Block additive functions on the Gaussian integers

by

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1. Introduction. Let \( q = -a + i \in \mathbb{Z}[i] \) for a positive integer \( a \) and
\( \mathcal{N} = \{0, 1, \ldots, a^2\} \).
Then every Gaussian integer \( z \in \mathbb{Z}[i] \) can be uniquely represented as
\[
z = \sum_{j \geq 0} \varepsilon_j(z)q^j
\]
with \( \varepsilon_j(z) \in \mathcal{N} \). Formally we set \( \varepsilon_j(z) = 0 \) for all negative integers \( j < 0 \). It
will be convenient sometimes to use infinite or even doubly infinite sequences
(filled with zeros) for the representation of Gaussian integers. We denote the
length of the expansion by
\[
\text{length}_q(z) = \max\{j \in \mathbb{N}_0 : \varepsilon_j(z) \neq 0\} + 1
\]
and \( \text{length}_q(0) = 0 \). (We denote the positive integers by \( \mathbb{N} \) and use \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \) for the non-negative integers.) Throughout the paper we will use the
notation \( \log_b \) for the logarithm to base \( b \). The following lemma was proved
in [13].

LEMMA 1. There exists a positive constant \( c \) such that for all \( z \in \mathbb{Z}[i] \),
\[
|\text{length}_q(z) - \log_q |z|| \leq c.
\]

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The fundamental domain of the base $q$ representation on $\mathbb{Z}[i]$ is defined by

$$\mathcal{F}_q = \left\{ \sum_{l=1}^{\infty} \frac{\varepsilon_l}{q^l} : \varepsilon_l \in \mathbb{N} \ \forall l \right\}.$$ 

This subset of $\mathbb{C}$ plays the same rôle for $q$-adic numeration as the unit interval does for classical number systems on the integers (cf. [8, 9, 13]). More generally, radix representations of elements of the ring of integers $\mathbb{Z}_K$ of a number field $K$ can be considered. A base $\alpha \in \mathbb{Z}_K$ together with the digit set $D = \{0, 1, \ldots, |N_K|Q(\alpha)| - 1\}$ is called a canonical number system (cf. [17, 18]) if every $\zeta \in \mathbb{Z}_K$ has a representation of the form

$$\zeta = \sum_{l=0}^{n} \varepsilon_l \alpha^l$$

with $\varepsilon_l \in D$ for $0 \leq l \leq n$. The point 0 is an inner point of $\mathcal{F}_q$. This follows from the general fact that $(\alpha, D)$ is a canonical number system if and only if the corresponding fundamental domain contains 0 as an inner point (cf. [1]).

Let $F : \mathbb{N}^{L+1} \to \mathbb{R}$ be any given function (for some $L \geq 0$) with $F(0, \ldots, 0) = 0$. Furthermore, set

$$s_F(z) = \sum_{j=-L}^{\infty} F(\varepsilon_j(z), \varepsilon_{j+1}(z), \ldots, \varepsilon_{j+L}(z)).$$

This means that we consider a weighted sum over all subsequent digital patterns of length $L + 1$ of the digital expansion of $z$. The function $s_F$ is called a block additive function of rank $L + 1$. This generalises the block additive digital functions studied in [4] for digital expansions on the rational integers. This definition readily extends to functions taking values in an arbitrary abelian group $A$. We will use this in the general considerations in Section 5.

For example, for $L = 0$ we obtain completely additive functions such as those studied in [16, Section 5], for instance for $F(\varepsilon) = \varepsilon$ we just have the sum-of-digits function studied in [10, 13, 14], or if $L = 1$ and $F(\varepsilon, \eta) = 1 - \delta_{\varepsilon, \eta}$ ($\delta_{x,y}$ denoting the Kronecker symbol) then $s_F(n)$ just counts the number of times that a digit is different from the preceding one etc.

2. Overview of the results. Our main objective is to get information on sums

$$S_N(x) = \sum_{|z|^2 < N} x^{s_F(z)},$$

where $x$ is a complex variable. It is clear that $S_N(x)$ encodes the distribution of $s_F(z)$. For example, if we assume that $s_F(z)$ has only non-negative integer
values then
\[ S_N(x) = \sum_{k \geq 0} \# \{ z \in \mathbb{Z}[i] : |z|^2 < N, \, s_F(z) = k \} x^k. \]

More generally, let \( Y_N \) denote the random variable induced by the distribution of \( s_F(z) \) for \( |z|^2 < N \), that is, the distribution function of \( Y_N \) is given by
\[ \mathbb{P}\{Y_N \leq y\} = \frac{1}{S_N(1)} \# \{ |z|^2 < N : s_F(z) \leq y \}. \]  
Then
\[ \mathbb{E} x^{Y_N} = \frac{1}{S_N(1)} \sum_{|z|^2 < N} x^{s_F(z)} = \frac{1}{S_N(1)} S_N(x). \]

In particular, the moment generating function \( \mathbb{E} e^{\lambda Y_N} \) and the characteristic function \( \mathbb{E} e^{it Y_N} \) of \( Y_N \) can be expressed with the help of \( S_N(x) \). (Note that \( S_N(1) = \pi N + O(N^{1/3}). \))

In what follows we will present three different methods to obtain asymptotic information for \( S_N(x) \). In Section 3 we use a measure-theoretic approach to show that for real numbers \( x \) sufficiently close to 1 we have
\[ S_N(x) = \Phi(x, \log |q| N) N^{\log |q|^2 \lambda(x)} (1 + O(N^{-\kappa})), \]
where \( \Phi(x, t) \) is a function that is analytic in \( x \) and periodic (with period 1) and Hölder continuous in \( t \), and \( \lambda(x) \) is the dominant eigenvalue of a certain matrix \( A(x) \) defined in (3.1). This representation directly implies that the random variable
\[ X_N = Y_N - \mu \log |q|^2 N \]
with
\[ \mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \frac{\lambda'(1)}{\lambda(1)} - \frac{\lambda'(1)^2}{\lambda(1)^2} \]
satisfies a central limit theorem and we have convergence of all moments. More precisely, we get (uniformly in \( y \))
\[ \frac{1}{\pi N} \# \{ |z|^2 < N : s_F(z) \leq \mu \log |q|^2 N + y \sqrt{\sigma^2 \log |q|^2 N} \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2} \, du + o(1), \]
and (for every \( L \geq 0 \))
\[ \frac{1}{\pi N} \sum_{|z|^2 < N} (s_F(z) - \mu \log |q|^2 N)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2} \, du + o(1). \]
The drawback of the method given in Section 3 is that it only works for real numbers \(x\). In Section 4 we present a method that is based on Dirichlet series that extends (2.4) to a complex neighbourhood of \(x = 1\). Furthermore, we provide upper bounds for \(S_N(x)\) for complex \(x\) with modulus close to 1. With the help of this extension we are able to provide more precise distributional results. Besides the central limit theorem we also get a local limit theorem, that is, asymptotic expansions for the numbers

\[
\# \{ z \in \mathbb{Z}[i] : |z|^2 < N, s_F(z) = k \}
\]

if \(k\) is close to the mean \(\mu = \log |q|^2 N\) and if \(s_F(z)\) is integer-valued. Furthermore, we obtain very precise asymptotic expansions of the moments.

Next we consider the sequence \(s_F(z)\) taking values in a compact abelian group \(A\). Then the closure of the set \(\{ s_F(z) : z \in \mathbb{Z}[i] \}\) is a subgroup of \(A\) denoted by \(A(F)\). The results on exponential sums obtained in Section 4 are used to prove uniform distribution of \((s_F(z))_{z \in \mathbb{Z}[i]}\) in the groups \(\mathbb{R}/\mathbb{Z}\) and \(\mathbb{Z}/M\mathbb{Z}\) with respect to the Haar measure \(\lambda_A\) under natural conditions. The method gives results on uniform distribution of the values of \(s_F\) in large circles, i.e.

\[
\lim_{N \to \infty} \frac{1}{\pi N} \# \{ z \in \mathbb{Z}[i] : |z|^2 < N, s_F(z) \in B \} = \lambda_A(B)
\]

for all \(B \subseteq A\) with \(\lambda_A(\partial B) = 0\).

In Section 5 we use an approach based on ergodic \(\mathbb{Z}[i]\)-actions and skew products to extend the distribution results for group-valued \(s_F\) to well uniform distribution with respect to Følner sequences \((Q_n)_{n \in \mathbb{N}}\), i.e.

\[
\lim_{n \to \infty} \frac{1}{\#Q_n} \# \{ z \in Q_n : s_F(z + y) \in B \} = \lambda_A(B)
\]

uniformly in \(y \in \mathbb{Z}[i]\). This generalises the results on uniform distribution of \((s_F(z))_{z \in \mathbb{Z}[i]}\) obtained in Section 4. On the other hand, methods from ergodic theory do not allow one to obtain error terms, which come as a natural by-product of the method used in Section 4.

### 3. A measure-theoretic method

In the following we use ideas developed in [11, 12]. The measure-theoretic approach to asymptotic questions about digital functions gives a smooth proof for a real version of the asymptotic representation (2.4) for \(S_N(x)\).

In order to formulate our results we have to introduce some notation.

For every block \(B = (\eta_0, \eta_1, \ldots, \eta_L) \in \mathcal{N}^{L+1}\) we set \(B' = (\eta_1, \ldots, \eta_L) \in \mathcal{N}^L\), that is, the block without the first digit, and \(\eta(B) = \eta_0\), the first digit of \(B\). Furthermore, set

\[
g_F(B) = \sum_{i=0}^{L} (F(0, \ldots, 0, \eta_0, \eta_1, \ldots, \eta_i) - F(0, \ldots, 0, 0, \eta_1, \ldots, \eta_i)).
\]

Note that \(g_F(B) = 0\) if \(\eta_0 = 0\).
By the definition of block additive function we directly get the following property.

**Lemma 2.** For \( z \in \mathbb{Z}[i] \) let \( B = B(z) = (\varepsilon_0(z), \ldots, \varepsilon_L(z)) \) be the block of the first \( L + 1 \) digits of the \( q \)-ary digital expansion of \( z \). Then

\[
s_F(z) = g_F(B) + s_F(v),
\]

where \( z = \varepsilon_0(z) + qv \).

Now define a matrix \( A(x) = (A_{B,C}(x))_{B,C \in \mathbb{N}^{L+1}} \) by

\[
A_{B,C}(x) = \begin{cases} 
x^{g_F(B)} & \text{if } C = (B', l) \text{ for some } l \in \mathbb{N}, \\
0 & \text{otherwise.}
\end{cases}
\]

Finally, let \( \lambda(x) \) be the dominant eigenvalue of the matrix \( A(x) \) that surely exists if \( x \) is close to the positive real axis, in particular, if \( x \) is close to 1 (cf. Lemma 4). Note that \( \lambda(1) = |q|^2 \).

**Theorem 1.** The following asymptotic relation holds uniformly for \( x \) in some interval \( I \) containing 1:

\[
\sum_{|z|^2 < N} x^{s_F(z)} = \Phi(x, \log|q|^2 N) N^{\log|q|^2 \lambda(x)} (1 + O(N^{-\kappa}))
\]

with some \( \kappa > 0 \), where \( \Phi(x, t) \) is 1-periodic and Hölder continuous in \( t \) and continuous in \( x \).

Before we present the proof of Theorem 1 we derive some direct corollaries.

**Corollary 1.** Set

\[
\mu = \frac{\lambda'(1)}{\lambda(1)} \quad \text{and} \quad \sigma^2 = \frac{\lambda''(1)}{\lambda(1)} + \mu - \mu^2.
\]

If \( \sigma^2 > 0 \) then uniformly for real \( y \),

\[
\frac{1}{\pi N} \# \{ |z|^2 < N : s_F(z) \leq \mu \log|q|^2 N + y \sqrt{\sigma^2 \log|q|^2 N} \} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-\frac{1}{2}u^2} du + o(1),
\]

and for every \( L \geq 0 \),

\[
\frac{1}{\pi N} \sum_{|z|^2 < N} (s_F(z) - \mu \log|q|^2 N)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2} du + o(1).
\]
Furthermore, we have exponential tail estimates of the form

\[
\frac{1}{\pi N} \# \left\{ |z|^2 < N : |s_F(z) - \mu \log |q|^2 N| \geq \eta \sqrt{\log |q|^2 N} \right\} \leq \min(e^{-c \eta}, e^{-c \eta^2 + O(\eta^3/\sqrt{\log N})})
\]

for some constant \(c > 0\).

