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Semi-invariants of binary forms and identities for Bernoulli, Euler and Hermite polynomials

by

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1. Introduction. The relationship between the theory of group representations and special functions is well known (see [V]). In this paper we establish a relationship between the classical invariant theory and the Bernoulli, Euler and Hermite polynomials.

The polynomials of Bernoulli $B_n(x)$, Euler $E_n(x)$ and Hermite $H_n(x)$, $n = 0, 1, 2, \ldots$, are defined by the following generating functions:

$$\frac{te^{xt}}{e^t - 1} = \sum_{i=0}^{\infty} B_n(x)\frac{t^n}{n!}, \quad \frac{2e^{xt}}{e^t + 1} = \sum_{i=0}^{\infty} E_n(x)\frac{t^n}{n!}, \quad e^{xt-t^2/2} = \sum_{i=0}^{\infty} H_n(x)\frac{t^n}{n!}.$$

In particular $B_0(x) = E_0(x) = H_0(x) = 1$. The numbers $B_n := B_n(0)$ are called the *Bernoulli numbers* and the numbers $E_n := E_n(0)$ are called the *Euler numbers*. These polynomials are special cases of the Appell polynomials $\mathcal{A} = \{A_n(x)\}$ (see [YY]), where $\deg(A_n(x)) = n$ and the polynomials satisfy the identity

$$A'_n(x) = nA_{n-1}(x), \qquad n = 0, 1, 2, \ldots. \tag{1.1}$$

It is clear that $\{x^n\}$ are also Appell polynomials. Denote by $\mathcal{B}$, $\mathcal{E}$, $\mathcal{H}$ the systems of Bernoulli, Euler and Hermite polynomials, respectively. Also, put $\mathcal{T} := \{1, x, x^2, \ldots\}$.

We are interested in finding all polynomial identities for Appell polynomials, i.e. identities of the form

$$F(A_0(x), A_1(x), \ldots, A_n(x)) = 0,$$

where F is some polynomial of $n + 1$ variables.

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First, consider motivating examples. Put

$$\Delta(x) := \begin{vmatrix} B_0(x) & B_1(x) \\ B_1(x) & B_2(x) \end{vmatrix} = B_0(x)B_2(x) - B_1(x)^2.$$

Taking into account (1.1) we have

$$\begin{aligned} \Delta(x)' &= (B_0(x)B_2(x) - B_1(x)^2)' \\ &= B_0(x)'B_2(x) + B_0(x)B_2(x)' - 2B_0(x)B_1(x) \\ &= 2B_0(x)B_1(x) - 2B_0(x)B_1(x) = 0. \end{aligned}$$

Thus, $\Delta(x)$ is a constant, equal to $\Delta(0)$. Substituting the corresponding Bernoulli polynomial, we find this constant and get the identity

$$B_0(x)B_2(x) - B_1(x)^2 = B_0B_2 - B_1^2 = -\frac{1}{12}.$$

Similarly, $E_0(x)E_2(x) - E_1(x)^2 = -1/4$ and $H_0(x)H_2(x) - H_1(x)^2 = -1$.

Consider now the differential operator

$$\mathcal{D} := a_0\frac{\partial}{\partial a_1} + 2a_1\frac{\partial}{\partial a_2} + \cdots + na_{n-1}\frac{\partial}{\partial a_n},$$

which acts on polynomials of the variables $a_0, a_1, \ldots, a_n$. The action is very similar to (1.1). Also, it is easy to see that

$$\mathcal{D}\left(\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}\right) = \mathcal{D}(a_0a_2 - a_1^2) = 0.$$

Consider the polynomial

$$\Delta_3 := \begin{vmatrix} a_0 & 3a_1 & 3a_2 & a_3 & 0 \\ 0 & a_0 & 3a_1 & 3a_2 & a_3 \\ 3a_0 & 6a_1 & 3a_2 & 0 & 0 \\ 0 & 3a_0 & 6a_1 & 3a_2 & 0 \\ 0 & 0 & 3a_0 & 6a_1 & 3a_2 \end{vmatrix}.$$

Note that Δ_3 is, up to a factor, the discriminant of the binary form of degree 3,

$$a_0X^3 + 3a_1X^2Y + 3a_2XY^2 + a_3Y^3.$$

By applying the determinant derivative rule we find that $\mathcal{D}(\Delta_3)$ equals the sum of five determinants each of them equal to zero. Thus $\mathcal{D}(\Delta_3) = 0$.

Similarly, for the determinant

$$\Delta_3(\mathcal{A}) := \begin{vmatrix} A_0(x) & 3A_1(x) & 3A_2(x) & A_3(x) & 0 \\ 0 & A_0(x) & 3A_1(x) & 3A_2(x) & A_3(x) \\ 3A_0(x) & 6A_1(x) & 3A_2(x) & 0 & 0 \\ 0 & 3A_0(x) & 6A_1(x) & 3A_2(x) & 0 \\ 0 & 0 & 3A_0(x) & 6A_1(x) & 3A_2(x) \end{vmatrix}$$

we obtain $\Delta_3(\mathcal{A})' = 0$. Therefore, for any Appell polynomials $\{A_n(x)\}$ the identity

$$\Delta_3(\mathcal{A}) = \text{const}$$

holds. By direct calculations we obtain

$$\Delta_3(\mathcal{B}) = \frac{1}{16}, \qquad \Delta_3(\mathcal{E}) = \frac{27}{16}, \qquad \Delta_3(\mathcal{H}) = 108.$$

These examples lead to the hypothesis that if a polynomial $S(a_0, a_1, \dots, a_n)$ satisfies the condition $\mathcal{D}(S(a_0, a_1, \dots, a_n)) = 0$, then the polynomial

$$S(A_0(x), A_1(x), \dots, A_n(x))$$

is a constant, thus it determines an identity between Appell polynomials.

