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Application of the explicit *abc*-conjecture to two Diophantine equations

by

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Dedicated to Professor T. N. Shorey on his 65th birthday

1. Introduction. For any integer $n \geq 1$, we denote by $\omega(n)$ the number of distinct prime factors of n, with $\omega(1) = 0$; by $P(n)$ the greatest prime factor of n, with $P(1) = 1$; and by $N(n)$ the radical of n, i.e. the product of distinct prime divisors of n, with $N(1) = 1$. We consider the equation

$$n(n+d)\dots(n+(k-1)d) = by^l \tag{1.1}$$

in positive integers n, d, y, b, $l \geq 2$, l prime, $k \geq 2$ and $\gcd(n, d) = 1$ and $P(b) \leq k$. Several authors have worked on this equation. Finiteness results and complete solutions have been found under various restrictions on the parameters involved. We refer to [8], [7], [11], [14] and [15] for an account of these results and further references. One of the main conjectures on this equation is due to Erdős:

Conjecture 1. *Equation* (1.1) *with* $d > 1$ *implies that* k *is bounded by an absolute constant.*

This conjecture is still open. A stronger conjecture states that solutions exist only when $(k, l) \in \{(3, 3), (4, 2), (3, 2)\}$. In these cases, in fact, there are infinitely many solutions.

Results under the *abc*-conjecture. It was shown by Shorey [15] that Conjecture 1 is true for $l > 3$ under the *abc*-conjecture. The cases $l = 2, 3$ also follow from the *abc*-conjecture for binary forms by an argument due to Granville (see [10]). Further, in 2004, Győry, Hajdu and Saradha [8] have shown that the *abc*-conjecture implies that (1.1) with $d > 1$, $k \geq 3$, $l > 4$ has only finitely many solutions in n, d, k, b, y and l. However, the

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bounds are not explicitly given due to the implicit constant involved in the *abc*-conjecture. In 2004, Baker [1] proposed an explicit version of the *abc*-conjecture as follows:

Suppose a, b, c are mutually coprime integers such that

$$a + b = c.$$

Then

$$\max(|a|, |b|, |c|) \le \frac{6}{5} N(abc) \frac{(\log N(abc))^{\omega(abc)}}{\omega(abc)!}. \tag{1.2}$$

Note that

$$\frac{6}{5}\frac{x^n}{n!} \le \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} < e^x \quad \text{for } x \ge \frac{n+1}{5}.$$

We put $x = \log N(abc)$ and $n = \omega(abc)$. Then

$$n \le 1.5 \log N(abc) \le 1.5x.$$

Hence we have the following explicit version of the *abc*-conjecture:

$$\max(|a|, |b|, |c|) \le (N(abc))^2. \tag{1.3}$$

This version can also be found in Granville and Tucker [6].

We shall apply this explicit version to show

THEOREM 1.1. *Under the explicit version* (1.3) *of the abc-conjecture, equation* (1.1) *with $k \ge 8$ implies that*

$$l \le 29.$$

Towards Conjecture 1, we have

THEOREM 1.2. *Under the explicit version* (1.3) *of the abc-conjecture, equation* (1.1) *implies that*

$$k \le \begin{cases} 8 & \text{if } l = 29, \\ 32 & \text{if } l = 19, 23, \\ 10^2 & \text{if } l = 17, \\ 10^7 & \text{if } l = 13, \\ e^{e^{280}} & \text{if } l = 7, 11. \end{cases}$$

When $b = 1$, i.e. the case of perfect powers, Győry, Hajdu and Pinter [7] have shown that (1.1) implies that $k \ge 35$ for $l \ge 3$. This is also true for $l = 2$ by the result of Hirata-Kohno, Laishram, Shorey and Tijdeman [9]. Thus Theorem 1.2 implies

COROLLARY 1.3. *Under the explicit version* (1.3) *of the abc-conjecture, equation* (1.1) *with $b = 1$ implies that*

$$l \le 17$$

and

$$k \le \begin{cases} 10^2 & \text{if } l = 17, \\ 10^7 & \text{if } l = 13, \\ e^{e^{280}} & \text{if } l = 7, 11. \end{cases}$$

The second equation that we will consider is the equation of Goormaghtigh

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \tag{1.4}$$

in integers $x > 1$, $y > 1$, $m > 2$, $n > 2$. We may assume without loss of generality that $x > y$. Then $m < n$. Thus we always consider (1.4) with

$$x > y > 1 \quad \text{and} \quad n > m > 2.$$

It is known that this equation has two solutions, viz.,

$$31 = \frac{5^3 - 1}{5 - 1} = \frac{2^5 - 1}{2 - 1} \quad \text{and} \quad 8191 = \frac{90^3 - 1}{90 - 1} = \frac{2^{13} - 1}{2 - 1}.$$

It has been conjectured that these are the only solutions of (1.4). A weaker conjecture is

Conjecture 2. *Equation* (1.4) *has only finitely many solutions in x, y, m and n.*

This conjecture is still open. It is known that (1.4) has only finitely many solutions if at least two of the variables are fixed. We refer to [16] for more details. It was shown in [15, p. 473] that the *abc*-conjecture implies Conjecture 2. Here we investigate equation (1.4) under the explicit *abc*-conjecture (1.3).

Theorem 1.4. *Under the explicit version* (1.3) *of the abc-conjecture, equation* (1.4) *implies one of the following:*

$$m \le 6; \qquad m = 7,\ 8 \le n \le 19; \qquad m = 8,\ 9 \le n \le 11.$$

It has been shown *unconditionally* that Conjecture 2 holds if x, y are fixed or composed of primes from a given set (see [16]). We show that under the explicit version (1.2) of the *abc*-conjecture, Conjecture 2 holds with a more relaxed condition. We have

Theorem 1.5. *Let the explicit version* (1.2) *of the abc-conjecture hold. Then Conjecture* 2 *holds for $m \ge 6$ and $\omega(xy(x-1)(y-1)(x-y))$ fixed.*

Remark. In the explicit version (1.3) of the *abc*-conjecture, we lose the arbitrariness of ϵ in the original *abc*-conjecture. Thus, due to the exponent 2 in (1.3), the values $l = 3, 5$ in Theorem 1.2 and $3 \le m \le 5$ in Theorem 1.5 are not covered.

