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Manin's conjecture for two quartic del Pezzo surfaces with $3\mathbf{A}_1$ and $\mathbf{A}_1 + \mathbf{A}_2$ singularity types

by

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Contents

1. Introduction. Let $V \subset \mathbb{P}^n$ be a singular del Pezzo surface defined over $\mathbb{Q}$ and anticanonically embedded, and let $U \subset V$ be the open subset formed by deleting the lines from V. Manin's conjecture [FMT89] predicts the asymptotic behaviour of the number of rational points of bounded height on U, namely of the quantity

$$N_{U,H}(B) = \#\{x \in U(\mathbb{Q}) \colon H(x) \leq B\}, \tag{1.1}$$

where $H : \mathbb{P}^n(\mathbb{Q}) \to \mathbb{R}_{>0}$ is the exponential height defined for $(x_0, \dots, x_n) \in \mathbb{Z}^{n+1}$ such that $\gcd(x_0, \dots, x_n) = 1$ by

$$H(x_0 : \dots : x_n) = \max\{|x_i| \colon 0 \leq i \leq n\}.$$

More precisely, if $\widetilde{V}$ denotes the minimal desingularization of V, it is expected that

$$N_{U,H}(B) = c_{V,H} B \log(B)^{\rho-1}(1 + o(1)), \tag{1.2}$$

where $c_{V,H}$ is a constant which has been given a conjectural interpretation by Peyre [Pey95] and where $\rho = \rho_{\widetilde{V}}$ is the rank of the Picard group of $\widetilde{V}$.

In this paper, we are interested in singular del Pezzo surfaces of degree four. These surfaces can be defined as the intersection of two quadrics in $\mathbb{P}^4$. Their classification is well-known and can be found in the work of Coray and

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Tsfasman [CT88]. Up to isomorphism over $\overline{\mathbb{Q}}$, there are fifteen types of such surfaces and they are categorized by their extended Dynkin diagrams, which describe the intersection behaviour of the negative curves on the minimal desingularizations (see for instance [Der06, Table 4]).

From now on, we restrict our attention to surfaces which are split over $\mathbb{Q}$. Manin's conjecture is already known to hold for seven surfaces of different types. Batyrev and Tschinkel have proved it for toric varieties [BT98] (which covers the three types $4\mathbf{A}_1$, $2\mathbf{A}_1+\mathbf{A}_2$ and $2\mathbf{A}_1+\mathbf{A}_3$) and Chambert-Loir and Tschinkel have proved it for equivariant compactifications of vector groups [CLT02] (which covers the type $\mathbf{D}_5$). In these two proofs, the conjecture follows from the study of the height Zeta function

$$Z_{U,H}(s) = \sum_{x \in U(\mathbb{Q})} H(x)^{-s},$$

which is well-defined for $\Re(s) \gg 1$, using techniques coming from harmonic analysis. Let us note that for a certain surface of type $\mathbf{D}_5$, la Bretèche and Browning have given an independent proof [BB07]. Furthermore, they have proved the following result, which is much stronger than (1.2). There exists a monic polynomial of degree $5 = \rho - 1$ such that for any fixed $\varepsilon > 0$,

$$N_{U,H}(B) = c_{V,H} B P(\log(B)) + O(B^{11/12+\varepsilon}). \tag{1.3}$$

Manin's conjecture has also been proved for three other surfaces, a surface of type $\mathbf{D}_4$ by Derenthal and Tschinkel [DT07], a surface of type $\mathbf{A}_1 + \mathbf{A}_3$ by Derenthal [Der09] and a surface of type $\mathbf{A}_4$ by Browning and Derenthal [BD09]. These proofs are intrinsically very different from those using harmonic analysis. They use a passage to universal torsors, which consists in defining a bijection between the set of points to be counted on U and a certain set of integral points on an affine variety of higher dimension. This can be done using only elementary techniques (see Section 4.1 for an example).

The aim of this paper is to give a proof of Manin's conjecture for two other surfaces which are split over $\mathbb{Q}$. The first, $V_1 \subset \mathbb{P}^4$, has singularity type $3\mathbf{A}_1$ and is defined as the intersection of the two quadrics

$$x_0x_1 - x_2^2 = 0, \qquad x_2^2 + x_1x_2 + x_3x_4 = 0.$$

We denote by U_1 the complement of the lines in V_1, and $N_{U_1,H}(B)$ is defined as in (1.1). There are six lines on V_1, given by $x_i = x_2 = x_j = 0$ and $x_0+x_2 = x_1 + x_2 = x_j = 0$ for $i \in \{0,1\}$ and $j \in \{3,4\}$. The three singularities of V_1 are $(1:0:0:0:0)$, $(0:0:0:1:0)$ and $(0:0:0:0:1)$. We see that V_1 is actually split over $\mathbb{Q}$ and thus, if $\widetilde{V_1}$ denotes the minimal desingularization of V_1, the Picard group of $\widetilde{V_1}$ has rank $\rho_1 = 6$. The universal torsor we use is an open subset of the hypersurface embedded in $\mathbb{A}^9 \simeq \operatorname{Spec}(\mathbb{Q}[\eta_1, \dots, \eta_9])$

and defined by

$$\eta_4\eta_5 + \eta_1\eta_6\eta_7 + \eta_8\eta_9 = 0. \tag{1.4}$$

Note that η_2 and η_3 do not appear in the equation.