Let now $\mathbb{K}[a_0, a_1, \dots, a_n]$ and $\mathbb{K}[x]$ be the algebras of polynomials over a field $\mathbb{K}$ of characteristic zero. Consider the substitution homomorphism $\varphi_{\mathcal{A}} : \mathbb{K}[a_0, a_1, \dots, a_n] \to \mathbb{K}[x]$ defined by $\varphi_{\mathcal{A}}(a_i) = A_i(x)$. Put

$$\ker^* \varphi_{\mathcal{A}} := \{S \in \mathbb{K}[a_0, a_1, \dots, a_n] \mid \varphi_{\mathcal{A}}(S) \in \mathbb{K}\}.$$

We will prove that any element $S(a_0, a_1, \dots, a_n)$ of the subalgebra $\ker^* \varphi_{\mathcal{A}}$ yields the identity

$$S(\mathcal{A}) = S(\mathcal{A})_0,$$

where

$$S(\mathcal{A}) := S(A_0(x), A_1(x), \dots, A_n(x)),$$
$$S(\mathcal{A})_0 := S(A_0(0), A_1(0), \dots, A_n(0)).$$

Therefore, the problem of describing all polynomial identities for Appell polynomials is reduced to that of describing the algebra $\ker^* \varphi_{\mathcal{A}}$. It will be shown that $\ker^* \varphi_{\mathcal{A}}$ is isomorphic to the algebra of covariants of a binary form of order n.

This idea can also be applied to find identities for different types of Appell polynomials. For instance, we have

$$\begin{vmatrix} B_0(x) & E_0(x) & H_0(x) \\ B_1(x) & E_1(x) & H_1(x) \\ B_2(x) & E_2(x) & H_2(x) \end{vmatrix}$$
$$= B_0(x)E_1(x)H_2(x) - B_0(x)H_1(x)E_2(x) - B_1(x)E_0(x)H_2(x) + B_1(x)H_0(x)E_2(x) + B_2(x)E_0(x)H_1(x) - B_2(x)H_0(x)E_1(x) = \frac{1}{12}.$$

The problem of describing all such polynomial identities for different Appell polynomials is reduced to that of describing the algebra of joint covariants for several binary forms. The algebra of covariants of a binary form and the algebra of joint covariants for several binary forms were an object of research in the classical invariant theory of the 19th century. In particular, efficient methods of finding elements of those algebras were developed.

We will deal mainly with the algebra of semi-invariants rather than the algebra of covariants. These algebras are isomorphic, but the former is a simpler object for computation. The aim of this paper is to bring together two areas of mathematics—the classical invariant theory and the theory of special functions. In this paper we give a brief introduction to the theory of covariants and semi-invariants of a binary form in the language of locally nilpotent derivations. Based on the classical invariant theory approach we prove that any identity for Appell polynomials is determined by some semi-invariant. Also, several types of identities for Appell polynomials are constructed.

2. Covariants and semi-invariants of binary forms. Let us recall that a *derivation* of a ring R is an additive map D satisfying the Leibniz rule:

$$D(r_1 r_2) = D(r_1)r_2 + r_1 D(r_2) \quad \text{for all } r_1, r_2 \in R.$$

A derivation D of a ring R is called *locally nilpotent* if for every $r \in R$ there is an $n \in \mathbb{N}$ such that $D^n(r) = 0$. The subring

$$\ker D := \{f \in R \mid D(f) = 0\}$$

is called the *kernel* of the derivation D.

Let us consider the algebra of polynomials $\mathbb{K}[a_0, a_1, \dots, a_n]$ over the field $\mathbb{K}$ of characteristic 0. Define the derivations $\mathcal{D}$, $\mathcal{D}^*$ and E of this algebra by

$$\mathcal{D}(a_i) = i a_{i-1}, \quad \mathcal{D}^*(a_i) = (n-i)a_{i+1}, \quad E(a_i) = (n-2i)a_i.$$

Note that $\mathcal{D}$, $\mathcal{D}^*$, E define a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{K})$.

Consider the derivations $\mathcal{D} - Y\frac{\partial}{\partial X}$ and $\mathcal{D}^* - X\frac{\partial}{\partial Y}$ of the polynomial algebra $\mathbb{K}[a_0, \dots, a_n, X, Y]$. It is clear that the intersection

$$\ker\left(\mathcal{D} - Y\frac{\partial}{\partial X}\right) \cap \ker\left(\mathcal{D}^* - X\frac{\partial}{\partial Y}\right)$$

is a subalgebra of $\mathbb{K}[a_0, \ldots, a_n, X, Y]$. Let us recall some concepts of the classical invariant theory.

DEFINITION 2.1. The homogeneous polynomial

$$\alpha(X, Y) := a_0 X^n + n a_1 X^{n-1} Y + \cdots + \binom{n}{i} a_i X^{n-i} Y^i + \cdots + a_n Y^n \tag{2.1}$$

is called the *generic binary form* of order n.

DEFINITION 2.2. The algebra $\mathcal{C}_n := \ker\big(\mathcal{D} - Y \frac{\partial}{\partial X}\big) \cap \ker\big(\mathcal{D}^* - X \frac{\partial}{\partial Y}\big)$ is called the *algebra of covariants* of the generic binary form (2.1).

The algebra $\mathcal{S}_n := \ker(\mathcal{D})$ is called the *algebra of semi-invariants* of the generic binary form.

The algebra $\mathcal{I}_n := \ker(\mathcal{D}) \cap \ker(\mathcal{D}^*)$ is called the *algebra of invariants* of the generic binary form.

The elements of the algebras $\mathcal{C}_n$, $\mathcal{S}_n$, $\mathcal{I}_n$ are called covariants, semi-invariants and invariants of the binary form (2.1), respectively. The following obvious inclusions hold: $\mathcal{I}_n \subset \mathcal{C}_n$ and $\mathcal{I}_n \subset \mathcal{S}_n$. It is well known that these algebras are finitely generated.

EXAMPLE 2.1. It is easy to check that the generic form $\alpha(X, Y)$ is itself a covariant and its leading coefficient a_0 (in the ordering $X > Y$) is a semi-invariant. Also, the element $a_0 a_2 - a_1^2$ is an invariant for $n = 2$.