**Remark 1.** The above result suggests that the distribution of \(s_F(z)\) for \(|z|^2 < N\) can be approximated by a sum of (weakly dependent) random variables. This is in fact a possible approach to this problem. Observe that the constant \(\mu\) can be explicitly calculated from

\[
\mu = \frac{\lambda'(1)}{\lambda(1)} = \frac{1}{|q|^{L+1}} \sum_{B \in N^{L+1}} s_F(B).
\]

Of course, this mean value corresponds to the contribution of one block of length \(L+1\) in the digital expansion of \(z\) that has approximately \(\log |q|^2 N\) digits. It is also possible to represent \(\sigma^2\) similarly, but this is much more involved.

**Proof of Corollary 1.** Let \(Y_N\) denote the random variable that is induced by the distribution of \(s_F(z)\) for \(|z|^2 < N\) given by (2.2). Then (by (2.3)) the moment generating function of \(Y_N\) is given by

\[
\mathbb{E} e^{tY_N} = \frac{1}{S_N(1)} S_N(e^t) = \frac{1}{\pi} \Phi(e^t, \log |q|^2 N) N^\log |q|^2 \lambda(e^t)^{-1}(1 + O(N^{-\eta})).
\]

Hence, by using the local expansion (recall that \(\lambda(1) = |q|^2\))

\[
\log \lambda(e^t) = \log |q|^2 + \mu t + \frac{\sigma^2}{2} t^2 + O(t^3)
\]

we directly see that the moment generating function of the normalised random variable

\[
Z_N = \frac{Y_N - \mu \log |q|^2 N}{\sqrt{\sigma^2 \log |q|^2 N}}
\]

is given by

\[
\mathbb{E} e^{tZ_N} = e^{-t(\mu/\sigma)} e^{\log |q|^2 N} e^{(t/\sqrt{\sigma^2 \log |q|^2 N}) Y_N} = e^{\frac{1}{2} t^2 + O(t^3/\sqrt{\log N})}.
\]

Of course, this directly translates to (3.3).

Furthermore, convergence of the moment generating function also implies convergence of all moments, that is, we get (3.4). Finally, the tail estimates (3.5) are a direct consequence of Chernov type inequalities.

The proof of Theorem 1 runs along the lines of [12, Sections 4 and 5] and is organised in four steps.
Step 1 defines a sequence of discrete measures, which are obtained by suitably rescaling point masses \( x^{s_F(z)} \). Let \( \delta_z \) denote the Dirac measure supported at \( \{z\} \). Then we define a family of measures (depending on \( n \) and \( x \)) by setting

\[
(3.6) \quad \mu_{n,x} = \frac{\sum_{z \in B_n} x^{s_F(z)} \delta_{z/q^n}}{\sum_{z \in B_n} x^{s_F(z)}},
\]

where

\[
B_n = \{z \in \mathbb{Z}[i] : \text{length}(z) \leq n\}.
\]

Using the matrix \( \mathbf{A}(x) \) introduced in (3.1), we can write the denominator in (3.6) as

\[
(x^{g_F(B)})_B \cdot \mathbf{A}(x)^n \cdot (\delta_{0,C})^T,
\]

\( \delta_{0,C} \) denoting the Kronecker symbol, and \( T \) the transposition.

Step 2 uses characteristic functions to show that the sequence \( \mu_{n,x} \) has a weak limit. The fact that the values \( x^{s_F(z)} \) are formed from the digital expansion of \( z \) can be used to express the Fourier transforms \( \hat{\mu}_{n,x} \) of the measures \( \mu_{n,x} \),

\[
(3.7) \quad \hat{\mu}_{n,x}(t) = \frac{\sum_{z \in B_n} x^{s_F(z)} e(\Re(t z/q^n))}{\sum_{z \in B_n} x^{s_F(z)}},
\]

in terms of matrix products. Here \( t \in \mathbb{C} \) and as usual \( e(\cdot) = e^{2\pi i\cdot} \). We define the matrix \( \mathbf{H}(x,t) \) by setting

\[
\mathbf{H}_{B,C}(x,t) = \mathbf{A}(x) e(\Re(t B_0)) \mathbf{C}.
\]

This allows us to write

\[
(3.8) \quad \hat{\mu}_{n,x}(t) = \frac{\mathbf{v}_1(x,t q^{-n}) \cdot \mathbf{H}(x,t/q^{n-1}) \cdots \mathbf{H}(x,t/q) \cdot \mathbf{v}_2}{\mathbf{v}_1(x,0) \cdot \mathbf{A}(x)^n \cdot \mathbf{v}_2}
\]

with

\[
\mathbf{v}_1(x,t) = (x^{s_F(B)} e(\Re(t B_0)))_B \quad \text{and} \quad \mathbf{v}_2 = (\delta_{0,C})^T.
\]

The matrices \((1/\lambda(x))\mathbf{H}(x,t)\) satisfy the conditions of [12, Lemma 5] \((mu-tatis\ mutandis)\) and therefore, the sequence of matrices

\[
\mathbf{P}_n(x,t) = \lambda(x)^{-n} \mathbf{H}(x,t/q^{n-1}) \cdots \mathbf{H}(x,t/q)
\]

converges to a limit \( \mathbf{P}(x,t) \) and

\[
(3.9) \quad \|\mathbf{P}_n(x,t) - \mathbf{P}_n(x,0)\| \ll |t| \quad \text{for } |t| \leq 1,
\]

\[
\|\mathbf{P}_n(x,t) - \mathbf{P}(x,t)\| \ll (1 + |t|) \eta(x)|q|^{-\eta(x)n} \quad \text{for all } t,
\]

where

\[
\eta(x) = \frac{\log \lambda(x) - \log |\lambda_1(x)|}{\log |q| + \log \lambda(x) - \log |\lambda_1(x)|}.
\]
with \( \lambda_1(x) \) denoting the second largest eigenvalue of \( A(x) \). These relations hold uniformly for \( x \) in compact subsets of \((0, \infty)\).

For \( |t| \geq 1 \), (3.9) together with (3.8) implies

\[
|\hat{\mu}_{n,x}(t) - \tilde{\mu}_n(t)| \ll |t|^\eta(x)q^{-n\eta(x)}.
\]

For \( |t| \leq 1 \) and \( L > K > l \) we estimate using (3.8):

\[
|\tilde{\mu}_{K,x}(t) - \tilde{\mu}_{L,x}(t)|
\]

\[
= \left| \lambda(x)^K \frac{v_1(x,tq^{-K}) \cdot P_{K-l}(q^{-l}t) P_l(t)v_2}{v_1(x,0) \cdot A(x)^K \cdot v_2}
- \lambda(x)^L \frac{v_1(x,tq^{-L}) \cdot P_{L-l}(q^{-l}t) P_l(t)v_2}{v_1(x,0) \cdot A(x)^L \cdot v_2} \right|
\]

\[
\ll \left| \lambda(x)^K \frac{v_1(x,tq^{-K}) \cdot P_{K-l}(0) P_l(t)v_2}{v_1(x,0) \cdot A(x)^K \cdot v_2}
- \lambda(x)^L \frac{v_1(x,tq^{-L}) \cdot P_{L-l}(0) P_l(t)v_2}{v_1(x,0) \cdot A(x)^L \cdot v_2} \right| + |q|^{-l}|t|
\]

\[
= \left| \lambda(x)^K \frac{v_1(x,tq^{-K}) \cdot P_{K-l}(0)(P_l(t) - P_l(0))v_2}{v_1(x,0) \cdot A(x)^K \cdot v_2}
- \lambda(x)^L \frac{v_1(x,tq^{-L}) \cdot P_{L-l}(0)(P_l(t) - P_l(0))v_2}{v_1(x,0) \cdot A(x)^L \cdot v_2} \right| + |q|^{-l}|t|
\]

\[
\ll |t| \left( \left( \frac{\lambda_1(x)}{\lambda(x)} \right)^{K-l} + |q|^{-l} \right) \ll |t| |q|^{-\eta(x)K},
\]

where we have chosen \( l = \lfloor \eta K \rfloor \). Letting \( L \) tend to infinity yields

\[
|\hat{\mu}_{n,x}(t) - \tilde{\mu}_n(t)| \ll |t|^\eta(x)q^{-n\eta(x)}
\]

for \( |t| \leq 1 \). In particular, (3.10) and (3.11) establish the existence of a (weak) limiting measure \( \mu_x \).

**Remark 2.** What we have proved up to now is enough to have the asymptotic relation (3.2) without error term for all \( x > 0 \).

**Step 3** establishes estimates for the measure dimension of \( \mu_x \), which will be needed in Step 4. We define the matrices \( I_\varepsilon \) by setting

\[
(I_\varepsilon)_{B,C} = \begin{cases} 
\delta_{B,C} & \text{if the block } B \text{ starts with the digit } \varepsilon, \\
0 & \text{otherwise}.
\end{cases}
\]

Clearly \( I_0 + I_1 + \cdots + I_{q^2} \) is the identity matrix. Furthermore, we have

\[
\mu_x \left( \frac{\varepsilon_1}{q^1} + \frac{\varepsilon_2}{q^2} + \cdots + \frac{\varepsilon_k}{q^k} + q^{-k}F \right) = \lim_{n \to \infty} \frac{v_1(x,0) \cdot I_{\varepsilon_1} A(x) I_{\varepsilon_2} A(x) \cdots I_{\varepsilon_k} A(x) A(x)^{n-k} \cdot v_2}{v_1(x,0) \cdot A(x)^n \cdot v_2}.
\]
The limit can be computed by the Perron–Frobenius theorem and equals
\[
\lambda(x)^{-k}v_1(x, 0) \cdot \mathbf{1}_{\epsilon_1} A(x) \mathbf{1}_{\epsilon_2} A(x) \cdots \mathbf{1}_{\epsilon_k} v(x),
\]
where \(v(x)\) denotes the (Perron–Frobenius) eigenvector of \(A(x)\) associated to the eigenvalue \(\lambda(x)\) normalised so that
\[
v_1(x, 0) \cdot v(x) = 1.
\]

Now we define
\[
\xi(x) = \max_{\epsilon} \max_B \frac{(A(x)\mathbf{1}_{\epsilon} v)_B}{(v(x))_B},
\]
(this is always finite, since all coordinates of \(v(x)\) are strictly positive). Clearly, \(\xi(x) < \lambda(x)\) and \(\xi(1) = 1\). By definition of \(\xi(x)\) we have the componentwise inequality
\[
A(x)\mathbf{1}_{\epsilon} v(x) \leq \xi(x) v_1(x, 0) \cdot v(x) = \xi(x)^k
\]
and
\[
(3.12) \quad \mu_x \left( \frac{\epsilon_1}{q} + \frac{\epsilon_2}{q^2} + \cdots + \frac{\epsilon_k}{q^k} + q^{-k} F \right) \ll \left( \frac{\xi(x)}{\lambda(x)} \right)^k.
\]

Since \((q, \{0, \ldots, a^2\})\) is a canonical number system, every ball \(B(z, r)\) can be covered by an absolutely bounded number of sets of the form
\[
\frac{\epsilon_1}{q} + \frac{\epsilon_2}{q^2} + \cdots + \frac{\epsilon_k}{q^k} + q^{-k} F
\]
for \(k = \lfloor -\log |q| r \rfloor\) and \(r < 1\). This together with (3.12) implies
\[
(3.13) \quad \mu_x(B(z, r)) \ll r^{\beta(x)}
\]
with
\[
\beta(x) = \frac{\log \lambda(x) - \log \xi(x)}{\log |q|}.
\]
Notice that \(\beta(1) = 2\), which is no surprise, since \(\mu_1\) is Lebesgue measure restricted to \(F\).

Furthermore, we need at most \(O(|q|^{2n})\) times the area of the annulus \(B(0, r + \epsilon + |q|^{-n}) \setminus B(0, r - |q|^{-n})\) copies of \(q^{-n} F\) to cover the annulus \(B(0, r + \epsilon) \setminus B(0, r)\). This together with (3.12) implies
\[
\mu_x(B(0, r + \epsilon) \setminus B(0, r)) \ll |q|^{-n \beta(x)} |q|^{2n} (2r + \epsilon)(\epsilon + |q|^{-n})
\]
for all \(n\). Setting \(n = -\lfloor \log |q| \epsilon \rfloor\) gives
\[
(3.14) \quad \mu_x(B(0, r + \epsilon) \setminus B(0, r)) \ll (r + \epsilon)^{e^{\beta(x)-1}}.
\]
This gives a reasonable estimate if \(\beta(x) > 1\) or equivalently \(\log \xi(x) < \log \lambda(x) - \log |q|\). Since this inequality is satisfied for \(x = 1\) and \(\beta(x)\) depends
continuously on $x$, there exists an interval $I$ around $x = 1$ such that $\beta(x) \geq \beta_0 > 1$ for some $\beta_0 < 2$.