The second surface $V_2 \subset \mathbb{P}^4$ has singularity type $\mathbf{A}_1 + \mathbf{A}_2$ and is defined as the intersection of the two quadrics

$$x_0x_1 - x_2x_3 = 0, \quad x_1x_2 + x_2x_4 + x_3x_4 = 0.$$

The open subset U_2, $N_{U_2,H}(B)$ and $\widetilde{V_2}$ are defined in a similar way. There are also six lines on V_2, given by $x_i = x_2 = x_j = 0$ for $i \in \{0,1\}$ and $j \in \{3,4\}$, $x_1 = x_3 = x_4 = 0$ and $x_0 = x_3 = x_1 + x_4 = 0$. The two singularities of V_2 are $(1:0:0:0:0)$ and $(0:0:0:0:1)$, of type $\mathbf{A}_2$ and $\mathbf{A}_1$ respectively. Just as before we have $\rho_2 = 6$. In this case, the universal torsor we use is an open subset of the hypersurface embedded in $\mathbb{A}^9 \simeq \mathrm{Spec}(\mathbb{Q}[\xi_1,\dots,\xi_9])$ and defined by

$$\xi_4\xi_5 + \xi_1^2\xi_6\xi_7 + \xi_8\xi_9 = 0. \tag{1.5}$$

We immediately see that the equations (1.4) and (1.5) are very much alike and it is not hard to imagine that the proofs have strong similarities; that is why we have decided to couple them in this paper.

This work has been motivated by a result of Browning [Bro07, Theorem 3]. Using the equation (1.4) of the universal torsor described above, he has proved the upper bound of the expected order of magnitude for $N_{U_1,H}(B)$, namely

$$N_{U_1,H}(B) \ll B\log(B)^5. \tag{1.6}$$

In most of the proofs of Manin's conjecture for del Pezzo surfaces using universal torsors, the first step consists in summing over two variables, viewing the torsor equation as a congruence and counting the number of integers lying in a prescribed region and satisfying this congruence. The novelty here is that we start by summing over three variables instead. In our two cases, this is linked to studying the distribution of the values of a certain restricted divisor function in arithmetic progressions. In this task, Weil's bound for Kloosterman sums plays a crucial role. Our result is the following.

Theorem 1. *For $i = 1,2$, as B tends to $+\infty$, we have the estimate*

$$N_{U_i,H}(B) = c_{V_i,H}B\log(B)^5\left(1 + O\left(\frac{\log(\log(B))}{\log(B)}\right)\right),$$

where $c_{V_1,H}$ and $c_{V_2,H}$ agree with Peyre's prediction.

Since $\rho_1 = \rho_2 = 6$, these estimates prove that Manin's conjecture holds for V_1 and V_2. Note that Derenthal has shown that V_1 and V_2 are not toric [Der06, Proposition 12], and Derenthal and Loughran have proved that they are not equivariant compactifications of $\mathbb{G}_a^2$ [DL10], so this work is not covered by the existing general results. In view of Theorem 1, it remains to deal with six types of split singular quartic del Pezzo surfaces among the list of fifteen.

In both cases, we have noted that the universal torsor is an open subset of a hypersurface embedded in $\mathbb{A}^9$. In [Der06], Derenthal has determined the del Pezzo surfaces whose universal torsors are hypersurfaces and it turns out that in the case of split quartic surfaces, Manin's conjecture has only been proved for surfaces whose universal torsors are either open affine subsets (which is equivalent to being toric), or open subsets of hypersurfaces. It would be interesting to prove Manin's conjecture for a surface which is in neither of these two classes.

2. Preliminary results

2.1. Equidistribution of the values of a restricted divisor function in arithmetic progressions. Let τ denote the divisor function. We start by recalling a classical fact about the sums of the values of τ in arithmetic progressions. For $a, q \in \mathbb{Z}_{\geq 1}$ two coprime integers and $X \geq 1$, define

$$D(X; q, a) = \sum_{\substack{n \leq X \\ n \equiv a \ (\mathrm{mod}\, q)}} \tau(n).$$

Then (see [HB79, Corollary 1] for instance) there exists an explicit quantity $D^*(X; q)$ independent of a such that for $q \leq X^{2/3}$,

$$D(X; q, a) - D^*(X; q) \ll X^{1/3+\varepsilon}.$$

We need a more general result since we have to consider a sum similar to $D(X; q, a)$ but with τ replaced by a function which only counts certain divisors. However, we will not determine a specific value of our main term and we will content ourselves with the value provided by averaging the estimate over a coprime to q.

