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On the distribution of kth powers of integral quaternions

by

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1. Introduction and statement of the main results. Concerning the distribution of kth powers of Gaussian integers (with $k \in \mathbb{N}$, $k \geq 2$), H. Müller and W. G. Nowak [7] showed that, as $X \to \infty$,

$$\#\{z^k \mid z \in \mathbb{Z}[i] \wedge |\mathrm{Re}(z^k)|, |\mathrm{Im}(z^k)| \leq X\} = \nu_k X^{2/k} + O(X^{46/(73k)+\varepsilon}),$$

where ν_k $(k = 2, 3, \ldots)$ are certain numerical constants. In [3] and [5] we studied three natural generalizations of this distribution question for $k = 2$ to integral quaternions, i.e. members of the Hurwitz subring $\mathbb{J} = \mathbb{Z}^4 \cup \left(\frac{1}{2} + \mathbb{Z}\right)^4$ of the division ring $\mathbb{H}$ of Hamilton's quaternions. Further, in [4] and [6] we investigated four natural questions concerning the distribution of squares of integral Cayley numbers.

The aim of the present paper is to treat the general case $k \geq 2$ for the special domain $\{q \in \mathbb{H} \mid |\mathrm{Re}(q)|, |\mathrm{Im}(q)| \leq X\} = [-X, X] \times \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 \leq X^2\}$, which corresponds to the easiest of the three distribution questions considered in [3] and [5].

In this connection one main problem is the irregular behaviour of the multiplicity of kth powers of integral quaternions. As an instance of the strange multiplicity of fourth powers we note that

$$\#\{q \in \mathbb{J} \mid q^4 = (5,3,4,0)^4\} = 60, \quad \#\{q \in \mathbb{J} \mid q^4 = (1,1,0,0)^4\} = 12,$$
$$\#\{q \in \mathbb{J} \mid q^4 = (0,0,0,1)^4\} = 8, \quad \#\{q \in \mathbb{J} \mid q^4 = (1,1,1,0)^4\} = 2.$$

Fortunately, these examples are an exception rather than the rule. Actually, for $k \geq 2$ let $\mathcal{S}_k$ be the smallest subset of $\mathbb{J}$ such that if $q_1 \in \mathbb{J} \setminus \mathcal{S}_k$, $q_2 \in \mathbb{J}$, and $q_1^k = q_2^k$ then $q_1 = q_2$ when k is odd and $q_1 = \pm q_2$ when k is even. This exceptional set $\mathcal{S}_k$ turns out to be relatively small when k is odd (in fact, $\mathcal{S}_k = \emptyset$ when $k \equiv \pm 1 \pmod 6$), whilst $\mathcal{S}_k$ is the union of a relatively small set and $\mathbb{J} \cap \{0\} \times \mathbb{R}^3 \setminus \{(0,0,0,0)\}$ when k is even.

In order to get these exceptional sets $\mathcal{S}_k$ under control it is appropriate to distinguish between a *distribution* problem and a *lattice point* problem.

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In the following we are going to develop asymptotic formulae for

$$\mathrm{A}_k(X) := \#\{q^k \mid q \in \mathbb{J} \wedge |\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X\} \tag{1.1}$$

and

$$\mathrm{B}_k(X) := \#\{q \in \mathbb{J} \mid |\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X\}. \tag{1.2}$$

In case $k = 2$ the distribution result has already been settled in [3] and it reads

$$\mathrm{A}_2(X) = 2\pi X^2 - \frac{2\pi}{3}\, X^{3/2} + O(X^{7/6}(\log X)^{19/4}) \quad (X \to \infty), \tag{1.3}$$

while it is easy to adapt the proof of this result in order to get

$$\mathrm{B}_2(X) = 4\pi X^2 + O(X^{7/6}(\log X)^{19/4}) \quad (X \to \infty). \tag{1.4}$$

Hence we may assume $k \ge 3$ throughout the paper. Now, the main results of the present paper are the following two theorems:

THEOREM 1. *For every natural $k \ge 3$ and positive real X let $\mathrm{B}_k(X)$ be defined by* (1.2). *Then, as $X \to \infty$,*

$$\mathrm{B}_k(X) = c_k X^{4/k} + O(X^{5/(2k)}),$$

where c_k $(k = 3, 4, 5, \ldots)$ are numerical constants with $\pi^2 < c_k < 2^{2/k}\pi^2$. More precisely,

$$c_k := \frac{4\pi}{3} \cdot \mathrm{area}\{u + iv^3 \in \mathbb{C} \mid u, v \in \mathbb{R} \wedge (u + iv)^k \in [-1, 1] + i[-1, 1]\}.$$

(*The O-constant depends on k.*)

THEOREM 2. *For every natural $k \ge 3$ and positive real X let $\mathrm{A}_k(X)$ be defined by* (1.1) *and $\mathrm{B}_k(X)$ be defined by* (1.2) *and $\nu(k) := (1 + (-1)^k)/2$. Then, as $X \to \infty$,*

$$\mathrm{A}_k(X) = \frac{1}{1 + \nu(k)}\, \mathrm{B}_k(X) - \nu(k)\, \frac{2\pi}{3}\, X^{3/k} + O(X^{2/k+\varepsilon}).$$

(*The O-constant depends on k and ε.*) *Moreover, if k is odd and not divisible by* 3 *then $\mathrm{A}_k(X) = \mathrm{B}_k(X)$ for every X.*

2. Raising a quaternion to the kth power. After having identified the quaternion algebra $\mathbb{H}$ with $\mathbb{R}^4$ we may identify the real space $\mathbb{R} \times \{0\}^3$ with $\mathbb{R}$ and the imaginary space $\mathrm{Im}\,\mathbb{H} := \{0\} \times \mathbb{R}^3$ with $\mathbb{R}^3$. Then every quaternion q has an unique representation $q = a + \vec{v}$ with $a \in \mathbb{R}$ and $\vec{v} \in \mathrm{Im}\,\mathbb{H}$. The number $a =: \mathrm{Re}(q)$ is the *real part* and the vector $\vec{v} =: \mathrm{Im}(q)$ is the *imaginary part* of the quaternion q. Further, addition and multiplication of two quaternions are defined formally with respect to $\vec{v}\lambda = \lambda\vec{v}$ and $\vec{v} \cdot \vec{w} = -\langle \vec{v}, \vec{w} \rangle + \vec{v} \times \vec{w}$, where $\langle \vec{v}, \vec{w} \rangle$ is the standard scalar product and $\vec{v} \times \vec{w}$ is the standard vector product in $\mathbb{R}^3$. Obviously, for every $q \in \mathbb{H} \setminus \mathbb{R}$ there are exactly two possibilities to write q as $a + b\vec{e}$, where $a, b \in \mathbb{R}$ and $\vec{e} \in \mathbb{R}^3$ is a *unit vector*, i.e. the Euclidean norm $|\vec{e}|$ is equal to 1. Now consider two quaternions with collinear imaginary parts, $a + b\vec{e}$ and $c + d\vec{e}$, where $\vec{e}$ is a unit vector in $\mathbb{R}^3$. Since $\vec{e} \cdot \vec{e} = -1$, we obviously have, doubtless a déjà vu,

$$(a + b\vec{e}) \cdot (c + d\vec{e}) = (ac - bd) + (ad + bc)\vec{e},$$

whence $\mathbb{R} + \mathbb{R}\vec{e}$ is a subalgebra of $\mathbb{H}$ and $a + b\vec{e} \mapsto a + bi$ settles a canonical isomorphism from the subalgebra $\mathbb{R} + \mathbb{R}\vec{e}$ to the field $\mathbb{C} = \mathbb{R} + \mathbb{R}i$ for every fixed unit vector $\vec{e} \in \mathbb{R}^3$. Therefore we immediately obtain the following lemma.

Lemma 1. *If any quaternion* (*including reals*) *is written as* $a + b\vec{e}$ *with* $a, b \in \mathbb{R}$ *and* $\vec{e} \in \mathbb{R}^3$, *where* $|\vec{e}| = 1$, *then for every* $k \in \mathbb{N}$,

$$(a + b\vec{e})^k = \Re((a + bi)^k) + \Im((a + bi)^k)\vec{e},$$

where $\Re(z)$ *is the real part and* $\Im(z)$ *is the imaginary part of* $z \in \mathbb{C}$.

Corollary 1. *Suppose that* π *is a permutation on* $\{1, 2, 3\}$, *i.e.* $\{\pi(1), \pi(2), \pi(3)\} = \{1, 2, 3\}$. *Further, for* $q = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbb{H}$, *define* $\pi[q] := (\alpha_0, \alpha_{\pi(1)}, \alpha_{\pi(2)}, \alpha_{\pi(3)})$. *Then* $(\pi[q])^k = \pi[q^k]$ *for all* $q \in \mathbb{H}$ *and* $k \in \mathbb{N}$.

