma_i$ fixes $K(\sqrt[d]{u})$. By construction, $\sigma_1, \ldots, \sigma_r$ generate $H$, so $K(\sqrt[d]{u}) = K$ by Galois theory. Thus $u \in U_K^d$. ■

**Proposition 3.3.** Let $P$ be a finite place of $\mathbb{F}_q(t)$. Then the natural map $U_K/U_K^d \to \prod_{v | P} U_v/U_v^d$ is injective if and only if for some place $v | P$ (hence for all $v | P$), $(v, L/K)$ is a $\mathbb{Z}/d\mathbb{Z}[G]$ generator of $H$.

**Proof.** Let $u$ be any unit of $K$. By Lemma 2.2,

$$u \in U_v^d, \forall v | P \Leftrightarrow (v, L/K) \text{ fixes } K(\sqrt[d]{u}), \forall v | P.$$ 

It is obvious that

$$u \in U_K^d \Leftrightarrow K(\sqrt[d]{u}) = K.$$ 

Thus, $U_K/U_K^d \to \prod_{v | P} U_v/U_v^d$ being injective is equivalent to

$$\forall u \in U_K, (v, L/K) \text{ fixes } K(\sqrt[d]{u}), \forall v | P \Rightarrow K(\sqrt[d]{u}) = K.$$ 

By Galois theory, this is equivalent to the subgroup generated by $(v, L/K)$ for all $v | P$ being equal to $H$. Recall the definition of the action of $G$ on $H$ in Section 2: $(g, L/K) = \tilde{g}(v, L/K)\tilde{g}^{-1} = g(v, L/K)$. This is also equivalent to $(v, L/K)$ being a $\mathbb{Z}/d\mathbb{Z}[G]$ generator of $H$ for any $v | P$. ■

**Theorem 3.4.** There exists a finite place $P$ of $\mathbb{F}_q(t)$ such that the map $U_K/U_K^d \to \prod_{v | P} U_v/U_v^d$ is injective if and only if $\text{Hom}(U_K/U_K^d, \mu_d)$ is a cyclic $\mathbb{Z}/d\mathbb{Z}[G]$-module.
Proof. Since \( H \) is isomorphic to \( \text{Hom}(U_K/U_K^d, \mu_d) \) as \( \mathbb{Z}/d\mathbb{Z}[G] \)-modules, the “only if” part follows easily from Proposition 3.3. Conversely, if \( H \) is a cyclic module, let \( \sigma \in H \) be a \( \mathbb{Z}/d\mathbb{Z}[G] \) generator of \( H \). By Chebotarev’s density theorem, there exists a finite place \( \mathfrak{p} \) such that \((\mathfrak{p}, L/K) = \sigma\). Also by Proposition 3.3, we conclude that \( U_K/U_K^d \to \prod_{g \in G} U_{g\mathfrak{p}}/U_{g\mathfrak{p}}^d \) is injective. \( \blacksquare \)

The following definition can be found on page 371 of [8].

**Definition 3.5.** An extension \( K \) of \( k = \mathbb{F}_q(t) \) is called **totally real** if the prime at infinity of \( k \) (which corresponds to \( 1/t \)) splits completely in \( K \).

**Lemma 3.6.** Let \( G \) be a finite group and \( V \) be a free \( \mathbb{Z}/d\mathbb{Z} \)-module of rank \( r = |G| \). Assume \( G \) acts on \( V \) linearly. Then \( V \) is a cyclic \( \mathbb{Z}/d\mathbb{Z}[G] \)-module if and only if \( V^* = \text{Hom}(V, \mathbb{Z}/d\mathbb{Z}) \) is a cyclic \( \mathbb{Z}/d\mathbb{Z}[G] \)-module.

**Theorem 3.7.** If \( K \) is a totally real geometric Galois extension of \( \mathbb{F}_q(t) \), there exists a finite place \( P \) of \( \mathbb{F}_q(t) \) such that the natural map

\[
U_K/U_K^d \to \prod_{v \mid P} U_v/U_v^d
\]

is injective if and only if \( U_K/U_K^d \) is a cyclic \( \mathbb{Z}/d\mathbb{Z}[G] \)-module.