The results stated in this section use several classical ideas which have for example been developed in Heath-Brown's investigation of the divisor function $\tau_3 := \tau * 1$ [HB86]. Let $\mathcal{I}$ and $\mathcal{J}$ be two ranges. We define

$$N(\mathcal{I}, \mathcal{J}; q, a) = \#\{(u, v) \in \mathcal{I} \times \mathcal{J} \cap \mathbb{Z}^2 \colon uv \equiv a \ (\mathrm{mod}\ q)\},$$
$$N^*(\mathcal{I}, \mathcal{J}; q) = \frac{1}{\varphi(q)} \#\{(u, v) \in \mathcal{I} \times \mathcal{J} \cap \mathbb{Z}^2 \colon \gcd(uv, q) = 1\}.$$

Lemma 1. *Let $\varepsilon > 0$ be fixed. Then*

$$N(\mathcal{I}, \mathcal{J}; q, a) - N^*(\mathcal{I}, \mathcal{J}; q) \ll q^{1/2+\varepsilon}.$$

Proof. Let e_q be the function defined by $e_q(x) = e^{2i\pi x/q}$. We detect the congruence using sums of exponentials:

$$\begin{aligned}
N(\mathcal{I},\mathcal{J};q,a) &= \sum_{\substack{\alpha,\beta=1\\ \alpha\beta\equiv a \,(\mathrm{mod}\, q)}}^{q} \#\{(u,v)\in\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2 \colon q \mid \alpha-u,\beta-v\}\\
&= \sum_{\substack{\alpha,\beta=1\\ \alpha\beta\equiv a \,(\mathrm{mod}\, q)}}^{q} \frac{1}{q^2}\Big(\sum_{u\in\mathcal{I}}\sum_{r=1}^{q} e_q(r\alpha-ru)\Big)\Big(\sum_{v\in\mathcal{J}}\sum_{s=1}^{q} e_q(s\beta-sv)\Big)\\
&= \frac{1}{q^2}\sum_{r,s=1}^{q} K(r,as,q)F_q(r,s),
\end{aligned}$$

where $K(r,as,q)$ is the Kloosterman sum defined by

$$K(r,as,q) = \sum_{\substack{\alpha=1\\ \gcd(\alpha,q)=1}}^{q} e_q(r\alpha+as\alpha^{-1}),$$

where α^{-1} denotes the inverse of α modulo q and where

$$F_q(r,s) = \Big(\sum_{u\in\mathcal{I}} e_q(-ru)\Big)\Big(\sum_{v\in\mathcal{J}} e_q(-sv)\Big).$$

Let $\|x\|$ denote the distance from x to the set of integers. If $r,s\neq q$, $F_q(r,s)$ is a product of two geometric sums and so

$$F_q(r,s) \ll \|r/q\|^{-1}\|s/q\|^{-1}.$$

Let $N(\mathcal{I},\mathcal{J};q)$ be the sum of the terms corresponding to $r=q$ or $s=q$. Since $\gcd(a,q)=1$, we see that $K(q,as,q)$ and $K(r,aq,q)$ are independent of a and thus $N(\mathcal{I},\mathcal{J};q)$ is also independent of a. We are therefore led to give a bound for $N(\mathcal{I},\mathcal{J};q,a)-N(\mathcal{I},\mathcal{J};q)$. Weil's bound for Kloosterman sums (see [Est61]) yields

$$\begin{aligned}
N(\mathcal{I},\mathcal{J};q,a)-N(\mathcal{I},\mathcal{J};q) &= \frac{1}{q^2}\sum_{r,s=1}^{q-1} K(r,as,q)F_q(r,s)\\
&\ll \frac{1}{q^2}\tau(q)q^{1/2}\sum_{r,s=1}^{q-1} \gcd(r,s,q)^{1/2}\|r/q\|^{-1}\|s/q\|^{-1}\\
&\ll \tau(q)q^{1/2}\sum_{0<|r|,|s|\leq q/2} \gcd(r,s,q)^{1/2}|r|^{-1}|s|^{-1}.
\end{aligned}$$

Let us bound the sum on the right-hand side:

$$\sum_{0<|r|,|s|\leq q/2} \gcd(r,s,q)^{1/2}|r|^{-1}|s|^{-1} \ll \sum_{d\mid q} d^{1/2}\sum_{\substack{r=1\\ d\mid r}}^{q} r^{-1}\sum_{\substack{s=1\\ d\mid r}}^{q} s^{-1} \ll \log(q)^2.$$

Since $N(\mathcal{I},\mathcal{J};q)$ does not depend on a, averaging over a coprime to q shows that we can replace $N(\mathcal{I},\mathcal{J};q)$ by $N^*(\mathcal{I},\mathcal{J};q)$. ∎

An immediate consequence of this lemma is the bound

$$N(\mathcal{I},\mathcal{J};q,a) \ll \frac{1}{\varphi(q)}\#(\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2)+q^{1/2+\varepsilon}. \tag{2.1}$$

Lemmas 2 and 3 below are respectively devoted to the treatment of the varieties V_1 and V_2.

Let $X,X_1,X_2,T,Z,L_1,L_2>0$. Let also $\mathcal{S}=\mathcal{S}(X,X_1,X_2,T,Z,L_1,L_2)$ be the set of $(x,y)\in\mathbb{R}^2$ such that

$$|xy|\le X, \tag{2.2}$$
$$|x|\,|xy+T|\le X_1, \tag{2.3}$$
$$|y|\le X_2, \tag{2.4}$$
$$Z\le |xy+T|, \tag{2.5}$$
$$L_1\le |x|, \tag{2.6}$$
$$L_2\le |y|. \tag{2.7}$$

Finally, we introduce

$$D(\mathcal{S};q,a)=\#\{(u,v)\in\mathcal{S}\cap\mathbb{Z}^2\colon uv\equiv a \pmod q\}, \tag{2.8}$$
$$D^*(\mathcal{S};q)=\frac{1}{\varphi(q)}\#\{(u,v)\in\mathcal{S}\cap\mathbb{Z}^2\colon \gcd(uv,q)=1\}. \tag{2.9}$$

Lemma 2. *Let $\varepsilon>0$ be fixed. If $T\le X$ then for $q\le X^{2/3}$,*

$$D(\mathcal{S};q,a)-D^*(\mathcal{S};q)\ll \frac{X^{2/3+\varepsilon}}{q^{1/2}}+\frac{X}{\varphi(q)}\left(\frac{1}{L_1}+\frac{1}{L_2}\right).$$

Note that the conditions $T\le X$, $|xy|\le X$ and $|xy+T|\ge Z$ imply $Z\le 2X$.