**Step 4** uses the estimates for the measure dimension of $\mu_x$ and a suitable version of the Berry–Esseen inequality to provide bounds for $|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))|$. Since $\mu_{n,x}(B(0,r))$ can be easily related to the sum occurring in (3.2), this gives the error term in (3.2).

We recall the following result obtained in [12]. The statement uses the notation $c(\phi) = (\cos \phi, \sin \phi)^T$.

**Proposition 1 ([12, Proposition 1]).** Let $\nu_1$ and $\nu_2$ be two probability measures in $\mathbb{R}^2$ with their Fourier transforms defined by

$$\hat{\nu}_k(t) = \int e(\langle x, t \rangle) \, d\nu_k(x).$$

Suppose that

$$\nu_2(B(0, r + \varepsilon) \setminus B(0, r)) \ll \varepsilon^\theta$$

for some $0 < \theta < 1$ and all $r \geq 0$. Then for all $r \geq 0$ and $T > 0$,

$$|\nu_1(B(0, r)) - \nu_2(B(0, r))|$$

$$\ll \int_0^{2\pi} K_r(t, T) |\hat{\nu}_1(t \cdot c(\phi)) - \hat{\nu}_2(t \cdot c(\phi))| t \, d\phi \, dt + T^{-2\theta/(\theta+2)},$$

where the kernel function $K_r(t, T)$ satisfies

$$K_r(t, T) \ll \frac{1}{T^2} + \min\left(\frac{r^2}{T^2}, \frac{r^{1/2}}{T^{3/2}}\right).$$

The implied constant in (3.16) depends only on the implied constant in (3.15).

Inserting (3.10) and (3.11) into (3.16) with $\theta = \beta(x) - 1$ yields

$$|\mu_{n,x}(B(0, r)) - \mu_x(B(0, r))|$$

$$\ll \frac{1}{n \eta(x)} + \frac{T}{1} K_r(t, T) t |q|^{-\eta(x)n} t \, dt + \int_0^T K_r(t, T) t^{\eta(x)} |q|^{-\eta(x)n} t \, dt + T^{-2\beta(x)/2 - 1}.$$

Using the bounds for $K_r(t, T)$ and setting

$$\log T = \frac{\eta(x)}{\eta(x) + \frac{1}{2} + 2\frac{\beta(x) - 1}{\beta(x)+1}} n \log |q|$$

yields

$$|\mu_{n,x}(B(0, r)) - \mu_x(B(0, r))| \ll |q|^{-2\kappa(x)n}$$
uniformly in \( r \) with
\[
\kappa(x) = \frac{\eta(x)(\beta(x) - 1)}{(\eta(x) + 1/2)(\beta(x) + 1) + 2\beta(x) - 2}.
\]
Choosing \( \kappa \) to be the minimum attained by \( \kappa(x) \) on a compact interval \( I \), where \( \beta(x) \geq \beta_0 > 1 \) for some \( \beta_0 < 2 \), gives
\[
|\mu_{n,x}(B(0,r)) - \mu_x(B(0,r))| \ll |q|^{-2\kappa n}
\]
for all \( x \in I \).

Now, by definition of \( \mu_{k,x} \), we have
\[
\sum_{|z|^2 < N} x^{s_F(z)} = v_1(x,0) \cdot A(x)^k \cdot v_2 \cdot \mu_{k,x}(B(0,|q|^{-k}\sqrt{N}))
\]
for \( k = \lfloor \log_{|q|2} N \rfloor + M \) and some integer constant \( M > 0 \), which is chosen so that \( B(0,|q|^{1-M}) \subset \mathcal{F} \). Inserting (3.18) and \( v_1(x,0) \cdot A(x)^k \cdot v_2 = C(x)\lambda(x)^k + \mathcal{O}(\lambda_1(x)^k) \) yields
\[
\sum_{|z|^2 < N} x^{s_F(z)} = C(x)\lambda(x)^k \mu_x(B(0,|q|^{-k}\sqrt{N})) + \mathcal{O}(\lambda_1(x)^k) + \mathcal{O}(|\lambda(x)^k|^{2\kappa})
\]
We observe that the measure \( \mu_x \) satisfies the self-similarity relation
\[
\mu_x(B(0,|q|r)) = \lambda(x)\mu_x(B(0,r))
\]
for \( r \) sufficiently small. Setting
\[
\Phi(x,t) = C(x)\lambda(x)^{t+M} \mu_x(B(0,q^{t-M}))
\]
for \( t < 1 \) and noting that (3.13) implies the Hölder continuity of \( \Phi \) as a function of \( t \) completes the proof of Theorem 1.

Remark 3. For complex values of \( x \) this method breaks down, because the weak limits \( \mu_x \) have infinite total variation and are therefore not complex measures.

4. A Dirichlet series method. The goal of this section is to generalise Theorem 1 to complex \( x \). The proof relies on Dirichlet series and Mellin–Perron techniques.

Theorem 2. There exists a complex neighbourhood of \( x = 1 \) (that is, \( |x - 1| \leq \delta \) for some \( \delta > 0 \)) such that uniformly
\[
\sum_{|z|^2 < N} x^{s_F(z)} = \Phi(x,\log_{|q|2} N)N^{\log_{|q|2} \lambda(x)}(1 + \mathcal{O}(N^{-\kappa}))
\]
with some \( \kappa > 0 \), where \( \Phi(x,t) \) is analytic in \( x \) and 1-periodic and Hölder continuous in \( t \).
Furthermore, if $F$ is integer-valued with the property that
\begin{equation}
 d = \gcd\{g_F(B) : B \in \mathcal{N}^{L+1}\} = 1,
\end{equation}
then uniformly for $|x - 1| \geq \delta$ and $|\Re(x) - 1| \leq \delta_2$,
\begin{equation}
 \sum_{|z|^2 < N} x^{s_F(z)} \ll N^{\log|q|^2 \lambda(|x|) - \kappa}
\end{equation}
with some $\kappa > 0$ and some $\delta_2$ with $0 < \delta_2 < \delta$.

**Remark 4.** We will also show that $\Phi(x, t)$ has an explicit representation (see (4.21)). For example, for the sum-of-digits function $s_q(z)$ we have
\begin{align*}
 \Phi(x, t) &= \frac{X^{-t}}{1 - X^{-1}} \sum_{l=1}^{a^2} x^l X^{(t - \log|q|^2 l^2)} \\
 &\quad + \frac{X^{-t}}{1 - X^{-1}} \sum_{l=0}^{a^2} x^l \sum_{z \neq 0} x^{s_q(z)} (X^{(t - \log|q|^2 \lceil qz + l \rceil^2)} - X^{(t - \log|q|^2 \lceil qz^2 \rceil^2)})
\end{align*}
where $X$ abbreviates
\[ X = \frac{x|q|^2 - 1}{x - 1}. \]

The asymptotic representations (4.1) and (4.3) can be used in various ways (cf. also [5] and [6]). We directly derive asymptotic expansions for moments (Corollary 2) and a refinement of the central limit theorem stated in Corollary 1, further a local limit theorem (Corollary 3), uniform distribution in residue classes (Corollary 4) and uniform distribution modulo 1 (Corollary 5).

**Corollary 2.** For every integer $r \geq 1$ we have
\begin{equation}
 \frac{1}{\pi N} \sum_{|z|^2 < N} s_F(z)^r = \mu^r (\log|q|^2 N)^r \\
 + \sum_{l=0}^{r-1} G_{r,l}(\log|q|^2 N)(\log|q|^2 N)^l + O(N^{-\kappa}),
\end{equation}
where the functions $G_{r,l}(t)$ ($0 \leq l < r$) are continuous and 1-periodic.

**Proof.** Since (4.1) is uniform in a neighbourhood of 1 and $\Phi(x, t)$ is analytic in $x$ one can take derivatives at $x = 1$ of arbitrary order by using the formula
\[ G^{(r)}(1) = \frac{r!}{2\pi i} \int_{|x-1|=\delta/2} \frac{G(x)}{(x-1)^{r+1}} \, dx. \]
Furthermore, note that $\Phi(1, t) = \pi$. Hence, the asymptotic leading term is given by $(\lambda'(1)/\lambda(1))^r (\log|q|^2 N)^r$ and has no periodic fluctuations. \[ \Box \]
Note that if we combine Corollaries 1 and 2 then we also get error terms for the central moments of the form

\[ \frac{1}{\pi N} \sum_{|z|^2 < N} (s_F(z) - \mu \log_{|q|^2} N)^L = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u^L e^{-\frac{1}{2}u^2} du + \mathcal{O}(N^{-\kappa}), \]

for every integer \( L \geq 0 \). Furthermore, if we use the characteristic function \( \mathbb{E} e^{itYN} = S_N(e^{it})/S_N(1) \) instead of the moment generating function \( \mathbb{E} e^{itYN} \), that is, if we set \( x = e^{it} \) in Theorem 2, combined with Berry–Esseen techniques we also get a central limit theorem with error terms:

\[ \frac{1}{\pi N} \sum_{|z|^2 < N} |z|^2 \leq \mu \log_{|q|^2} N + y \sum_{|z|^2 < N} \lambda \left( \sum_{|z|^2 < N} \lambda \right) = 1 + \mathcal{O}(\log N)^{-1/2}. \]

**Corollary 3.** Suppose that \( F \) is integer-valued and that (4.2) holds. Set

\[ \mu(x) = \frac{x\lambda'(x)}{\lambda(x)} \quad \text{and} \quad \sigma^2(x) = \frac{x^2\lambda''(x)}{\lambda(x)} + \mu(x) - \mu(x)^2. \]

Furthermore, for \( k \in K(N) = \mathbb{Z} \cap [\mu(1-\delta_2) \log_{|q|^2} N, \mu(1+\delta_2) \log_{|q|^2} N] \) we define \( x_{k,N} \) by \( \mu(x_{k,N}) = k/\log_{|q|^2} N \), where \( \delta \) and \( \delta_2 \) are from Theorem 2. Then uniformly for \( k \in K(N) \),

\[ \# \{ z \in \mathbb{Z}[i] : |z|^2 < N, s_F(z) = k \} = \frac{\Phi(x_{k,N}, \log_{|q|^2} N)}{\sqrt{2\pi\sigma^2(x_{k,N}) \log_{|q|^2} N}} N^{\log_{|q|^2} \lambda(x_{k,N})} x_{k,N}^{-k} \left( 1 + \mathcal{O}\left( \frac{1}{\log N} \right) \right). \]

Furthermore, if \( |k - \mu \log_{|q|^2} N| \leq C \sqrt{\log_{|q|^2} N} \) (for some \( C > 0 \)) then also

\[ \# \{ z \in \mathbb{Z}[i] : |z|^2 < N, s_F(z) = k \} = \frac{\pi N}{\sqrt{2\pi\sigma^2 \log_{|q|^2} N}} \exp\left( -\frac{(k - \mu \log_{|q|^2} N)^2}{2\sigma^2 \log_{|q|^2} N} \right) \left( 1 + \mathcal{O}\left( \frac{1}{\sqrt{\log N}} \right) \right). \]

Note that \( \mu = \mu(1) \) and \( \sigma^2 = \sigma^2(1) \).

**Proof.** We apply (4.1) and (4.3) and use Cauchy’s formula:

\[ \# \{ z \in \mathbb{Z}[i] : |z|^2 < N, s_F(z) = k \} = \frac{1}{2\pi i} \int_{|x| = x_{k,N}} \left( \sum_{|z|^2 < N} x^{s_F(z)} \right) x^{-k-1} dx, \]

where \( x_{k,N} \) is the saddle point of the asymptotic leading term of the integrand:

\[ N^{\log_{|q|^2} \lambda(x)} x^{-k} = e^{\log \lambda(x) \cdot \log_{|q|^2} N - k \log x}. \]
We do not work out the details of standard saddle point techniques. We just refer to [6], where problems of almost the same kind have been discussed.

**Corollary 4.** Suppose that $F$ is integer-valued and that (4.2) holds. Then for every integer $M \geq 1$ and all $m \in \{0, 1, \ldots, M - 1\}$ we have

$$\frac{1}{\pi N} \#\{|z|^2 < N : s_F(z) \equiv m \mod M\} = \frac{1}{M} + O(N^{-\eta})$$

for some $\eta > 0$.