Unconditional results. In 2002, Shorey and Tijdeman [19] showed that equation (1.1) implies that

$$k \le C(l, \omega(d))$$

where $C(l, \omega(d))$ is an effectively computable number depending only on l and $\omega(d)$. Thus if l and $\omega(d)$ are fixed then Conjecture 1 follows. In [11], Laishram and Shorey explicitly calculated

$$C(2, \omega(d)) = 2\omega(d)2^{\omega(d)}.$$

For $l \ge 3$, we have the following result.

Theorem 1.6. *Equation* (1.1) *implies that*

$$k \le \begin{cases} \max(10^{28}, 20 \cdot l^{\omega(d)}) & \textit{if } l \ge 11, \\ \max(10^{50}, 20 \cdot 7^{\omega(d)}) & \textit{if } l = 7, \\ \max(10^{3000}, 23 \cdot 5^{\omega(d)}) & \textit{if } l = 5, \\ \max(10^{10^{500}}, 16 \cdot (3.005)^{\omega(d)}) & \textit{if } l = 3. \end{cases}$$

Suppose $l = 7$. Then $20 \cdot l^{\omega(d)} \le 10^{50}$ for $\omega(d) \le 59$. Thus we conclude from Theorem 1.6 that

$$k \le 10^{50} \quad \text{if } l = 7 \text{ and } \omega(d) \le 59.$$

This is better than the bound given in Theorem 1.2. Similar results can be obtained for other values of l in Theorem 1.6. The proof of Theorem 1.6 depends mainly on the method of Erdős and repeated application of the box principle, as in Shorey and Tijdeman [19]. Further it relies on the result of Evertse [5] on the number of solutions of an equation of the form $AX^l - BY^l = C$ or CZ with some condition on Z. The case $l = 3$ also uses the result of Saradha and Shorey [13] that equation (1.1) implies

$$d > .13k^{1/3}.$$

As we are interested in making the method explicit, we are not economical with the various constants involved. Thus it is possible to improve the bounds given in Theorem 1.6. The large constants that occur reflect the limitations of the method. In Theorem 1.2, the bounds for k when $l \ge 13$ follow easily from the proof of Theorem 1.1. To get a bound for k when $l \in \{7, 11\}$ we use Theorem 1.6. Further we also use a result of Saradha and Shorey [14] that $d \ge 10^8$ for $l \ge 7$.

2. Notation and preliminaries. We assume that equation (1.1) holds. For $0 \le i < k$, we can write

$$n + id = a_i x_i^l \quad \text{where } a_i \text{ is } l\text{th power free and } P(a_i) \le k$$

and

$$n + id = A_i X_i^l \quad \text{with } P(A_i) \le k \text{ and } \quad \gcd\Big(\prod_{p\le k} p, X_i\Big) = 1.$$

By a result of Shorey and Tijdeman [18], the product $n(n+d)\dots(n+(k-1)d)$ is divisible by a prime $> k$ except when $(n,d,k) = (2,7,3)$. We shall assume throughout that $(n,d,k) \neq (2,7,3)$. Then

$$(2.1) \qquad n + (k-1)d > k^l \quad \text{and} \quad n + id > k^{l-1} \quad \text{for } 1 \le i < k-1.$$

Let

$$S = \{A_0, A_1, \dots, A_{k-1}\} \quad \text{and} \quad S' = \{A_i \mid 0 < i < k,\ A_i \le k^{l-1}\}.$$

Then for any $A_i \in S'$, we have

$$k^{l-1} < n + id = A_i X_i^l \le k^{l-1} X_i^l,$$

implying that $X_i > 1$. Then by the definition of X_i, we get $X_i > k$. Thus

$$(2.2) \qquad X_i > k \quad \text{for } A_i \in S'.$$

Further, note that X_i's are coprime to each other since $\gcd(n,d) = 1$. Hence there exists an X_i with $A_i \in S'$ such that X_i is divisible by a prime $\ge p_{\pi(k)+|S'|}$. By well known estimates for the nth prime p_n and the prime counting function $\pi(n)$ (see Lemma 3.1 below) it follows that

$$(2.3) \qquad n + (k-1)d > \left(k\left(1 - \frac{2}{\log k}\right) + |S'|(\log k - 2)\right)^l \quad \text{for } k \ge 17.$$

Fix $1 \le l' < l$. For any tuple $(i_1, \dots, i_{l'})$ with $0 < i_1 \le \dots \le i_{l'} < k$, we call $X_{i_1} \dots X_{i_{l'}}$ an *X-product* and $A_{i_1} \dots A_{i_{l'}}$ an *A-product*. If necessary, we will mention the number l' of terms in these products.

3. Some estimates and combinatorial lemmas. We begin with a lemma collecting some estimates from prime number theory. For inequalities (ii)–(v) below we refer to (3.5), (3.13), (3.10) and (3.26) in Rosser and Schoenfeld [12]. For (i) see Dusart [2].

Lemma 3.1.

(i) $$\pi(n) < \frac{n}{\log n}\left(1 + \frac{1.2762}{\log n}\right) \qquad \textit{for } n > 1,$$

(ii) $$\pi(n) > \frac{n}{\log n} \qquad \textit{for } n \ge 17,$$

(iii) $$p_n < n(\log n + \log\log n) \qquad \textit{for } n \ge 6,$$

(iv) $$p_n > n(\log n + \log\log n - 3/2) \qquad \textit{for } n \ge 2,$$

(v) $$\prod_{p\le z}\left(1 - \frac{1}{p}\right) \le \frac{e^{-\gamma}}{\log z}\left(1 + \frac{1}{2(\log z)^2}\right)$$

where γ is the Euler constant.