Proof. The result is true if $\mathcal{S}\cap\mathbb{Z}^2_{\neq 0}=\emptyset$ so assume $\mathcal{S}\cap\mathbb{Z}^2_{\neq 0}\neq\emptyset$. Let $0<\delta\le 1$, to be selected in due course, and $\zeta=1+\delta$. Let also U and V be variables running over the set $\{\pm\zeta^n\colon n\in\mathbb{Z}_{\ge -1}\}$ and let $\mathcal{I}=]U,\zeta U]$ if $U>0$ ($\mathcal{I}=[\zeta U,U[$ if $U<0$) and $\mathcal{J}=]V,\zeta V]$ if $V>0$ ($\mathcal{J}=[\zeta V,V[$ if $V<0$). We have

$$D(\mathcal{S};q,a)-\sum_{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\subset\mathcal{S}}N(\mathcal{I},\mathcal{J};q,a)\ll \sum_{\substack{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathcal{S}\\ \mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathbb{R}^2\setminus\mathcal{S}}}N(\mathcal{I},\mathcal{J};q,a).$$

We define

$$D(\mathcal{S};q)=\sum_{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\subset\mathcal{S}}N^*(\mathcal{I},\mathcal{J};q).$$

Note that since $N^*(\mathcal{I},\mathcal{J};q)$ is independent of a, so is $D(\mathcal{S};q)$. Furthermore,

$$\sum_{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\subset\mathcal{S}} N(\mathcal{I},\mathcal{J};q,a)-D(\mathcal{S};q)\ll\frac{X^{\varepsilon}q^{1/2+\varepsilon}}{\delta^2},$$

using Lemma 1 and noticing that the number of rectangles $\mathcal{I}\times\mathcal{J}$ with $\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\subset\mathcal{S}$ is less than $(2\log(X)/\log(\zeta))^2\ll X^{\varepsilon}\delta^{-2}$ since $\delta\leq 1$. Since $q^{\varepsilon}\leq X^{\varepsilon}$, we have obtained

$$D(\mathcal{S};q,a)-D(\mathcal{S};q)\ll\sum_{\substack{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathcal{S}\\ \mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathbb{R}^2\setminus\mathcal{S}}} N(\mathcal{I},\mathcal{J};q,a)+\frac{X^{\varepsilon}q^{1/2}}{\delta^2}.$$

Using the bound (2.1), we finally deduce

$$D(\mathcal{S};q,a)-D(\mathcal{S};q)\ll\frac{1}{\varphi(q)}\sum_{\substack{\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathcal{S}\\ \mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathbb{R}^2\setminus\mathcal{S}}}\#(\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2)+\frac{X^{\varepsilon}q^{1/2}}{\delta^2},$$

since the number of rectangles $\mathcal{I}\times\mathcal{J}$ such that $\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathcal{S}$ and $\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2\not\subset\mathbb{R}^2\setminus\mathcal{S}$ is also $\ll X^{\varepsilon}\delta^{-2}$. The sum on the right-hand side is over all the rectangles $\mathcal{I}\times\mathcal{J}$ for which $(\zeta^{s_1}U,\zeta^{s_2}V)\in\mathbb{Z}^2\cap\mathcal{S}$ and $(\zeta^{t_1}U,\zeta^{t_2}V)\in\mathbb{Z}^2\setminus\mathcal{S}$ for some $(s_1,s_2)\in\,]0,1]^2$ and $(t_1,t_2)\in\,]0,1]^2$. This implies that one of the inequalities defining $\mathcal{S}$ is not satisfied by $(\zeta^{t_1}U,\zeta^{t_2}V)$ and we need to estimate the contribution coming from each condition among (2.2)–(2.7). Note that (2.2), (2.6) and (2.7) together imply

$$|U|\ll X/L_2, \tag{2.10}$$

$$|V|\ll X/L_1. \tag{2.11}$$

In the following, we could sometimes write strict inequalities instead of non-strict ones but this would not change anything in our reasoning. Let us start by treating the case of (2.2). For the rectangles $\mathcal{I}\times\mathcal{J}$ described above, we have $\zeta^{s_1+s_2}|UV|\leq X$ and $\zeta^{t_1+t_2}|UV|>X$ for some $(s_1,s_2)\in\,]0,1]^2$ and $(t_1,t_2)\in\,]0,1]^2$. These two inequalities imply

$$\zeta^{-2}X<|UV|\leq X. \tag{2.12}$$

Going back to the variables u and v, we get $\zeta^{-2}X<|uv|\leq\zeta^2X$. Therefore, the error we aim to estimate is bounded by

$$\begin{aligned}\sum_{(2.11),\,(2.12)}\#(\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2)&\ll\#\left\{(u,v)\in\mathbb{Z}^2_{\neq 0}:\begin{array}{l}\zeta^{-2}X<|uv|\leq\zeta^2X\\ |v|\ll X/L_1\end{array}\right\}\\ &\ll\sum_{|v|\ll X/L_1}(\delta X/|v|+1)\ll\delta X^{1+\varepsilon}+X/L_1.\end{aligned}$$

