ACTA ARITHMETICA
139.1 (2009)

# $p$-adic polylogarithms and irrationality

by

PIERRE BEL (Bordeaux)

**1. Results.** We denote by $\mathcal{L}i_s$ the $p$-adic polylogarithm function defined for an integer $s$ and $p$-adic number $x \in \mathbb{C}_p$ by

$$\mathcal{L}i_s(x) = \sum_{k=1}^{+\infty} \frac{x^k}{k^s},$$

for $|x|_p < 1$. We denote by $\mathrm{Li}_s(z)$ the complex polylogarithm defined by the same series and for complex numbers $z$ and $s$ such that $|z| < 1$.

The $p$-adic polylogarithms have applications to number fields (cf. [Col]) and $p$-adic $L$-functions (cf. [Fu]).

In the archimedean case, we have the following diophantine results. The results of M. Hata (cf. [Ha]) improved by G. Rhin and C. Viola (cf. [Rh]) give

THEOREM 1. *For any integer $q$ such that $|q| \geq 6$, the number* $\mathrm{Li}_2(1/q)$ *is irrational.*

M. Hata also gives explicit conditions on the integer $m$ and the rational number $x$ for $\mathrm{Li}_m(x)$ to be an irrational number.

In [Ri], T. Rivoal proves

THEOREM 2. *Let $x$ be a rational number such that $|x| < 1$. The set $\{\mathrm{Li}_s(x)\}_{s\in\mathbb{N}}$ contains infinitely many irrational numbers linearly independent over $\mathbb{Q}$.*

R. Marcovecchio proved this result for $x$ an algebraic number (cf. [Ma]).

In the $p$-adic case, the diophantine results are fewer than in the archimedean case. In this paper, we prove the following new results.

THEOREM 3. *Let $\mathbb{K} = \mathbb{Q}(\delta)$ be a number field and $p$ a prime number. Consider $\mathbb{K}$ as embedded into $\mathbb{C}_p$ and denote by $\mathbb{K}_p$ the completion of this embedding. Suppose that $|\delta|_p > 1$ and let $d(\delta)$ be the denominator of $\delta$. For*

2000 *Mathematics Subject Classification*: 11J72, 11J61.
*Key words and phrases*: $p$-adic, irrationality, linear independence, polylogarithm.

DOI: 10.4064/aa139-1-4  

*any integer $A \geq 2$, the dimension $\tau$ of the $\mathbb{K}$-vector space spanned by* 1 *and* $(\mathcal{L}i_s(\delta^{-1}))_{s\in[1,A]}$ *satisfies*

$$\tau \geq \frac{X_2 - \sqrt{X_2^2 - 2X_1X_3}}{X_1}, \tag{1}$$

*where*

$$\begin{aligned} X_1 &= [\mathbb{K}:\mathbb{Q}],\\ X_2 &= ((A+1)\log(A+1) + \log d(\delta) + A(1+\log 2))\\ &\quad + \sum_{v\in\mathcal{V}_\infty} \eta_v \log\max(1,|\delta|_v) + \frac{1}{2}[\mathbb{K}:\mathbb{Q}],\\ X_3 &= \eta_p(A+1)\log|\delta|_p. \end{aligned}$$

Remark 1. Under the hypotheses of Theorem 3, we have the lower bound

$$\tau \geq \frac{[\mathbb{K}_p:\mathbb{Q}_p](A+1)\log|\delta|_p}{[\mathbb{Q}(\delta):\mathbb{Q}]((A+1)\log(A+1)+\log d(\delta)+A(1+\log 2)) + \sum_{v\in\mathcal{V}_\infty}\eta_v\log\max(1,|\delta|_v)}, \tag{2}$$

which follows from (1).

Corollary 1. *For any integer $s \geq 2$ and any integer $a > 0$, if the prime number $p$ sastisfies*

$$a\log p > \frac{s}{2} + s\log(s+1) + s^2\log(s+1) + \frac{s^2}{2} + s^2\log 2 =: f(s),$$

*then the number $\mathcal{L}i_s(p^a)$, which belongs to $\mathbb{Q}_p$, is irrational.*

*Proof.* We apply the inequality (1) with a fixed integer $A = s$ and $\delta = p^{-a}$. In this case $[\mathbb{K}_p:\mathbb{Q}_p] = [\mathbb{K}:\mathbb{Q}] = 1$, $\log|p^{-a}|_p = \log d(p^{-a}) = a\log p$ and $\log(\max(1,|p^{-a}|)) = 0$. We thus have

$$\lim_{p^a\to+\infty} \frac{X_2 - \sqrt{X_2^2 - 2X_1X_3}}{X_1} = A+1.$$

The equation $\frac{X_2 - \sqrt{X_2^2-2X_1X_3}}{X_1} = A$ has one solution in $\mathbb{R}^+$ which is $p^a = e^{f(A)}$. We obtain

$$\dim_{\mathbb{K}} \mathrm{Vect}(1,(\mathcal{L}i_s(\delta^{-1}))_{s\in[1,A]}) > A$$

for $a\log p > f(A)$, which completes the proof.