Lemma 3.2. *For $d > 3$, we have*

$$\omega(d) \le \frac{3\log d}{\log\log d}.$$

Proof. Suppose d is an integer with $l(d) > \log d$, where $l(d)$ is the least prime divisor of d. Then $(l(d))^{\omega(d)} \le d$, implying

$$\omega(d) \le \frac{\log d}{\log l(d)} < \frac{\log d}{\log\log d}. \tag{3.1}$$

Hence the conclusion is true in this case. For any d, let us write $d = d_1 d_2$ such that if $p \mid d_1$, then $p \le \log d$, and if $p \mid d_2$, then $p > \log d$. So $l(d_2) > \log d \ge \log d_2$. Hence by (3.1),

$$\omega(d_2) < \frac{\log d_2}{\log\log d_2}. \tag{3.2}$$

Also

$$\omega(d_1) \le \pi(\log d) < \frac{\log d}{\log\log d} + \frac{1.2762\log d}{(\log\log d)^2} < 1.836\frac{\log d}{\log\log d} \tag{3.3}$$

for $d \ge 100$. Now for $20 \le d < 100$, $\omega(d_1) \le 3$, hence the inequality in (3.3) is satisfied. For $3 < d < 20$, one can check that $\omega(d_1)$ is either 0 or 1, and again (3.3) is satisfied. Combining (3.1)–(3.3) we get the assertion of the lemma. ∎

In Lemmas 3.3–3.8, below, we assume that equation (1.1) holds.

Lemma 3.3. *Let $1 \le l' < l$. Suppose $(i_1, \ldots, i_{l'})$ and $(j_1, \ldots, j_{l'})$ are two distinct tuples with $0 < i_1 \le \cdots \le i_{l'} < k$ and $0 < j_1 \le \cdots \le j_{l'} < k$. Assume that*

$$n + (k-1)d > k^{\frac{ll'+l-l'}{l-l'}} \left(\max\left(\frac{1}{i_1}, \frac{1}{j_1}\right)\right)^{\frac{l'(l-1)}{l-l'}}.$$

Then the corresponding A-products are distinct whenever the respective X-products are congruent modulo d.

Proof. Suppose for two distinct tuples $(i_1, \ldots, i_{l'})$ and $(j_1, \ldots, j_{l'})$ we have

$$A_{i_1} \ldots A_{i_{l'}} = A_{j_1} \ldots A_{j_{l'}}.$$

Consider

$$\Delta = (n + i_1 d) \ldots (n + i_{l'} d) - (n + j_1 d) \ldots (n + j_{l'} d).$$

First we show $\Delta \ne 0$. Suppose not. Then

$$n + i_1 d \le \gcd(n + i_1 d, n + j_1 d) \ldots \gcd(n + i_1 d, n + j_{l'} d) < k^{l'}.$$

Thus $n+(k-1)d < k(n+i_1d) < k^{l'+1} \le k^l$. This contradicts (2.1). Thus $\Delta \neq 0$. We may assume that $\Delta > 0$. Now

(3.4)

$$\begin{aligned}\Delta &= (n+i_1d)\ldots(n+i_{l'}d) - (n+j_1d)\ldots(n+j_{l'}d)\\ &\le (n+(k-1)d)^{l'} - n^{l'}\\ &= l'(k-1)dn^{l'-1} + \binom{l'}{2}((k-1)d)^2n^{l'-2} + \binom{l'}{3}((k-1)d)^3n^{l'-3} + \cdots\\ &\le l'(k-1)d\Big\{n^{l'-1} + (l'-1)(k-1)dn^{l'-2}\\ &\qquad + \frac{(l'-1)(l'-2)}{2.1}((k-1)d)^2n^{l'-3} + \cdots\Big\}\\ &\le l'kd(n+(k-1)d)^{l'-1}.\end{aligned}$$

On the other hand,

$$\begin{aligned}\Delta &= A_{i_1}\ldots A_{i_{l'}}(X_{i_1}\ldots X_{i_{l'}})^l - A_{j_1}\ldots A_{j_{l'}}(X_{j_1}\ldots X_{j_{l'}})^l\\ &= A_{j_1}\ldots A_{j_{l'}}(X^l - Y^l)\end{aligned}$$

where $X = X_{i_1}\ldots X_{i_{l'}}$, $Y = X_{j_1}\ldots X_{j_{l'}}$. Now

$$\Delta = A_{j_1}\ldots A_{j_{l'}}(X-Y)\frac{X^l-Y^l}{X-Y} \neq 0.$$

Hence by assumption, $X - Y \ge d$. Since $j_1 > 0$, we get

$$\begin{aligned}\Delta &\ge ldA_{j_1}\ldots A_{j_{l'}}Y^{l-1} \ge ld(A_{j_1}X_{j_1}^l\ldots A_{j_{l'}}X_{j_{l'}}^l)^{(l-1)/l}\\ &\ge ld\bigg(\frac{j_1}{k-1}(n+(k-1)d)\bigg)^{l'(l-1)/l}.\end{aligned} \tag{3.5}$$

Thus from (3.4) and (3.5) we get

$$\bigg(\frac{j_1}{k}(n+(k-1)d)\bigg)^{l'(l-1)/l} \le \frac{l'k}{l}(n+(k-1)d)^{l'-1},$$

i.e.

$$(n+(k-1)d)^{1-l'/l} \le \frac{l'}{l}\,\frac{k^{(ll'+l-l')/l}}{j_1^{(ll'-l')/l}},$$

which gives

$$n+(k-1)d \le \frac{k^{(ll'+l-l')/(l-l')}}{j_1^{(ll'-l')/(l-l')}}.$$

This contradicts our assumption. ∎

Note that in the above proof we only need that the difference between the respective X-products exceed d. The next two lemmas are based on the arguments of Lemma 8 of [19].