We now deal with the other conditions in a similar fashion. Let us treat (2.3). In this case, for some $(s_1, s_2) \in]0,1]^2$ and $(t_1, t_2) \in]0,1]^2$,

$$\zeta^{s_1} |U| \, |\zeta^{s_1+s_2} UV + T| \leq X_1, \tag{2.13}$$

$$\zeta^{t_1} |U| \, |\zeta^{t_1+t_2} UV + T| > X_1. \tag{2.14}$$

Note that since $|UV| \leq X$ and $T \leq X$, the inequality (2.14) gives

$$|U| \gg X_1/X. \tag{2.15}$$

The inequalities (2.13) and (2.14) imply

$$\zeta^{-3} \frac{X_1}{|U|} - (1-\zeta^{-2})T < |UV + T| \leq \frac{X_1}{|U|} + (1-\zeta^{-2})T. \tag{2.16}$$

Going back to the variables u and v, we easily get

$$\big| |uv + T| - |UV + T| \big| \leq |uv - UV| \leq 3\delta |UV| \leq 3\delta (X_1/|U| + T),$$

using (2.13). Since $1 - \zeta^{-2} \leq 2\delta$, the inequality (2.16) gives

$$(\zeta^{-3} - 3\delta) \frac{X_1}{|U|} - 5\delta T < |uv + T| \leq (1 + 3\delta) \frac{X_1}{|U|} + 5\delta T,$$

and therefore

$$(\zeta^{-3} - 3\delta) \frac{X_1}{|u|} - 5\delta T < |uv + T| \leq \zeta (1 + 3\delta) \frac{X_1}{|u|} + 5\delta T. \tag{2.17}$$

Note that we have not tried to sharpen this inequality because this is useless for our purpose. Thus in this case, the error is bounded by

$$\sum_{\substack{(2.10) \\ (2.15), (2.16)}} \#(\mathcal{I} \times \mathcal{J} \cap \mathbb{Z}^2) \ll \# \left\{ (u,v) \in \mathbb{Z}^2_{\neq 0} : \begin{array}{l} (2.17) \\ |u| \gg X_1/X \\ |u| \ll X/L_2 \end{array} \right\}$$

$$\ll \sum_{\substack{|u| \gg X_1/X \\ |u| \ll X/L_2}} \left(\frac{\delta X_1}{u^2} + \frac{\delta T}{|u|} + 1 \right) \ll \delta X^{1+\varepsilon} + X/L_2,$$

since $T \leq X$. In the case of (2.4), the condition which plays the role of (2.12) and (2.16) in the previous two cases is

$$\zeta^{-1} X_2 < |V| \leq X_2. \tag{2.18}$$

Combined with $|UV| \leq X$, this gives

$$|U| \ll X/X_2. \tag{2.19}$$

Moreover, in terms of v, we have $\zeta^{-1} X_2 < |v| \leq \zeta X_2$. Therefore, this contribution is bounded by

$$\sum_{\substack{(2.10)\\(2.18),(2.19)}} \#(\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2) \ll \#\left\{(u,v)\in\mathbb{Z}^2_{\neq 0}\colon \begin{array}{l}\zeta^{-1}X_2<|v|\le\zeta X_2\\ |u|\ll X/X_2\\ |u|\ll X/L_2\end{array}\right\}$$
$$\ll \sum_{\substack{|u|\ll X/X_2\\ |u|\ll X/L_2}} (\delta X_2+1) \ll \delta X + X/L_2.$$

Let us now deal with (2.5). Here, reasoning as we did to deduce (2.16) from (2.13) and (2.14), we get

$$\zeta^{-2}Z-(1-\zeta^{-2})T\le |UV+T| < Z+(1-\zeta^{-2})T, \tag{2.20}$$

and following the reasoning we made to derive (2.17) from (2.16), we obtain

$$(\zeta^{-2}-3\delta)Z-5\delta T\le |uv+T|<(1+3\delta)Z+5\delta T. \tag{2.21}$$

Therefore, this contribution is bounded by

$$\sum_{(2.10),(2.20)} \#(\mathcal{I}\times\mathcal{J}\cap\mathbb{Z}^2) \ll \#\left\{(u,v)\in\mathbb{Z}^2_{\neq 0}\colon \begin{array}{l}(2.21)\\ |u|\ll X/L_2\end{array}\right\}$$
$$\ll \sum_{|u|\ll X/L_2}\left(\frac{\delta Z}{|u|}+\frac{\delta T}{|u|}+1\right) \ll \delta X^{1+\varepsilon}+X/L_2,$$

since $T\le X$ and $Z\le 2X$. Mimicking what we have done for (2.4), we find that the contributions corresponding to (2.6) and (2.7) are respectively $\ll \delta X+X/L_1$ and $\ll \delta X+X/L_2$. Writing $1/\varphi(q)\ll X^{\varepsilon}/q$ and rescaling ε, we have finally proved that

$$D(\mathcal{S};q,a)-D(\mathcal{S};q)\ll X^{\varepsilon}\left(\frac{\delta X}{q}+\frac{q^{1/2}}{\delta^2}\right)+\frac{X}{\varphi(q)}\left(\frac{1}{L_1}+\frac{1}{L_2}\right).$$