Let

$$\varkappa : \mathcal{C}_n \to \mathcal{S}_n$$

be the $\mathbb{K}$-linear map that takes each homogeneous covariant to its leading coefficient. In other words, for $f(a_0, \ldots, a_n, X, Y) \in \mathcal{C}_n$,

$$\varkappa(f(a_0, a_1, \ldots, a_n, X, Y)) = f(a_0, a_1, \ldots, a_n, 1, 0).$$

The following theorem holds:

THEOREM 2.3 ([R], [O]). *The map $\varkappa$ is a homomorphism of algebras.*

The inverse map $\varkappa^{-1} : \mathcal{S}_n \to \mathcal{C}_n$ can be defined as follows:

$$\varkappa^{-1}(s) = \sum_{i=0}^{\operatorname{ord}(s)} \frac{(\mathcal{D}^*)^i(s)}{i!} X^{\operatorname{ord}(s)-i} Y^i,$$

where

$$\operatorname{ord}(s) = \max\{k \mid (\mathcal{D}^*)^k(s) \neq 0\}.$$

The natural number $\operatorname{ord}(s)$ is called the *order* of the semi-invariant s. The degree of a covariant with respect to the variables X, Y is called the *order* of the covariant, and its degree in the coefficients of the generic binary form is called the *degree* of the covariant.

Similarly, we can define the algebras of joint covariants, semi-invariants and invariants of several generic binary forms. Let us consider the following three generic binary forms of order n:

$$\beta(X,Y) := b_0X^n + nb_1X^{n-1}Y + \cdots + \binom{m}{i} b_iX^{n-i}Y^i + \cdots + b_nY^n,$$

$$\gamma(X,Y) := c_0X^n + nc_1X^{n-1}Y + \cdots + \binom{m}{i} c_iX^{n-i}Y^i + \cdots + c_nY^n,$$

$$\delta(X,Y) := d_0X^n + nd_1X^{n-1}Y + \cdots + \binom{n}{i} d_iX^{n-i}Y^i + \cdots + d_nY^n.$$

Extend the derivations $\mathcal{D}$, $\mathcal{D}^*$ to the polynomial algebra

$$\mathbb{K}[a_0,\ldots,a_n,b_0,\ldots,b_n,c_0,\ldots,c_n,d_0,\ldots,d_n]$$

by $\mathcal{D}(b_i) = ib_{i-1}$, $\mathcal{D}^*(b_i) = (n-i)b_{i+1}$, $\mathcal{D}(c_i) = ic_{i-1}$, $\mathcal{D}^*(c_i) = (n-i)c_{i+1}$ and $\mathcal{D}(d_i) = id_{i-1}$, $\mathcal{D}^*(d_i) = (n-i)d_{i+1}$.

Then the subalgebra

$$\ker\left(\mathcal{D} - Y\frac{\partial}{\partial X}\right) \cap \ker\left(\mathcal{D}^* - X\frac{\partial}{\partial Y}\right)$$

of $\mathbb{K}[a_0,\ldots,d_n,X,Y]$ is called the *algebra of joint covariants* of the forms $\alpha(X,Y)$, $\beta(X,Y)$, $\gamma(X,Y)$, $\delta(X,Y)$. The algebras of joint semi-invariants and joint invariants can be defined similarly.

The main computational tool of the classical invariant theory is the transvectant.

Definition 2.4. The covariant

$$(f,g)^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\partial^r f}{\partial X^{r-i}\partial Y^i} \frac{\partial^r g}{\partial X^i \partial Y^{r-i}}, \qquad r \le \min(n,m).$$

is called the rth *transvectant* of the covariants f and g.

For instance, the transvectants $(f,g)^1$ and $(f,f)^2$ are equal to the Jacobian $J(f,g)$ and the Hessian $\mathrm{Hes}(f)$ respectively. If Q is a generic binary form, then starting from Q one can obtain new covariants by taking transvectants of covariants already constructed. In this way, one can generate all covariants. It is well known (see [O]) that each covariant can be represented by transvectants.

To generate semi-invariants, in [B] we introduced the semi-transvectant as an analogue of the transvectant.

Definition 2.5. The semi-invariant

$$[p,q]^r := \varkappa((\varkappa^{-1}(p),\varkappa^{-1}(q))^r), \qquad r \le \min(\mathrm{ord}(p),\mathrm{ord}(q)),$$

is called the rth *semi-transvectant* of the semi-invariants $p,q \in \mathbb{K}[a_0,\ldots,a_n]$.

We have

$$[p,q]^r = \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{(\mathcal{D}^*)^i(p)}{[\mathrm{ord}(p)]_i} \frac{(\mathcal{D}^*)^{r-i}(q)}{[\mathrm{ord}(q)]_{r-i}}, \tag{2.2}$$

where $[m]_i = m(m-1)\dots(m-(i-1))$ is the falling factorial.

Directly from the definition we get the following properties:

$$[p,q]^0 = pq,$$
$$[f,g]^k = (-1)^k [g,f]^k, \quad \text{so} \quad [f,f]^k = 0 \quad \text{if } k \text{ is odd.}$$

EXAMPLE 2.2.

$$[p,q]^1 := [p,q] = p\frac{\mathcal{D}^*(q)}{\mathrm{ord}(q)} - q\frac{\mathcal{D}^*(p)}{\mathrm{ord}(p)}, \qquad \text{the } \textit{semi-Jacobian} \text{ of } p \text{ and } q,$$

$$[p,p]^2 = 2p\frac{(\mathcal{D}^*)^2(p)}{[\mathrm{ord}(p)]_2} - 2\frac{\mathcal{D}^*(p)}{[\mathrm{ord}(p)]}\,\frac{\mathcal{D}^*(p)}{[\mathrm{ord}(p)]}, \qquad \text{the } \textit{semi-Hessian} \text{ of } p.$$

Up to the constant factor $\frac{1}{72}$, the semi-Hessian of the semi-covariant a_0 equals

$$\frac{1}{2}[a_0,a_0]^2 = a_0a_2 - a_1^2 = \begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix},$$

DEFINITION 2.6. A homogeneous polynomial F is called *isobaric* if it is an eigenvector of the operator E, i.e. $E(F) = \omega(F)F$ for some $\omega(F) \in \mathbb{Z}$.