LEMMA 3.4. *Let T denote a set of h numbers from among the A_i's. For any integer $r > 0$ with $2r < l$, assume that*

$$n + (k-1)d > k^{\frac{2rl+l-2r}{l-2r}}.$$

Suppose T_r is the maximal subset of T such that all the $A(r)$-products $A_{i_1} \dots A_{i_r}$ with A's in T are distinct. Then

$$\gamma_1 l^{\omega(d)} + \gamma_2 \geq h - |T_r|,$$

where

$$(\gamma_1, \gamma_2) = \begin{cases} (2,6) & \text{if } l = 3, \\ (1,2) & \text{if } l = 5, \\ (1,1) & \text{if } l \geq 7. \end{cases}$$

Proof. If $|T_r| = h$ or $h-1$ then the assertion is trivially true. So we shall assume that $|T_r| \leq h-2$. Let $A_\mu \in T \setminus T_r$. Then there exist $A_{\mu_2}, \dots, A_{\mu_r}$ and $A_{\mu'_1}, \dots, A_{\mu'_r}$ in T_r such that

$$A_\mu A_{\mu_2} \dots A_{\mu_r} = A_{\mu'_1} \dots A_{\mu'_r}.$$

Let $I = \{\mu \mid A_\mu \in T \setminus T_r\}$. Consider the map $f : I \to \mathbb{Z}/(d)$ defined as

$$f(\mu) = \overline{X}_\mu \overline{Y}_\mu^{-1} \pmod d,$$

where $\overline{X}_\mu$ and $\overline{Y}_\mu$ are the $X(r)$-products given by

$$\overline{X}_\mu = X_\mu X_{\mu_2} \dots X_{\mu_r} \quad \text{and} \quad \overline{Y}_\mu = X_{\mu'_1} \dots X_{\mu'_r}.$$

Suppose for some $\mu \neq \nu$, $\mu, \nu \in I$, we have $f(\mu) = f(\nu)$. Then

$$\overline{X}_\mu \overline{Y}_\mu^{-1} \equiv \overline{X}_\nu \overline{Y}_\nu^{-1} \pmod d \quad \text{or} \quad \overline{X}_\mu \overline{Y}_\nu \equiv \overline{X}_\nu \overline{Y}_\mu \pmod d.$$

Also note that

$$A_\mu A_{\mu_2} \dots A_{\mu_r} A_{\nu'_1} \dots A_{\nu'_r} = A_{\mu'_1} \dots A_{\mu'_r} A_\nu A_{\nu_2} \dots A_{\nu_r}.$$

Then by Lemma 3.3 with $l' = 2r$, we get a contradiction. Hence f is 1-1. Further

$$(\overline{X}_\mu \overline{Y}_\mu^{-1})^l \equiv 1 \pmod d.$$

Thus $\overline{X}_\mu \overline{Y}_\mu^{-1}$ is a solution of $Z^l \equiv 1 \pmod d$. The number of solutions of this congruence is $\leq \gamma_1 l^{\omega(d)} + \gamma_2$, as shown by Evertse [5]. Thus

$$\gamma_1 l^{\omega(d)} + \gamma_2 \geq |I| = h - |T_r|. \ \blacksquare$$

REMARK. By taking $r = 1$ in Lemma 3.4, we see that the number of distinct elements in T, viz., $|T_1|$ satisfies

$$|T_1| \geq h - \gamma_1 l^{\omega(d)} - \gamma_2 \tag{3.6}$$

provided $n + (k-1)d > k^{3+4/(l-2)}$, which is true for $l \geq 5$. For $l = 3$, we have the following lemma.

Lemma 3.5. *Let* $l = 3$ *and* $k \geq 10^{1000}$. *Let*

$$T' = \{A_i \mid A_i \in T,\, i \geq k/16\}.$$

Then

$$|T_1| \geq h - k/16 - 2l^{\omega(d)} - 6$$

provided there exists a subset $T'' \subseteq T'$ *having at least* $k/16$ *elements for which* $X_i \neq 1$.

Proof. By Lemma 3.4 and the proof of Lemma 3.3 we need to satisfy

$$n + (k-1)d > \left(\frac{2}{3}\right)^3 \frac{k^7}{j^4}.$$

We take A_j's in T'. Thus the above inequality implies that we need

$$n + (k-1)d \geq 19420k^3. \tag{3.7}$$

Suppose (3.7) is not valid. Then

$$d \leq 19420k^2. \tag{3.8}$$

Further, since T'' has at least $k/16$ elements with $X_i \neq 1$, we find that

$$A_i \leq 19420$$

for at least $k/16$ of the A_i's in T'. Hence there are at least $.000003k$ pairs (A_i, A_j) which are equal. Now we follow the argument in [19, pp. 335–336]. Let $\zeta = e^{2\pi i/3}$. Then, in $\mathbb{Q}(\zeta)$, we can write

$$[d/3] = D_1 D_2 D_3$$

with

$$D_1 \mid [X_i - X_j], \qquad D_2 \mid [X_i - \zeta X_j], \qquad D_3 \mid [X_i - \zeta^2 X_j] \tag{3.9}$$

for any pair (A_i, A_j). There are at most $3^{\omega(d)}$ ways of writing $[d/3]$ as above. By Lemma 3.2, (3.8) and $k \geq 10^{1000}$, we see that $3^{\omega(d)} \leq k^{.9}$. The total number of such pairs which is at least $.000003k$ exceeds $k^{.9}$. Hence there are two pairs (A_i, A_{j_1}), (A_i, A_{j_2}) such that (3.9) holds with $j = j_1$ and j_2. This leads to

$$|X_{j_1} - X_{j_2}| > d/3.$$

On the other hand, since $A_{j_1} = A_{j_2}$, we get

$$(j_1 - j_2)d \geq A_{j_1}(X_{j_1} - X_{j_2})X_{j_1}^2,$$

implying $k \geq (n + j_1 d)^{2/3} \geq k^{4/3}$, a contradiction. ∎

Let $S_1 = \{A_1, \ldots, A_{k-1}\}$. For every $p \leq k$, we remove an element, say A_{i_p}, in which p appears to the maximum power. Let S_2 be the set of remaining A_i's. Thus there are at least $k - 1 - \pi(k)$ A_i's in S_2. Further

$$\prod_{A_i \in S_2} A_i \mid (k-1)!. \tag{3.10}$$

Let

$$S_3 = \{A_i \in S_2 \mid A_i \le \alpha k\} \quad \text{with some } \alpha > 1.$$