Corollary 2. *The numbers $\mathcal{L}i_2(234281)$ and $\mathcal{L}i_2(2^{18})$, which belongs to $\mathbb{Q}_{234281}$ and $\mathbb{Q}_2$ respectively, are irrational.*

**2. Notations and conventions.** In this paper, $\mathbb{K}$ represents a number field and $\mathcal{O}(\mathbb{K})$ its ring of algebraic integers. For an algebraic number $\beta$, we denote by $d(\beta)$ the denominator of $\beta$, which is defined as the least positive integer $l$ for which $l\beta$ is an algebraic integer.

We set $d_n = \operatorname{lcm}(1,\dots,n)$. The prime number theorem gives the estimate $d_n = e^{n+o(n)}$.

For a prime number $p$, we denote by $v_p$ the $p$-adic valuation over $\mathbb{Q}$ and $|\cdot|_p = p^{-v_p(\cdot)}$ the $p$-adic norm.

Let $v$ be a place of the number field $\mathbb{K}$. Then $\mathbb{K}_v$ and $\mathbb{Q}_v$ denote the completions of $\mathbb{K}$ and $\mathbb{Q}$ at this place and $\eta_v$ stands for the index $[\mathbb{K}_v : \mathbb{Q}_v]$. $\mathcal{V}$, $\mathcal{V}_\infty$ and $\mathcal{V}_f$ represent the sets of places, of infinite places and of finite places respectively.

For any $\alpha \in \mathbb{K}^*$, we have the product formula

$$\sum_{v\in\mathcal{V}} \eta_v \log|\alpha|_v = 0.$$

Moreover,

$$\sum_{v\in\mathcal{V}_\infty} \eta_v = [\mathbb{K} : \mathbb{Q}].$$

If $\alpha$ is an element of $\mathcal{O}(\mathbb{K}) \setminus \{0\}$ and $\mathfrak{p}$ a finite place, as $|\alpha|_v \le 1$ for any finite place $v$ of $\mathbb{K}$, we have

$$\eta_{\mathfrak{p}} \log|\alpha|_{\mathfrak{p}} + \sum_{v\in\mathcal{V}_\infty} \eta_v \log|\alpha|_v \ge 0.$$

**3. A criterion of linear independence.** This criterion is an adaptation in the $p$-adic case of the criterion used in the complex case by R. Marcovecchio (cf. [Ma]). The author did not find any statement in this form in the mathematical literature.

Let $m$ be a positive integer, $L = (\ell_1,\dots,\ell_m) \in \mathbb{K}^m$, $\theta = (\theta_1,\dots,\theta_m) \in \mathbb{C}_p^m$ and $(L,\theta) = \ell_1\theta_1 + \dots + \ell_m\theta_m$. For any place $v$ of $\mathbb{K}$, we define $\|L\|_v = \max_{1\le j\le m} |\ell_j|_v$.

LEMMA 1. *Let $p$ be a prime number and $\mathbb{K}$ a number field. Fix an embedding of $\mathbb{K}$ into $\mathbb{C}_p$ and denote by $\mathbb{K}_p = \mathbb{Q}_p(\mathbb{K})$ its completion. Let $\theta = (\theta_1,\dots,\theta_m)$ be a nonzero vector of $\mathbb{K}_p^m$. Suppose that there exist real positive numbers $(c_v)_{v\in\mathcal{V}_\infty}$, a real number $\rho$, and $m$ sequences $(L_n^{(i)}) = ((\ell_{n,j}^{(i)})_{j\in[1,m]})$, with $n \in \mathbb{N}$ and $1 \le i \le m$, of vectors in $(\mathcal{O}(\mathbb{K}))^m$ such that for all $n$, the $m$ vectors $L_n^{(i)}$ are linearly independent over $\mathbb{K}$ and enjoy the following properties*:

(i) *for any place $v \in \mathcal{V}_\infty$, $\limsup_n n^{-1} \log\|L_n^{(i)}\|_v \le c_v$,*

(ii) *$\limsup_n n^{-1} \log|(L_n^{(i)},\theta)|_p \le -\rho$.*

*Then*

$$\tau = \dim_{\mathbb{K}} \operatorname{Vect}(\theta_1,\dots,\theta_m) \ge \frac{\rho\,[\mathbb{K}_p : \mathbb{Q}_p]}{\sum_{v\in\mathcal{V}_\infty} \eta_v c_v}. \tag{3}$$