The eigenvalue $\omega(F)$ is called the *weight* of F.

THEOREM 2.7 ([B]).

(i) $\omega(a_0^{k_0} a_1^{k_1} \cdots a_n^{k_n}) = n(k_0 + k_1 + \cdots + k_n) - 2(k_1 + \cdots + k_n)$,
(ii) *if s is a homogeneous isobaric semi-invariant then* $\mathrm{ord}(s) = \omega(s)$,
(iii) *if p, q are homogeneous isobaric semi-invariants then* $\omega([p,q]^i) = \omega(p) + \omega(q) - 2i$.

Throughout this paper a semi-invariant means an isobaric homogeneous semi-invariant.

DEFINITION 2.8. A semi-invariant S of the generic binary form of order n is called *proper* if $\partial S/\partial a_n \neq 0$.

PROBLEM. Find all irreducible proper semi-invariants of the generic binary form of order n.

3. The main theorems. The following theorem is crucial for the constructions of identities for Appell polynomials.

THEOREM 3.1. *Let $\varphi_{\mathcal{A}} \colon \mathbb{K}[a_0, a_1, \dots, a_n] \to \mathbb{K}[x]$ be the substitution homomorphism*

$$\varphi_{\mathcal{A}}(a_i) = A_i(x).$$

Then

$$\ker^* \varphi_{\mathcal{A}} = \mathcal{S}_n.$$

Proof. First we shall show that the homomorphism $\varphi_{\mathcal{A}}$ intertwines $\mathcal{D}$ with the derivative operator d/dx, i.e.

$$\varphi_{\mathcal{A}}(\mathcal{D}(h(a_0, a_1, \dots, a_n))) = \frac{d}{dx}(\varphi_{\mathcal{A}}(h(a_0, a_1, \dots, a_n)))$$

for all $h(a_0, a_1, \dots, a_n) \in \mathbb{K}[a_0, a_1, \dots, a_n]$. The proof is by induction on the degree of the polynomial $h(a_0, a_1, \dots, a_n)$.

First, the statement holds for all polynomials of degree 1:

$$\varphi_{\mathcal{A}}(\mathcal{D}(a_i)) = \varphi_{\mathcal{A}}(ia_{i-1}) = iA_{i-1}(x) = \frac{d}{dx}A_i(x) = \frac{d}{dx}\varphi_{\mathcal{A}}(a_i).$$

Assume that it holds for all $f \in \mathbb{K}[a_0, a_1, \dots, a_n]$ with $\deg(f) \leq k$:

$$\varphi_{\mathcal{A}}(\mathcal{D}(f)) = \frac{d}{dx}\varphi_{\mathcal{A}}(f).$$

Then for all i we have

$$\begin{aligned}
&\varphi_{\mathcal{A}}(\mathcal{D}(a_i f)) \\
&\quad = \varphi_{\mathcal{A}}(\mathcal{D}(a_i)f) + \varphi_{\mathcal{A}}(a_i\mathcal{D}(f)) = \varphi_{\mathcal{A}}(\mathcal{D}(a_i))\varphi_{\mathcal{A}}(f) + \varphi_{\mathcal{A}}(a_i)\varphi_{\mathcal{A}}(\mathcal{D}(f)) \\
&\quad = \frac{d}{dx}\varphi_{\mathcal{A}}(a_i)\varphi_{\mathcal{A}}(f) + \varphi_{\mathcal{A}}(a_i)\frac{d}{dx}\varphi_{\mathcal{A}}(f) = \frac{d}{dx}(\varphi_{\mathcal{A}}(a_i)\varphi_{\mathcal{A}}(f)) = \frac{d}{dx}(\varphi_{\mathcal{A}}(a_i f)).
\end{aligned}$$

The linearity of the derivations $\mathcal{D}$, d/dx and the linearity of the homomorphism $\varphi_{\mathcal{A}}$ imply that the statement holds for all polynomials of degree $k+1$.

Thus, by induction $\varphi_{\mathcal{A}}$ intertwines $\mathcal{D}$ with the derivative d/dx.

We now show that $\mathcal{S}_n \subset \ker^* \varphi_{\mathcal{A}}$. For $h(a_0, a_1, \dots, a_n) \in \mathcal{S}_n$ we have

$$\begin{aligned}
&\frac{d}{dx}(h(A_0(x), \dots, A_n(x))) \\
&\qquad = \mathcal{D}\varphi_{\mathcal{A}}(h(A_0(x), \dots, A_n(x))) = \mathcal{D}(h(a_0, \dots, a_n)) = 0.
\end{aligned}$$

Therefore, $h(A_0(x), \dots, A_n(x))$ is a constant as claimed.

Conversely, assume $g(A_0(x), \dots, A_n(x)) \in \mathbb{K}$. Then

$$\mathcal{D}(g(a_0, \dots, a_n)) = \frac{d}{dx}g(A_0(x), \dots, A_n(x)) = 0.$$

Thus $g(a_0, \dots, a_n) \in \mathcal{S}_n$ and $\mathcal{S}_n = \ker^* \varphi_{\mathcal{A}}$. ∎

So, any semi-invariant $S(a_0, \dots, a_n)$ yields the identity

$$S(\mathcal{A}) = S(\mathcal{A})_0$$

for the sequence of Appell polynomials $\{A_n(x)\}$.