**Remark 5.** Alternatively to condition (4.2) we can assume that $s_F$ attains a value that is relatively prime to $M$. Then the same assertion holds (cf. Corollary 8).

**Proof of Corollary 4.** We use (4.3) for all $M$th roots of unity $x = \exp(2\pi im/M)$ and apply simple discrete Fourier techniques.

**Corollary 5.** Let $s_F$ be a block-additive function which attains one irrational value. Then the sequence $(s_F(z))_{z \in \mathbb{Z}[i]}$ is uniformly distributed modulo 1.

**Remark 6.** Note that Corollary 5 in particular applies to sequences of the kind $(\alpha s_F(z))_{z \in \mathbb{Z}[i]}$ if $s_F$ is integer-valued and $\alpha$ is irrational.

**Proof of Corollary 5.** We only have to prove that there exists a block $B$ of length $L + 1$ such that $g_F(B)$ is irrational. For this purpose we find a $z_0 \in \mathbb{Z}[i]$ with $s_F(z_0)$ irrational and with base $q$ representation of minimal length. Then by Lemma 2 we write $z_0 = \epsilon_0 + qv$ and $g_F(B) = s_F(z_0) - s_F(v)$. Since the base $q$ representation of $v$ has one digit less than the representation of $z_0$, $s_F(v)$ is rational, and therefore $g_F(B)$ is irrational.

Choosing $x^{g_F(B)} = \exp(hg_F(B))$ for $h \in \mathbb{Z} \setminus \{0\}$ gives a matrix $A(x)$ with eigenvalues strictly less than $|q|^2$. By Weyl’s criterion this implies the assertion.

We now turn to the proof of Theorem 2. For this purpose we will consider the Dirichlet series

$$G_B(x, s) = \sum_{z \in \mathbb{Z}[i] \setminus \{0\}, (\epsilon_0(z), \ldots, \epsilon_L(z)) = B} \frac{x^{s_F(z)}}{|z|^{2s}}$$

for $B \in \mathcal{N}^{L+1}$. It is easy to see that these series are well defined in a certain range. Set $A_1 = \max_{B \in \mathcal{N}^{L+1}} F(B)$ and $A_2 = \min_{B \in \mathcal{N}^{L+1}} F(B)$. Then we have $A_2 \log |q|^2 |z| - O(1) \leq s_F(z) \leq A_2 \log |q|^2 |z| + O(1)$. Hence, if $|x| \geq 1$ then $G_B(x, s)$ is surely absolutely convergent for $\Re(s) > 1 + \frac{1}{2} A_1 \log |q|^2 |x|$. Similarly, if $|x| \leq 1$ then $G_B(x, s)$ is absolutely convergent for $\Re(s) > 1 - \frac{1}{2} A_2 \log |q|^2 (1/|x|)$.

Next we provide a representation for $G_B(x, s)$ that can be used for analytic continuation.
Lemma 3. Define the vectors $G(x, s) = (G_B(x, s))_{B \in \mathcal{N}^L + 1}$ and $H(x, s) = (H_B(x, s))_{B \in \mathcal{N}^L + 1}$, where

$$H_B(x, s) = \begin{cases} 0 & \text{if } \eta(B) = 0, \\ \frac{x^s \sigma F(\eta_0)}{|\eta_0|^{2s}} + \frac{x^s \sigma F(B)}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, (\varepsilon(v), \ldots, \varepsilon_{L-1}(v)) = (0, \ldots, 0)} \frac{x^s F(v)}{|v + \eta_0/q|^{2s}} - \frac{x^s F(v)}{|v|^{2s}} & \text{if } \eta_0 = \eta(B) \neq 0 \text{ and } B' = (0, \ldots, 0), \\ \frac{x^s \sigma F(B)}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, (\varepsilon(v), \ldots, \varepsilon_{L-1}(v)) = B'} \frac{x^s F(v)}{|v + \eta_0/q|^{2s}} - \frac{x^s F(v)}{|v|^{2s}} & \text{if } \eta_0 = \eta(B) \neq 0 \text{ and } B' \neq (0, \ldots, 0). \end{cases}$$

Then $H_B(x, s)$ is absolutely convergent for $\Re(s) > \frac{1}{2} + \frac{1}{2} A_1 \log |q| |x|$ if $|x| \geq 1$ and for $\Re(s) > \frac{1}{2} - \frac{1}{2} A_2 \log |q| (1/|x|)$ if $|x| \leq 1$. More precisely, in that range

$$H(x, \sigma + it) \ll \begin{cases} (1 + |t|)^{2(1-\sigma) + A_1 \log |q|^2 |x|} & \text{if } |x| \geq 1, \\ (1 + |t|)^{2(1-\sigma) - A_2 \log |q|^2 (1/|x|)} & \text{if } |x| \leq 1, \end{cases}$$

and a meromorphic continuation of $G(x, s) = (G_B(x, s))_{B \in \mathcal{N}^L + 1}$ is given by

$$G(x, s) = \left( I - \frac{1}{|q|^{2s}} A(x) \right)^{-1} H(x, s),$$

where $A(x)$ is defined in (3.1).

Proof. We use the substitution $z = \eta_0 + qv$. If $\varepsilon_0(z) = \eta_0 = 0$ we have $s_F(z) = s_F(q)$ and consequently

$$G_B(x, s) = \frac{1}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, (\varepsilon_0(v), \ldots, \varepsilon_{L}(v)) = B'} \frac{x^s F(v)}{|v|^{2s}} = \frac{1}{|q|^{2s}} \sum_{l=0}^{a^2} G(B', l)(x, s).$$

Similarly, if $\eta_0 > 0$ and $B' = (0, \ldots, 0)$ we get

$$G_B(x, s) = \frac{x^s \sigma F(\eta_0)}{|\eta_0|^{2s}} + \frac{x^s \sigma F(B)}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, (\varepsilon_0(v), \ldots, \varepsilon_{L-1}(v)) = (0, \ldots, 0)} \frac{x^s F(v)}{|v + \eta_0/q|^{2s}} - \frac{x^s F(v)}{|v|^{2s}}$$

$$= \frac{x^s \sigma F(B)}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, (\varepsilon_0(v), \ldots, \varepsilon_{L-1}(v)) = (0, \ldots, 0)} \frac{x^s F(v)}{|v|^{2s}} + H_B(x, s)$$

$$= \frac{x^s \sigma F(B)}{|q|^{2s}} \sum_{l=0}^{a^2} G(0, \ldots, 0, l)(x, s) + H_B(x, s).$$

Finally, if $\eta_0 > 0$ and $B' \neq (0, \ldots, 0)$ then the case $v = 0$ cannot appear and
we also get
\[ G_B(x, s) = \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{v \in \mathbb{Z}[i] \setminus \{0\}, \langle \varepsilon_0(v), \ldots, \varepsilon_{L-1}(v) \rangle = B'} |v + \eta_0/q|^{2s} \]
\[ = \frac{x^{g_F(B)}}{|q|^{2s}} \sum_{i=0}^a G(B', l)(x, s) + H_B(x, s). \]
Now with $A(x) = (A_{B,C}(x))_{B,C \in \mathbb{N}^{L+1}}$ this directly translates to
\[ G(x, s) = \frac{1}{|q|^{2s}} A(x) G(x, s) + H(x, s), \]
which implies (4.8).

Set $s = \sigma + it$. Since
\[ ||v + l/\eta_0||^{2s} - |v|^{2s} \ll |v|^{2\sigma} \min \left(1, \frac{1}{|v|} \right) \]
it easily follows that $H_B(x, s)$ is absolutely convergent for $\Re(s) > \frac{1}{2} + \frac{1}{2} A_1 \log|q|^2 |x|$ if $|x| \geq 1$ and for $\Re(s) > \frac{1}{2} - \frac{1}{2} A_2 \log|q|^2 (1/|x|)$ if $|x| \leq 1$, and that $H(x, s)$ is bounded by (4.7).

If we set $a_n = \sum_{|z|^2 = n} x^{s_F(z)}$ then $G(s, x) = \sum_{n \geq 1} a_n n^{-s}$ and Mellin–Perron’s formula gives (for non-integral $N$)
\[ \sum_{n < N} a_n = \sum_{0 \leq |z|^2 < N} x^{s_F(z)} = \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} G(x, s) N^s s ds \]
for any sufficiently large $c$ such that the line $\Re(s) = c$ is contained in the half-plane of convergence of $G(x, s)$.

We will first use this representation to get upper bounds for the sum $\sum_{0 \neq |z|^2 < N} x^{s_F(z)}$. For this purpose we have to know something on the dominant eigenvalue $\lambda(x)$ of $A(x)$.

**Lemma 4.** If $x$ is sufficiently close to the positive real axis then $\lambda(x)$ is a simple eigenvalue of $A(x)$ and all other eigenvalues have smaller modulus. Furthermore, if $F$ is integer-valued such that (4.2) holds and if $x \neq 0$ is not a positive real number then all eigenvalues $\beta$ of $A(x)$ satisfy
\[ |\beta| < \lambda(|x|). \]

**Proof.** Suppose first that $x$ is a positive real number. Then it easily follows that $A(x)$ is a primitive irreducible non-negative matrix. We just have to observe that for every pair of blocks $B, C \in \mathcal{N}_{L+1}$ there exists a Gaussian integer $z$ such that both $B$ and $C$ occur in the $q$-ary digital expansion of $z$. Hence, all elements of $A(x)^{L+1}$ are positive and consequently by [22, Theorem 2.1, p. 49], $A(x)$ is primitive and irreducible. Thus, $\lambda(x) > 0$.
is simple and all other eigenvalues have smaller modulus. By continuity, this property remains true if \( x \) is sufficiently close to the positive real axis.

Next, suppose that \( x = |x|e^{i\varphi} \) with \( 0 < \varphi < 2\pi \). Since \(|x^{g_F(B)}| = |x|^{g_F(B)}\), [22, Theorem 2.1, p. 36] implies that all eigenvalues \( \beta \) of \( A(x) \) satisfy \(|\beta| \leq \lambda(|x|)\). Furthermore, the equality \(|\beta| = \lambda(|x|)\) holds if and only if there exists a complex number \( \mu \) with \(|\mu| = 1\) and a diagonal matrix \( D = \text{diag}(\mu_B)_{B \in \mathcal{N}_{L+1}} \) with complex numbers \( \mu_B \) of modulus \(|\mu_B| = 1\) such that

\[
A(x) = \lambda DA(|x|)D^{-1}.
\]

Without loss of generality we may assume that \( \mu_0...0 = 1\).

We now show that in this case \( \mu = 1 \) and \( \mu_B = 1 \) for all \( B \in \mathcal{N}_{L+1}, \) resp. \( A(x) = A(|x|) \). First observe that \( A_{0...0,0...0}(x) = 1 \) (for all \( x \)). Thus, \( \mu = 1 \). Furthermore, observe that \( A_{B,C}(x) = A_{B,C}(|x|) \neq 0 \) implies \( \mu_B = \mu_C \). Obviously, we have \( A_{B,C}(x) = A_{B,C}(|x|) \neq 0 \) if \( C = (B', l) \) (for some \( l \)) and \( \eta_B = 0 \). Thus, if \( B = (\eta_1, \ldots, \eta_L) \) is any block in \( \mathcal{N}_{L+1} \) then we can consider the sequence of blocks

\[
B_0 = (0, \ldots, 0), \quad B_1 = (0, \ldots, 0, \eta_1), \quad B_2 = (0, \ldots, 0, \eta_1, \eta_2), \ldots, \quad B_L = B
\]

and conclude inductively that

\[
1 = \mu_{B_0} = \mu_{B_1} = \cdots = \mu_B.
\]

However, if (4.2) holds then for every \( 0 < \varphi < 2\pi \) there exists \( B \in \mathcal{N}_{L+1} \) with \( e^{i\varphi g_F(B)} \neq 1 \), and thus \( x^{g_F(B)} \neq |x|^{g_F(B)} \). Consequently, all eigenvalues \( \beta \) of \( A(|x|e^{i\varphi}) \) are strictly bounded by \(|\beta| < \lambda(|x|)\).