*Moreover, if $d_n^{j-1}$ divides $\ell_{n,j}^{(i)}$ for all $(i,j) \in [1,m]^2$, we have more precisely*

$$\tau \geq \frac{\sum_{v\in\mathcal{V}_\infty} \eta_v c_v + \frac{1}{2}[\mathbb{K}:\mathbb{Q}] - \sqrt{\Big(\sum_{v\in\mathcal{V}_\infty} \eta_v c_v + \frac{1}{2}[\mathbb{K}:\mathbb{Q}]\Big)^2 - 2\rho\eta_p[\mathbb{K}:\mathbb{Q}]}}{[\mathbb{K}:\mathbb{Q}]}. \tag{4}$$

*Proof.* By swapping the indices of $(\theta_i)_{i\in[1,m]}$, we can suppose that $\theta_1$ is nonzero. Furthermore, replacing $(\theta_j)_{j\in[1,m]}$ by $(\theta_j/\theta_1)_{j\in[1,m]}$, we assume that $\theta_1 = 1$.

If $\tau$ is the dimension of the $\mathbb{K}$-vector space spanned by the $\theta_j$, then there exist $m-\tau$ vectors $(A^{(i)})_{i\in[\tau+1,m]}$ of $(\mathcal{O}(\mathbb{K}))^m$, linearly independent over $\mathbb{K}$, such that $(A^{(i)},\theta) = 0$ for all $i \in [\tau+1,m]$.

By permutation of $i$, we can suppose that for all $n \in \mathbb{N}$, the vectors $(L_n^{(1)},\dots,L_n^{(\tau)},A^{(\tau+1)},\dots,A^{(m)})$ are linearly independent.

Let $M_n$ be the matrix whose rows are the vectors

$$(L_n^{(1)},\dots,L_n^{(\tau)},A^{(\tau+1)},\dots,A^{(m)}),$$

i.e. $L_n^{(i)} = (\ell_{n,1}^{(i)},\dots,\ell_{n,m}^{(i)})$ and $A^{(i)} = (a_1^{(i)},\dots,a_m^{(i)})$,

$$M_n = \begin{pmatrix} \ell_{n,1}^{(1)} & \ell_{n,2}^{(1)} & \cdots & \ell_{n,m}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ \ell_{n,1}^{(\tau)} & \ell_{n,2}^{(\tau)} & \cdots & \ell_{n,m}^{(\tau)} \\ a_1^{(\tau+1)} & a_2^{(\tau+1)} & \cdots & a_m^{(\tau+1)} \\ \cdots & \cdots & \cdots & \cdots \\ a_1^{(m)} & a_2^{(m)} & \cdots & a_m^{(m)} \end{pmatrix}. \tag{5}$$

Since the matrix is nonsingular, we have

$$\Lambda_n = \det(M_n) \neq 0. \tag{6}$$

Since $\Lambda_n$ belongs to $\mathcal{O}(\mathbb{K})$, we deduce from (6) that

$$0 \leq \eta_p \log|\Lambda_n|_p + \sum_{v\in\mathcal{V}_\infty} \eta_v \log|\Lambda_n|_v. \tag{7}$$

For the infinite places, the expansion of the determinant (5) gives

$$|\Lambda_n|_v \leq m!\Big(\max_{\substack{j\in[1,\tau]\\ j\in[1,n]}} |\ell_{n,i}^{(j)}|_v\Big)^\tau \Big(\max_{\substack{j\in[\tau+1,m]\\ j\in[1,n]}} |a_i^{(j)}|_v\Big)^{m-\tau}.$$

By using assumption (i), this implies that

$$\limsup_n \frac{\log|\Lambda_n|_v}{n} \leq \tau c_v. \tag{8}$$

By multi-linearity of $\Lambda_n$, we can add the $j$th column multiplied by $\theta_j$ to the first. We obtain

$$\Lambda_n = \begin{vmatrix} (L_n^{(1)}, \theta) & \ell_{n,2}^{(1)} & \cdots & \ell_{n,m}^{(1)} \\ \cdots & \cdots & \cdots & \cdots \\ (L_n^{(\tau)}, \theta) & \ell_{n,2}^{(\tau)} & \cdots & \ell_{n,m}^{(\tau)} \\ 0 & a_2^{(\tau+1)} & \cdots & a_m^{(\tau+1)} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & a_2^{(m)} & \cdots & a_m^{(m)} \end{vmatrix}.$$