Assume that S_3 has at most βk elements with $\beta < 1$. Thus there are at least $k - 1 - \pi(k) - \beta k$ A_i's in S_2 which exceed αk. From (3.10) we get

$$(3.11) \qquad (\alpha k + 1) \ldots (\alpha k + k - 1 - \pi(k) - [\beta k]) \\ \le |S_3|!(\alpha k + 1) \ldots (\alpha k + k - 1 - \pi(k) - [\beta k]) \le (k-1)!,$$

which implies

$$|S_3|!(\alpha k)^{k-1-\pi(k)-|S_3|}\left(1 + \frac{1}{\alpha k}\right) \ldots \left(1 + \frac{k - 1 - \pi(k) - \beta k}{\alpha k}\right) \le (k-1)!.$$

Using Stirling's formula $n! > n^n e^{-n}$ and $n! < n^{n+1/2}e^{-n}(2\pi)^{1/2}$ and the fact that $(y/x)^y$ is a decreasing function of y whenever $y/x < 1/e$, we find that

$$\left(\frac{\beta k}{e\alpha k}\right)^{\beta k}(\alpha k)^{k-1-\pi(k)}\left(1 + \frac{1}{\alpha k}\right) \ldots \left(1 + \frac{k - 1 - \pi(k) - \beta k}{\alpha k}\right) \\ \le (k-1)! \le k^{k-1/2}e^{-k+1}(2\pi)^{1/2}.$$

This gives

$$(3.12) \quad \beta^\beta e^{1-\beta-1/k}\alpha^{1-\beta-1/k-\pi(k)/k}\left(\left(1+\frac{1}{\alpha k}\right) \ldots \left(1+\frac{k-1-\pi(k)-\beta k}{\alpha k}\right)\right)^{1/k} \\ \le k^{1/2k+\pi(k)/k}(2\pi)^{1/2k}.$$

This also implies that

$$(3.13) \qquad \beta^\beta e^{1-\beta-1/k}\alpha^{1-\beta-1/k-\pi(k)/k} \le k^{1/2k+\pi(k)/k}(2\pi)^{1/2k}.$$

Lemma 3.6. *Let $k \ge 33$. Then there exists a subset $S_4 \subseteq S_1$ having at least $\frac{1}{4}k$ elements with $A_i \le 8.5k$.*

Proof. In the previous discussion, we take $\alpha = 8.5$ and $\beta = 1/4$. We use the upper bound for $\pi(n)$ from Lemma 3.1(i) in (3.13) to conclude that $k \le 80$. Now we check that (3.11) is not satisfied for $33 \le k \le 80$ with exact value of $\pi(k)$. This gives the assertion of the lemma. ∎

Lemma 3.7. *Let $14 \le k \le 32$. Then there exist at least three A_i's with $A_i \le 8.5k$, $i > 0$.*

Proof. Suppose there are only at most two A_i's with $A_i \le 8.5k$. Then we apply (3.11) with $\alpha = 8.5$, $\beta = 2/k$ to get a contradiction. ∎

Lemma 3.8. *Let $8 \le k \le 13$. Then there exist at least three A_i's with*

$$A_i \le C = \begin{cases} 120 & \text{if } 9 \le k \le 13, \\ 420 & \text{if } k = 8. \end{cases}$$

Proof. For a given k, we count the A_i's with $i > 0$ which are divisible only by 2, 3, 5 and 7 at most to the powers a, b, c and d, respectively, where

$$(a,b,c,d) = \begin{cases} (3,1,1,0) & \text{for } 9 \le k \le 13, \\ (2,1,1,1) & \text{for } k = 8. \end{cases}$$

For these choices of (a, b, c, d), we get at least three A_i's which are bounded by $2^a 3^b 5^c 7^d$. This proves the result. ∎

4. Proofs of the theorems

Proof of Theorem 1.6. Let $l > 3$ and $k \ge 10^{20}$. We may assume that

$$l^{\omega(d)} \le k/16. \tag{4.1}$$

Let $k \ge 33$. By Lemma 3.6, there exists a subset $S_4 \subseteq S_1$ having at least $\frac{1}{4}k$ elements with $A_i \le 8.5k$. Let $T \subseteq S_4$ be a set having h elements. We shall specify T later. Let T_1 be the set of distinct elements of T, and T_2 the maximal subset of T_1 such that all the products $A_i A_j$ with $A_i, A_j \in T_1$ are distinct. Then by (3.6) and (4.1) we have

$$|T_1| \ge h - k/16 - 2. \tag{4.2}$$

Further by Lemma 3.4,

$$l^{\omega(d)} \ge |T_1| - |T_2| - 2 \tag{4.3}$$

provided

$$n + (k-1)d > k^{(5l-4)/(l-4)}. \tag{4.4}$$

It is known by a result of Erdős [4] (see also Shorey and Tijdeman [17] for a neat proof) that

$$|T_2| \le \pi(8.5k) + (8.5k)^{7/8} + (8.5k)^{3/4} + (8.5k)^{1/2}.$$

Thus

$$l^{\omega(d)} \ge |T_1| - \pi(8.5k) - (8.5k)^{7/8} - (8.5k)^{3/4} - (8.5k)^{1.2} - 2 \tag{4.5}$$

provided (4.4) holds.