Lemma 1 is the clue to get the asymptotic formulae for $\mathrm{A}_k(X)$ and $\mathrm{B}_k(X)$ because on the one hand it enables us to compare kth powers, and on the other hand an immediate consequence of Lemma 1 reads

$$|\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X \quad \text{iff} \quad (\mathrm{Re}(q) + |\mathrm{Im}(q)|i)^k \in [-X, X] + i[-X, X] \subset \mathbb{C}.$$

Thus, if for $X \ge 1$ and $k \in \mathbb{N}$ a domain $\mathcal{D}_k(X) \subset \mathbb{R}^2$ is defined via

$$\mathcal{D}_k(X) := \{(x, y) \in \mathbb{R}^2 \mid |\Re((x + yi)^k)|, |\Im((x + yi)^k)| \le X\} \tag{2.1}$$

and the domain $\mathcal{B}_k(X) \subset \mathbb{H}$ is given by

$$\mathcal{B}_k(X) := \{q \in \mathbb{H} \mid |\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X\}, \tag{2.2}$$

so that $\mathrm{B}_k(X) = \#(\mathcal{B}_k(X) \cap \mathbb{J})$, then for every $q \in \mathbb{H}$,

$$q \in \mathcal{B}_k(X) \quad \text{iff} \quad (\mathrm{Re}(q), |\mathrm{Im}(q)|) \in \mathcal{D}_k(X). \tag{2.3}$$

Further we claim that for every $q \in \mathbb{H}$ and $X \ge 1$,

$$|\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X \ \Rightarrow\ |\mathrm{Re}(q)|, |\mathrm{Im}(q)| \le |q| \le 2^{1/(2k)} X^{1/k}. \tag{2.4}$$

This is certainly true because $|\mathrm{Re}(q^k)|^2 + |\mathrm{Im}(q^k)|^2 = |q^k|^2 = (|q|^k)^2 = |q|^{2k}$.

We conclude this section with basic facts on kth powers of pure imaginary quaternions.

Lemma 2. *Suppose that* $q \in \mathrm{Im}\,\mathbb{H}$. *Then* $q^k \in \mathbb{R}$ *when* k *is even*, *and* $q^k \in \mathrm{Im}\,\mathbb{H}$ *when* k *is odd. More precisely*,

(i) $q^k = (-1)^{k/2}|q|^k$ *when* k *is even*,
(ii) $q^k = (-1)^{(k-1)/2}|q|^{k-1}q$ *when* k *is odd.*

Proof. For $q \in \operatorname{Im}\mathbb{H}$ we have $q^2 = -|q|^2$. Now we only have to look at $q^k = (q^2)^{k/2}$ when k is even and at $q^k = q^{k-1}q$ when k is odd.

As an immediate consequence of Lemma 2 we obtain

COROLLARY 2. *For every $q \in \operatorname{Im}\mathbb{H}$ and $X \geq 1$,*

$$|\operatorname{Re}(q^k)|, |\operatorname{Im}(q^k)| \leq X \quad \textit{iff} \quad |q| \leq X^{1/k}.$$

3. On the multiplicity of kth powers of integral quaternions

LEMMA 3. *Let $q_1, q_2 \in \mathbb{J}$ be such that $q_1^k = q_2^k \notin \mathbb{R}$.*

(i) *If k is odd and $|\operatorname{Im}(q_1)|, |\operatorname{Im}(q_2)| \notin \frac{\sqrt{3}}{2} \cdot \mathbb{Z}$ then $q_1 = q_2$.*

(ii) *If $k \equiv \pm 1 \pmod 6$ then $q_1 = q_2$.*

(iii) *If k is even and $|\operatorname{Im}(q_1)|, |\operatorname{Im}(q_2)| \notin \frac{\sqrt{3}}{2} \cdot \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$ then $q_1 = q_2$ or $q_1 = -q_2$.*

Proof. Let q_1 and q_2 be two integral quaternions with $q_1^k = q_2^k$ and $\operatorname{Im}(q_1^k) = \operatorname{Im}(q_2^k) \neq 0$. Then Lemma 1 tells us that the vectors $\operatorname{Im}(q_1)$ and $\operatorname{Im}(q_2)$ must be collinear, so that we can write $q_1 = a_1 + b_1\vec{e}$ and $q_2 = a_2 + b_2\vec{e}$ with *one* unit vector $\vec{e} \in \mathbb{R}^3$. Due to $q_1, q_2 \in \mathbb{J}$ we have $2a_1, 2a_2 \in \mathbb{Z}$ and $2b_1\vec{e}, 2b_2\vec{e} \in \mathbb{Z}^3$, whence we can write $2q_1 = \alpha_1 + \beta_1\sqrt{d_1}\vec{e}$, $2q_2 = \alpha_2 + \beta_2\sqrt{d_2}\vec{e}$ with $\alpha_i, \beta_i \in \mathbb{Z}$ and $d_1, d_2 \in \mathbb{N}$, each d_i either squarefree or equal to 1. Although the multiplication in $\mathbb{H}$ is not commutative, $q_1^k = q_2^k$ is clearly equivalent to $(2q_1)^k = (2q_2)^k$. Consequently, by Lemma 1 we have $\Im((\alpha_1 + \beta_1 i\sqrt{d_1})^k) = \Im((\alpha_2 + \beta_2 i\sqrt{d_2})^k)$. Hence, by the binomial theorem, $n_1\sqrt{d_1} = n_2\sqrt{d_2}$ for some $n_1, n_2 \in \mathbb{Z}$, whence $d_1 = d_2 =: d$. It now follows from Lemma 1 that $(\alpha_1 + \beta_1 i\sqrt{d})^k = (\alpha_2 + \beta_2 i\sqrt{d})^k$ and thus the number $\zeta = (\alpha_1 + \beta_1 i\sqrt{d})/(\alpha_2 + \beta_2 i\sqrt{d})$ is both a kth root of unity and a member of the quadratic field $\mathbb{Q}[\sqrt{-d}]$, where either $d = 1$ or d is squarefree. Now we distinguish the cases of k even and odd.

First assume that k is odd. Then either $\zeta = 1$, i.e. $q_1 = q_2$, or $d = 3$. This finishes the proof of (i) because $d = 3$ implies $|\operatorname{Im}(q_1)|, |\operatorname{Im}(q_2)| \in \frac{\sqrt{3}}{2} \cdot \mathbb{Z}$. When k is odd and not divisible by 3, then ζ cannot be a third or sixth root of unity $\neq 1$, whence in case $d = 3$ we must have $\zeta = 1$ as well. This proves clause (ii).

Now assume that k is even. Due to $|\operatorname{Im}(q_1)|, |\operatorname{Im}(q_2)| \notin \frac{\sqrt{3}}{2} \cdot \mathbb{Z} \cup \frac{1}{2}\mathbb{Z}$ we must have $d \neq 1, 3$ and therefore the kth root of unity ζ lies in a quadratic field different from $\mathbb{Q}[i]$ and $\mathbb{Q}[\sqrt{-3}]$. So we must have $\zeta = 1$ or $\zeta = -1$, and this concludes the proof of Lemma 2.

REMARK. When $k \equiv 3 \pmod 6$ and $d = 3$ then the multiplicity need not equal 3. For example, $(-2, 7, -1, 5)$ is the unique solution of the equation $q^3 = (442, -441, 63, -315)$ in $\mathbb{J}$. On the other hand, the equation $q^3 =$

$(442000, -441000, 63000, -315000)$ has the three solutions

$$q_1 = (-20, 70, -10, 50), \quad q_2 = (-65, -49, 7, -35), \quad q_3 = (85, -21, 3, -15).$$

As an obvious consequence of the proof of Lemma 3 we obtain

COROLLARY 3. *If $a \in \mathbb{H} \setminus \mathbb{R}$ then among all solutions $x \in \mathbb{H}$ of the equation $x^k = a$, whose total number is k and which can easily be computed by Lemma 1, there are at most six integral quaternions.*

LEMMA 4. *Let $q_1, q_2 \in \operatorname{Im} \mathbb{H}$ (not necessarily integral) with $q_1^k = q_2^k$ and k odd. Then $q_1 = q_2$.*

Proof. Recall that $|q^n| = |q|^n$ is always true, and apply Lemma 2(ii).

LEMMA 5. *If $k \equiv \pm 1 \pmod 6$ then the map $q \mapsto q^k$ is injective on $\mathbb{J}$.*

Proof. In view of Lemma 3(ii) it suffices to show that for every $q \in \mathbb{J}$ we have $q^k \in \mathbb{R}$ if and only if $q \in \mathbb{R}$. Hence, in view of Lemma 1 it is enough to show that for $m, n \in \mathbb{Z}$ and $n \geq 0$ we have $(m + \sqrt{n} i)^k \in \mathbb{R}$ if and only if $n = 0$, which is clearly true because $(m + \sqrt{n} i)^k \in \mathbb{R}$ implies $(m - \sqrt{n} i)^k = \overline{(m + \sqrt{n} i)^k} = (m + \sqrt{n} i)^k$, and a kth root of unity in any quadratic field (or $\mathbb{Q}$) cannot be different from 1 when $k \equiv \pm 1 \pmod 6$.

REMARK. As a consequence of Lemma 5 we have $\mathrm{A}_k(X) = \mathrm{B}_k(X)$ for every $X > 0$ when $k \equiv \pm 1 \pmod 6$, so that the last statement of Theorem 2 is true.