By expansion along the first column we obtain

$$|\Lambda_n|_p \le \max_{j\in[1,\tau]} |(L_n^{(j)}, \theta)|_p \Big( \max\Big( \max_{\substack{j\in[1,\tau]\\ j\in[1,n]}} |\ell_{n,i}^{(j)}|_p, \max_{\substack{j\in[\tau+1,m]\\ j\in[1,n]}} |a_i^{(j)}|_p \Big)\Big)^{m-1}.$$

Since $\ell_{n,i}^{(j)}$ and $a_i^{(j)}$ are algebraic integers, we deduce

$$(\Lambda_n)_p \le \max_{j\in[1,\tau]} |(L_n^{(j)}, \theta)|$$

and an application of (i) implies that

$$\limsup_n \frac{\log|\Lambda_n|_p}{n} \le -\rho. \tag{9}$$

Dividing (7) by $n$ and using (8) and (9), we have

$$0 \le -\rho\eta_p + \tau \sum_{v\in\mathcal{V}_\infty} \eta_v c_v.$$

This proves (3), the first lower bound of Lemma 1.

Moreover, if $d_n^{j-1}$ divides $\ell_{n,j}^{(i)}$ for all $(i,j) \in [1,m]^2$, the expansion of the determinant of (5) shows that $d_n^{1+2+3+\cdots+(\tau-1)} = d_n^{\tau(\tau-1)/2}$ divides $\Lambda_n$. Thus

$$\Lambda_n = d_n^{\tau(\tau-1)/2}\lambda_n$$

with $\lambda_n \in \mathcal{O}(\mathbb{K})$.

When $n$ tends to $+\infty$, the following asymptotics hold:

$$\log|d_n| \sim n, \qquad \log|d_n|_p = o(n).$$

This implies that

$$\lim_{n\to+\infty} \frac{1}{n}\log|\lambda_n|_p = \lim_{n\to+\infty} \frac{1}{n}\log|\Lambda_n|_p \tag{10}$$

and

$$\lim_{n\to+\infty} \frac{1}{n}\log|\lambda_n|_v = \lim_{n\to+\infty} \frac{1}{n}\log|\Lambda_n|_p - \frac{\tau(\tau-1)}{2}, \tag{11}$$

for any infinite place $v$. Thus

$$0 \leq \eta_p \log |\lambda_n|_p + \sum_{v \in \mathcal{V}_\infty} \eta_v \log |\lambda_n|_v.$$

By dividing by $n$ and using (8)–(11), we conclude that

$$0 \leq -\rho\eta_p + \tau \sum_{v \in \mathcal{V}_\infty} \eta_v c_v - \frac{\tau(\tau-1)}{2} \sum_{v \in \mathcal{V}_\infty} \eta_v,$$

and thus

$$0 \leq -\rho\eta_p + \tau \sum_{v \in \mathcal{V}_\infty} \eta_v c_v - \frac{\tau(\tau-1)}{2} [\mathbb{K} : \mathbb{Q}].$$

This proves (4), the second lower bound of Lemma 1.

**4. Simultaneous Padé approximants of** $(\mathrm{Li}_s(z))_{s\in[0,A]}$**.** The results of this section and the next are adapted from the article by T. Rivoal (cf. [Ri]). We construct explicitly the simultaneous Padé approximants of polylogarithms. These approximations provide us with the linear form used to apply the linear independence criterion.

For any integers $A$, $n$ and $q$ which satisfy $n > 0$, $A \geq 2$ and $0 \leq q \leq A$, we define

$$R_{n,q}(k) = \frac{(k-An)_{An}}{(k)_n^A (k+n)^q}.$$

The $R_{n,q}(k)$ are rational fractions in $k$ of degree $-q$. By partial fraction expansion, we have

$$R_{n,q}(k) = \sum_{j=0}^{n} \sum_{s=1}^{A} \frac{r_{j,s,n,q}}{(k+j)^s} + \delta_{0,q},$$

where $\delta$ is the Kronecker symbol.

For $s \in [1, A]$, we set

$$P_{s,n,q}(z) = \sum_{j=0}^{n} r_{j,s,n,q} z^j$$

and

$$P_{0,n,q}(z) = -\sum_{s=1}^{A} \sum_{j=0}^{n} r_{j,s,n,q} \sum_{k=1}^{j} \frac{z^{j-k}}{k^s} + \delta_{0,q} \frac{1}{z-1}.$$

We introduce a class of functions $S_{n,q}(z)$ defined by

$$S_{n,q}(z) = \sum_{k=1}^{+\infty} R_{n,q}(k) z^{-k}.$$

PROPOSITION 1. *The fractions $(P_{s,n,q}(z))_{s\in[0,A]}$ and the formal series $S_{n,q}(z)$ in $\mathbb{Q}((z^{-1}))$ satisfy*

$$S_{n,q}(z) = P_{0,n,q}(z) + \sum_{s=1}^{A} P_{s,n,q}(z)\,\mathrm{Li}_s(z^{-1})$$

*and*

$$\operatorname{ord} S_{n,q}(z) = An + 1.$$

REMARK 2. For $q = 0$, $S_{n,q}(z)$ is not a Padé approximant, because $P_{0,n,0}(z)$ is not a polynomial, but it is the case of $(z-1)S_{n,q}(z)$.