EXAMPLE 3.1, Let $\Gamma(a_0,a_1,a_2)=\frac{1}{2}[a_0,a_0]^2$ be the semi-Hessian. Then

$$\begin{aligned}
\Gamma(\mathcal{B}) &= B_0(x)B_2(x)-B_1(x)^2=\frac{1}{6}+x^2-x-\left(x-\frac{1}{2}\right)^2=-\frac{1}{12},\\
\Gamma(\mathcal{E}) &= E_0(x)E_2(x)-E_1(x)^2=x^2-x-\left(x-\frac{1}{2}\right)^2=-\frac{1}{4},\\
\Gamma(\mathcal{H}) &= H_0(x)H_2(x)-H_1(x)^2=-1+x^2-x^2=-1,\\
\Gamma(\mathcal{T}) &= 0.
\end{aligned}$$

THEOREM 3.2. *The semi-invariant $S(a_0,a_1,\ldots,a_n)$ determines the identity $S(1,\ldots,1)=0$.*

Proof. It is easy to see that for a homogeneous isobaric polynomial $S(a_0,a_1,\ldots,a_n)$ we have

$$S(\mathcal{T})=S(1,x,x^2,\ldots,x^n)=x^mS(1,1,\ldots,1)$$

for some integer m. Therefore, $S(\mathcal{T})_0=0$. On the other hand, the identity $S(\mathcal{T})=S(\mathcal{T})_0$ implies

$$x^mS(1,\ldots,1)=0$$

for all x. Thus $S(1,\ldots,1)=0$. ■

For the algebra of joint semi-invariants one can easily formulate and prove similar theorems.

4. Identities for a single Appell sequence. To describe identities for Appell polynomials of the same type let us describe the low degree proper semi-invariants of the binary form $\alpha(X,Y)$. The formula (2.2) generates semi-invariants of degree 2, namely $[a_0,a_0]^i$, $i=0,\ldots,n$. Therefore, the semi-transvectant

$$[a_0,a_0]^n=\sum_{i=0}^{n}(-1)^i\binom{n}{i}a_ia_{n-i}$$

is a proper semi-invariant of degree 2. Denote it by $\mathrm{Dv}_n(a_0)$ and its image $\varphi_{\mathcal{A}}(\mathrm{Dv}_n(a_0))$ by $\mathrm{Dv}_n(\mathcal{A})$:

$$\mathrm{Dv}_n(\mathcal{A}):=\sum_{i=0}^{n}(-1)^i\binom{n}{i}A_i(x)A_{n-i}(x).$$

It is easy to check that the variable a_i can be expressed by a_0 as follows: $a_i=(\mathcal{D}^*)^i(a_0)/[n]_i$. So, for simplicity of notation, we write $\mathrm{Dv}_n(a_0)$ instead of $\mathrm{Dv}_n(a_0,a_1,\ldots,a_n)$.

EXAMPLE 4.1. For the Bernoulli polynomials we have

$$\mathrm{Dv}_n(\mathcal{B})=\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i(x)B_{n-i}(x).$$

Now

$$\mathrm{Dv}_n(\mathcal{B})_0=\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i(0)B_{n-i}(0)=\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_iB_{n-i}.$$

On the other hand, by direct calculation one can show [S] that

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_iB_{n-i}=(1-n)B_n.$$

Therefore we obtain an identity for the Bernoulli polynomials,

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i(x)B_{n-i}(x)=(1-n)B_n.$$

Theorem 3.2 implies the well-known binomial identity $\sum_{i=0}^{n}(-1)^i\binom{n}{i}=0$. In [B] we found another proper semi-invariant of degree 2:

$$W_n(a_0):=\sum_{i=1}^{n}(-1)^i\binom{n}{i}a_{n-i}a_1^ia_0^{n-i-1}+(n-1)(-1)^{n+1}a_1^n.$$

To construct a proper semi-invariant of degree 3 use the semi-Hessian

$$\frac{1}{2}[a_0,a_0]^2=\begin{vmatrix}a_0 & a_1\\ a_1 & a_2\end{vmatrix}$$

of the semi-invariant a_0. Denote $\mathrm{Tr}_n(a_0):=[a_0,\frac{1}{2}[a_0,a_0]^2]^n$.

Theorem 4.1. *For $n\geq 4$,*

$$\mathrm{Tr}_n(a_0):=\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i}{[2n-4]_i}\binom{n}{i}\binom{i}{j}a_{n-i}\begin{vmatrix}[n]_ja_j & [n-1]_{i-j}a_{i-j+1}\\ [n-1]_ja_{j+1} & [n-2]_{i-j}a_{i-j+2}\end{vmatrix},$$

where $[n]_i=n(n-1)\dots(n-(i-1))$.

Proof. Since the semi-Hessian has weight $2n-4$, the semi-transvectant $[a_0,\frac{1}{2}[a_0,a_0]^2]^n$ has order $n+2n-4-2n=n-4$. Thus it is well-defined for $n\geq 4$. We have

$$(\mathcal{D}^*)^i(a_k)=[n-k]_i\,a_{i+k}.$$

By the determinant derivative rule we have

$$\begin{aligned}(\mathcal{D}^*)^i\left(\begin{vmatrix}a_0 & a_1\\ a_1 & a_2\end{vmatrix}\right)&=\sum_{j=0}^{i}\binom{i}{j}\begin{vmatrix}(\mathcal{D}^*)^j(a_0) & (\mathcal{D}^*)^{i-j}(a_1)\\ (\mathcal{D}^*)^j(a_1) & (\mathcal{D}^*)^{i-j}(a_2)\end{vmatrix}\\ &=\sum_{j=0}^{i}\binom{i}{j}\begin{vmatrix}[n]_ja_j & [n-1]_{i-j}a_{i-j+1}\\ [n-1]_ja_{j+1} & [n-2]_{i-j}a_{i-j+2}\end{vmatrix}.\end{aligned}$$

By (3) we get

$$[a_0,[a_0,a_0]^2]^n = \sum_{i=0}^{n} \frac{(-1)^i}{[n]_{n-i}[2n-4]_i}\binom{n}{i}(\mathcal{D}^*)^{n-i}(a_0)(\mathcal{D}^*)^i\left(\begin{vmatrix} a_0 & a_1 \\ a_1 & a_2 \end{vmatrix}\right)$$
$$= \sum_{i=0}^{n} \frac{(-1)^i}{[2n-4]_i}\binom{n}{i}\sum_{j=0}^{i}\binom{i}{j}a_{n-i}\begin{vmatrix} [n]_j a_j & [n-1]_{i-j}a_{i-j+1} \\ [n-1]_j a_{j+1} & [n-2]_{i-j}a_{i-j+2} \end{vmatrix}. \blacksquare$$