**Proof.** Suppose that \([K : \mathbb{F}_q(t)] = n\). By Definition 3.5, \( K \) has \( n \) infinite places. By Dirichlet’s unit theorem, \( U_K/U_K^d \) is a free \( \mathbb{Z}/d\mathbb{Z} \)-module of rank \( n \). By Theorem 3.4, the injectivity in question is equivalent to \( \text{Hom}(U_K/U_K^d, \mu_d) \) being a cyclic \( \mathbb{Z}/d\mathbb{Z}[G] \)-module. Applying Lemma 3.6 to \( V = U_K/U_K^d \), we get the desired result. \( \blacksquare \)

A unit \( u \) is called a **Minkowski unit** if its Galois conjugates generate a subgroup of finite index in the whole unit group. We know that such units always exist (see [9, Lemma 5.27], the proof is the same for global function fields).

**Corollary 3.8.** Let \( K/\mathbb{F}_q(t) \) be a totally real geometric Galois extension. There exists a finite place \( P \) of \( \mathbb{F}_q(t) \) such that the natural map

\[
U_K/U_K^d \to \prod_{v \mid P} U_v/U_v^d
\]

is injective if and only if there exists a Minkowski unit \( \epsilon \) such that the index of \( \mathbb{Z}[G]\epsilon \) in \( U_K \) is relatively prime to \( d \).

**Proof.** By Theorem 3.4, the existence of such \( P \) is equivalent to \( U_K/U_K^d \) being a cyclic \( \mathbb{Z}/d\mathbb{Z}[G] \)-module. This means that there exists a unit \( \epsilon \) such that \( U_K = U_K^d(\mathbb{Z}[G]\epsilon) \). Let \( E = \mathbb{Z}[G]\epsilon \). We get

\[
U_K = EU_K^d \iff U_K/E = (U_K/E)^d \iff (\#U_K/E, d) = 1.
\]

This completes the proof of the corollary. \( \blacksquare \)
4. The case of quadratic function fields. In this section, we assume $K$ is a quadratic extension of $k = \mathbb{F}_q(t)$ and $2 \nmid q$. We will use the theory developed in Section 3 to investigate the situation of quadratic function fields. Such fields can be written as $k(\sqrt{D})$, where $D$ is a square free polynomial of $\mathbb{F}_q[t]$. They were systematically studied by E. Artin [1].

Fix a generator $g$ of $\mathbb{F}_q^\times$. Then we can assume that the leading coefficient of $D$ is 1 or $g$. The infinite place $(1/t)$ is splitting, inertial, or ramified in $K$ when, respectively: the degree of $D$ is even and $\text{sgn}(D) = 1$; the degree of $D$ is even and $\text{sgn}(D) = g$; or the degree of $D$ is odd. Then the field $K$ is called real, inertial imaginary, or ramified imaginary respectively, according to E. Artin [1]. When $K$ is real, we let $\epsilon_0$ be the fundamental unit of $K$. Any fundamental unit is determined only up to multiplication by a constant, thus its norm is either a square or $g$ times a square. So multiplying $\epsilon_0$ by an appropriate constant we can assume $N(\epsilon_0)$ is 1 or $g$.

Now we state the main theorem of this section.

**Theorem 4.1.** Let the notations be as above. There exists a finite place $P$ of $\mathbb{F}_q(t)$ such that

$$U_K/U_K^d \to \prod_{v|P} U_v/U_v^d$$

is injective if and only if either $K$ is imaginary, or $K$ is real and $d$ is odd, or $K$ is real, $d$ is even and $N(\epsilon_0) = g$.

**Proof.** If $K$ is imaginary, then $U_K = \mathbb{F}_q^*$ and $U_K/U_K^d = \mathbb{F}_q^*/\mathbb{F}_q^d$ is a cyclic group. Thus $\text{Hom}(U_K/U_K^d, \mathbb{Z}/d\mathbb{Z})$ is a cyclic $\mathbb{Z}/d\mathbb{Z}[G]$-module. By Theorem 3.4, there exists a finite place $P$ of $\mathbb{F}_q(t)$ such that

$$U_K/U_K^d \to \prod_{v|P} U_v/U_v^d$$

is injective.