Averaging over a coprime to q and using the fact that $D(\mathcal{S};q)$ does not depend on a, we can replace $D(\mathcal{S};q)$ by $D^*(\mathcal{S};q)$. Furthermore, the choice $\delta=q^{1/2}X^{-1/3}$ is allowed provided that $q\le X^{2/3}$. ∎

We emphasize that the average effect which yields the term $1/\varphi(q)$ in $D^*(\mathcal{S};q)$ is the key step of the proof. Note that the estimate of Lemma 2 is actually true for $q\le X$ but the error term is no longer better than the trivial error term $X^{1+\varepsilon}/q$ when $q\ge X^{2/3}$.

For given $X_3>0$, let $\mathcal{S}_1=\mathcal{S}_1(X,X_1,X_2,X_3,T,Z,L_1,L_2)$ be the set of $(x,y)\in\mathbb{R}^2$ satisfying (2.2)–(2.7) and

$$|xy+T|\le X, \tag{2.22}$$
$$|x|y^2\le X_3. \tag{2.23}$$

Let also $\mathcal{S}_2 = \mathcal{S}_2(X, X_1, X_2, X_3, T, Z, L_1, L_2)$ be the set of $(x, y) \in \mathbb{R}^2$ satisfying (2.2), (2.5)–(2.7), (2.22) and

$$|x| \leq X_1, \tag{2.24}$$

$$|y|\,|xy + T| \leq X_2, \tag{2.25}$$

$$|x| y^2 |xy + T| \leq X_3. \tag{2.26}$$

Finally, $D(\mathcal{S}_1; q, a)$ and $D(\mathcal{S}_2; q, a)$ are defined as in (2.8), and $D^*(\mathcal{S}_1; q)$ and $D^*(\mathcal{S}_2; q)$ as in (2.9).

Lemma 3. *Let $\varepsilon > 0$ be fixed. If $T \leq 2X$ then for $q \leq X^{2/3}$,*

$$D(\mathcal{S}_1; q, a) - D^*(\mathcal{S}_1; q) \ll \frac{X^{2/3+\varepsilon}}{q^{1/2}} + \frac{X}{\varphi(q)}\left(\frac{1}{L_1} + \frac{1}{L_2}\right),$$

$$D(\mathcal{S}_2; q, a) - D^*(\mathcal{S}_2; q) \ll \frac{X^{4/5+\varepsilon}}{q^{7/10}} + \frac{X}{\varphi(q)}\left(\frac{1}{L_1} + \frac{1}{L_2}\right).$$

To prove Lemma 3, we can proceed almost exactly as in the proof of Lemma 2 except that the condition (2.26) is more complicated than the others. Indeed, it is the only condition where both x and y appear with powers greater than or equal to 2. To solve this problem, we need the following result.

Lemma 4. *Let $0 < \delta \leq 1$, $Y \in \mathbb{R}_{>0}$ and $A, Y' \in \mathbb{R}$ be such that $0 < Y - Y' \ll \delta M^2$ where $M = \max(|A|, Y^{1/2})$. Let $\mathcal{R}$ be the set of real numbers y subject to*

$$Y' < |y^2 + 2Ay| \leq Y. \tag{2.27}$$

Then

$$\#(\mathcal{R} \cap \mathbb{Z}) \ll \delta^{1/2} M + 1.$$

Proof. It clearly suffices to show that $\operatorname{meas}(\mathcal{R}) \ll \delta^{1/2} M$. If we set $z = y + A$, the condition (2.27) can be rewritten as $Y' < |z^2 - A^2| \leq Y$.

Let us treat first the case where $z^2 - A^2 > 0$. If $Y' + A^2 > 0$ then

$$(Y' + A^2)^{1/2} < |z| \leq (Y + A^2)^{1/2}.$$

Therefore,

$$\begin{aligned}\operatorname{meas}(\mathcal{R} \cap \{y \in \mathbb{R} : (y + A)^2 > A^2\}) &\ll (Y + A^2)^{1/2} - (Y' + A^2)^{1/2} \\ &= \frac{Y - Y'}{(Y + A^2)^{1/2} + (Y' + A^2)^{1/2}} \ll \delta M,\end{aligned}$$

which is satisfactory. Now if $Y' + A^2 \leq 0$ then $Y + A^2 \leq Y - Y' \ll \delta M^2$ and thus

$$\operatorname{meas}(\mathcal{R} \cap \{y \in \mathbb{R} : (y + A)^2 > A^2\}) \ll (Y + A^2)^{1/2} \ll \delta^{1/2} M.$$