As above, direct calculation of the nth semi-transvectant ($n \geq 4$) of a two semi-Hessians yields a semi-invariant of degree 4:

$$\mathrm{Ch}_n(a_0) := \left[\frac{1}{2}[a_0,a_0]^2, \frac{1}{2}[a_0,a_0]^2\right]^n = \sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{n-i} \frac{(-1)^i\binom{n}{i}\binom{i}{j}\binom{n-i}{k}}{[2n-4]_i[2n-4]_{n-i}} A_{i,j,k},$$

where

$$A_{i,j,k} := \begin{vmatrix} [n]_k a_k & [n-1]_{n-i-k}a_{n-i-k+1} \\ [n-1]_k a_{k+1} & [n-2]_{n-i-k}a_{n-i-k+2} \end{vmatrix}\begin{vmatrix} [n]_j a_j & [n-1]_{i-j}a_{i-j+1} \\ [n-1]_j a_{j+1} & [n-2]_{i-j}a_{i-j+2} \end{vmatrix}.$$

Now, consider the discriminant and the catalecticant of a binary form. The discriminant is a well known invariant which can be defined as the $(2n-1)\times(2n-1)$ determinant of the Sylvester matrix of the binary form $\alpha(X,Y)$:

$$\mathrm{Discr}_n(a_0) := \begin{vmatrix}
a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\
0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\
\multicolumn{8}{c}{\dotfill} \\
0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2}a_2 & \cdots & a_n \\
na_0 & (n-1)na_1 & \ldots & na_{n-1} & 0 & 0 & \cdots & 0 \\
0 & na_0 & \cdots & \ldots & na_{n-1} & 0 & \cdots & 0 \\
\multicolumn{8}{c}{\dotfill} \\
0 & 0 & \cdots & na_0 & (n-1)na_1 & (n-2)\binom{n}{2}a_2 & \ldots & na_{n-1}
\end{vmatrix}.$$

The corresponding identities have the form $\mathrm{Discr}_n(\mathcal{A}) = \mathrm{Discr}_n(\mathcal{A})_0$.

The catalecticant of a binary form of even degree, $n = 2k$, can be written as the $(k+1) \times (k+1)$ determinant

$$\mathrm{Cat}_n(a_0) := \begin{vmatrix} a_0 & a_1 & a_2 & \cdots & a_k \\ a_1 & a_2 & a_3 & \cdots & a_{k+1} \\ \multicolumn{5}{c}{\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots} \\ a_{k-1} & a_k & a_{k-1} & \cdots & a_{2k-1} \\ a_k & a_{k+1} & a_{k+2} & \cdots & a_{2k} \end{vmatrix}.$$

The corresponding identities have the form $\mathrm{Cat}_n(\mathcal{A}) = \mathrm{Cat}_n(\mathcal{A})_0$.

EXAMPLE 4.2. Let $\mathcal{A} = \mathcal{B}$, $n = 3$. Simplifying the identity

$$\mathrm{Discr}_3(\mathcal{B}) = \mathrm{Discr}_3(\mathcal{B})_0,$$

we obtain

$$-27B_3(x)^2B_0(x)^2 + 162B_3(x)B_0(x)B_1(x)B_2(x) + 81B_2(x)^2B_1(x)^2 - 108B_2(x)^3B_0(x) - 108B_1(x)^3B_3(x) = \frac{1}{16}.$$

CONJECTURE. $\mathrm{Discr}_n(\mathcal{H})_0 = \prod_{k=1}^n k^k, \mathrm{Cat}_n(\mathcal{H})_0 = (-1)^n n!!$.

Thus, we get the following five types of identities for Appell polynomials.

THEOREM 4.2. *Let $\mathcal{A} = \{A_n(x)\}$ be Appell polynomials. Then the following identities hold:*

$$\mathrm{Dv}_n(\mathcal{A}) = \mathrm{Dv}_n(\mathcal{A})_0, \tag{4.1}$$

$$\mathrm{Tr}_n(\mathcal{A}) = \mathrm{Tr}_n(\mathcal{A})_0, \tag{4.2}$$

$$\mathrm{Ch}_n(\mathcal{A}) = \mathrm{Ch}_n(\mathcal{A})_0, \tag{4.3}$$

$$\mathrm{Discr}_n(\mathcal{A}) = \mathrm{Discr}_n(\mathcal{A})_0, \tag{4.4}$$

$$\mathrm{Cat}_n(\mathcal{A}) = \mathrm{Cat}_n(\mathcal{A})_0, \tag{4.5}$$

$$W_n(\mathcal{A}) = W_n(\mathcal{A})_0. \tag{4.6}$$

By applying Theorem 3.2 to the above identities we derive the corresponding binomial identities, for instance:

$$\sum_{i=0}^{n}\sum_{j=0}^{i} \frac{(-1)^i}{[2n-4]_i}\binom{n}{i}\binom{i}{j}\begin{vmatrix} [n]_j & [n-1]_{i-j} \\ [n-1]_j & [n-2]_{i-j} \end{vmatrix} = 0, \tag{4.7}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\sum_{k=0}^{n-i} \frac{(-1)^i\binom{n}{i}\binom{i}{j}\binom{n-i}{k}}{[2n-4]_i[2n-4]_{n-i}}\begin{vmatrix} [n]_k & [n-1]_{n-i-k} \\ [n-1]_k & [n-2]_{n-i-k} \end{vmatrix} \times \begin{vmatrix} [n]_j & [n-1]_{i-j} \\ [n-1]_j & [n-2]_{i-j} \end{vmatrix} = 0. \tag{4.8}$$

5. Joint identities. Let us find joint proper semi-invariants of the binary forms $\alpha(X,Y)$ and $\beta(X,Y)$.

First of all we consider the nth semi-transvectant of the semi-invariants a_0 and b_0:

$$\mathrm{Dv}_n(a_0,b_0) := [a_0,b_0]^n = \sum_{i=0}^{n}(-1)^i\binom{n}{i}a_i b_{n-i}.$$