*Proof.* We have

$$\begin{aligned}
S_{n,q}(z) &= \sum_{k=1}^{+\infty} R_{n,q}(k) z^{-k} = \sum_{k=1}^{+\infty}\Bigg(\delta_{0,q} + \sum_{s=1}^{A}\sum_{j=1}^{n} \frac{r_{j,s,n,q}}{(k+j)^s}\Bigg) z^{-k} \\
&= \frac{\delta_{0,q}}{z-1} + \sum_{s=1}^{A}\sum_{j=1}^{n} z^j r_{j,s,n,q} \sum_{k=1}^{+\infty} \frac{z^{-(k+j)}}{(k+j)^s} \\
&= \frac{\delta_{0,q}}{z-1} + \sum_{s=1}^{A}\sum_{j=1}^{n}\Bigg[z^j r_{j,s,n,q}\,\mathrm{Li}_s(z^{-1}) - \sum_{k=1}^{j} \frac{z^{-k}}{k^s}\Bigg] \\
&= P_{0,n,q}(z) + \sum_{s=1}^{A} P_{s,n,q}\,\mathrm{Li}_s(z^{-1}).
\end{aligned}$$

The first assertion is proved. Since $R_{n,q}(k)$ vanishes for $k$ between 1 and $An$ and $R_{n,q}(An+1)$ is nonzero, we deduce the second assertion.

## 5. Auxiliary results.

We keep the notation of Section 4.

REMARK 3. By construction, for $s \geq 1$, the $P_{s,n,q}$ are polynomials of degree at most $n$, and at most $n-1$ if $s > q$. Moreover, for $q \geq 1$, the $r_{n,q,n,q} = (-(A+1)n)_{An}/(-n)_n^A$ do not vanish, hence $P_{q,n,q}(z)$ is a polynomial of degree $n$.

PROPOSITION 2. *For all $s \in [1, A]$, we have*

$$d_n^{A-s}\, P_{s,n,q}(z) \in \mathbb{Z}[z] \quad \textit{and} \quad d_n^{A}\Bigg(P_{0,n,q}(z) - \delta_{0,q}\,\frac{1}{z-1}\Bigg) \in \mathbb{Z}[z].$$

*Proof.* It is sufficient to show that $d_n^{A-s}\, r_{j,s,n,q}$ is an integer. We suppose that $j \in [0, n-1]$ (the case $j = n$ is similar, with $s \leq q$). We have

$$r_{j,s,n,q} = \frac{(-1)^{A-s}}{(A-s)!}\bigg(\frac{d}{dl}\bigg)^{A-s} [R_{n,q}(-l)(j-l)^A]_{|l=j}.$$

We can write

$$R_{n,q}(-l)(j-l)^A = \frac{(-l-An)_{An}}{(-l)_n^A(n-l)^q}\,(j-l)^A = \Big(\prod_{c=1}^{A} F_c(l)\Big)\,H(l),$$

where $F_c(l) = \frac{(-l-cn)_n}{(-l)_{n+1}}(j-l)$ and $H(l) = (-l+n)^{A-q}$.

By partial fraction expansion of $F_c(l)$, we obtain

$$F_c(l) = 1 + \sum_{\substack{i\neq j\\ 0\le i\le n}} \frac{(j-i)f_{i,c}}{i-l},$$

where

$$f_{i,c} = \frac{(-i-cn)_n}{\prod_{\substack{h\neq i\\ 0\le h\le n}}(h-i)} = (-1)^{i+n}\binom{cn+i}{n}\binom{n}{i}. \tag{12}$$

We deduce from (12) that the $f_{i,c}$ are integers. Setting $D_\lambda = \frac{1}{\lambda!}\left(\frac{d}{dl}\right)^\lambda$, we have, for all $\lambda \ge 0$,

$$D_\lambda(F_c(l)) = \delta_{0,\lambda} + \sum_{\substack{i\neq j\\ 0\le i\le n}} \frac{(j-i)f_{i,c}}{(i-l)^{\lambda+1}}.$$

We have shown that the $d_n^\lambda D_\lambda(F_c(l))_{|l=j}$ are integers for all $\lambda \ge 0$. Moreover, $D_\lambda(H(l))_{|l=j}$ is an integer.