Let $l \ge 11$. Then we see that (4.4) holds by (2.1). We take $T = S_4$ as given in Lemma 3.6. Then $|T_1| \ge 3k/16 - 2$ by (4.2) and using (4.5) we get

$$l^{\omega(d)} \ge .05k \tag{4.6}$$

for $k \ge 10^{28}$. Thus $k \le 20 \cdot l^{\omega(d)}$, which gives the assertion.

Next we consider $l = 7$. Let

$$S_5 = \{A_i \in S_4 \mid i \ge k/16\}$$

and take $T = S_5$. By (4.2), Lemma 3.4 and the proof of Lemma 3.3, we have

$$l^{\omega(d)} \ge |S_5| - |T_2| - 2 \ge |T_1| - |T_2| - k/16 - 2 \ge k/8 - |T_2| - 4 \tag{4.7}$$

provided

$$n+(k-1)d > k^{31/3}(16/k)^8 = 16^8k^{7/3},$$

which is satisfied by (2.1). Thus from (4.7) we get

$$l^{\omega(d)} \geq \frac{1}{8}k - \pi(8.5k) - (8.5k)^{7/8} - (8.5k)^{3/4} - (8.5k)^{1/2} - 4 \geq .05k$$

for $k \geq 10^{50}$. Thus $k \leq 20 \cdot l^{\omega(d)}$, which gives the assertion.

Now we consider $l = 5$. Define

$$S_6 = \left\{ A_i \in S_4 \,\middle|\, i \geq \frac{9}{64}k \right\}$$

and take $T = S_6$. As in the case $l = 7$, we have

$$l^{\omega(d)} \geq |S_6| - |T_2| - 2$$

provided

$$n+(k-1)d > k^{21}\left(\frac{64}{9k}\right)^{16} = k^5\left(\frac{64}{9}\right)^{16}. \tag{4.8}$$

Let $k > 10^{3000}$. From (2.3), since $|S'| > |S_6|$, we get

$$n+(k-1)d > \left(\frac{3}{64}(.99)k\log k\right)^5.$$

Thus (4.8) is satisfied if

$$(\log k)^5 > \left(\frac{64}{9}\right)^{16}\left(\frac{64}{3}\right)^5(1.0003)^5.$$

Then we find as in the case of $l = 7$ that $k \leq 23 \cdot l^{\omega(d)}$, which gives the assertion. This completes the proof of Theorem 1.6 for $l \geq 5$.

Lastly, we take up the case $l = 3$. Let $k \geq 10^{1000}$. We take $T = S_5$. Since X_i's exceed 1 for all A_i's in S_5, we apply Lemma 3.5 and assume (4.1) to get

$$|T_1| \geq k/16 - 6.$$

We split the proof into two lemmas. The first is based on the method of Erdős [3].

Lemma 4.1. *Let* $0 < \theta < 1$. *Let* $b_1, \ldots, b_s$ *be integers in* $(k/(\log k)^\theta, 8.5k]$ *such that every proper divisor of* b_i *is* $\leq k/(\log k)^\theta$. *Then there exists an equation*

$$NX^l - MY^l = Ld$$

with $N, M, L \leq 8.5(\log k)^\theta$ *having at least* $h/(2(8.5)^2(\log k)^{3\theta})$ *solutions* (X, Y) *where* $h = k/16 - k/(\log k)^\theta - s - 6$.

Proof. Note that all integers in the interval $(k/(\log k)^\theta, 8.5k]$ of the form p^α with p prime and α minimal belong to the set of b_i's. Suppose an integer

n in $(k/(\log k)^\theta, 8.5k]$ is not divisible by any of the b_i. Then n is a product of at least two distinct primes exceeding $k/(\log k)^\theta$, implying that

$$8.5k \geq n > k^2/(\log k)^{2\theta},$$

which gives $k \leq 4$. Hence every integer in $(k/(\log\ k)^\theta, 8.5k]$, is divisible by some b_i. Let S_7 be the set of A_i's in this interval. We observe that among these A_i's there can be at most s such that if some b_i divides an A_i, it does not divide any other A_j. After removing these A_i's, we are left with a set $S_8 \subseteq S_7$ of A_i's such that

$$|S_8| \geq |T_1| - \frac{k}{(\log k)^\theta} - s \geq \frac{k}{16} - \frac{k}{(\log k)^\theta} - s - 6 \tag{4.9}$$

and there are at least $|S_8|/2$ pairs (A_i, A_j) such that $D_{i,j} = \gcd(A_i, A_j) > k/(\log k)^\theta$. For each pair (A_i, A_j) we have the equality

$$A_i' X_i^l - A_j' X_j^l = D_{i,j}' d$$

where

$$A_i' = \frac{A_i}{D_{i,j}}, \qquad A_j' = \frac{A_j}{D_{i,j}}, \qquad D_{i,j}' = \frac{i-j}{D_{i,j}}.$$

Hence $A_i', A_j', D_{i,j}'$ are all bounded by $8.5(\log k)^\theta$. Thus the distinct pairs (X_i, X_j), which are $|S_8|/2$ in number, satisfy the Thue equations $A_i' X^l - A_j' Y^l = D_{i,j}' d$. The number of such equations is at most $(8.5)^2(\log k)^{3\theta}$. Hence there must be an equation

$$NX^l - MY^l = Dd$$

with N, M, D not exceeding $8.5(\log k)^\theta$ having at least

$$\frac{|S_8|}{2(8.5)^2(\log k)^{3\theta}}$$

solutions. Now the assertion follows from (4.9). ∎

In the next lemma we bound s.