Next note that the inverse matrix \((I - uA(x))^{-1}\) can be written as

\[
(I - uA(x))^{-1} = \frac{1}{\det(I - uA(x))} (P_{BC}(u, x))_{B,C \in \mathcal{N}_{L+1}}
\]

with polynomials \( P_{BC}(u, x) \) having degree in \( u \) smaller than \( D := |\mathcal{N}_{L+1}| = |q|^{2L+2} \). As above let \( \lambda(x) \) be the dominating eigenvalue of \( A(x) \) and \( \lambda_2(x), \ldots, \lambda_D(x) \) the remaining ones (where we assume that \( x \) is sufficiently close to the real axis and that all roots are simple). Then by the partial fraction decomposition we have

\[
P_{BC}(u, x) = \frac{C_{BC}(x)}{1 - u\lambda(x)} + \sum_{j=2}^{D} \frac{C_{j,BC}(x)}{1 - u\lambda_j(x)}
\]

for certain (analytic) functions \( C_{BC}(x) \) and \( C_{j,BC}(x) \). This also shows that \( G(x, s) \) can be represented as

\[
G(x, s) = \frac{K(x, s)}{1 - |q|^{2x}\lambda(x)} + \sum_{j=2}^{D} \frac{K_j(x, s)}{1 - |q|^{2x}\lambda_j(x)}.
\]
where $K(x, s)$ and $K_j(x, s)$ are linear combinations of the functions $H_B(x, s)$ with coefficients that are analytic in $x$ (cf. also (4.20)).

This shows that (4.8) provides an analytic continuation of $G(s, x)$ to the range $\Re(s) > \log_{|q|^2} |\lambda(x)|$ if $x$ is sufficiently close to 1, say $|x - 1| < \delta$. Furthermore, if $|x - 1| \geq \delta$ and $|\Re(x) - 1| \leq \delta_2$ then Lemma 4 shows that all eigenvalues $\beta$ of $A(x)$ satisfy $|\beta| < \lambda(|x|) - \eta'$ for some $\eta'$. Consequently, for all $x$ in that range the function $G(x, s)$ is analytic in the half-plane $\Re(s) > \log_{|q|^2} (\lambda(|x|) - \eta')$.

With this knowledge we are now ready to prove the second part of Theorem 2. The argument is close to that of [14].

**Lemma 5.** Suppose that $F$ is integer-valued and that (4.2) holds. Then there exist $\delta, \kappa > 0$ such that

$$
\sum_{|z|^2 < N} x^{s_F(z)} \ll N^{\log_{|q|^2} \lambda(|x|) - \kappa}
$$

uniformly for $|x - 1| \geq \delta$ and $|\Re(x) - 1| \leq \delta_2$.

**Proof.** Our starting point is formula (4.9). Observe that the integral there is not absolutely convergent. However, a slight variation of the Mellin–Perron formula gives

$$
S^{(2)}_N(x) = \sum_{0 \neq |z|^2 < N} x^{s_q(z)} \left(1 - \frac{|z|^2}{N}\right) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} G(x, s) \frac{N^s}{s(s+1)} ds
$$

with an integral that will be absolutely convergent in the range of interest.

Suppose now that $|x - 1| \geq \delta$ and $|\Re(x) - 1| \leq \delta_2$. Then we already know that $G(x, s)$ is analytic for $\Re(s) > \log_{|q|^2} (\lambda(|x|) - \eta')$ and that

$$
|G(x, s)| \ll (1 + |t|)^{2(1-\sigma)+\eta''}
$$

if $\sigma = \Re(s) \geq \log_{|q|^2} (\lambda(|x|) - \eta'/2) > \log_{|q|^2} \lambda(|x|) - \eta''$. It follows that

$$
S^{(2)}_N(x) \ll N^{\log_{|q|^2} \lambda(|x|) - \eta''}.
$$

It is now easy to derive proper upper bounds for

$$
S_N(x) = \sum_{0 \neq |z|^2 < N} x^{s_q(z)}.
$$

Observe that for every factor $\varrho > 1$ we have

$$
S_N(x) = \frac{\varrho S^{(2)}_N(x) - S^{(2)}_N(x)}{\varrho - 1} + \frac{1}{\varrho - 1} \sum_{N \leq |z|^2 < \varrho N} x^{s_F(z)} \left(1 - \frac{|z|^2}{N}\right).
$$

Set $c = \log_{|q|^2} \lambda(|x|) - \eta''$. By adjusting $\delta_2$ we can assume that $c < 1$. Finally, with

$$
\varrho = 1 + N^{-(1-c)/2}
$$
it follows that
\[
S_N(x) \ll N^{(1+c)/2} N^\max(A_1 \log |q|^2 (1+\delta_2), A_2 \log |q|^2 (1-\delta_2)).
\]
Since \(\delta_2\) can be chosen arbitrarily small it finally follows that
\[
S_N(x) \ll N^{\log |q|^2 \lambda(|x|)-\eta}
\]
for some \(\eta > 0\).

In order to prove the asymptotic expansion (4.1) for complex \(x\) (close to 1) we will use the following properties (see also [2, p. 243]).

**Lemma 6.** Suppose that \(a\) and \(c\) are positive real numbers. Then

\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - 1 \right| \leq \frac{a^c}{\pi T \log a} \quad (a > 1),
\]

\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} \right| \leq \frac{a^c}{\pi T \log(1/a)} \quad (0 < a < 1),
\]

\[
\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} - \frac{1}{2} \right| \leq \frac{C}{T} \quad (a = 1).
\]

**Proof.** Suppose first that \(a > 1\). By considering the contour integral of the function \(F(s) = a^s/s\) around the rectangle with vertices \(-A - iT, c - iT, c + iT, -A + iT\) and passing \(A\) to infinity one directly gets the representation

\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} a^s \frac{ds}{s} = \text{Res}(a^s/s; s = 0)
\]

\[+ \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x+iT}}{x+iT} dx + \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x-iT}}{x-iT} dx.
\]

Since

\[
\left| \frac{1}{2\pi i} \int_{-\infty}^{c} \frac{a^{x \pm iT}}{x \pm iT} dx \right| \leq \frac{a^c}{\pi T \log a}
\]

we directly obtain the bound in the case \(a > 1\).

The case \(0 < a < 1\) can be handled in the same way. Finally, in the case \(a = 1\) the integral can be explicitly calculated (and estimated).

For the formulation of the next lemma we use Iverson’s notation \([p]\) which is 1 if \(p\) is a true proposition and 0 otherwise.

**Lemma 7.** Suppose that \(l\) is a positive real number, \(\lambda\) a non-zero complex number, \(c\) a real number with \(c > \log |q|^2 |\lambda|\). Then for all real \(N > l^2\),

\[
[b(a)] = \frac{a}{\pi T \log a}
\]

for some \(\eta > 0\).
\[ (4.18) \quad \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{\frac{1}{T^s}}{1 - \frac{1}{|q|^{2s}} \lambda} \frac{N^s}{s} ds = \lambda^{[\log |q|^2(N/l^2)] + 1} - \frac{1}{2} \lambda^{[\log |q|^2(N/l^2)]} [\log |q|^2(N/l^2) \in \mathbb{Z}]. \]

Furthermore, if \( c > \max\{1, \log |q|^2 |\lambda|\} \) and \( x \) is sufficiently close to 1 then for every set of \( S \) of Gaussian integers with \( 0 \not\in S \) and all irrational numbers \( N > 1 \),

\[ (4.19) \quad \frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \sum_{z \in S} x^{s_F(z)} \left( \frac{\frac{1}{|qz + l|^2s}}{1 - \frac{1}{|q|^{2s}} \lambda} - \frac{1}{2} \lambda^{[\log |q|^2(N/l^2)]} [\log |q|^2(N/l^2) \in \mathbb{Z}] \right) \frac{N^s}{s} ds \]

\[ = \sum_{z \in S} x^{s_F(z)} \lambda^{[\log |q|^2(N/l^2)]} [\log |q|^2(N/l^2) \in \mathbb{Z}] + \mathcal{O}(1). \]

**Proof.** By assumption we have \( |\lambda/|q|^2| < 1 \). Thus, by using a geometric series expansion and Lemma 6, for all \( N > 1 \) such that \( \log |q|^2(N/l^2) \) is not an integer we get

\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\frac{1}{T^s}}{1 - \frac{1}{|q|^{2s}} \lambda} \frac{N^s}{s} ds = \sum_{k \geq 0} \lambda^k \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( \frac{N}{|q|^{2k l^2}} \right)^s \frac{ds}{s} \]

\[ = \sum_{k \leq \log |q|^2(N/l^2)} \lambda^k + \mathcal{O} \left( \frac{1}{T} \sum_{k \geq 0} \frac{|\lambda|^k (\frac{N}{|q|^{2k l^2}})^c}{\log (\frac{N}{|q|^{2k l^2}})} \right) \]

\[ = \lambda^{[\log |q|^2(N/l^2)]} + \mathcal{O} \left( \frac{1}{T} \sum_{k \geq 0} \frac{|\lambda|^k (\frac{N}{|q|^{2k l^2}})^c}{\log (\frac{N}{|q|^{2k l^2}})} \right). \]

If \( \log |q|^2(N/l^2) \) is an integer, we can proceed similarly. Of course, this implies (4.18).

Next assume that neither \( \log |q|^2(N/l^2) \) nor \( \log |q|^2(N/|q|^2) \) are integers for all \( z \in S \). Hence, if \( N > |qz + l|^2 \) then

\[ \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\frac{1}{|qz + l|^2s}}{1 - \frac{1}{|q|^{2s}} \lambda} \frac{N^s}{s} ds \]

\[ = \lambda^{[\log |q|^2(N/|qz + l|^2)]} + \mathcal{O} \left( \frac{1}{T} \sum_{k \geq 0} \frac{|\lambda|^k (\frac{N}{|q|^{2k}|qz + l|^2})^c}{\log (\frac{N}{|q|^{2k}|qz + l|^2})} \right) \]
and if \( N < |qz + l|^2 \) then we just have
\[
\frac{1}{2\pi i} \int_{c-iT}^{c+iT} 1 - \frac{1}{|q|^2 \lambda s} \frac{Ns}{s} \, ds = O \left( \frac{1}{T} \sum_{k \geq 0} \frac{|\lambda|^k}{|q|^{2k}|qz + l|^2} \right).
\]
Furthermore, for given \( N \) there are only finitely many pairs \((k, z)\) with
\[
\left| \frac{N}{|q|^{2k}|qz + l|^2} - 1 \right| < \frac{1}{2}.
\]
Hence, the series
\[
\sum_{z \in S} \sum_{k \geq 0} \frac{|\lambda|^k}{|q|^{2k}|qz + l|^2} < \frac{1}{2}
\]
is convergent if \( x \) is sufficiently close to 1. Consequently, we get
\[
\frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \sum_{z \in S} x^{s_F(z)} \left( \frac{1}{|qz + l|^2} - \frac{1}{|qz|^2} \right) \frac{Ns}{s} \, ds
\]
\[
= \frac{1}{1 - \lambda^{-1}} - \sum_{z \in S, |z|^2 < N} x^{s_F(z)} \left( \lambda^{\log |q|^2(N/|qz + l|^2)} - \lambda^{\log |q|^2(N/|qz|^2)} \right) + O(1).
\]
Finally, since \( |qz + l|^2 = |qz|^2(1 + O(1/|z|)) \) it follows that for \( x \) sufficiently close to 1 we have
\[
\sum_{z \in S, |z|^2 \geq N} x^{s_F(z)} \left( \lambda^{\log |q|^2(N/|qz + l|^2)} - \lambda^{\log |q|^2(N/|qz|^2)} \right) = O(1).
\]
This proves (4.19) if neither \( \log |q|^2(N/|qz + l|^2) \) nor \( \log |q|^2(N/|qz|^2) \) are integers. It is, however, easy to adapt the above calculation in the general case.