Let us now deal with the case where $z^2 - A^2 \leq 0$. Under this assumption, we have $A^2 - Y \leq z^2 < A^2 - Y'$ so we can assume that $A^2 - Y' > 0$. First if $A^2 - Y \geq 0$ then

$$\operatorname{meas}(\mathcal{R} \cap \{y \in \mathbb{R} \colon (y+A)^2 \leq A^2\}) \ll (A^2 - Y')^{1/2} - (A^2 - Y)^{1/2} \ll \delta^{1/2} M,$$

where we have used $a^{1/2} - b^{1/2} \leq (a-b)^{1/2}$, valid for any $a \geq b \geq 0$. Finally, if $A^2 - Y < 0$ then $A^2 - Y' < Y - Y' \ll \delta M^2$ and thus

$$\operatorname{meas}(\mathcal{R} \cap \{y \in \mathbb{R} \colon (y+A)^2 \leq A^2\}) \ll \delta^{1/2} M. \blacksquare$$

Proof of Lemma 3. We proceed as in the proof of Lemma 2. The only thing we have to do is to repeat for all our new conditions what we have done for (2.2)–(2.7). By symmetry between (2.24), (2.25) and (2.3), (2.4), it suffices to consider the cases of the conditions (2.22), (2.23) and (2.26). Reasoning as for (2.5), we see that the contribution corresponding to (2.22) is $\ll \delta X^{1+\varepsilon} + X/L_2$. In the case of (2.23), we have

$$\zeta^{-3} X_3 < |U| V^2 \leq X_3. \tag{2.28}$$

Combined with $|UV| \leq X$, this implies

$$|U| \ll X^2/X_3. \tag{2.29}$$

In terms of u and v, we have $\zeta^{-3} X_3 < |u| v^2 \leq \zeta^3 X_3$. Therefore,

$$\sum_{\substack{(2.10)\\(2.28),(2.29)}} \#(\mathcal{I} \times \mathcal{J} \cap \mathbb{Z}^2) \ll \# \left\{ (u,v) \in \mathbb{Z}^2_{\neq 0} \colon \begin{array}{l} \zeta^{-3} X_3 < |u| v^2 \leq \zeta^3 X_3 \\ |u| \ll X^2/X_3 \\ |u| \ll X/L_2 \end{array} \right\}$$

$$\ll \sum_{\substack{|u| \ll X^2/X_3\\ |u| \ll X/L_2}} \left(\frac{\delta X_3^{1/2}}{|u|^{1/2}} + 1 \right) \ll \delta X + X/L_2.$$

Finally, let us deal with the case of (2.26). For some $(s_1, s_2) \in\,]0,1]^2$ and $(t_1, t_2) \in\,]0,1]^2$, we have

$$\zeta^{s_1 + 2s_2} |U| V^2 |\zeta^{s_1+s_2} UV + T| \leq X_3, \tag{2.30}$$

$$\zeta^{t_1 + 2t_2} |U| V^2 |\zeta^{t_1+t_2} UV + T| > X_3. \tag{2.31}$$

Since $|UV| \leq X$ and $T \leq 2X$, the condition (2.31) gives

$$|V| \gg X_3/X^2. \tag{2.32}$$

For $t \in \mathbb{R}_{\neq 0}$, we set

$$M(t) = \max(X_3^{1/2}/|t|^{3/2}, T/|t|).$$

The condition (2.30) shows that $|U| \leq 2M(V)$. The inequalities (2.30) and (2.31) imply

$$\zeta^{-5} \frac{X_3}{|U| V^2} - (1 - \zeta^{-2}) T < |UV + T| \leq \frac{X_3}{|U| V^2} + (1 - \zeta^{-2}) T. \tag{2.33}$$

To go back to u and v, we can proceed as we did to deduce (2.17) from (2.16) in the proof of Lemma 2. We get

$$(\zeta^{-5} - 3\delta)\frac{X_3}{|U|V^2} - 5\delta T < |uv + T| \leq (1 + 3\delta)\frac{X_3}{|U|V^2} + 5\delta T,$$

and thus, multiplying by $|U|$ and using $|U| \leq 2M(V)$, we obtain

$$(\zeta^{-5} - 3\delta)\frac{X_3}{V^2} - 10\delta M(V)T < |u|\,|uv + T| \leq \zeta(1 + 3\delta)\frac{X_3}{V^2} + 10\zeta\delta M(V)T.$$

Setting $c = 10\zeta^{3/2}$ for short and using $M(V) \leq \zeta^{3/2}M(v)$ and $T/|v| \leq M(v)$, we finally see that

$$\begin{aligned} (2.34)\qquad (\zeta^{-5} - 3\delta)\frac{X_3}{|v|^3} &- c\delta M(v)^2 \\ &< |u|\left|u + \frac{T}{v}\right| \leq \zeta^3(1 + 3\delta)\frac{X_3}{|v|^3} + \zeta c\delta M(v)^2. \end{aligned}$$

Applying Lemma 4 to count the number of u subject to (2.34), we get

$$\begin{aligned} \sum_{\substack{(2.11)\\(2.32),(2.33)}} \#(\mathcal{I} \times \mathcal{J} \cap \mathbb{Z}^2) &\ll \#\left\{(u,v) \in \mathbb{Z}^2_{\neq 0} : \begin{array}{c} (2.34) \\ |v| \gg X_3/X^2 \\ |v| \ll X/L_1 \end{array}\right\} \\ &\ll \sum_{\substack{|v| \gg X_3/X^2\\|v| \ll X/L_1}} \left(\frac{\delta^{1/2}X_3^{1/2}}{|v|^{3/2}} + \frac{\delta^{1/2}T}{|v|} + 1\right) \\ &\ll \delta^{1/2}X^{1+\varepsilon} + X/L_1, \end{aligned}$$

since $T \leq X$. In the case of $\mathcal{S}_1$, the proof can be completed as that of Lemma 2. In the case of $\mathcal{S}_2$, the optimal choice of δ is seen to be $\delta = q^{3/5}X^{-2/5}$, which yields the result claimed. ∎