Lemma 4.2. *Let $k \geq 10^{10^{500}}$ and $1/4 \leq \theta < 1$. Let $b_1, \ldots, b_s$ be integers as in Lemma 4.1. Then*

$$s \leq 60 \frac{k}{\log\log k}.$$

Proof. Suppose some $b_i > k/(\log k)^{\theta/2}$. Then all its proper divisors are $> (\log k)^{\theta/2}$. By Brun's sieve, the number of such b_i's is

$$\leq 8.5k \prod_{p \leq (\log k)^{\theta/2}} (1 - 1/p) + 2^{\pi((\log k)^{\theta/2})}.$$

Now we apply estimates (i) and (v) from Lemma 3.1 to get the assertion. ∎

Concluding the proof of Theorem 1.6 for $l = 3$. Let $k \ge 10^{10^{500}}$ and $\theta = 1/4$. Combining Lemmas 4.1 and 4.2 shows that there exists an equation

$$NX^l - MY^l = Dd$$

with N, M, D not exceeding $8.5(\log k)^\theta$ having at least

$$\frac{\frac{k}{16} - \frac{k}{(\log k)^\theta} - \frac{60k}{\log\log k} - 6}{2(8.5)^2(\log k)^{3\theta}}$$

solutions. But by Corollary 1(ii) of Evertse [5] such an equation has at most $4 \cdot 3^{\omega(d)} + 3$ solutions provided $D \le d^{1/5}$. From Saradha and Shorey [13], we find that

$$d > .13k^{1/3}.$$

Since $D \le 8.5(\log k)^{1/4}$, we see that $D \le d^{1/5.}$ Thus we have

$$4 \cdot 3^{\omega(d)} + 3 \ge k\left\{\frac{\frac{1}{16} - \frac{1}{(\log k)^{1/4}} - \frac{60}{\log\log k} - \frac{6}{k}}{2(8.5)^2(\log k)^{3/4}}\right\}.$$

We check that the term in the curly bracket above is $> k^{-1/1001}$ for $k \ge 10^{10^{500}}$. Thus we find that

$$3^{\omega(d)} \ge k^{1-1/1000}.$$

From this and our assumption (4.1) we get the assertion of the theorem. ∎

Proof of Theorem 1.1. Let $0 < f < g < h < k$ be any three indices. Then we have the identity

$$(g-f)A_hX_h^l + (h-g)A_fX_f^l = (h-f)A_gX_g^l. \tag{4.10}$$

Let G be the gcd of $(g-f)A_hX_h^l$, $(h-g)A_fX_f^l$ and $(h-f)A_gX_g^l$. We know that X_f, X_g, X_h are coprime to each other. Hence

$$G \,|\, (g-f)A_h, \quad G \,|\, (h-g)A_f, \quad G \,|\, (h-f)A_g.$$

Let $G = g_1g_2$ with $g_1 \,|\, (g-f)$ and $g_2 \,|\, A_h$. We write $g_2 = g_2^{(1)}g_2^{(2)}$ with $g_2^{(1)} \,|\, (h-g)$ and $g_2^{(2)} \,|\, A_f$. Thus $g_2^{(2)}$ divides both A_f and A_h, and hence $g_2^{(2)} \,|\, (h-f)$. Thus

$$G = g_1g_2^{(1)}g_2^{(2)} \le (g-f)(h-g)(h-f). \tag{4.11}$$

We divide (4.10) by G and put

$$a = \frac{(g-f)A_hX_h^l}{G}, \quad b = \frac{(h-g)A_fX_f^l}{G}, \quad c = \frac{(h-f)A_gX_g^l}{G}.$$

Note that by (4.10),

$$X_h^l \le (h-f)A_gX_g^l, \quad X_f^l \le (h-f)A_gX_g^l. \tag{4.12}$$

Thus

$$N(abc) \le N((g-f)(h-g))(h-f)^{1+2/l}A_fA_hA_g^{1+2/l}X_g^3.$$

Hence, by (1.3), we get

$$(h-f)\frac{A_g X_g^l}{G} \le (N((g-f)(h-g)))^2 (h-f)^{2+4/l} A_f^2 A_h^2 A_g^{2+4/l} X_g^6.$$

Thus, using (4.11), we get

$$X_g^{l-6} \le (g-f)(h-g)(h-f)^{2+4/l}(N((g-f)(h-g)))^2 A_f^2 A_h^2 A_g^{1+4/l}. \tag{4.13}$$

Case 1: $k \ge 33$. We divide the interval $[0,k)$ into $[k/8]$ equal subintervals. If each subinterval contains only at most two indices i,j with $A_i, A_j \in S_4$, then S_4 has at most $k/4$ elements. This contradicts Lemma 3.6. Hence there exists a subinterval containing at least three indices $0 < f < g < h < k$ such that A_f, A_g, A_h are in S_4 and $h - f \le 9$. Also note that $X_f, X_g, X_h > 1$ since $S_4 \subseteq S'$. Hence by (2.2), $X_f, X_g, X_h > k$. Further since $h - f \le 9$, we have

$$(g-f)(h-g) \le 20 \quad \text{and} \quad N((g-f)(h-g)) \le 15.$$

Thus from (4.13) we get

$$k^{l-6} \le X_g^{l-6} \le 2 \cdot 10^{11} k^{5+4/l}, \tag{4.14}$$

implying that

$$(l - 11 - 4/l)\log k \le \log(2 \cdot 10^{11}) \le 26.03.$$

Thus we get

$$l \le 17. \tag{4.15}$$

Further for $l = 13$ and 17, we also have $k \le 10^7$ and 10^2, respectively.