We now come back to the representation (4.12) for \( G(s, x) \). We already mentioned that \( K(s, x) \) and \( K_j(s, x) \) are linear combinations of the functions \( H_B(x, y) \) with coefficients that are analytic in \( x \). We make this explicit for \( K(s, x) \) in the following form:

\[
K(s, x) = \sum_{l=1}^{a^2} \frac{c_l(x)}{l^{2s}}
\]
\[
+ \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}} c''_{l, B'}(x) \sum_{\substack{z \in \mathbb{Z} \setminus \{0\} \atop (\varepsilon_0(z), \ldots, \varepsilon_{L-1}(z)) = B'}} x^{s_F(z)} \left( \frac{1}{|qz + l|^2} - \frac{1}{|qz|^2} \right).
\]
Hence, for \( N > 1 \) we obtain
\[
\frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{K(x, s)}{1 - \frac{1}{|q|^2 s}} \lambda(x) - \frac{N^s}{s} ds = \frac{1}{1 - \lambda(x)^{-1}} \sum_{l=1}^{a^2} c_l'(x) \lambda(x)^{\log |q|^2 \frac{N}{2}}
\]
\[
+ \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} \frac{c_{l,B'}'(x)}{1 - \lambda(x)^{-1}} \sum_{z \neq 0} x^{s_F(z)} \left( \lambda(x)^{\log |q|^2 \frac{N}{|qz+l|^2}} - \lambda(x)^{\log |q|^2 \frac{N}{|q|^2}} \right)
\]
\[
- \frac{1}{2} \sum_{l=1}^{a^2} c_l'(x) \lambda(x)^{\log |q|^2 \frac{N}{2}} \left[ \log |q|^2 \frac{N}{|q|^2} \in \mathbb{Z} \right]
\]
\[
- \frac{1}{2} \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} c_{l,B'}'(x) \sum_{z \neq 0} x^{s_F(z)} \lambda(x)^{\log |q|^2 \frac{N}{|qz+l|^2}} \left[ \log |q|^2 \frac{N}{|qz+l|^2} \in \mathbb{Z} \right]
\]
\[
+ \frac{1}{2} \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} c_{l,B'}'(x) \sum_{z \neq 0} x^{s_F(z)} \lambda(x)^{\log |q|^2 \frac{N}{|qz|^2}} \left[ \log |q|^2 \frac{N}{|qz|^2} \in \mathbb{Z} \right] + \mathcal{O}(1),
\]
where the \(\mathcal{O}(1)\)-term is uniform for \(N > 1\) and for \(x\) in a complex neighbourhood of \(x = 1\). Note that the correction terms vanish if \(N\) is, for example, irrational. Actually, we will prove in Lemma 8 that these correction terms can always be neglected since they sum up to zero in all cases.

Furthermore, note that the right hand side of this representation is of order \(\mathcal{O}(N^{\log |q|^2 \Re(\lambda(x))})\). Thus, if we do corresponding calculations for \(K_j(x, s)\) and \(\lambda_j(x)\) we also get
\[
\frac{1}{2\pi i} \lim_{T \to \infty} \int_{c-iT}^{c+iT} \frac{K_j(x, s)}{1 - \frac{1}{|q|^2 s}} \lambda_j(x) \frac{N^s}{s} ds = \mathcal{O}(N^{\log |q|^2 \Re(\lambda_j(x))}).
\]
Hence, setting
\[
\Phi(x, t) = \frac{\lambda(x)^{-t}}{1 - \lambda(x)^{-1}} \sum_{l=1}^{a^2} c_l'(x) \lambda(x)^{\log |q|^2 t^2}
\]
\[
+ \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} \lambda(x)^{-t} c_{l,B'}'(x) \sum_{z \neq 0} x^{s_F(z)} \lambda(x)^{\log |q|^2 |qz+l|^2} - \lambda(x)^{\log |q|^2 |qz|^2}
\]
and
\[
\Phi(x, t) = \frac{\lambda(x)^{-t}}{2} \sum_{l=1}^{a^2} c_l'(x) \lambda(x)^{\log |q|^2 t^2} \left[ t - \log |q|^2 t^2 \in \mathbb{Z} \right]
\]
\[
- \frac{\lambda(x)^{-t}}{2} \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} c_{l,B'}'(x) \sum_{z \neq 0} x^{s_F(z)} \lambda(x)^{\log |q|^2 |qz+l|^2} \left[ t - \log |q|^2 |qz+l|^2 \in \mathbb{Z} \right]
\]
\[
+ \frac{\lambda(x)^{-t}}{2} \sum_{l=1}^{a^2} \sum_{B' \in \mathcal{L}^L} c_{l,B'}'(x) \sum_{z \neq 0} x^{s_F(z)} \lambda(x)^{\log |q|^2 |qz|^2} \left[ t - \log |q|^2 |qz|^2 \in \mathbb{Z} \right]
\]
we end up with the representation

\[(4.23) \quad S_N(x) = (\overline{\Phi}(x, \log_{|q|^2} N) + \overline{\Phi}(x, \log_{|q|^2} N)) \times N^{\log_{|q|^2} \lambda(x)}(1 + O(N^{-\kappa}))\]

where \(\kappa > 0\) is just the minimal difference between \(\Re(\lambda(x))\) and \(\Re(\lambda_j(x))\) \((j \geq 2)\) when \(x\) varies in a sufficiently small neighbourhood of \(x = 1\). By definition it is clear that \(\overline{\Phi}(x,t) = \overline{\Phi}(x,t+1)\), \(\overline{\Phi}(x,t) = \overline{\Phi}(x,t+1)\) and that \(\overline{\Phi}(x,t)\) and \(\overline{\Phi}(x,t)\) represent analytic functions in \(x\) if \(t\) is fixed. However, \(\overline{\Phi}(x,\log_{|q|^2} N) = 0\) if \(N\) is irrational. Thus, it is natural to expect that \(\overline{\Phi}(x,t) = 0\) for all \(t\) which is in fact true. The next lemma provides this fact and also the continuity of \(\overline{\Phi}(x,t)\), thus completing the proof of Theorem 2.

**Lemma 8.** The function \(\overline{\Phi}(x,t)\) is Hölder continuous in \(t\) and analytic for \(x\) in a complex neighbourhood of \(x = 1\). Furthermore, \(\overline{\Phi}(x,t) = 0\) for all \(t\).

**Remark 7.** In particular this shows that \(\Phi(x,t)\) from Theorem 1 equals \(\overline{\Phi}(x,t)\) for real \(x\).

**Proof of Lemma 8.** First assume that \(x\) is real. By considering \(N = |q|^{2(n+t)}\) for \(n = 0, 1, 2, \ldots\) it follows from Theorem 1 and (4.23) that \(\Phi(x,t) = \overline{\Phi}(x,t) + \overline{\Phi}(x,t)\). Furthermore, we have \(\overline{\Phi}(x,t) = 0\) if \(t\) is not of the form \(t = \log_{|q|^2} m - k\) for some positive integers \(m\) and \(k\). (This occurs, for example, if \(t = \log_{|q|^2} T\) for some irrational number \(T\).) Since the numbers \(t\) with this property are dense in \([0,1)\) it follows that \(\overline{\Phi}(x,t)\) is continuous in \(t\) if and only if \(\overline{\Phi}(x,t) = 0\) for all \(t\). This observation can also be deduced from the inequality (4.24) below which is also true for complex \(x\). Hence, continuity of the mapping \(t \mapsto \overline{\Phi}(x,t)\) follows from \(\overline{\Phi}(x,t) = 0\) even if \(x\) is a complex number.

We now suppose that \(t \in [0,1)\) is of the form \(t = \log_{|q|^2} m - k\) for some positive integers \(m\) and \(k\) where we assume that \(k\) is chosen to be minimal. If \(s \neq t\) is also of that form, that is, \(s = \log_{|q|^2} n - j \in [0,1)\) for positive integers \(n\) and \(j\), then (for a properly chosen constant \(c > 0\)) we have

\[|s - t| \geq c|q|^{2s} - |q|^{2t} \geq \frac{1}{|q|^{2(k+j)}}.\]

In particular,

\[|q|^{2j} \geq \frac{c}{|q|^{2k}|s - t|}.\]

Observe that only terms of the form \(\lambda(x)^{-j}\) contribute to \(\overline{\Phi}(x,s)\); notice that for these values of \(t\) we have \(n = |qz + l|^{2}\) for some \(z \in \mathbb{Z}[i]\) and \(l \in \{0, \ldots, a^{2}\}\). Thus, if we fix some \(\varepsilon > 0\) there exists \(\delta > 0\) such that

\[|\overline{\Phi}(x,s)| < \varepsilon\] for all \(s\) with \(|s - t| < \delta\).
Next observe that if \( t = \log_{|q|^2} m - k \) then for \( 0 < \theta < 1 \),
\[
\frac{\lambda(x)^{(t+\theta)-\log_{|q|^2}l^2} - \lambda(x)^{(t-\theta)-\log_{|q|^2}l^2}}{1 - 1/\lambda(x)} = \frac{\lambda^{-k} - \lambda^{-k-1}}{1 - 1/\lambda(x)} = \lambda^{-k} = \lambda(x)^{(t-\log_{|q|^2}l^2)}.
\]
Thus, by a similar reasoning we also get
\[
(4.24) \quad \left| \Phi(x,t + \theta) - \Phi(x,t - \theta) + \frac{1}{2} \Phi(x,t) \right| < \varepsilon
\]
if \( 0 < \theta < \delta \). Furthermore, by continuity of \( \Phi(x,t) = \Phi(x,t) + \Phi(x,t) \),
\[
|\Phi(x,t + \theta) - \Phi(x,t - \theta)| = \left| \Phi(x,t + \theta) + \Phi(x,t + \theta) - \Phi(x,t - \theta) + \Phi(x,t - \theta) \right| < \varepsilon
\]
in that range. Consequently,
\[
|\overline{\Phi}(x,t)| \leq 2|\overline{\Phi}(x,t + \theta) - \overline{\Phi}(x,t - \theta)| + 2\varepsilon \\
\leq 2|\overline{\Phi}(x,t + \theta) - \overline{\Phi}(x,t - \theta)| + 6\varepsilon < 7\varepsilon.
\]
Since \( \varepsilon > 0 \) can be chosen arbitrarily small it follows that \( \overline{\Phi}(x,t) = 0 \).

Thus, we have shown that \( \overline{\Phi}(x,t) = 0 \) for all \( t \) if \( x \) is a real number close to 1. Since \( \overline{\Phi}(x,t) \) is an analytic function in \( x \) we also obtain \( \overline{\Phi}(x,t) = 0 \) for complex \( x \) close to 1. As mentioned above, this implies that \( \overline{\Phi}(x,t) \) is continuous in \( t \) even if \( x \) is a complex number close to 1.

Similarly we show that \( \Phi(x,t) \) is Hölder continuous in \( t \). Here we just have to use a quantified version of (4.24). We leave the details to the reader. ■

5. A method based on ergodic \( \mathbb{Z}[i] \)-actions and skew products.

In this section we will consider block additive functions \( s_F \) taking values in an abelian group \( A \), hence \( F : A^{L+1} \to A \). The neutral element will be denoted by \( 0_A \). We assume that \( A \) is compact metrisable, equipped with its Haar measure \( \lambda_A \), and we introduce the metrisable compact space \( \Omega := A^{\mathbb{Z}[i]} \). The shift \( \mathbb{Z}[i] \)-action \( \Sigma : \zeta \mapsto \Sigma \zeta \) on \( \Omega \) is defined by setting, for all \( \omega : z \mapsto \omega_z \) and all \( \zeta \in \mathbb{Z}[i] \),
\[
(\Sigma \zeta(\omega))_z := \omega_{\zeta + z}.
\]
For any \( \omega \in \Omega \), consider its orbit closure \( K_\omega \) which is the topological closure of its orbit
\[
O_\omega := \{ \Sigma \zeta(\omega) : \zeta \in \mathbb{Z}[i] \}
\]
under the shift. Clearly \( K_\omega \) is a compact subspace of \( \Omega \) and \( \Sigma \zeta(K_\omega) = K_\omega \) for all \( \zeta \in \mathbb{Z}[i] \). The restriction of \( \Sigma \zeta \) on \( K_\omega \), still denoted by \( \Sigma \zeta \), is a
homeomorphism of $K_\omega$, defining the shift $\mathbb{Z}[i]$-action $\Sigma : \zeta \mapsto \Sigma \zeta$ on $K_\omega$. By definition, the couple $\mathcal{K}_\omega := (\Sigma, K_\omega)$ is the flow associated to $\omega$.

The function $s_F$ can be viewed as an element of the compact space $\Omega := A^\mathbb{Z}[i]$. For short we write $K(F)$ (resp. $K_A(F)$) for $Ks_F$ (resp. $K_{s_F}$) and we set $I(F) := \{s_F(z) : z \in \mathbb{Z}[i]\}$.

**Lemma 9.** Assume that $A$ is a compact metrisable group. Then the closure $A(F)$ of the set $I(F)$ is a subgroup of $A$.