4. The contribution of the imaginary space. In the following we make use of the arithmetic functions d and r_2 and r_3 given by

$$\begin{aligned} d(n) &:= \#\{m \in \mathbb{N} \mid m \mid n\} \quad (n \in \mathbb{N}), \\ r_2(n) &:= \#\{(x, y) \in \mathbb{Z}^2 \mid x^2 + y^2 = n\} \quad (n \in \mathbb{Z}), \\ r_3(n) &:= \#\{(x, y, z) \in \mathbb{Z}^3 \mid x^2 + y^2 + z^2 = n\} \quad (n \in \mathbb{Z}). \end{aligned}$$

(Notice that $r_2(0) = r_3(0) = 1$ and $r_2(n) = r_3(n) = 0$ for $n < 0$.) It is well known (cf. [9, p. 38 and p. 102]) that

$$d(n), r_2(n) \ll n^{\varepsilon} \quad (n \to \infty), \tag{4.1}$$

whence

$$r_3(n) = \sum_{m \in \mathbb{Z}} r_2(n - m^2) \ll n^{1/2 + \varepsilon} \quad (n \to \infty). \tag{4.2}$$

Further (cf. [1, Cor. 4]),

$$\#\{n \in \mathbb{N}_0 \mid n \leq t \wedge r_3(n) > 0\} = \frac{5}{6} t + O(\log t) \quad (t \to \infty). \tag{4.3}$$

Finally, it is well known (cf. [9]) that for

$$R_3(t) := \sum_{0\le n\le t} r_3(n),$$

so that $R_3(x) = \#\{(x,y,z)\in\mathbb{Z}^3 \mid x^2+y^2+z^2 \le t\} = \#\{\vec{x}\in\mathbb{Z}^3 \mid |\vec{x}| \le \sqrt{t}\}$,

$$R_3(t) = \frac{4\pi}{3}\, t^{3/2} + O(t^{2/3}(\log t)^6) \quad (t\to\infty). \tag{4.4}$$

The given O-term is not best possible but good enough for our purpose. We also deal with

$$\widetilde{R}_3(t) := \#\big\{(x,y,z)\in\big(\tfrac12+\mathbb{Z}\big)^3 \,\big|\, x^2+y^2+z^2\le t\big\},$$

for which, as argued in [3], we also have

$$\widetilde{R}_3(t) = \frac{4\pi}{3}\, t^{3/2} + O(t^{2/3}(\log t)^6) \quad (t\to\infty). \tag{4.5}$$

Proposition 1. *Let* $\mathcal{B}_k(X)$ *be as in* (2.2) *and* $\mathrm{A}_k^\circ(X) := \#\{q^k \mid q\in \mathcal{B}_k(X)\cap\mathbb{J}\cap\mathrm{Im}\,\mathbb{H}\}$ *and* $\mathrm{B}_k^\circ(X) := \#(\mathcal{B}_k(X)\cap\mathbb{J}\cap\mathrm{Im}\,\mathbb{H})$. *Then for every* $k\ge 3$ *and* $X\ge 1$,

$$\text{(i)}\qquad \mathrm{B}_k^\circ(X) = \frac{4\pi}{3}\, X^{3/k} + O(X^{4/(3k)}\log^6 X) \quad (X\to\infty).$$

Further,

(ii) *if* k *is odd then* $\mathrm{A}_k^\circ(X) = \mathrm{B}_k^\circ(X)$,
(iii) *if* k *is even then* $\mathrm{A}_k^\circ(X) = \frac56 X^{2/k} + O(\log X)$ $(X\to\infty)$.

Proof. First note that $\mathbb{J}\cap\mathrm{Im}\,\mathbb{H} = \mathbb{Z}^4\cap\mathrm{Im}\,\mathbb{H}$. Hence, (i) follows from Corollary 2 and (4.4). Further, (ii) is an immediate consequence of Lemma 4. Finally, if k is even then for $q\in\mathbb{J}\cap\mathrm{Im}\,\mathbb{H}$, $q^k = (-|q|^2)^{k/2}$ with $-|q|^2 = q^2\in\mathbb{Z}$ (notice that $\mathbb{Z}^4$ is a subring of $\mathbb{J}$). Hence we obviously have $\mathrm{A}_k^\circ(X) = \#\{m^{k/2}\in[0,X] \mid m\in\mathbb{N}_0\wedge r_3(m)>0\}$, so that (iii) follows from (4.3).

5. Proof of Theorem 2. Let $\mathcal{B}_k(X)$ be as in (2.2) and

$$\mathrm{B}_k^*(X) := \#(\mathcal{B}_k(X)\cap\mathbb{J}\setminus\mathrm{Im}\,\mathbb{H}),$$
$$\mathrm{A}_k^*(X) := \#\{q^k \mid q\in\mathcal{B}_k(X)\cap\mathbb{J}\setminus\mathrm{Im}\,\mathbb{H}\}.$$

Then, with $\mathrm{A}_k^\circ(X)$ and $\mathrm{B}_k^\circ(X)$ as in Proposition 1, for every $k\ge 3$ and $X\ge 1$ we have

$$\mathrm{A}_k(X) = \mathrm{A}_k^*(X) + \mathrm{A}_k^\circ(X), \qquad \mathrm{B}_k(X) = \mathrm{B}_k^*(X) + \mathrm{B}_k^\circ(X).$$

Now, in view of Proposition 1, Theorem 2 is an immediate consequence of the following proposition.

PROPOSITION 2. *For natural $k \geq 3$ let $\nu(k) := (1+(-1)^k)/2$. Then, as $X \to \infty$,*

$$\mathrm{A}_k^*(X) = \frac{1}{1+\nu(k)}\,\mathrm{B}_k^*(X) + O(X^{2/k+\varepsilon}).$$

Proof. Obviously, as $X \to \infty$,

$$\mathrm{A}_k^*(X) - \frac{1}{1+\nu(k)}\,\mathrm{B}_k^*(X) \ll 1 + \mathrm{E}_k(X) + \#\mathcal{F}_k(X),$$

where $\mathrm{E}_k(X)$ and $\mathcal{F}_k(X)$ are given by

$$\mathrm{E}_k(X) := \#\{q \in \mathbb{J} \setminus \operatorname{Im}\mathbb{H} \mid |\mathrm{Re}(q^k)| \leq X \wedge \operatorname{Im}(q^k) = 0\}$$

and

$$\mathcal{F}_k(X) := \{q \in \mathbb{J} \setminus \operatorname{Im}\mathbb{H} \mid |\mathrm{Re}(q^k)|, |\operatorname{Im}(q^k)| \leq X \wedge \operatorname{Im}(q^k) \neq 0 \wedge \\ \exists q_1 \in \mathbb{J} \setminus (\operatorname{Im}\mathbb{H} \cup \{q, -q\}) : q_1^k = q^k\}.$$

(Notice that $\mathcal{F}_k(X) = \emptyset$ for all X when $k \equiv \pm 1, \pm 2, \pm 5 \pmod{12}$.)

First we estimate $\mathrm{E}_k(X)$, so that we count all integral quaternions $q \notin \operatorname{Im}\mathbb{H}$ with $q^k \in [-X, X]$. By (2.4) there are $\ll X^{1/k}$ possible values for $\mathrm{Re}(q)$. Further, by Lemma 1, for a quaternion q with $\operatorname{Im}(q^k) = 0$ we have $(\mathrm{Re}(q) + |\operatorname{Im}(q)|i)^k \in \mathbb{R}$, whence we certainly have

$$(\mathrm{Re}(q) + |\operatorname{Im}(q)|i) \in \bigcup_{m=1}^{k} \mathbb{R} \cdot e^{2\pi i m/k}.$$

Consequently, since $\mathrm{Re}(q) \neq 0$, for every choice of $\mathrm{Re}(q)$ there are at most k choices for $|\operatorname{Im}(q)| = \frac{1}{2}\sqrt{m}, m \in \mathbb{N}_0$, each one to be multiplied by a factor $\ll r_3(m)$. By (2.4) and (4.2), all these factors are uniformly $\ll X^{1/k+\varepsilon}$. Consequently, $\mathrm{E}_k(X) \ll X^{2/k+\varepsilon}$.

Next we investigate the set $\mathcal{F}_k(X)$. If $q \in \mathcal{F}_k(X)$ then there is a $q_1 \in \mathcal{F}_k(X)$ such that $q_1 \neq \pm q$ and $q_1^k = q^k \notin \mathbb{R}$. Consequently, the two vectors $\operatorname{Im}(q)$ and $\operatorname{Im}(q_1)$ are both non-zero and, by Lemma 1, they are collinear, whence q and q_1 are both members of the two-dimensional subalgebra $\mathcal{S}(q) := \mathbb{R} + \mathbb{R} \cdot \operatorname{Im}(q)$ of the quaternion algebra $\mathbb{H}$. Since $\mathcal{S}(q)$ is isomorphic to $\mathbb{C}$ we have $qq_1 = q_1 q$, whence $\varrho := q_1/q$ is a *rational* quaternion, i.e. $\varrho \in \mathbb{Q}^4$, with $\varrho \neq \pm 1$ and $\varrho^k = 1$. Further, by Lemma 3, we either have $|\operatorname{Im}(q)| \in \frac{\sqrt{3}}{2} \cdot \mathbb{Z}$ or $|\operatorname{Im}(q)| \in \frac{1}{2}\mathbb{Z}$. Hence $2q, 2q_1 \in \mathbb{Z} + \sqrt{d}\,\mathbb{Z} \cdot \vec{e} \subset \mathcal{S}(q)$, where $\vec{e} := \operatorname{Im}(q)/|\operatorname{Im}(q)|$ is a unit vector, and either $d = 3$ or $d = 1$. Obviously, $\mathbb{Z} + \sqrt{d}\,\mathbb{Z} \cdot \vec{e}$ is a ring which is isomorphic to the order $\mathbb{Z}[\sqrt{-d}]$, whose quotient field equals the imaginary quadratic field $\mathbb{Q}[\sqrt{-d}]$. Consequently, the rational quaternion ϱ lies in the field $\mathbb{Q} + \sqrt{d}\,\mathbb{Q} \cdot \vec{e}$ because in its turn this field is isomorphic to $\mathbb{Q}[\sqrt{-d}]$. Since ϱ corresponds to a kth root of unity $\neq \pm 1$ in the field $\mathbb{Q}[\sqrt{-d}]$, we must have $\varrho = \pm\frac{1}{2} \pm \frac{\sqrt{3}}{2} \cdot \vec{e}$ when $d = 3$ and $\varrho = \pm\vec{e}$ when $d = 1$. On the other hand, $\operatorname{Im}(\varrho) = \lambda \operatorname{Im}(q)$ for some $\lambda \in \mathbb{R}$. Clearly, $\lambda \in \mathbb{Q}$ since