Using the Leibniz identity, we have

$$D_{A-s}[R_{n,q}(-l-x)(j-l)^A] = \sum_\nu (D_{\nu_0}(F_1))\cdots(D_{\nu_{A-1}}(F_A))(D_{\nu_A}(H))$$

($\nu \in \mathbb{N}^{A+1}$ with $\nu_0+\cdots+\nu_A = A-s$). We deduce from this that $d_n^{A-s}\, r_{j,s,n,q}$ is an integer and thus $d_n^{A-s}\, P_{s,n,q}(z)$ is an element of $\mathbb{Z}[z]$.

Proposition 3. *If $\beta$ is an element of the number field $\mathbb{K}$ and if $v$ is an infinite place of $\mathbb{K}$, then for $s \in [0, A]$, we have*

$$\limsup_n |P_{s,n,q}(\beta)|_v^{1/n} \le (A+1)^{A+1}2^A \max(1, |\beta|_v).$$

*Proof.* It is sufficient to bound $r_{j,s,n,q}$. We have

$$\begin{aligned} r_{j,s,n,q} &= \frac{1}{2\pi i}\int_{|t+j|=1/2} R_{n,q}(t)(t+j)^{s-1}\,dt \\ &= \frac{1}{2\pi i}\int_{|t+j|=1/2} \frac{(t-An)_{An}}{(t)_n^A(t+n)^q}\,(t+j)^{s-1}\,dt. \end{aligned}$$

Hence

$$|r_{j,s,n,q}| \le \frac{1}{2\pi}\,\pi \sup_{|t+j|=1/2} \frac{|(t-An)_{An}|\,|(t+j)^{s-1}|}{|(t)_n^A(t+n)^q|}$$
$$\le 2^{-s} \sup_{|t+j|=1/2} \frac{|(t-An)_{An}|}{|(t)_n^A(t+n)^q|}.$$

As $|t+j| = 1/2$, we have

$$|(t-An)_{An}| = \prod_{k=1}^{An} |t-k| = \prod_{k=1}^{An} |t+j-k-j|$$
$$\le \prod_{k=1}^{An}\left(\frac{1}{2}+|-k-j|\right) \le \prod_{k=1}^{An}(1+|k+j|),$$

so

$$|(t-An)_{An}| \le \frac{(An+j+1)!}{(j+1)!}, \tag{13}$$

and

$$|(t)_n| = \prod_{k=0}^{n-1} |t+k| = \prod_{k=0}^{n-1} |t+j-j+k|$$
$$\ge \prod_{k=0}^{n-1}\left(\left|-\frac{1}{2}+|k-j|\right|\right) \ge \frac{1}{8} \prod_{\substack{0\le k\le n-1\\ k\notin\{j-1,j,j+1\}}} (-1+|k-j|),$$

so

$$|(t)_n| \ge \frac{1}{8n^3}\, j!\,(n-j)!, \tag{14}$$

and

$$|(t+n)^q| = |(t+j-j+n)|^q \ge \left||n-j|-\frac{1}{2}\right|^q \ge 2^{-A}. \tag{15}$$

We deduce from (13)–(15) that

$$|r_{j,s,n,q}| \le 2^{4A-s} n^{3A}\, \frac{(An+j+1)!}{(j+1)!\, j!^A\,(n-j)!^A}.$$

Thus

$$|r_{j,s,n,q}| \le 2^{4A} n^{3A}\, \frac{(An)!}{n!^A} \binom{n}{j}^A \binom{An+j+1}{An}. \tag{16}$$

The multinomial series

$$(x_1+\cdots+x_m)^{km} = \sum_{\substack{n_1,\ldots,n_m\ge 0\\ n_1+\cdots+n_m=km}} \frac{(km)!}{n_1!\cdots n_m!}\, x_1^{n_1}\cdots x_m^{n_m}$$

applied to $x_1=\cdots=x_m=1$ and $n_1=\cdots=n_m=k$ gives $(km)!/k!^m \le m^{km}$.