If $K$ is real, then $U_K = \langle \epsilon_0 \rangle \times \mathbb{F}_q^*$. By Corollary 3.8, the existence of such places is equivalent to the existence of a Minkowski unit $\epsilon$ such that $(\#U_K/\mathbb{Z}[G] \epsilon, d) = 1$. If $N(\epsilon_0) = g$, we can take $\epsilon = \epsilon_0$, and then $U_K = \mathbb{Z}[G] \epsilon$. If $N(\epsilon_0) = 1$ and $d$ is odd, we can take $\epsilon = g \epsilon_0$, and then

$$\mathbb{Z}[G] \epsilon = \mathbb{Z} \epsilon \oplus \langle N(\epsilon) \rangle = \mathbb{Z} \epsilon \oplus \langle g^2 \rangle.$$  

Thus $2 | \#U_K/\mathbb{Z}[G] \epsilon = 2$ is prime to $d$. If $N(\epsilon_0) = 1$ and $d$ is even, for any Minkowski unit $\epsilon$, write $\epsilon = c_k^l g^l$, $k, l \in \mathbb{Z}, k \neq 0$. As above,

$$\mathbb{Z}[G] \epsilon = \mathbb{Z} \epsilon \oplus \langle N(\epsilon) \rangle = \mathbb{Z} \epsilon \oplus \langle g^{2l} \rangle \subset \mathbb{Z} \epsilon \oplus \langle g^2 \rangle.$$  

Thus $2 | \#U_K/\mathbb{Z}[G] \epsilon$, so $2 | (\#U_K/\mathbb{Z}[G] \epsilon, d)$. The proof is complete.
5. **The case of cyclotomic function fields.** Before stating the main theorem of this section, we must introduce some notation. Write $k = \mathbb{F}_q(t)$ and $R = \mathbb{F}_q[t]$. Let $k^{\text{ac}}$ be the algebraic closure of $k$. In order to construct the explicit class field theory for $k$, Carlitz [2] introduced an $R$-module structure on $k^{\text{ac}}$, called the *Carlitz module* (see also [4]). Let $\text{End}(k^{\text{ac}})$ be the ring of $\mathbb{F}_q$-algebra endomorphisms of $k^{\text{ac}}$. Let

$$\rho : R \rightarrow \text{End}(k^{\text{ac}}), \quad M \mapsto \rho_M,$$

be a ring homomorphism defined by

$$\rho_a(\alpha) = a\alpha, \quad \rho_t(\alpha) = t\alpha + \alpha^q,$$

where $a \in \mathbb{F}_q$ and $\alpha \in k^{\text{ac}}$. Let

$$\Lambda_M = \{\alpha \in k^{\text{ac}} \mid \rho_M(\alpha) = 0\},$$

which is called the *$M$-torsion module* of $k^{\text{ac}}$. If $M$ is monic, $k(\Lambda_M)$ is called the $M$th cyclotomic function field. Chapter 12 of [7] gives a nice exposition of the theory of cyclotomic function fields. Let $S_\infty(k(\Lambda_M))$ be the set of infinite places of $k(\Lambda_M)$ and $U_M$ be the group of $S_\infty(k(\Lambda_M))$-units of $k(\Lambda_M)$. For simplicity, let $P(3)$ denote the following property: there exists a finite place $P$ in $\mathbb{F}_q(t)$ such that $U_K/U_K^d \rightarrow \prod_{v \mid P} U_v/U_v^d$ is injective, where $K$ is a geometric Galois extension of $\mathbb{F}_q(t)$. Now we can state the main theorem of this section.

**Theorem 5.1.** Let $A$ be a monic irreducible polynomial in $\mathbb{F}_q[t]$, $K = k(\Lambda_A)$ and $K^+$ be the maximal real subfield of $K$ (for the definitions, see Theorem 12.14 of [7]). Let $h_A$ be the class number of $O_K$ and $h^+_A$ be the class number of $O_K^+$. If $d \mid q - 1$ and $(h^+_A, d) = 1$, then $P(3)$ holds for $K$ and $K^+$.