$\varrho \in \mathbb{Q}^4$ and $q \in \mathbb{J}$. Now let $q = (\alpha_0, \alpha_1, \alpha_2, \alpha_3)$ so that $\alpha_i \in \frac{1}{2}\mathbb{Z}$. Notice that $\alpha_0 \neq 0$ and, in view of (2.4), $2|\alpha_i| \leq 3X^{1/k}$ $(i = 0,1,2,3)$. Then, $\varrho = \lambda(q - \alpha_0) \pm (d-1)/4$. Further, by applying the well-known equation $q^2 = 2\,\mathrm{Re}(q)q - |q|^2$ we derive

$$\varrho q = \lambda(q^2 - \alpha_0 q) \pm \frac{d-1}{4}\, q = \left(\lambda\alpha_0 \pm \frac{d-1}{4}\right) q - \lambda|q|^2.$$

Due to $\varrho q = q_1 \in \mathbb{J}$ we have

$$\mathrm{Im}(\varrho q) = \left(\left(\lambda\alpha_0 \pm \frac{d-1}{4}\right)\alpha_1, \left(\lambda\alpha_0 \pm \frac{d-1}{4}\right)\alpha_2, \left(\lambda\alpha_0 \pm \frac{d-1}{4}\right)\alpha_3\right) \in \frac{1}{2}\,\mathbb{Z}^3,$$

whence $4\lambda\alpha_0\alpha_i \in \mathbb{Z}$ for $i = 1,2,3$. Now, due to $|\mathrm{Im}(\varrho)| = |\lambda|\,|\mathrm{Im}(q)|$ and $|\mathrm{Im}(q)| = \frac{\sqrt{d}}{2} s$ $(s \in \mathbb{N})$ and $|\mathrm{Im}(\varrho)| = \sqrt{3}/2$ when $d = 3$ and $|\mathrm{Im}(\varrho)| = 1$ when $d = 1$, we have either $\lambda = \pm 1/s$ or $\lambda = \pm 2/s$ where $s \in \mathbb{N}$ and either $s \mid 4\alpha_0\alpha_i$ $(i = 1,2,3)$ or $s \mid 8\alpha_0\alpha_i$ $(i = 1,2,3)$. Moreover, $0 \neq (2\alpha_1)^2 + (2\alpha_2)^2 + (2\alpha_3)^2 = 4|\mathrm{Im}(q)|^2 = ds^2$. Finally, referring to Corollary 1 it is obvious that for every permutation π on $\{1,2,3\}$ we have $\pi[q] \in \mathcal{F}_k(X)$ if and only if $q \in \mathcal{F}_k(X)$. Summing up, for every $q \in \mathcal{F}_k(X)$ there is a permutation π on $\{1,2,3\}$ and a natural number s such that $\pi[q] = (a/2, b/2, u/2, v/2)$ with $a,b,u,v \in \mathbb{Z}$ and $0 < |a|, |b| \leq 3X^{1/k}$ and $s \mid 2ab$ and $u^2 + v^2 = ds^2 - b^2$ for $d = 1$ or $d = 3$.

Therefore, by symmetry and by Corollary 3 and by taking all permutations π into account, the total number of all integral quaternions which are distinguished as members of the set $\mathcal{F}_k(X)$ is certainly not greater than

$$144 \sum_{d \in \{1,3\}} \sum_{0 < a \leq 3X^{1/k}} \sum_{0 < b \leq 3X^{1/k}} \sum_{s \mid 2ab} r_2(ds^2 - b^2),$$

so that by (4.1) we obtain $\#\mathcal{F}_k(X) \ll X^{2/k+\varepsilon}$ and the proof is finished.

6. Preparation of the main proof. Throughout the paper, for the sake of simplicity we make the following arrangement which perhaps seems artificial but actually is standard in axiomatic set theory (cf. [8]).

Arrangement. *Any real function is identified with its graph, so that* $f \subset \mathbb{R}^2$ *is a real function if and only if for every* $x \in \mathbb{R}$ *the set* $\{x\} \times \mathbb{R} \cap f$ *equals either the empty set or a singleton* $\{(x,y)\}$ *with* $y \in \mathbb{R}$. *The set* $\mathrm{dom}(f) := \{x \mid \exists y : (x,y) \in f\}$ *is the domain of* f. *We write* $f : A \to \mathbb{R}$ *iff* f *is a real function and* $A = \mathrm{dom}(f)$. *If* $x \in \mathrm{dom}(f)$ *then* $f(x)$ *equals the unique* $y \in \mathbb{R}$ *such that* $(x,y) \in f$. *But if* $x \in \mathbb{R} \setminus \mathrm{dom}(f)$ *we set* $f(x) = 0$ (1).

(1) Admittedly, this appointment seems strange because it has the apparently contradictory consequence that $f(x)$ is always *defined* for every $x \in \mathbb{R}$ although possibly $\mathrm{dom}(f) \neq \mathbb{R}$! We lay emphasis on the fact that this appointment is an immediate consequence of set-theoretical standard definitions. Indeed, in classical set theory without

Particularly, $\emptyset$ *is a real function with* $\mathrm{dom}(\emptyset) = \emptyset$ *whence* $\emptyset(x) = 0$ *for all* $x \in \mathbb{R}$.

In order to prove Theorem 1 we will make use of (2.3), so that first we have to look carefully at the domain $\mathcal{D}_k(X)$ given by (2.1). Obviously, $\mathcal{D}_k(X)$ can be obtained by applying a homothetic dilatation to the basic domain $\mathcal{D}_k(1)$. More precisely, $\mathcal{D}_k(X) = X^{1/k} \cdot \mathcal{D}_k(1)$. Consequently, in the following we are going to collect important facts on the basic domain

$$\mathcal{D}_k := \mathcal{D}_k(1) = \{(x, y) \in \mathbb{R}^2 \mid |\Re((x + yi)^k)|, |\Im((x + yi)^k)| \leq 1\}. \tag{6.1}$$

After having identified $\mathbb{R}^2$ with $\mathbb{C}$, the boundary $\partial\mathcal{D}_k$ of $\mathcal{D}_k$ can be parametrized via $\varrho(\theta)e^{i\theta}$ $(0 \leq \theta < 2\pi)$, where

$$\varrho(\theta) := (\max\{|\cos(k\theta)|, |\sin(k\theta)|\})^{-1/k} \qquad (\theta \in \mathbb{R}). \tag{6.2}$$

Obviously, $\varrho(\theta)$ is periodic with minimal period $\pi/(2k)$. Further, $1 \leq \varrho(\theta) \leq 2^{1/(2k)}$ with $\varrho(\theta) = 1$ iff $\theta \in \frac{\pi}{2k}\mathbb{Z}$, and $\varrho(\theta) = 2^{1/(2k)}$ iff $\theta \in \frac{\pi}{4k} + \frac{\pi}{2k}\mathbb{Z}$. The function $\theta \mapsto \varrho(\theta)$ is everywhere continuous, infinitely differentiable on $\mathbb{R} \setminus \left(\frac{\pi}{4k} + \frac{\pi}{2k}\mathbb{Z}\right)$, and both the right and left derivatives exist everywhere. If we set

$$\theta_n := \frac{\pi}{4k} + n \cdot \frac{\pi}{2k} \qquad (n = 0, 1, 2, \ldots, 4k - 1), \tag{6.3}$$

then the points $P_n := (2^{1/(2k)} \cos(\theta_n), 2^{1/(2k)} \sin(\theta_n))$ are cuspidal points of the curve $\partial\mathcal{D}_k$. Except for these $4k$ cuspidal points the curve is smooth. Clearly, the domain $\mathcal{D}_k$ is axially and centrally symmetric with respect to both axes and the origin of the coordinate system.

Now, as a representative segment of $\partial\mathcal{D}_k$ we consider the arc $\varrho(\theta)e^{i\theta}$ where $-\pi/(4k) \leq \theta \leq \pi/(4k)$.

A direction vector of the tangent of the curve through the point $\varrho(\theta)e^{i\theta}$ is given by

$$\vec{t}(\theta) := \begin{pmatrix} \sin((k-1)\theta) \\ \cos((k-1)\theta) \end{pmatrix} \qquad (-\pi/(4k) \leq \theta \leq \pi/(4k)). \tag{6.4}$$

Here we may include the two cuspidal points at $|\theta| = \pi/(4k)$ since it is natural to speak of *two* tangents through every cuspidal point of the whole curve $\partial\mathcal{D}_k$. (By the definition of $\mathcal{D}_k$, these two tangents are always orthogonal because the mapping $z \mapsto z^k$ is conformal.)

urelements it is common to define the number zero as the empty set and to define $a(b) := \{c \mid (\exists d : c \in d \wedge (b, d) \in a) \wedge (\forall d_1, d_2 : (b, d_1), (b, d_2) \in a \Rightarrow d_1 = d_2)\}$ for arbitrary sets a, b. Hence, if f and x are sets such that there is not a unique set y with $(x, y) \in f$ then $f(x) = \emptyset$ automatically. As an example of the utility of that appointment consider the function $f :]0, \infty[\to \mathbb{R}$, $x \mapsto 1/x$. On the one hand we can say that f is continuous, on the other hand we can make use of the equation $f(0) = 0$.