Using the upper bounds $(An)!/n!^A \leq A^{An}$, $\binom{n}{j} \leq 2^n$ and $\binom{An+j+1}{An} \leq \binom{An+n+1}{An}$, we deduce that

$$|r_{j,s,n,q}| \leq A^{An} 2^{4A-s} 2^{n(A+1)} n^{3A} \binom{An+n+1}{An}.$$

By Stirling's formula,

$$\lim_{n\to+\infty} \left(\binom{An+n+1}{An}\right)^{1/n} = \frac{(A+1)^{A+1}}{A^A}.$$

Hence

$$|r_{j,s,n,q}| \leq ((A+1)^{A+1}2^A)^{n+o(n)}.$$

Thus for $s \geq 1$,

$$|P_{s,n,q}(\beta)|_v \leq \sum_{j=0}^{n} |r_{j,s,n,q}|\, |\beta|_v^j \leq (n+1)((A+1)^{A+1}2^A)^{n+o(n)} \max(1, |\beta|_v)^n$$

and

$$\begin{aligned} |P_{0,n,q}(\beta)|_v &\leq \sum_{s=1}^{A}\sum_{j=0}^{n} |r_{j,s,n,q}| \sum_{k=1}^{j} \frac{|\beta|_v^{j-k}}{k^s} + \delta_{0,q}\left|\frac{1}{\beta-1}\right|_v \\ &\leq A(n+1)((A+1)^{A+1}2^A)^{n+o(n)} \max(1,|\beta|_v)^n + \delta_{0,q}\left|\frac{1}{\beta-1}\right|_v. \end{aligned}$$

This yields the conclusion.

**6. Independence of linear forms.** The results of this section are adapted from an article of Marcovecchio (cf. [Ma]). We set

$$M_n(z) = (P_{s,n,q}(z))_{\substack{s\in[0,A]\\ q\in[0,A]}}. \tag{17}$$

PROPOSITION 4. *There exists a constant $\gamma \in \mathbb{Q}^*$ such that*

$$\det M_n(z) = \gamma(z-1)^{-1}.$$

*Proof.* For $(s,q) \neq (0,0)$, $P_{s,n,q}$ is a polynomial whereas $P_{0,n}^{(0)}$ is a rational fraction with one simple pole at $z=1$. Hence the determinant of (17) is a rational fraction with at most a simple pole at $z=1$. By multi-linearity of determinant, we add the $j$th column multiplied by $\mathrm{Li}_s(z^{-1})$ to the first. We obtain

$$\det M_n(z) = \begin{vmatrix} S_{n,0}(z) & P_{1,n,0}(z) & \cdots & \cdots & P_{A,n,0}(z) \\ \vdots & & \ddots & & \vdots \\ S_{n,A}(z) & P_{1,n,A}(z) & \cdots & \cdots & P_{A,n,A}(z) \end{vmatrix}.$$

The elements of the first column are formal power series in $z^{-1}$ of valuation $An+1$ (Proposition 1). The other columns are polynomials of degree at most $n$ in $z$ (Remark 3). We deduce that the determinant is a rational fraction in $z$ of degree at most $-1$. Remark 3 shows that the elements above the diagonal have degree at most $n-1$ in $z$. Hence only the product of diagonal elements can be of degree $-1$ in $z$, the others have a strictly lower degree. Remark 3 also implies that $P_{q,n,q}(z)$ is exactly of degree $n$ in $z$. We thus have an element of degree exactly $-1$. The degree of $\det M_n(z)$ in $z$ is $-1$, proving the assertion.

## 7. Transfer from complex to $p$-adic and proof of Theorem 3

PROPOSITION 5. *Let $\alpha \in \mathbb{C}_p$ with $|\alpha|_p > 1$ and set*

$$U_{n,q}(\alpha) = d_n^A\Big(P_{0,n,q}(\alpha) + \sum_{s=1}^{A} P_{s,n,q}(\alpha)\mathcal{L}i_s(\alpha^{-1})\Big).$$

*Then*

$$\limsup_n \frac{1}{n}\log|U_{n,q}(\alpha)|_p \le -A\log|\alpha|_p.$$

We will prove this proposition using the following two lemmas.

LEMMA 2. *We have*

$$U_{n,q}(\alpha) = \sum_{k=0}^{+\infty} u_{k,n}\alpha^{-k}$$

*where $(u_{k,n})$ is a sequence of rational numbers independent of $\alpha$, with $u_{k,n} = 0$ for all $k \le An$.*

*Proof.* In the field $\mathbb{Q}((X^{-1}))$ of Laurent series, we have

$$U_{n,q}(X) = d_n^A S_{n,q}(X).$$

Proposition 1 proves that this series has valuation at least $An+1$ in $X$. We can write

$$U_{n,q}(X) = \sum_{k=An+1}^{+\infty} u_{k,n}X^{-k}.$$

Moreover, the Laurent series $U_{n,q}(X)$ is convergent on $\mathbb{C}_p$ for $|X|_p > 1$, since $\mathcal{L}i_s(X^{-1})$ is convergent on the same domain and $U_{n,q}(\alpha)$ is the sum of this series for $X = \alpha$.