**Proof.** It is clear that the neutral element $0_A$ of $A$ belongs to $I(F)$ so that, due to compactness, it is enough to prove that $a + a' \in A(F)$ for any $a$ and $a'$ in $A(F)$. Let $U$ be any neighbourhood of $0_A$ and let $V$ be another neighbourhood of $0_A$ such that $V + V \subseteq U$. By assumption there exist Gaussian integers $z$ and $z'$ such that $s_F(z) - a \in V$ and $s_F(z') - a' \in V$. Setting $z'' = z + q^\text{length}_q(z) + L + 1 z'$ one gets $s_F(z'') - (a + a') = s_F(z) - a + s_F(z') - a' \in V + V$. Hence $s_F(z'') - (a + a') \in U$, proving that $a + a' \in A(F)$.

In the next theorem we make use of the following simple result:

**Lemma 10.** For any neighbourhood $V$ of $0_A$ in $A$ there exists a finite set $B = B(V)$ of $\mathbb{Z}[i]$ such that for all $r \in \mathbb{Z}[i]$ there exists $b \in B$ such that $s_F(r + b) \in V$.

**Proof.** We may assume that $V = -V$ otherwise replace $V$ by $V \cap (-V)$. Since $I(F)$ is dense in $A(F)$ and $A(F)$ is compact there exists an integer $N = N(V)$ such that

$$A(F) \subseteq \bigcup_{z, \text{length}_q(z) \leq N} s_F(z) + V.$$ 

Given any Gaussian integer $r$, we use the $q$-adic expansion of $r$ to write the decomposition $r = r' + q^{N+L+1}t$ with $\text{length}_q(r') \leq N + L + 1$ and choose $r''$ with $\text{length}_q(r'') \leq N$ such that $-s_F(t) \in V + s_F(r'')$. With $b = -r' + r''$ we get

$$s_F(r + b) = s_F(r'' + q^{N+L+1}t) = s_F(r'') + s_F(t) \in V.$$ 

In addition, from Lemma 1,

$$\text{length}(b) \leq c + \frac{\log(|r'| + |r''|)}{\log |q|} \leq c + \frac{\log 2 + \log |q|(c + N + L + 1)}{\log |q|} \leq c' + N + L + 1.$$ 

The proof ends by taking $B := \{z \in \mathbb{Z}[i] : \text{length}_q(z) \leq c' + N + L + 1\}$. ■

We are ready to prove the main result on the topological structure of $\mathcal{K}(F)$. 


**Theorem 3.** The flow $\mathcal{K}(F)$ is minimal, that is, if $M$ is a non-empty compact subspace of $\mathcal{K}(F)$ such that $\Sigma_\zeta(M) \subset M$ for all $\zeta \in \mathbb{Z}[i]$ then $M = \mathcal{K}(F)$.

**Proof.** Since $\mathcal{K}(F)$ is the orbit closure of $s_F$, it is enough to prove that $s_F$ is uniformly recurrent (see [7, Section 4]). To this end we have to show that for any neighbourhood $W$ of $0$, the set $S(W) := \{ u \in \mathbb{Z}[i] : \Sigma_u(s_F) - s_F \in W \}$ is syndetic, that is, there is a finite set $E$ such that $\mathbb{Z}[i] = S(W) + E$. We may restrict ourselves to fundamental neighbourhoods of the form $W(M, U) = \bigcap_{\text{length}_q(z) \leq M} \{ \omega \in \Omega : \omega_z \in U \}$ where $U$ is any neighbourhood of $0_A$. Choose a neighbourhood $V$ of $0$ such that $V + V \subset U$ and a finite subset $B = B(V)$ of $\mathbb{Z}[i]$ as in Lemma 10 and let $h = \max\{\text{length}_q(b) : b \in B\}$. Fix any Gaussian integer $z$ and decompose it as $z = z' + q^{M+L+1} r$ with $\text{length}_q(z') \leq M+L+1$. By Lemma 1, $\text{length}(-z') \leq 2c + M + L + 1$ and there exists $r' \in B$ such that $s_F(r + r') \in V$.

Now set $\zeta = -z' + q^{M+L+1} r'$. By construction $z + \zeta = q^{M+L+1} (r + r')$, which implies $s_F(z + \zeta + t) - s_F(t) \in V$ for all Gaussian integers $t$ of length at most $M$. This means that $z + \zeta \in S(W)$ with

$$\text{length}_q(\zeta) \leq c + \frac{\log(|z'| + |r'| |q|^{M+L+1})}{\log |q|} \leq c'' + M + L + 1 + h$$

where $c''$ is an absolute constant. Therefore $\zeta$ belongs to a finite subset of $\mathbb{Z}[i]$ and consequently $S(W)$ is syndetic. ■

Now we introduce tools from ergodic theory to prove rather general distribution results on block-additive functions. We will use ideas discussed in more detail in [14] and refer to that paper for a detailed exposition of the method.

The general idea of the approach motivated by ergodic theory is to build a dynamical system $(X, T, \mu)$ from the underlying digital expansion. The space $X$ is then a suitably chosen compactification of $\mathbb{Z}[i]$, the action $T : \mathbb{Z}[i] \to \text{Aut}(X)$ is simply addition by elements of $\mathbb{Z}[i]$. Since the compactification $X$ carries a natural group structure in our case, $\mu$ is chosen as the Haar measure on this group. Since no non-trivial block additive function can be extended to a continuous or even measurable function on $X$ (see Remark 9 below), we use a trick developed by T. Kamae [15], which overcomes this problem by constructing a suitable cocycle (we will introduce this notion below). The fact that the additive function has no extension to $X$ is then reflected by the non-triviality of the cocycle.
Consider the infinite product space
\[ K_q = \{0, 1, \ldots, a^2\}^{\mathbb{N}_0} \]
and embed \( \mathbb{Z}[i] \) by \( q \)-adic digital expansion
\[ \iota : \mathbb{Z}[i] \to K_q, \quad z \mapsto (\varepsilon_0(z), \varepsilon_1(z), \ldots, \varepsilon_L(z), 0, 0, \ldots). \]
Then it was proved in [14] that addition in \( \mathbb{Z}[i] \) can be extended continuously to \( K_q \). By this construction \( K_q \) inherits a group structure by
\[ K_q = \text{proj lim}_{n \to \infty} \mathbb{Z}[i]/q^n \mathbb{Z}[i]. \]
The corresponding Haar measure \( \mu \) is the infinite product measure of uniform distribution on the digits. The cylinder set of base \((x_0, \ldots, x_n) \in \{0, \ldots, a^2\}^{n+1}\) is given by
\[ [x_0, \ldots, x_n] := x_0 + x_1 q + \cdots + x_n q^n + q^{n+1} K_q \]
\[ = \{ z \in K_q : \varepsilon_0(z) = x_0, \ldots, \varepsilon_n(z) = x_n \}. \]
The Haar measure of such sets is given by \( \mu([x_0, \ldots, x_n]) = |q|^{-n-1} \). The Gaussian integers \( \mathbb{Z}[i] \) act on \( K_q \) by addition
\[ T : \mathbb{Z}[i] \to \text{Aut}(K_q), \quad z \mapsto (x \mapsto x + z). \]
This continuous action is uniquely ergodic.

**Definition 1.** A sequence \((Q_n)_{n \in \mathbb{N}}\) of finite subsets of \( \mathbb{Z}[i] \) is called a \textit{Følner sequence} if it has the following properties:

1. \( Q_n \subset Q_{n+1} \) for all \( n \);
2. There exists a constant \( K \) such that \( \#(Q_n - Q_n) \leq K \#Q_n \) for all \( n \);
3. \( \lim_{n \to \infty} \frac{\#(Q_n \triangle (g + Q_n))}{\#Q_n} = 0 \) for all \( g \in \mathbb{Z}[i] \).

(\( \triangle \) denotes symmetric difference.)

Classical examples of such sequences are the sequence of balls of radius \( \sqrt{n} \), \( Q_n = \{ z \in \mathbb{Z}[i] : |z|^2 < n \} \), or the squares \( Q_n = \{ z \in \mathbb{Z}[i] : |\Re(z)| < n, |\Im(z)| < n \} \). Another example more connected to digital expansions is the “discrete \( q \)-adic dragons” \( Q_n = \{ z \in \mathbb{Z}[i] : \text{length}_q(z) \leq n \} \).

We recall that a point \( x \in X \) is called \( (T, \mu) \)-generic (or simply generic, if the underlying action is clear) if
\[ \forall f \in C(X) : \lim_{n \to \infty} \frac{1}{\#Q_n} \sum_{z \in Q_n} f \circ T_z(x) = \int_X f \, d\mu \]
for a Følner sequence \((Q_n)_{n \in \mathbb{N}}\). By Tempel’man’s ergodic theorem (cf. [19, Chapter 6, Theorem 4.4]) \( \mu \)-almost all points are generic. Clearly, for a uniquely ergodic continuous action every point is generic, and even more: the convergence in (5.1) is uniform in \( x \).
For uniquely ergodic non-continuous actions we need additional conditions, which will be developed below, to have the same conclusion. To this end we introduce the following definition.

**Definition 2.** Let $X$ be a compact metrisable space and $T : \mathbb{Z}[i] \times X \to X$ a Borel measurable $\mathbb{Z}[i]$-action. A subset $A \subset X$ is called *uniformly $T$-negligible* if

$$\forall \varepsilon > 0 \ \exists g \in C(X), g \geq 1_A : \limsup_{n \to \infty} \left\| \frac{1}{\# Q_n} \sum_{z \in Q_n} g \circ T_z \right\|_\infty < \varepsilon$$

for a Følner sequence $(Q_n)_{n \in \mathbb{N}}$.

**Definition 3.** Let $X$ be a compact metrisable space and $T : \mathbb{Z}[i] \times X \to X$ a Borel measurable $\mathbb{Z}[i]$-action. The action $T$ is called *uniformly quasi-continuous* if for every $z \in \mathbb{Z}[i]$ the set of discontinuity points of $T_z$ is uniformly $T$-negligible.

**Remark 8.** If $T$ is uniformly quasi-continuous and $\mu$ is a $T$-invariant Borel probability measure on $X$, then $T$ is $\mu$-continuous.

The following theorem is an adapted version of [21, Annexe, Théorème]. The proof is slightly simplified by the fact that the action is invertible.

**Theorem 4.** Let $T$ be a uniformly quasi-continuous $\mathbb{Z}[i]$-action on the compact metric space $X$ and assume that $T$ is uniquely ergodic with invariant measure $\lambda$. Then for any $\lambda$-continuous function $f$ we have

$$\lim_{n \to \infty} \frac{1}{\# Q_n} \sum_{z \in Q_n} f \circ T_z(x) = \int_X f \, d\lambda$$

uniformly in $x$.

**Proof.** Let $\mathcal{R}_\lambda$ denote the Banach space of real-valued $\lambda$-continuous functions on $X$ equipped with the uniform norm and let

$$E = \{ \{g - g \circ T_z : g \in \mathcal{R}_\lambda, z \in \mathbb{Z}[i] \} \}.$$ 

Then $\lambda$ defines a linear form on $\mathcal{R}_\lambda$ with $\ker(\lambda) \subseteq E$. We will show that we have equality in fact.

Let $L : \mathcal{R}_\lambda \to \mathbb{R}$ be a continuous linear form with $E \subseteq \ker(L)$ and $L(1) = 1$. For $f \geq 0$ define

$$|L|(f) = \sup \{ L(g) : g \in \mathcal{R}_\lambda, |g| \leq f \}.$$ 

Then $|L|$ can be extended to a continuous positive linear form on $\mathcal{R}_\lambda$. Thus $|L|$ determines a measure $\ell$ on $X$.

We will now prove that $|L|$ and therefore $\ell$ is $T$-invariant. By definition we have, for $f \geq 0$,
|L|(f \circ T_z) = \sup \{L(g) : g \in \mathcal{R}_\lambda, |g| \leq f \circ T_z \}
\geq \sup \{L(g \circ T_z) : g \circ T_z \in \mathcal{R}_\lambda, |g \circ T_z| \leq f \circ T_z \} \geq |L|(f),

where we have used $L(g) = L(g \circ T_z)$ since $E \subseteq \ker(L)$. Applying the same inequality to $f \circ T_{-z}$ shows the $T$-invariance.

By unique ergodicity we have $\ell = \lambda$. On the other hand, $|L| - L$ is also a $T$-invariant positive linear form. Thus we have $|L| - L = a\lambda$ with $a \geq 0$. Hence $L = (1-a)\lambda$ and as $L(1) = 1$ we get $a = 0$, and we have $E = \ker(L)$ by the Hahn–Banach theorem.