The absolute value of the curvature $\kappa(\theta)$ at the point $\varrho(\theta)e^{i\theta}$, which again has a natural meaning at the two cuspidal points at $|\theta| = \pi/(4k)$, is given by

$$|\kappa(\theta)| = (k-1)(\cos(k\theta))^{1+1/k} \qquad (-\pi/(4k) \le \theta \le \pi/(4k)). \tag{6.5}$$

Since $|\kappa(-\pi/(4k))| = |\kappa(\pi/(4k))| = (k-1)2^{-1/2-1/(2k)}$ the value $|\kappa(\theta)|$ is well defined on the whole curve $\varrho(\theta)e^{i\theta}$ $(0 \le \theta < 2\pi)$ and we have

$$(k-1)/2 \le |\kappa(\theta)| \le k-1 \qquad (0 \le \theta \le 2\pi). \tag{6.6}$$

In order to get tangent vectors at every point of $\partial\mathcal{D}_k$ we consider the rotation ϕ given by

$$\mathbb{R}^2 \ni \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \phi\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) := \begin{pmatrix} \cos\left(\frac{\pi}{2k}\right) & -\sin\left(\frac{\pi}{2k}\right) \\ \sin\left(\frac{\pi}{2k}\right) & \cos\left(\frac{\pi}{2k}\right) \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix}$$

or, equivalently, by $\mathbb{C} \ni z \mapsto \phi(z) := e^{\pi i/(2k)} \cdot z$, so that $\phi^n(\mathcal{D}_k) = \mathcal{D}_k$ for every $n \in \mathbb{N}$. Then, with

$$\mathcal{C}_k^n := \{\varrho(\theta)e^{i\theta} \mid \theta_{n-1} \le \theta \le \theta_n\} \qquad (n \in \mathbb{N}), \tag{6.7}$$

so that $\bigcup_{n=0}^{4k-1} \mathcal{C}_k^n = \partial\mathcal{D}_k$, we have $\phi^n(\mathcal{C}_k^0) = \mathcal{C}_k^n$ for every $n \in \mathbb{N}$.

Now, referring to (6.4), it is plain that a tangent vector at any point of each arc $\mathcal{C}_k^n$ is given by $\vec{t}(\theta) = \phi^n\big(\vec{t}\big(\theta - n\frac{\pi}{2k}\big)\big)$. Since we always have

$$\begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}^n \cdot \begin{pmatrix} \sin\beta \\ \cos\beta \end{pmatrix} = \begin{pmatrix} \sin(\beta - n\alpha) \\ \cos(\beta - n\alpha) \end{pmatrix},$$

a complete set of tangent vectors of $\partial\mathcal{D}_k$ is given by

$$\vec{t}(\theta) = \begin{pmatrix} \sin\big((k-1)\theta - n\frac{\pi}{2}\big) \\ \cos\big((k-1)\theta - n\frac{\pi}{2}\big) \end{pmatrix} \qquad (\theta_{n-1} \le \theta \le \theta_n), \tag{6.8}$$

where $n = 0, 1, \ldots, 4k-1$. Of course this notation is a bit sloppy because we have a unique $\vec{t}(\theta)$ only if $\theta \notin \frac{\pi}{4k} + \frac{\pi}{2k}\mathbb{Z}$, but *two* tangent vectors at each cuspidal point P_n $(n = 0, 1, \ldots, 4k-1)$.

Referring to (6.8) it is easy to determine all points $Q \in \partial\mathcal{D}_k$ where the tangent through Q is parallel to the y-axis. A point $Q = \varrho(\theta)e^{i\theta}$ with $\theta_{n-1} \le \theta \le \theta_n$ for $0 \le n \le 4k-1$ has a vertical tangent if and only if $\sin\big((k-1)\theta - n\frac{\pi}{2}\big) = 0$.

Because of symmetry it suffices to consider only the points Q in the first quadrant $[0,\infty[^2$, so that we consider only the arcs $\mathcal{C}_k^n$ for $0 \le n \le k$. Obviously, there is no vertical tangent point $Q \in \mathcal{D}_k$ lying in the sector $\pi/4 \le \theta \le \pi/2$ with the only exception of the cuspidal point $(2^{1/(2k)-1/2}, 2^{1/(2k)-1/2})$ at $\theta = \pi/4$ if k is odd. (For even k there is no cuspidal point at $\theta = \pi/4$.) Hence we may restrict the indices n to $0 \le n \le k/2$. Now, the arc $\mathcal{C}_k^n$ contains a vertical tangent point at $\theta \in [\theta_{n-1}, \theta_n]$ if and only if $(k-1)\theta - n\pi/2 \in \pi\mathbb{Z}$.

We observe that

$$[(k-1)\theta_{n-1} - n\pi/2, (k-1)\theta_n - n\pi/2] \cap \pi\mathbb{Z} = \{0\}$$

for $0 \le n \le (k-1)/2$, whilst for $n = k/2$ in case that k is even,

$$[(k-1)\theta_{n-1} - n\pi/2, (k-1)\theta_n - n\pi/2] \cap \pi\mathbb{Z} = \emptyset.$$

Consequently, precisely for $n = 0, 1, \ldots, [(k-1)/2]$ there is exactly one vertical tangent point Q_n on the arc $\mathcal{C}_k^n$, and the coordinates of $Q_n = (u_n, v_n)$ are

$$u_n = \varrho\left(\frac{n}{k-1}\frac{\pi}{2}\right)\cos\left(\frac{n}{k-1}\frac{\pi}{2}\right), \quad v_n = \varrho\left(\frac{n}{k-1}\frac{\pi}{2}\right)\sin\left(\frac{n}{k-1}\frac{\pi}{2}\right). \tag{6.9}$$

Further, a vertical tangent point Q_n is a cuspidal point if and only if k is odd and $n = (k-1)/2$. If we consider the x-coordinates $\xi_n := 2^{1/(2k)}\cos(\theta_n)$ of the cuspidal points P_n, then trivially $\xi_0 > \xi_1 > \ldots > \xi_{k-1}$. For obvious geometrical reasons we have $u_{n-1} > u_n$ for $1 \le n \le (k-1)/2$ and $\xi_{n-1} > \xi_n \ge u_n$ for $n = 1, 2, \ldots, [(k-1)/2]$ with $\xi_n = u_n$ only for $n = (k-1)/2$ in case k is odd.

Now, considering all arcs $\mathcal{C}_k^n$ $(n = 0, 1, \ldots, k)$ we define functions φ_n and ψ_n by

$$\varphi_n := \begin{cases} \mathcal{C}_k^n \cap \mathbb{R} \times [v_n, \infty[& \\ & \text{when } n \in \{0, 1, \ldots, [(k-1)/2]\} \setminus \{(k-1)/2\}, \\ \emptyset & \text{when } (k-1)/2 \le n \le k, \end{cases} \tag{6.10}$$

$$\begin{gathered} \psi_n := \begin{cases} \mathcal{C}_k^n \cap \mathbb{R} \times]-\infty, v_n] & \text{when } n \in \{1, \ldots, [(k-1)/2]\}, \\ \mathcal{C}_k^n & \text{when } (k-1)/2 < n < k, \end{cases} \\ \psi_k := \mathcal{C}_k^k \cap [0, \infty[^2, \qquad \psi_0 := \emptyset. \end{gathered} \tag{6.11}$$

If $\varphi_n \ne \emptyset$ then $\mathrm{dom}(\varphi_n) = [u_n, \xi_n]$ and the function φ_n is continuous and strictly increasing with a continuous and strictly decreasing derivative on $]u_n, \xi_n[$. For $n \in \{1, \ldots, [(k-1)/2]\}$ we have $\mathrm{dom}(\psi_n) = [u_n, \xi_{n-1}]$ and the function ψ_n is continuous and strictly decreasing with a continuous and strictly increasing derivative on $]u_n, \xi_{n-1}[$. In all these cases the derivatives φ_n' and ψ_n' are unbounded near u_n, $\psi_n(x) < \varphi_n(x)$ for $u_n < x \le \xi_n$ and $\psi_n(u_n) = \varphi_n(u_n)$. For $(k-1)/2 < n < k$ we have $\mathrm{dom}(\psi_n) = [\xi_n, \xi_{n-1}]$ and the function ψ_n is continuous, the union of a decreasing and an increasing function, and its derivative is continuous, strictly increasing and bounded on $]\xi_n, \xi_{n-1}[$. Finally, $\mathrm{dom}(\psi_k) = [0, \xi_{k-1}]$ and the function ψ_k is continuous and strictly decreasing with a continuous, strictly increasing and bounded derivative on $]0, \xi_{k-1}[$. As an obvious consequence we have

$$\begin{aligned} \psi_{n+1}(x) > \varphi_n(x) \quad & \text{for } u_n \le x < \xi_n \text{ and} \\ & n \in \{0, 1, \ldots, [(k-1)/2]\} \setminus \{(k-1)/2\}, \end{aligned} \tag{6.12}$$

since for all these n, $\varphi_n(\xi_n) = \psi_{n+1}(\xi_n)$.