Example 5.1. For the Bernoulli and Euler polynomials we have

$$\mathrm{Dv}_n(\mathcal{B},\mathcal{E}) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i(x)E_{n-i}(x).$$

By direct calculations we get

$$\mathrm{Dv}_1(\mathcal{B},\mathcal{E})_0 = 0,\quad \mathrm{Dv}_2(\mathcal{B},\mathcal{E})_0 = -\frac{1}{3},\quad \mathrm{Dv}_3(\mathcal{B},\mathcal{E})_0 = 0,\quad \mathrm{Dv}_4(\mathcal{B},\mathcal{E})_0 = \frac{7}{15}.$$

Similarly, the identity

$$\mathrm{Dv}_n(\mathcal{B},\mathcal{T}) = \mathrm{Dv}_n(\mathcal{B},\mathcal{T})_0,$$

implies that $\mathrm{Dv}_n(\mathcal{B},\mathcal{T}) = \sum_{i=0}^{n}(-1)^i\binom{n}{i}B_i(x)x^{n-i}$. It follows that

$$\mathrm{Dv}_n(\mathcal{B},\mathcal{T})_{|x=0} = \mathrm{Dv}_n(\mathcal{B},\mathcal{T})_0 = B_n(0) = B_n.$$

After simplification we get an identity for the Bernoulli polynomials,

$$B_n(x) = \sum_{i=0}^{n-1}(-1)^{i+1}\binom{n}{i}B_i(x)x^{n-i} + B_n.$$

In the same way we get identities for the Euler and Hermite polynomials:

$$E_n(x) = \sum_{i=0}^{n-1}(-1)^{i+1}\binom{n}{i}E_i(x)x^{n-i} + E_n,$$

$$H_n(x) = \sum_{i=0}^{n-1}(-1)^{i+1}\binom{n}{i}H_i(x)x^{n-i} + H_n(0).$$

The semi-transvectants $[a_0,[a_0,b_0]^i]^n$ for $2i \le n$ are proper joint semi-invariants of degree 3. By direct calculations for $i=1$ we get

$$\mathrm{Tr}_n(a_0,b_0) := [a_0,[a_0,b_0]^1]^n = \left[a_0, \begin{vmatrix} a_0 & b_0 \\ a_1 & b_1 \end{vmatrix}\right]^n$$

$$= \sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i}{[2n-2]_i}\binom{n}{i}\binom{i}{j}a_{n-i}\begin{vmatrix} [n]_j a_j & [n]_{i-j}b_{i-j} \\ [n-1]_j a_{j+1} & [n-1]_{i-j}b_{i-j+1} \end{vmatrix},$$

and

$$\overline{\mathrm{Tr}}_n(a_0, b_0) := [a_0, [a_0, b_0]^2]^n = [a_0, a_0 b_2 - 2a_1 b_1 + a_2 b_0]^n$$
$$= \sum_{i=0}^{n} \sum_{j=0}^{i} \frac{(-1)^i}{[2n-2]_i} \binom{n}{i} \binom{i}{j} a_{n-i} A_{i,j},$$

where

$$A_{i,j} := [n]_i [n-2]_{i-j} a_i b_{i-j+2} - 2[n-1]_i [n-1]_{i-j} a_{i+1} b_{i-j+1} + [n-2]_i [n]_{i-j} a_{i+2} b_{i-j}.$$

The resultant of two binary forms is a well known joint covariant. The corresponding semi-invariant $\mathrm{sRes}_n(a_0, b_0)$ has the form

$$\mathrm{sRes}_n(a_0, b_0) := \begin{vmatrix} a_0 & na_1 & \cdots & a_n & 0 & \cdots & \cdots & 0 \\ 0 & a_0 & \cdots & na_{n-1} & a_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & a_0 & na_1 & \binom{n}{2} a_2 & \cdots & a_n \\ b_0 & nb_1 & \cdots & b_n & 0 & \cdots & \cdots & 0 \\ 0 & b_0 & \cdots & nb_{n-1} & b_n & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & 0 & b_0 & b_1 & \binom{n}{2} b_2 & \cdots & b_n \end{vmatrix}.$$

EXAMPLE 5.2. The semi-resultant of two binary forms of order 2 leads to the following identity for the Bernoulli and Euler polynomials:

$$\mathrm{sRes}_2(\mathcal{B}, \mathcal{E}) = \mathrm{sRes}_2(\mathcal{B}, \mathcal{E})_0.$$

We expand the determinants to get

$$B_2(x)^2 E_0(x)^2 - 2B_2(x)E_0(x)E_2(x)B_0(x) + E_2(x)^2 B_0(x)^2$$
$$-4B_1(x)B_2(x)E_1(x)E_0(x) - 4B_1(x)E_1(x)E_2(x)B_0(x) + 4E_2(x)B_1(x)^2 E_0(x)$$
$$+ 4B_0(x)B_2(x){E_1}^2(x) = \frac{1}{36}.$$

Let us find joint proper semi-invariants of the binary forms $\alpha(X, Y)$, $\beta(X, Y)$ and $\gamma(X, Y)$. Since the semi-Jacobian $[b_0, c_0]$ has weight $2n-2$, the semi-transvectant $[a_0, [b_0, c_0]]^n$ is well-defined. We have

$$(\mathcal{D}^*)^i(b_k) = [n-k]_i b_{i+k}, \qquad (\mathcal{D}^*)^i(c_k) = [n-k]_i c_{i+k}.$$

Therefore

$$(\mathcal{D}^*)^i\left(\begin{vmatrix} b_0 & c_0 \\ b_1 & c_1 \end{vmatrix}\right) = \sum_{j=0}^{i} \binom{i}{j} [n]_j [n-1]_{i-j} \begin{vmatrix} b_j & c_j \\ b_{i-j+1} & c_{i-j+1} \end{vmatrix}.$$