LEMMA 3. *The terms $u_{k,n}$ satisfy*

$$|u_{k,n}|_p \le (k+n+1)^A.$$

*Proof.* The $p$-adic absolute value of the $k$th term of the expansion of $\mathcal{L}i_s(X^{-1})$ in $\mathbb{Q}((X^{-1}))$ is at most $k^s$. As

- $U_{n,q}(X) = d_n^A(P_{0,n,q}(X) + \sum_{s=1}^{A} P_{s,n,q}(X)\mathcal{L}i_s(X^{-1}))$,
- $d_n^A P_{s,n,q}$ is an element of $\mathbb{Z}[X]$ of degree at most $n$, for $v$ an infinite place,
- $d_n^A P_{0,n,q}$ is an element of $\mathbb{Z}[[X^{-1}]][X]$ of degree at most $n$,

we infer that

$$|u_{k,n}|_p \le (k+n+1)^A.$$

*Proof of Proposition 5.* Using Lemmas 2 and 3, we find

$$|U_{n,q}(\alpha)|_p \le \sup_{k\ge An+1} (k+n+1)^A|\alpha|_p^{-k} = ((A+1)n+2)^A|\alpha|_p^{-An}$$

for $n$ sufficiently large (indeed $k \mapsto (k+n+1)^A|\alpha|_p^{-k}$ is a decreasing function on $[An+1, +\infty[$). Proposition 5 is thus proved.

*Proof of Theorem 3.* Using Proposition 2 and Remark 3, we find that $d(\alpha)^{n+1}(\alpha-1)d_n^A P_{s,n,q}(\alpha)$ is an algebraic integer.

Using Proposition 3, we have

$$\begin{aligned}(18)\qquad &\limsup_n \frac{1}{n}\log|d(\alpha)^{n+1}(\alpha-1)d_n^A P_{s,n,q}(\alpha)|_v\\ &\le (A+1)\log(A+1) + A(1+\log 2) + \log d(\alpha) + \log\max(1,|\alpha|_v) = c_v\end{aligned}$$

for any infinite place $v$.

For the $p$-adic absolute values, using Proposition 5 and the inequality $|d(\alpha)|_p \le |\alpha|_p^{-1}$, we obtain

$$-\limsup_n \frac{1}{n}\log|d(\alpha)^{n+1}(\alpha-1)d_n^A U_{n,q}(\alpha)|_p \ge (A+1)\log|\alpha|_p = \rho.$$

Proposition 4 gives the linear independence of the linear forms $(U_{n,q})_{q\in[0,A]}$ in $1,\dots,\mathcal{L}i_A(\alpha^{-1})$. Since $\sum_{v\in\mathcal{V}_\infty}\eta_v = [\mathbb{K}:\mathbb{Q}]$ and the hypotheses of Lemma 1 are checked, we obtain

$$\begin{aligned}&\dim_{\mathbb{Q}(\alpha)}\mathrm{Vect}(1,(\mathcal{L}i_s(\alpha^{-1}))_{s\in[1,A]})\\ &\ge \frac{[\mathbb{Q}_p(\alpha):\mathbb{Q}_p](A+1)\log|\alpha|_p}{[\mathbb{Q}(\alpha):\mathbb{Q}]((A+1)\log(A+1)+A\log 2+A+\log d(\alpha)) + \sum_{v\in\mathcal{V}_\infty}\eta_v\log\max(1,|\alpha|_v)}.\end{aligned}$$

Inequality (2) is thus proved. Using Proposition 2, we can apply inequality (4) of Lemma 1 to obtain (1).

**Acknowledgements.** I would like to thank the referee for helpful suggestions on improvement of the text.

### References

[Col] R. Coleman, *Dilogarithms, regulators and p-adic L-functions*, Invent. Math. 69 (1982), 171–208.

[Fu] H. Furusho, *$p$-adic multiple zeta values. I. $p$-adic multiple polylogarithms and the $p$-adic KZ equation*, ibid. 155 (2004), 253–286.

[Ha] M. Hata, *On the linear independence of the values of polylogarithmic functions*, J. Math. Pures Appl. 69 (1990), 133–173.

[Ma] R. Marcovecchio, *Linear independence of forms in polylogarithms*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. 5 (2006), 1–11.

[Rh] G. Rhin and C. Viola, *The permutation group method for the dilogarithm*, ibid. 4 (2005), 389–437.

[Ri] T. Rivoal, *Indépendance linéaire de valeurs des polylogarithmes*, J. Théor. Nombres Bordeaux 15 (2003), 551–559.

Institut de Mathématiques de Bordeaux UMR 5251
Université Bordeaux 1
351, cours de la Libération
F-33405 Talence Cedex, France
E-mail: pierre.bel1@free.fr

*Received on 3.9.2008*
*and in revised form on 23.12.2008* (5790)