By definition, $\mathcal{C}_k^n = \varphi_n \cup \psi_n$ for $0 < n < k$, $\mathcal{C}_k^0 \cap [0,\infty[^2 = \varphi_0$ and $\mathcal{C}_k^k \cap [0,\infty[^2 = \psi_k$.

Consequently,

$$\partial\mathcal{D}_k \cap [0,\infty[^2 = \bigcup_{n=0}^{k} \psi_n \cup \varphi_n.$$

For every $n = 0, 1, \ldots, k-1$ we have $\mathrm{dom}(\varphi_n) \subset \mathrm{dom}(\psi_{n+1})$, whence, referring to (6.12) and our arrangement on real functions,

$$\psi_{n+1}(x) \geq \varphi_n(x) \quad \text{for all } n = 0, 1, \ldots, k-1 \text{ and all } x \in \mathbb{R}. \tag{6.13}$$

Hence we can write

$$\mathcal{D}_k \cap [0,\infty[^2 = \bigcup_{0\leq x\leq \xi_0} \bigcup_{n=0}^{k-1} \{x\} \times [\varphi_n(x), \psi_{n+1}(x)]. \tag{6.14}$$

We conclude this section with a proposition on the order of magnitude of the derivatives and the difference quotients of the functions φ_n and ψ_n whenever they are unbounded.

PROPOSITION 3. *For every $n = 0, 1, \ldots, [(k-1)/2]$ where $\varphi_n \neq \emptyset$ and $\psi_n \neq \emptyset$, respectively, for sufficiently small positive λ we have*

$$\text{(i)} \qquad \varphi_n'(u_n+\lambda),\ \psi_n'(u_n+\lambda) \ll \frac{1}{\sqrt{\lambda}} \quad (\lambda \to 0^+).$$

Moreover, we always have

$$\text{(ii)} \qquad \frac{\varphi_n(u_n+\lambda)-\varphi_n(u_n)}{\lambda},\ \frac{\psi_n(u_n+\lambda)-\psi_n(u_n)}{\lambda} \ll \frac{1}{\sqrt{\lambda}} \quad (\lambda \to 0^+).$$

Proof. We consider the circle through the vertical tangent point $Q_n = (u_n, v_n)$ given by $\{(x,y) \in \mathbb{R}^2 \mid (x-(u_n+r))^2 + (y-v_n)^2 = r^2\}$, whence the circle tangent through Q_n is vertical as well. Referring to (6.5) we fix the radius r of our circle independent of n and large enough so that the circle encloses not only the osculating circle of the curve $\mathcal{C}_k^n$ through Q_n but also the curve $\mathcal{C}_k^n$ itself. In particular, the two arcs φ_n and ψ_n lie within our circle. Consequently, for small $\lambda > 0$,

$$0 < \varphi_n'(u_n+\lambda) < \frac{\varphi_n(u_n+\lambda)-v_n}{\lambda} < \frac{\mathit{\Delta}(\lambda)}{\lambda},$$
$$0 > \psi_n'(u_n+\lambda) > \frac{\psi_n(u_n+\lambda)-v_n}{\lambda} > -\frac{\mathit{\Delta}(\lambda)}{\lambda},$$

where $\mathit{\Delta}(\lambda)$ is the vertical distance between the point $(u_n+\lambda, v_n)$ and our circle. We have $\mathit{\Delta}(\lambda) = \sqrt{r^2-(r-\lambda)^2} = \sqrt{2r\lambda-\lambda^2} < \sqrt{2r}\sqrt{\lambda}$ and this concludes the proof.

7. Proof of Theorem 1

Notation. Let f be a real function whose domain $\mathrm{dom}(f)$ is a bounded subset of $\mathbb{R}$. Then for $a, b \in \mathbb{R}$ we set

$$\sum_{a<m\le b}^{*} f(m) := \sum_{2a<m\le 2b} f\left(\frac{m}{2}\right),$$

so that the dummy index in a "star sum" always runs through $\frac{1}{2}\mathbb{Z}$. Further, this sum is always well defined (and finite), even when $]a,b] \not\subset \mathrm{dom}(f)$. (Recall our arrangement that $f(x) = 0$ for any $x \notin \mathrm{dom}(f)$.)

For "star sums" the Euler summation formula (cf. [2, Theorem 1.3]) appears in the following shape.

LEMMA 6. *Let τ denote the row-of-teeth function given by $\tau(x) = 2x - [2x] - 1/2$ $(x \in \mathbb{R})$. Further let f be a real function with* $\mathrm{dom}(f) = [\alpha,\beta] \subset \mathbb{R}$ *such that f is continuous on $[\alpha,\beta]$ and continuously differentiable on $]\alpha,\beta[$. Then*

$$\sum_{\alpha<m\le\beta}^{*} f(m) = 2\int_{\alpha}^{\beta} f(x)\,dx + \tau(\alpha)f(\alpha) - \tau(\beta)f(\beta) + \int_{\alpha}^{\beta} f'(x)\tau(x)\,dx.$$

By applying four times the second mean-value theorem and by making use of the estimate $|\int_a^b \tau(x)\,dx| \le 1/4$ we obtain the following lemma.

LEMMA 7. *Let f and g be real functions with* $\mathrm{dom}(f) = \mathrm{dom}(g) = [\alpha,\beta] \subset \mathbb{R}$ *such that f and g are continuous, g is monotonic, and f is the union of two monotonic functions. Then*

$$\left|\int_{\alpha}^{\beta} f(x)g(x)\tau(x)\,dx\right| \le 2(\max_{\alpha\le x\le\beta}|f(x)|)(\max_{\alpha\le x\le\beta}|g(x)|).$$

Now we are ready to prove Theorem 1. In view of (5.1) and Proposition 1 it remains to look carefully at $\mathrm{B}_k^*(X)$. By symmetry and referring to (2.3) we have

$$\begin{aligned} (7.1)\quad & \mathrm{B}_k^*(X) \\ &= 2\cdot\#\big\{(a,\vec{b}) \in \mathbb{N}\times\mathbb{Z}^3 \cup \left(\tfrac{1}{2}+\mathbb{N}_0\right)\times\left(\tfrac{1}{2}+\mathbb{Z}\right)^3 \,\big|\, (a,|\vec{b}|) \in \mathcal{D}_k(X)\big\}.\end{aligned}$$

Now for $a \in \frac{1}{2}\mathbb{Z}$ and $Y \ge 0$ let

$$\begin{aligned} L_a(Y) &:= \#\{\vec{b} \in (a+\mathbb{Z})^3 \mid |\vec{b}| \le Y\}, \\ L_a^{\circ}(Y) &:= \#\{\vec{b} \in (a+\mathbb{Z})^3 \mid |\vec{b}| < Y\}. \end{aligned}$$

Then, by (4.2), (4.4) and (4.5), for every $a \in \frac{1}{2}\mathbb{Z}$,

$$L_a(Y), L_a^{\circ}(Y) = \frac{4\pi}{3}Y^3 + O(1+Y^{7/5}) \tag{7.2}$$

independently of a and uniformly in Y. (The exponent $7/5$ has been chosen as a house number greater than $4/3$ in order to get rid of the logarithmic factor.)

Recall that $\mathcal{D}_k(X) = X^{1/k} \cdot \mathcal{D}_k$. Consequently, referring to (6.14), we have, for $X \geq 1$,

$$\mathcal{D}_k(X) \cap [0,\infty[^2 = \bigcup_{0\leq x\leq \xi_0} \bigcup_{n=0}^{k-1} \{xX^{1/k}\} \times [X^{1/k}\varphi_n(x), X^{1/k}\psi_{n+1}(x)]$$

and thus

$$\begin{aligned}\frac{1}{2}\,\mathrm{B}_k^*(X) = {} & \sum_{0<a\leq \xi_{k-1}X^{1/k}}^{*} L_a(X^{1/k}\psi_k(aX^{-1/k})) \\ & + \sum_{n=[(k+1)/2]}^{k-1} \sum_{\xi_n X^{1/k}<a\leq \xi_{n-1}X^{1/k}}^{*} L_a(X^{1/k}\psi_n(aX^{-1/k})) \\ & + \sum_{n=1}^{[(k-1)/2]} \sum_{u_n X^{1/k}<a\leq \xi_{n-1}X^{1/k}}^{*} L_a(X^{1/k}\psi_n(aX^{-1/k})) \\ & - \sum_{n=0}^{[(k-1)/2]} \sum_{u_n X^{1/k}<a\leq \xi_{n}X^{1/k}}^{*} L_a^{\circ}(X^{1/k}\varphi_n(aX^{-1/k})).\end{aligned}$$

(Notice that $\varphi_{[(k-1)/2]} = \emptyset$ when k is odd.)