Case 2: $14 \le k \le 32$. In this case by Lemma 3.7, we have three indices $0 < f < g < h < k$ such that $A_f, A_g, A_h \le 8.5k$. We apply (4.13) to get

$$k^{l-6} \le (8.5)^{5+4/l} k^{13+8/l},$$

which implies that

$$(l - 19 - 5/l)\log k \le (5 + 4/l)\log 8.5.$$

This is not valid for any $l \ge 29$. Thus in this case we get

$$l \le 23. \tag{4.16}$$

Case 3: $8 \le k \le 13$. We apply Lemma 3.8 to (4.13). We get

$$k^{l-6} \le C^{5+4/l} k^{8+4/l},$$

which implies that $l \le 23$ if $k \ge 9$ and $l \le 29$ if $k = 8$. Thus Theorem 1.1 follows from (4.15) and (4.16). ■

Proof of Theorem 1.2. From the proof of Theorem 1.1, it is clear that

$$k \le 8 \quad \text{if } l = 29, \qquad k \le 32 \quad \text{if } l = 19, 23.$$

These bounds together with the bounds for $l = 13, 17$ in Case 1 prove Theorem 1.2 for $l \ge 13$. We shall now derive a bound for k when $l = 7, 11$. In this case we may assume that $k > 34$. So we are in Case 1 of the proof of Theorem 1.1. From (4.14) we get

$$X_g^{l-6} \le 2 \cdot 10^{11} k^{5+4/l}.$$

Thus

$$A_g X_g^l \le 8.5k(2 \cdot 10^{11} k^{5+4/l})^{\frac{l}{l-6}} \le 8.5 \cdot 2^{\frac{l}{l-6}} 10^{\frac{11l}{l-6}} k^{\frac{6l-2}{l-6}}.$$

This implies that

$$d \le n + d \le A_g X_g^l \le 8.5 \cdot 2^{\frac{l}{l-6}} 10^{\frac{11l}{l-6}}\ k^{\frac{6l-2}{l-6}}. \tag{4.17}$$

We combine Lemma 3.2 and Theorem 1.6 to give a bound for k in terms of l. We state it as a lemma.

LEMMA 4.3. *Let $l \ge 7$. Suppose $d \le k^\delta$ with $\delta > 1$. Then*

$$k \le l^{l^{4\delta}}.$$

Proof. Let $k \ge 10^{50}$. By Theorem 1.6, we have $k \le 20 l^{\omega(d)}$. Consequently, $\omega(d) \ge 13$ and

$$\begin{aligned} \log d &\le \delta \log k \le \delta(\log 20 + \omega(d) \log l) \\ &\le \frac{\delta \log d}{\log\log d}(\log l)\left(3 + \frac{(\log 20)(\log\log d)}{\log d}\right). \end{aligned}$$

by Lemma 3.2. By the result of [14], we may assume that $d \ge 10^8$ for $l \ge 7$. Hence from the above inequality we get

$$\log\log d \le 3.48\delta \log l, \quad \text{implying} \quad \log d \le l^{3.48\delta}.$$

Thus

$$\begin{aligned} \log k \le \log 20 + \omega(d)\log l &= (\omega(d)\log l)\left(1 + \frac{\log 20}{\omega(d)\log l}\right) \\ &\le 1.2\omega(d)\log l \le 1.2\log d \cdot \log l \le 1.2 l^{3.48\delta}\log l. \end{aligned}$$

Hence

$$k \le l^{1.2 l^{3.48\delta}} \le l^{l^{4\delta}}.$$

Note that this estimate also holds when $k < 10^{11}$. Thus $k \le l^{l^{4\delta}}$ always. ∎

Continuation of the proof of Theorem 1.2. Let $l = 7, 11$. Let $k \ge 10^{120}$. From (4.17), we get

$$d \le \begin{cases} k^{41} & \text{if } l = 7, \\ k^{14} & \text{if } l = 11. \end{cases}$$

We apply Lemma 4.3 with $\delta = 41, 14$ for $l = 7, 11$, respectively, to get

$$k \leq \begin{cases} 7^{7^{164}} & \text{if } l = 7, \\ 11^{11^{56}} & \text{if } l = 11. \end{cases}$$

This proves the assertion of the theorem. ∎

Proof of Theorem 1.4. From (1.4) we see that $x^{m-1} < 2y^{n-1}$, implying

$$x < 2^{\frac{1}{m-1}} y^{\frac{n-1}{m-1}}. \tag{4.18}$$

Further

$$y^n(x-1) = x^m(y-1) + (x-y). \tag{4.19}$$

Let $G = \gcd(y^n(x-1), x^m(y-1), x-y)$. Then $G \leq x - y < x$. Take

$$a = x^m(y-1)/G, \qquad b = (x-y)/G, \qquad c = y^n(x-1)/G.$$

Applying (1.3), we get

$$y^n(x-1)/G \leq (xy(x-1)(y-1)(x-y)/G)^2,$$

which by (4.18) gives

$$y^{n-4} \leq 2^{5/(m-1)} y^{5(n-1)/(m-1)}.$$

Thus

$$y^{mn-4m-6n+9} \leq 2^5.$$

Hence we derive that $m \leq 9$. Further we have the following possibilities:

(i) $m = 9$: $n = 10$ and $y = 2, 3$,
(ii) $m = 8$: $n = 14$, $y = 2$; $n = 13$, $y \leq 32$; $n = 12$, $y \leq 32$,
(iii) $m = 8$: $n \leq 11$,
(iv) $m = 7$: $n \leq 19$,
(v) $m \leq 6$.

The first two possibilities are excluded by a direct verification of (1.4). The possibilities (iii)–(v) give the assertion of the theorem. ∎

Proof of Theorem 1.5. We may assume that y is large and $m > 6$. Further let $\omega(xy(x-1)(y-1)(x-y))$ be bounded by, say, h. Applying (1.2) we get

$$\frac{y^n(x-1)}{G} \leq \frac{6}{5}(xy(x-1)(y-1)(x-y))/G)(\log(x^3y^2))^h/h!,$$

which implies that

$$y^n \ll y^{\frac{2m+2n-4}{m-1}} \left(\frac{3n+2m}{m-1} \log y\right)^h$$

or

$$y^{\frac{n(m-5)}{m-1}} \ll_h n^h (\log y)^h,$$

giving

$$y^{\frac{n(m-6)}{m-1}} \ll_h (\log y)^h.$$

Thus y and n are bounded for $m > 6$, showing also that x and m are bounded. This proves Theorem 1.5. ∎

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