Summing up, for every $f \in \mathcal{R}_\lambda$ and every $\varepsilon > 0$ there exist $k \in \mathbb{N}$, $g_1, \ldots, g_k \in \mathcal{R}_\lambda$, and $z_1, \ldots, z_k \in \mathbb{Z}[i]$ such that

$$\left\|f - \lambda(f) - \sum_{m=1}^{k} (g_m - g_m \circ T_{z_m})\right\|_\infty < \varepsilon.$$  

Applying the ergodic means to this inequality and using (3) Definition 1 finishes the proof.  

We recall the definition of a cocycle:

**Definition 4.** Let $(X,T,\mu)$ be a $\mathbb{Z}[i]$-action on $X$ and $A$ an abelian group. A $T$-cocycle (or simply a cocycle, if the underlying action $T$ is fixed) is a Borel map $a : \mathbb{Z}[i] \times X \rightarrow A$ such that

(i) $a(g + h, x) = a(g, T_h x) + a(h, x)$ $\mu$-a.e.,

(ii) $\mu(\bigcup_{g \in \mathbb{Z}[i]}(\{x : T_g x = x\} \cap \{x : a(g, x) \neq 0_A\})) = 0$.

If we assume that $T$ is aperiodic, i.e. $\mu(\{x : \exists g \neq 0, T_g x = x\}) = 0$, then condition (ii) is always satisfied.

A cocycle $a$ is called a coboundary if there exists a Borel map $f : X \rightarrow A$ such that

$$\forall x \in X, g \in \mathbb{Z}[i] : a(g, x) = f(T_g x) - f(x).$$

The skew product $(X \times A, T^a, \mu \otimes \lambda_A)$ corresponding to the cocycle $a$ is given by

$$T^a : \mathbb{Z}[i] \rightarrow \text{Aut}(X \times A), \ z \mapsto ((x, b) \mapsto (x + z, b + a(z, x))).$$

**Definition 5.** An element $\alpha \in A$ is said to be an essential value of the cocycle $a$ if for every neighbourhood $N(\alpha)$ of $\alpha$ in $A$ and for every $B \in \mathcal{B}(X)$ (Borel sets) with $\mu(B) > 0$,

$$\mu\left( \bigcup_{g \in \mathbb{Z}[i]} (B \cap T_{-g}^{-1}(B) \cap \{x : a(g, x) \in N(\alpha)\}) \right) > 0.$$  

Let

$$E(a) = \{\alpha \in A : \alpha \text{ is an essential value of } a\}.$$
This definition does not require ergodicity of $T$. We have the following proposition.

**Proposition 2** (cf. [23]). Let $a : \mathbb{Z}[i] \times X \to A$ be a cocycle. Then the following properties hold:

1. If $b : \mathbb{Z}[i] \times X \to A$ is a coboundary then $E(a + b) = E(a)$.
2. $E(a)$ is a closed subgroup of $A$.
3. $a$ is a coboundary $\iff E(a) = \{0_A\}$.

Let $I$ be the set of $T^a$-invariant elements in $\mathcal{B} \otimes \mathcal{B}_A$ and put

$$I(a) = \{\beta \in A : \mu \otimes h_A(\tau_\beta B \triangle B) = 0 \text{ for every } B \in I\}$$

where $\tau_\beta : X \times A \to X \times A$ is given by $\tau_\beta(x,\alpha) = (x,\alpha + \beta)$.

The set of essential values is directly related to the ergodicity of the skew product action $T^a$ by the following theorem of K. Schmidt.

**Theorem 5** ([23, Theorem 5.2]). Let $T$ be an ergodic action on $(X, \mathcal{B}, \mu)$ which is assumed to be non-atomic. Then for any cocycle $a : G \times X \to A$,

$$E(a) = I(a).$$

**Corollary 6.** If $T$ is ergodic, then $T^a$ is ergodic $\iff E(a) = A$.

The cocycle suitable for our purposes is defined as

$$a_F(z,x) = \begin{cases} \lim_{w \to z} (s_F(w + z) - s_F(w)) & \text{if the limit exists,} \\
0 & \text{otherwise.} \end{cases} \tag{5.5}$$

The limit exists if the carry propagation in the addition $x + z$ terminates after finitely many steps. It was proved in [14] that for almost all $x \in \mathcal{K}_q$ the addition $x + z$ produces only finitely many carries. Thus $a_F(z,x)$ is defined for $\mu$-almost all $x$. Furthermore, since $a_F(z,\cdot)$ is constant on cylinder sets defined by the different possible carries in the addition $x + z$ (cf. [14]), $a_F$ is also $\mu$-continuous. Moreover, the set of discontinuity points of $a_F(z,\cdot)$ is closed, hence it is also uniformly $T$-negligible by the unique ergodicity of the continuous action $T$. Thus we have proved

**Lemma 11.** The skew product action $T^{a_F}$ given by (5.3) is uniformly quasi-continuous.

We naturally define

$$V(a_F) = \{a_F(z,x) : x \in \mathcal{K}_q, z \in \mathbb{Z}[i]\},$$

the closed subgroup consisting of the values of $a_F$. Recalling the definition of the group $A(F) = \{s_F(z) : z \in \mathbb{Z}[i]\}$, we readily have
Proposition 3. The groups generated by the values of $s_F$ and $a_F$ are equal:

$$V(a_F) = A(F) = \{s_F(z) : z \in \mathbb{Z}[i]\}.$$  

Proposition 4. Let $s_F$ be a block additive function on $\mathbb{Z}[i]$ and $a_F$ be the corresponding cocycle on $K_q$. Then the set of essential values of $a_F$ equals the closed subgroup $A(F)$ of $A$ generated by the values of $s_F$:

$$E(a_F) = A(F).$$

Proof. We need the following lemma which is the analog of [3, Lemma 12] but in the case of cocycles for a $\mathbb{Z}[i]$-action.

Lemma 12. Let $\alpha \in A$ and assume that for any neighbourhood $V = V(\alpha)$ of $\alpha$ in $A$ there exists a constant $\kappa > 0$ such that for every non-empty cylinder set $C$ of $K_q$ there exists $\zeta \in \mathbb{Z}[i]$ such that

$$\mu(C \cap T_\zeta(C) \cap \{x \in K_q : a_F(\zeta, x) \in V\}) \geq \kappa \mu(C).$$

Then $\alpha \in E(a_F)$.

Proof of Lemma 12. Set for short $W(V, \zeta) := \{x \in K_q : a_F(\zeta, x) \in V\}$. If $B$ is a Borel subset of $K_q$, then due to the regularity of the Haar measure, for any $\varepsilon > 0$ (and $\varepsilon < 1$), there exists a non-empty cylinder set $C$ such that $\mu(B \cap C) \geq (1 - \varepsilon)\mu(C)$, hence $\mu(C \setminus (B \cap C)) \leq \varepsilon \mu(C)$, leading to

$$\mu(B \cap T_\zeta(B) \cap W(V, \zeta)) \geq \mu((B \cap C) \cap T_\zeta(B \cap C) \cap W(V, \zeta)) \geq \mu(C \cap T_\zeta(C) \cap W(V, \zeta)) - 2\varepsilon \mu(C).$$

Choose $\zeta$ such that $\mu(C \cap T_\zeta(C) \cap W(V, \zeta)) \geq \kappa \mu(C)$ and $\varepsilon < \kappa/2$. Then we get $\mu(B \cap T_\zeta(B) \cap W(V, \zeta)) > 0$. Hence $\zeta \in E(a_F)$ as expected. 

Going back to the proof of Proposition 4, it is enough to prove that $a_F(y, z_0) \in E(a_F)$ for all $y, z_0 \in \mathbb{Z}[i]$, where $y = (y_0, y_1, \ldots, y_t)_q$. Let $C$ be any non-empty cylinder set, say

$$C = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k].$$

Set $\zeta = q^{k+L+3}z_0$ and consider

$$C_0 = [\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_k, \underbrace{0, \ldots, 0}_{L+2}, y_0, y_1, \ldots, y_t, \underbrace{0, \ldots, 0}_{M}]$$

with $M = 4 + \max(0, \text{length}_q(z_0) - t)$. One has $\mu(C_0) = \kappa \mu(C)$ with $\kappa = 1/|q|^{M+t+L+2}$. The $M$ digits 0 at the end ensure that there is no carry propagation beyond the $k + L + t + M + 4$ fixed digits. This means that for any $x \in C_0$,

$$a_F(\zeta, x) = a_F(z_0, y)$$

and $C_0 \subset C \cap T_\zeta^{-1}(C)$. 

This implies that for any neighbourhood $V$ of $a_F(z_0, y)$,
\[ \mu(C \cap T^{-1}_\zeta(C) \cap W(V, \zeta)) \geq \kappa \mu(C), \]
and Lemma 12 gives $a_F(z_0, y) \in E(a_F)$. 

**Remark 9.** By considering both Proposition 3 and Proposition 2(3) one sees that if $s_F$ can be extended to a measurable map on $K_q$, then the cocycle $\alpha_F$ is a coboundary, hence $s_F$ is trivial, i.e., $s_F(z) = 0_A$ for all $z \in \mathbb{Z}[i]$.

Putting together Proposition 4, Corollary 6, and Lemma 11 we obtain

**Proposition 5.** Let $s_F$ be a block additive function taking its values in the compact abelian metrisable group $A$, let $\alpha_F$ be the corresponding cocycle defined by (5.5), and assume that $A(F) = A$. Then the skew product $T^{a_F}$ is uniquely ergodic and more precisely, for all $\mu \otimes \lambda_A$-continuous maps $f : X \times A \to \mathbb{C}$,
\[ \lim_{n \to \infty} \frac{1}{\#Q_n} \sum_{z \in Q_n} f \circ T_{z}^{a_F}(x, g) = \int_{X \times A} f \, d(\mu \otimes \lambda_A) \]
uniformly in $(x, g) \in K_q$.

**Corollary 7.** Let $s_F$ be a real-valued block additive function which attains an irrational value. Then $(s_F(z))_{z \in \mathbb{Z}[i]}$ is well uniformly distributed modulo 1 with respect to any Følner sequence $(Q_n)_{n \in \mathbb{N}}$, i.e.,
\[ \lim_{n \to \infty} \frac{1}{\#Q_n} \#\{z \in Q_n : \{s_F(z + y)\} \in I\} = \lambda(I) \]
for every interval $I \subset [0, 1]$ ($\{\cdot\}$ denotes the fractional part), uniformly in $y \in \mathbb{Z}[i]$.

**Proof.** The assumption that $s_F$ attains an irrational value clearly implies that $V(a_F \pmod{1}) = \mathbb{R}/\mathbb{Z}$. By Weyl’s criterion (cf. [20]) the assertion is equivalent to
\[ \forall k \in \mathbb{Z} \setminus \{0\} : \lim_{n \to \infty} \frac{1}{\#Q_n} \sum_{z \in Q_n} e(ks_F(z + y)) = 0 \]
uniformly in $y \in \mathbb{Z}[i]$. The points $(y, 0)$ are uniformly generic for $T^{a_F}$ by Proposition 5. Now, by definition of $T^{a_F}$ we have
\[ T_z^{a_F}(y, 0) = (y + z, a_F(z, y)) = (y + z, s_F(y + z) - s_F(y)) \pmod{1}. \]
Genericity of $(y, 0)$ implies
\[ \lim_{n \to \infty} \frac{1}{\#Q_n} \left| \sum_{z \in Q_n} \chi_0 \otimes e_k(T_z^{a_F}(y, 0)) \right| = \lim_{n \to \infty} \frac{1}{\#Q_n} \left| \sum_{z \in Q_n} e(ks_F(y + z)) \right| = 0, \]
where $\chi_0$ denotes the trivial character of $K_q$ and $e_k(\cdot) = e(k \cdot)$. The convergence is uniform in $y \in \mathbb{Z}[i]$.

**Corollary 8.** Let $s_F$ be an integer-valued block additive function. Then for any integer $M \geq 2$ for which there exists a value $s_F(z)$ that is coprime to $M$ the sequence $(s_F(z))_{z \in \mathbb{Z}[i]}$ is well uniformly distributed in residue classes modulo $M$ with respect to any Følner sequence $(Q_n)_{n \in \mathbb{N}}$, i.e.

$$\lim_{n \to \infty} \frac{1}{\#Q_n} \# \{ z \in Q_n : s_F(z + y) \equiv m \pmod{M} \} = \frac{1}{M}$$

for $m \in \{0, 1, \ldots, M - 1\}$, uniformly in $y \in \mathbb{Z}[i]$.

**Proof.** After observing that $V(a_F \pmod{M}) = \mathbb{Z}/M\mathbb{Z}$, the proof runs along the same lines as the proof of Corollary 7. ■

**References**


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