By applying (7.2) and Lemma 6, after an obvious substitution we derive

$$\mathrm{B}_k^*(X) = c_k X^{4/k} + \frac{8\pi}{3}\,X^{3/k}T_k(X) + 8\pi X^{2/k}J_k(X) + O(X^{12/(5k)}), \tag{7.3}$$

where (with respect to our arrangement on real functions)

$$c_k := \frac{16\pi}{3}\sum_{n=1}^{k}\int_0^{\xi_0}(\psi_n(t)^3 - \varphi_{n-1}(t)^3)\,dt, \tag{7.4}$$

$$\begin{aligned}J_k(X) := \sum_{n=1}^{k}\int_0^{\xi_0 X^{1/k}} & (\psi_n(uX^{-1/k})^2\psi_n'(uX^{-1/k}) \\ & - \varphi_{n-1}(uX^{-1/k})^2\varphi_{n-1}'(uX^{-1/k}))\tau(u)\,du,\end{aligned}$$

$$T_k(X) := \sum_{n=1}^{k}(f_n(X) - g_{n-1}(X))$$

with

$$\begin{aligned}f_n(X) := {} & \tau(X^{1/k}\cdot \inf\mathrm{dom}(\psi_n))\cdot\psi_n(\inf\mathrm{dom}(\psi_n))^3 \\ & - \tau(X^{1/k}\cdot \sup\mathrm{dom}(\psi_n))\cdot\psi_n(\sup\mathrm{dom}(\psi_n))^3\end{aligned}$$

for $n = 1, \ldots, k$ and

$$g_m(X) := \tau(X^{1/k} \cdot \inf \mathrm{dom}(\varphi_m)) \cdot \varphi_m(\inf \mathrm{dom}(\varphi_m))^3 - \tau(X^{1/k} \cdot \sup \mathrm{dom}(\varphi_m)) \cdot \varphi_m(\sup \mathrm{dom}(\varphi_m))^3$$

for $m \in \{0, 1, \ldots, [(k-1)/2]\} \setminus \{(k-1)/2\}$ and $g_m(X) = 0$ otherwise.

Obviously and fortunately all terms occurring in $T_k(X)$ are annihilated except the first summand of $f_k(X)$, so that we obtain

$$T_k(X) = \tau(0)\psi_k(0)^3 = -1/2. \tag{7.5}$$

It remains to estimate $J_k(X)$ and we claim

$$J_k(X) \ll X^{1/(2k)} \quad (X \to \infty). \tag{7.6}$$

In order to verify (7.6) let $h \in \{\psi_n \mid n=1, \ldots, k\} \cup \{\varphi_n \mid n=0, 1, \ldots, k\} \setminus \{\emptyset\}$ with $\mathrm{dom}(h) = [\sigma, \omega]$. If h' is bounded we have

$$\int\limits_{\sigma X^{1/k}}^{\omega X^{1/k}} h(uX^{-1/k})^2 h'(uX^{-1/k}) \tau(u)\, du \ll 1$$

by Lemma 7. If h' is unbounded near σ we choose X large enough so that $\sigma X^{1/k} + 1 \le \omega X^{1/k}$, whence we can write

$$\int\limits_{\sigma X^{1/k}}^{\omega X^{1/k}} h(uX^{-1/k})^2 h'(uX^{-1/k}) \tau(u)\, du = \int\limits_{\sigma X^{1/k}}^{\sigma X^{1/k}+1} h^2 h' \tau + \int\limits_{\sigma X^{1/k}+1}^{\omega X^{1/k}} h^2 h' \tau.$$

From Proposition 3(ii) with $\lambda = X^{-1/k}$ we derive

$$\begin{aligned} \Big| \int\limits_{\sigma X^{1/k}}^{\sigma X^{1/k}+1} & (h^2 h')(uX^{-1/k}) \tau(u)\, du \Big| \\ & \le \big(\max_{\sigma \le x \le \omega} h(x)^2 \big) \cdot \int\limits_{\sigma X^{1/k}}^{\sigma X^{1/k}+1} |h'(uX^{-1/k})|\, du \\ & = \big(\max_{\sigma \le x \le \omega} h(x)^2 \big) \cdot X^{1/k} \cdot |h(\sigma + X^{-1/k}) - h(\sigma)| \ll X^{1/(2k)}. \end{aligned}$$

From Lemma 7 and Proposition 3(i) with $\lambda = X^{-1/k}$ we derive

$$\begin{aligned} \Big| \int\limits_{\sigma X^{1/k}+1}^{\omega X^{1/k}} & (h^2 h')(uX^{-1/k}) \tau(u)\, du \Big| \\ & \le 2 \cdot \big(\max_{\sigma \le x \le \omega} h(x)^2 \big) \cdot |h'(\sigma + X^{-1/k})| \ll X^{1/(2k)}. \end{aligned}$$

Now we insert (7.4), (7.5) and (7.6) into (7.3), so that by (5.1) and Proposition 1 we obtain the asymptotic formula of Theorem 1; hence it remains to verify the statements on the constants c_k in Theorem 1. The stated area formula for c_k is equivalent to

$$(7.7)\qquad c_k := \frac{16\pi}{3}\cdot \operatorname{area}\{(u,v^3)\in\mathbb{R}^2 \mid (u,v)\in\mathcal{D}_k\cap[0,\infty[^2\},$$

which is an obvious consequence of (7.4). Now, the domain $\mathcal{D}_k$ contains the unit circle $x^2+y^2=1$ and lies within the circle $x^2+y^2=2^{1/k}$. Therefore, by (7.7) we surely have

$$c_k > \frac{16\pi}{3}\int_0^1 (\sqrt{1-x^2})^3\,dx = \pi^2,$$

$$c_k < \frac{16\pi}{3}\int_0^{2^{1/(2k)}} (\sqrt{2^{1/k}-x^2})^3\,dx = 2^{2/k}\pi^2,$$

and this concludes the proof of Theorem 1.

8. On the distribution of kth powers of Cayley integers. Considering hypercomplex numbers as members of certain quadratic algebras (cf. [4]), it is plain that Lemma 1 remains unchanged when the quaternions are replaced by hypercomplex numbers, i.e. when the unit vector $\vec{e}\in\mathbb{R}^3$ is replaced by a unit vector $\vec{e}\in\mathbb{R}^{s-1}$, where s is the order of the algebra. Consequently, it is not difficult to adapt the proof of Theorem 1 to derive an asymptotic formula for the number of hypercomplex integers of order s whose kth powers lie in the cylinder $[-X,X]\times\{\vec{x}\in\mathbb{R}^{s-1}\mid |\vec{x}|\le X\}$. Since the concept of *real* and *imaginary part* is common only for complex numbers, quaternions and octaves, instead of formulating an s-dimensional analogue (2) of Theorem 2 we give the corresponding distribution formula for the special case of the Cayley algebra $\mathbb{O}=\mathbb{R}^8$ and its integral domain $\Gamma=\mathbb{Z}^8$.

THEOREM 3. *For natural $k\ge 2$ and positive real X let*

$$\mathrm{C}_k(X) := \#\{a^k \mid a\in\Gamma\wedge|\mathrm{Re}(a^k)|,|\mathrm{Im}(a^k)|\le X\}.$$

Then, as $X\to\infty$,

$$\mathrm{C}_k(X) = \frac{2}{3+(-1)^k}\,d_k X^{8/k} - \frac{4(1+(-1)^k)\pi^3}{105}\,X^{7/k} + O(X^{13/(2k)}),$$

where

$$d_k := \frac{8\pi^3}{105}\cdot\operatorname{area}\{u+iv^7\in\mathbb{C}\mid u,v\in\mathbb{R}\wedge(u+iv)^k\in[-1,1]+i[-1,1]\},$$

(2) Notice that it is not necessary that the hypercomplex algebra is a *division* algebra. If a_1 and a_2 are hypercomplex integers such that $a_1^k=a_2^k\notin\mathbb{R}$ then a_1 and a_2 lie in a two-dimensional subalgebra which is isomorphic to the complex number field. Hence a_1/a_2 is well-defined and $(a_1/a_2)^k=1$.

whence we always have

$$\frac{1}{24}\,\pi^4 < d_k < \frac{2^{4/k}}{24}\,\pi^4.$$

REMARK. As shown in [4, Theorem 2], in the case $k = 2$ the error term is even $O(X^3)$. Further, $d_2 = \frac{2}{9}\pi^3$.

9. On the distribution of all powers of integral quaternions. In this final section we give an asymptotic formula for the number of all quaternions p such that $|\mathrm{Re}(p)|, |\mathrm{Im}(p)| \le X$, and p can be written as $p = q^k$, where q is any integral quaternion and $k \ge 2$ is any natural number.

THEOREM 4. *For positive real X let*

$$\mathrm{P}(X) := \#\{p \in \mathbb{H} \mid |\mathrm{Re}(p)|, |\mathrm{Im}(p)| \le X \wedge \exists k \in \mathbb{N}, k \ge 2, \exists q \in \mathbb{J} : p = q^k\}.$$

Then, as $X \to \infty$,

$$\mathrm{P}(X) = 2\pi X^2 - \frac{2\pi}{3}\,X^{3/2} + c_3 X^{4/3} + O(X^{7/6}(\log X)^{19/4}),$$

where $c_3 = 11.53735238\ldots$ is the same constant as in Theorem 1.