Before proving the above theorem, we briefly recall Galovich and Rosen’s work on Sinnott’s cyclotomic units in cyclotomic function fields [3].

**Definition 5.2.** Let $M$ be a monic polynomial in $\mathbb{F}_q[t]$, and $\lambda$ be a primitive $M$-torsion element. Define

$$S = \{\rho_B(\lambda)/\lambda \mid B \text{ is a monic polynomial, } 0 < \deg B < \deg M, (B, M) = 1\}$$

(obviously, $S \subset U_{k(\Lambda_M)}^+$). The elements in the subgroup generated by $S$ are called the *Kummer–Hilbert circular units*, denoted by $C_q(k(\Lambda_M)^+)$. Let $G$ be the multiplicative subgroup of $k(\Lambda_M)^*$ generated by $\mathbb{F}_q^*$ and $\Lambda_M^* = \Lambda_M - \{0\}$. The elements of $C = G \cap U_{k(\Lambda_M)}$ and $C^+ = C \cap U_{k(\Lambda_M)}^+$ are called the *Sinnott circular units* of $k(\Lambda_M)$ and $k(\Lambda_M)^+$, respectively.

**Remark 5.3.** Since $A$ is irreducible, from [3] we know that

$$U_K = U_{K^+}, \quad C = C^+ \quad \text{and} \quad C^+ = \mathbb{F}_q^*C_q(K^+).$$
In this case, Galovich and Rosen proved (see [3])

**Theorem 5.4 (Galovich–Rosen).** \([U_K : C] = [U_{K^+} : C^+] = h_A^+\).

Now we can start the proof of Theorem 5.1.

**Proof of Theorem 5.1.** Let

\[ G = \text{Gal}(K/k) \quad \text{and} \quad G^+ = \text{Gal}(K^+/k). \]

From Remark 5.3, we have \(U_K/U_K^d = U_{K^+}/U_{K^+}^d\). Thus \(\text{Hom}(U_K/U_K^d, \mu_d)\) is a cyclic \(\mathbb{Z}/d\mathbb{Z}[G]\)-module if and only if \(\text{Hom}(U_{K^+}/U_{K^+}^d, \mu_d)\) is a cyclic \(\mathbb{Z}/d\mathbb{Z}[G^+]\)-module. By Theorem 3.4, P(3) holds for \(K\) if and only if P(3) holds for \(K^+\). From Theorem 5.4, \(h_A^+ = [U_{K^+} : C^+]\), so \(\langle [U_{K^+} : C^+], d \rangle = 1\) by assumption. If we can show that \(C^+\) is a cyclic \(\mathbb{Z}/d\mathbb{Z}[G^+]\)-module, then by Corollary 3.8, we will complete the proof. Suppose \(M\) is a generator of \((\mathbb{F}_q[t]/A\mathbb{F}_q[t])^*\). Since by Remark 5.3, \(C^+ = \mathbb{F}_q^*C_y(K^+), C^+\) is generated by the set

\[ \tilde{S} = \{\rho_{M_i}(\lambda)/\lambda \mid 1 \leq i \leq q^{\deg A} - 1\}. \]

For each polynomial \(W\) relatively prime to \(A\) there is a unique element \(\sigma_W \in G\) such that \(\sigma_W(\lambda) = \rho_W(\lambda)\) where \(\lambda\) is a primitive \(A\)-torsion element (see Theorem 12.8 of [7]). Using the definition of the group ring action, the multiplicity of \(\sigma_N\), and cancellation in a telescoping product, we have

\[
\frac{\rho_{M^i+1}(\lambda)}{\lambda} = \frac{\sigma_{M^i+1}(\lambda)}{\lambda} = (1 + \sigma_M + \sigma_M^2 + \cdots + \sigma_M^i) \frac{\sigma_M(\lambda)}{\lambda}.
\]

Thus

\[ C^+ = \mathbb{Z}[G] \frac{\sigma_M(\lambda)}{\lambda} = \mathbb{Z}[G^+] \frac{\sigma_M(\lambda)}{\lambda} \]

is a cyclic module, and the proof is finished. \(\blacksquare\)

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