Thus

$$[a_0,[b_0,c_0]]^n=\sum_{i=0}^{n}\frac{(-1)^i}{[n]_{n-i}[2n-2]_i}\binom{n}{i}(\mathcal{D}^*)^{n-i}(a_0)(\mathcal{D}^*)^i\left(\begin{vmatrix}b_0 & c_0\\ b_1 & c_1\end{vmatrix}\right)$$

$$=\sum_{i=0}^{n}\frac{(-1)^i}{[n]_{n-i}[2n-2]_i}\binom{n}{i}[n]_{n-i}a_{n-i}\sum_{j=0}^{i}\binom{i}{j}[n]_j[n-1]_{i-j}\begin{vmatrix}b_j & c_j\\ b_{i-j+1} & c_{i-j+1}\end{vmatrix}.$$

This implies

$$\begin{aligned}\mathrm{Tr}_n(a_0,b_0,c_0)&:=[a_0,[b_0,c_0]]^n\\ &=\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i[n]_j[n-1]_{i-j}}{[2n-2]_i}\binom{n}{i}\binom{i}{j}a_{n-i}\begin{vmatrix}b_j & c_j\\ b_{i-j+1} & c_{i-j+1}\end{vmatrix}.\end{aligned}$$

Finally, let us find joint proper semi-invariants of the four binary forms $\alpha(X,Y)$, $\beta(X,Y)$, $\gamma(X,Y)$ and $\delta(X,Y)$. It is easy to see that the determinant

$$\varDelta:=\begin{vmatrix}b_0 & c_0 & d_0\\ b_1 & c_1 & d_1\\ b_2 & c_2 & d_2\end{vmatrix}$$

is a semi-invariant with weight $3n-6$. Then the semi-transvectant $[d_0,\varDelta]^n$ is well defined for $n\geq 3$. As above, we obtain

$$\begin{aligned}\mathrm{Ch}_n(a_0,b_0,c_0,d_0)&:=[d_0,\varDelta]^n=\sum_{i=0}^{n}\frac{(-1)^i}{[3n-6]_i}\binom{n}{i}a_{n-i}\\ &\times\sum_{i_1+i_2+i_3=i}\frac{i!}{i_1!i_2!i_3!}\begin{vmatrix}[n]_{i_1}b_{i_1} & [n]_{i_2}c_{i_2} & [n]_{i_3}d_{i_3}\\ [n-1]_{i_1}b_{i_1+1} & [n-1]_{i_2}c_{i_2+1} & [n-1]_{i_3}d_{i_3+1}\\ [n-2]_{i_1}b_{i_1+2} & [n-2]_{i_2}c_{i_2+2} & [n-2]_{i_3}d_{i_3+2}\end{vmatrix}.\end{aligned}$$

Therefore we get the following identities for Appell polynomials of different series $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$:

THEOREM 5.1.

$$\mathrm{Dv}_n(\mathcal{A}_1,\mathcal{A}_2)=\mathrm{Dv}_n(\mathcal{A}_1,\mathcal{A}_2)_0, \tag{5.1}$$

$$\mathrm{Tr}_n(\mathcal{A}_1,\mathcal{A}_2)=\mathrm{Tr}_n(\mathcal{A}_1,\mathcal{A}_2)_0, \tag{5.2}$$

$$\overline{\mathrm{Tr}}_n(\mathcal{A}_1,\mathcal{A}_2)=\overline{\mathrm{Tr}}_n(\mathcal{A}_1,\mathcal{A}_2)_0, \tag{5.3}$$

$$\mathrm{Ch}_n(\mathcal{A}_1,\mathcal{A}_2)=\mathrm{Ch}_n(\mathcal{A}_1,\mathcal{A}_2)_0, \tag{5.4}$$

$$\mathrm{sRes}_n(\mathcal{A}_1,\mathcal{A}_2)=\mathrm{sRes}_n(\mathcal{A}_1,\mathcal{A}_2)_0, \tag{5.5}$$

$$\mathrm{Tr}_n(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)=\mathrm{Tr}_n(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3)_0, \tag{5.6}$$

$$\mathrm{Ch}_n(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4)=\mathrm{Ch}_n(\mathcal{A}_1,\mathcal{A}_2,\mathcal{A}_3,\mathcal{A}_4)_0. \tag{5.7}$$

By using Theorem 3.3 we get the binomial identities

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i}{[2n-2]_i}\binom{n}{i}\binom{i}{j}\begin{vmatrix}[n]_j & [n]_{i-j}\\ [n-1]_j & [n-1]_{i-j}\end{vmatrix}=0, \tag{5.8}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{i}\frac{(-1)^i}{[2n-2]_i}\binom{n}{i}\binom{i}{j}([n]_i[n-2]_{i-j}-2[n-1]_i[n-1]_{i-j} + [n-2]_i[n]_{i-j})=0, \tag{5.9}$$

$$\sum_{i=0}^{n}\frac{(-1)^i}{[3n-6]_i}\binom{n}{i}\sum_{i_1+i_2+i_3=i}\frac{i!}{i_1!i_2!i_3!}\begin{vmatrix}[n]_{i_1} & [n]_{i_2} & [n]_{i_3}\\ [n-1]_{i_1} & [n-1]_{i_2} & [n-1]_{i_3}\\ [n-2]_{i_1} & [n-2]_{i_2} & [n-2]_{i_3}\end{vmatrix}=0. \tag{5.10}$$

References

[B] L. Bedratyuk, *On complete system of invariants for the binary form of degree 7*, J. Symbolic Comput. 42 (2007), 935–947.

[O] P. Olver, *Classical Invariant Theory*, Cambridge Univ. Press, 1999.

[R] M. Roberts, *The covariants of a binary quantic of the nth degree*, Quart. J. Math. 4 (1861), 168–178.

[S] L. H. Sun, private communications.

[V] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, Transl. Math. Monogr. 22, Amer. Math. Soc., Providence, 1968.

[YY] Y. Yang and H. Youn, *Appell polynomial sequences: A linear algebra approach*, JP J. Algebra Number Theory Appl. 13 (2009), 65–98.

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