Proof. First, we note that for $X \ge 3$ and $q \in \mathbb{H}$,

$$|q| \ge 2 \wedge |\mathrm{Re}(q^k)|, |\mathrm{Im}(q^k)| \le X \;\Rightarrow\; k \le 2\log X, \tag{9.1}$$

which follows immediately from $|q|^{2k} = |q^k|^2 = |\mathrm{Re}(q^k)|^2 + |\mathrm{Im}(q^k)|^2$. Further it is easy to check

$$\#\{q \in \mathbb{J} \mid |q| < 2\} = 145. \tag{9.2}$$

Now let

$$\mathcal{P}_{2,3}(X) := \{p \in \mathbb{H} \mid |\mathrm{Re}(p)|, |\mathrm{Im}(p)| \le X \wedge \exists q_1, q_2 \in \mathbb{J} : p = q_1^2 = q_2^3\}.$$

Then by (9.1) and (9.2) we have

$$\mathrm{P}(X) = \mathrm{A}_2(X) + \mathrm{A}_3(X) - \#\mathcal{P}_{2,3}(X) + O\Big(1 + \sum_{5 \le k \le 2\log X} \mathrm{A}_k(X)\Big),$$

because every fourth power in $\mathbb{J}$ is already counted as a square. Thus, in view of (1.3) and Theorems 1 and 2, the proof of Theorem 4 is finished by showing that

$$\#\mathcal{P}_{2,3}(X) \ll X^{5/6+\varepsilon} \qquad (X \to \infty). \tag{9.3}$$

The story would be quite simple if, as one might expect, $\#\mathcal{P}_{2,3}(X) = \mathrm{A}_6(X)$. But unfortunately this is not true. In the world of whole numbers or Gaussian integers a number is both a square and a third power if and only if it is a sixth power. In the world of integral quaternions the situation is different. For example, the integral quaternion $(-1917, 2646, 378, 756)$ is the square of

$(27, 49, 7, 14)$ and also the cube of $(3, -14, -2, -4)$, but it cannot be written as the sixth power of an integral quaternion.

Indeed, the six solutions of the equation $x^6 = (-1917, 2646, 378, 756)$ are

$$\pm\Big(-3, \frac{7}{3}, \frac{1}{3}, \frac{2}{3}\Big), \qquad \pm\Big(\frac{3\sqrt{2}}{2} \mp \frac{3}{2}, \frac{7\sqrt{2}}{4} \pm \frac{7}{6}, \frac{\sqrt{2}}{4} \pm \frac{1}{6}, \frac{\sqrt{2}}{2} \pm \frac{1}{3}\Big).$$

Thus we will not get (9.3) without investigating the set $\mathcal{P}_{2,3}(X)$ carefully.

It suffices to consider only those $p \in \mathcal{P}_{2,3}(X)$ where $\mathrm{Im}(p) \neq 0$ because the total number of all $p \in \mathcal{P}_{2,3}(X)$ with $\mathrm{Im}(p) = 0$ is not greater than $2X^{1/3} + 1$. In fact, if $q \in \mathbb{J}$ is such that $\mathrm{Im}(q^3) = 0$ (and hence $q^3 \in \mathbb{Z}$) then either $q \in \mathbb{Z}$ or $|\mathrm{Im}(q)| = |\mathrm{Re}(q)|\sqrt{3}$ by Lemma 1. Therefore, either $q \in \mathbb{Z}$ or $q^3 = -8\,\mathrm{Re}(q)^3$, whence we always have $q^3 = n^3$ for some $n \in \mathbb{Z}$.

Now suppose that p is a quaternion with $\mathrm{Im}(p) \neq 0$ and $p = q_1^2 = q_2^3$ for some $q_1, q_2 \in \mathbb{J}$. Then, by Lemma 1, the vectors $\mathrm{Im}(q_1)$ and $\mathrm{Im}(q_2)$ must be collinear, whence $q_1 q_2 = q_2 q_1$. (Note that this is not true when $\mathrm{Im}(p) = 0$. For example, if we choose $q_1 = (0, 1, 0, 0)$ and $q_2 = \big(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\big)$ then $q_1^2 = q_2^3 = -1$, but $q_1 q_2 \neq q_2 q_1$.) Consequently, $(q_1/q_2)^2 = q_2$ and $(q_1/q_2)^3 = q_1$ and $(q_1/q_2)^6 = p$. Since q_1/q_2 is clearly a rational quaternion, in order to verify (9.3) it is enough to count all $\varrho \in \mathbb{Q}^4$ such that

$$\varrho^2 \in \mathbb{J} \setminus \mathbb{R} \quad \text{and} \quad |\mathrm{Re}(\varrho^6)|, |\mathrm{Im}(\varrho^6)| \leq X. \tag{9.4}$$

Now, for $\alpha = (a_0, a_1, a_2, a_3) \in \mathbb{H}$ let $N(\alpha) := a_0^2 + a_1^2 + a_2^2 + a_3^2$ denote the *norm* of the quaternion α, so that we always have $N(\alpha) = |\alpha|^2$ and $N(\alpha^n) = N(\alpha)^n$ for every $n \in \mathbb{N}$. Further, $N(q) \in \mathbb{Z}$ for every $q \in \mathbb{J}$, so that, considering $\mathbb{Z}$ as a subring of $\mathbb{J}$, we have $N(q) \in \mathbb{J}$ for every $q \in \mathbb{J}$.

If ϱ is a rational quaternion which satisfies (9.4) then on the one hand we have $N(\varrho)^6 = N(\varrho^6) = \mathrm{Re}(\varrho^6)^2 + |\mathrm{Im}(\varrho^6)|^2 \leq 2X^2$, whence $4N(\varrho) \leq 5X^{1/3}$. On the other hand, due to $N(\varrho)^2 = N(\varrho^2)$ and $\varrho^2 \in \mathbb{J}$, we have $N(\varrho)^2 \in \mathbb{Z}$, so that $N(\varrho) \in \mathbb{Z}$ since $N(\varrho) \in \mathbb{Q}$. Moreover, from $\varrho^2 \in \mathbb{J} \setminus \mathbb{R}$ and the well known equation $\varrho^2 = 2\,\mathrm{Re}(\varrho)\varrho - N(\varrho)$ we derive $\mathrm{Re}(\varrho) \neq 0$ and $2\,\mathrm{Re}(\varrho)\varrho \in \mathbb{J}$. Hence, $2\,\mathrm{Re}(\varrho)^2 \in \frac{1}{2}\mathbb{Z}$, so that $2\,\mathrm{Re}(\varrho) \in \mathbb{Z} \setminus \{0\}$.

Hence it suffices to count all $\varrho \in \mathbb{Q}^4$ which satisfy $4N(\varrho) \leq 5X^{1/3}$ and

$$2\,\mathrm{Re}(\varrho), N(\varrho) \in \mathbb{Z} \setminus \{0\}, \qquad 4\,\mathrm{Re}(\varrho)\varrho \in \mathbb{Z}^4. \tag{9.5}$$

In order to do that we write $\varrho = (a/2, r_1/s, r_2/s, r_3/s)$ with $a, s, r_1, r_2, r_3 \in \mathbb{Z}$ and $a \neq 0$ and $s > 0$. In view of (9.5), we may restrict the denominator s so that $s \mid 2a$ and $s^2 \mid 4(r_1^2 + r_2^2 + r_3^2)$. To see this, let $r_i/s = m_i/n_i$ with m_i and n_i relatively prime. Then $4\,\mathrm{Re}(\varrho)\varrho \in \mathbb{Z}^4$ implies $n_i \mid 2am_i$, whence $s \mid 2a$ if s equals the least common multiple of n_1, n_2, n_3.

Consequently, in view of $a^2 \leq 4N(\varrho)$ and by applying (4.1) and (4.2), the total number of all $\varrho \in \mathbb{Q}^4$ which satisfy $4N(\varrho) \leq 5X^{1/3}$ and (9.5) is not greater than

$$2 \sum_{0<a\le 3X^{1/6}} \sum_{s|2a} \sum_{m\le 5X^{1/3}-a^2} r_3(ms^2)$$
$$\ll 3X^{1/6}\cdot(6X^{1/6})^{\varepsilon}\cdot 5X^{1/3}\cdot(5X^{1/3}\cdot(6X^{1/6})^2)^{1/2+\varepsilon} \ll X^{5/6+\varepsilon}.$$

This finishes the proof of (9.3). Note that the total number of all *integral* $\varrho \notin \operatorname{Im}\mathbb{H}$ which satisfy $4N(\varrho) \le 5X^{1/3}$ (and (9.5) anyway) is obviously $\asymp X^{2/3}$. Finally, by (7.7) and referring to Section 6, the constant c_3 is equal to

$$\frac{16\pi}{3}\int_{\pi/2}^{0} \frac{\sin^3 t}{\max\{|\cos(3\theta)|,|\sin(3\theta)|\}} \times\left(\frac{d}{dt}((\max\{|\cos(3\theta)|,|\sin(3\theta)|\})^{-1/3}\cos t)\right)dt,$$

so that, with electronic support, it is plain to calculate the numerical value of c_3.

Concerning the distribution of all powers of Cayley integers the situation is rather simple because the error term of the contribution coming from the squares dominates the contributions coming from all other powers. By applying Theorem 3 and [4, Theorem 2] we obtain the following result we conclude this paper with.

THEOREM 5. *For positive real X let*

$$\widetilde{\mathrm{P}}(X) := \#\{p\in\mathbb{O} \mid |\mathrm{Re}(p)|,|\mathrm{Im}(p)|\le X \wedge \exists k\in\mathbb{N}, k\ge 2, \exists q\in\Gamma : p=q^k\}.$$

Then, as $X\to\infty$,

$$\widetilde{\mathrm{P}}(X) = \frac{\pi^3}{9}X^4 - \frac{8\pi^3}{105}X^{7/2} + O(X^3).$$

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