2.2. Arithmetic functions. Let us introduce the following arithmetic functions:

$$\varphi^*(n) = \prod_{p \mid n}\left(1 - \frac{1}{p}\right), \tag{2.35}$$

$$\varphi^\dagger(n) = \prod_{p \mid n}\left(1 - \frac{1}{p^2}\right), \tag{2.36}$$

$$\varphi'(n) = \prod_{p \mid n}\left(1 - \frac{1}{p}\right)^{-1}\left(1 + \frac{1}{p} - \frac{1}{p^2}\right)^{-1}. \tag{2.37}$$

Define also, for $a,b,c\geq 1$,

$$\psi_{a,b,c}(n)=\begin{cases}\varphi^*(n)^2\varphi^*(\gcd(n,a))^{-1}\varphi^*(\gcd(n,b))^{-1} & \text{if } \gcd(n,c)=1,\\ 0 & \text{otherwise.}\end{cases}$$

Finally, for $\sigma>0$, let

$$\varphi_\sigma(n)=\sum_{k\mid n}2^{\omega(k)}k^{-\sigma}, \tag{2.38}$$

where $\omega(k)$ denotes the number of prime numbers dividing k. The next two lemmas are built following the reasoning of [BD09, Section 3].

Lemma 5. *Let $0<\sigma\leq 1$ be fixed. Then*

$$\sum_{n\leq X}\psi_{a,b,c}(n)=\mathcal{P}\varPsi(a,b,c)X+O_\sigma(\varphi_\sigma(c)X^\sigma),$$

where

$$\mathcal{P}=\prod_p\varphi'(p)^{-1},\qquad \varPsi(a,b,c)=\varphi^*(c)\frac{\varphi^\dagger(abc)}{\varphi^\dagger(\gcd(a,b)c)}\varphi'(abc). \tag{2.39}$$

Proof. Let us calculate the Dirichlet convolution of $\psi_{a,b,c}$ with the Möbius function μ. We have

$$(\psi_{a,b,c}*\mu)(n)=\sum_{d\mid n}\psi_{a,b,c}\left(\frac{n}{d}\right)\mu(d)=\prod_{p^\nu\|n}(\psi_{a,b,c}(p^\nu)-\psi_{a,b,c}(p^{\nu-1})).$$

Moreover $\psi_{a,b,c}(1)=1$ and for all $\nu\geq 1$,

$$\psi_{a,b,c}(p^\nu)=\psi_{a,b,c}(p)=\begin{cases}(1-1/p)^2 & \text{if } p\nmid abc,\\ 1-1/p & \text{if } p\nmid c,\ p\mid ab \text{ and } p\nmid\gcd(a,b),\\ 1 & \text{if } p\nmid c \text{ and } p\mid\gcd(a,b),\\ 0 & \text{if } p\mid c.\end{cases}$$

Thus, we get

$$(\psi_{a,b,c}*\mu)(n)=\mu(n)2^{\omega(n)-\omega(\gcd(n,abc))}\frac{\gcd(c,n)}{n}\prod_{p\mid n,\,p\nmid abc}\left(1-\frac{1}{2p}\right)$$

whenever $\gcd(a,b,n)\mid c$, and $(\psi_{a,b,c}*\mu)(n)=0$ otherwise. Writing $\psi_{a,b,c}=(\psi_{a,b,c}*\mu)*1$ yields

$$\sum_{n\leq X}\psi_{a,b,c}(n)=\sum_{n\leq X}\sum_{d\mid n}(\psi_{a,b,c}*\mu)(d)=\sum_{d=1}^{+\infty}(\psi_{a,b,c}*\mu)(d)\left[\frac{X}{d}\right].$$

Let $0 < \sigma \le 1$ be fixed. Let us use the elementary estimate $[t] = t + O(t^\sigma)$ for $t = X/d$. Since

$$\sum_{d=1}^{+\infty} \frac{|(\psi_{a,b,c} * \mu)(d)|}{d^\sigma} \le \sum_{d=1}^{+\infty} 2^{\omega(d)} \frac{\gcd(c,d)}{d^{1+\sigma}} \ll \varphi_\sigma(c),$$

we have shown that

$$\sum_{n \le X} \psi_{a,b,c}(n) = X \sum_{d=1}^{+\infty} \frac{(\psi_{a,b,c} * \mu)(d)}{d} + O(\varphi_\sigma(c) X^\sigma).$$

A straightforward calculation finally yields

$$\begin{aligned}\sum_{d=1}^{+\infty} \frac{(\psi_{a,b,c} * \mu)(d)}{d} &= \prod_{p \mid c} \left(1 - \frac{1}{p}\right) \prod_{\substack{p \nmid c,\, p \mid ab \\ p \nmid \gcd(a,b)}} \left(1 - \frac{1}{p^2}\right) \\ &\quad \times \prod_{p \nmid c,\, p \nmid ab} \left(1 - \frac{2}{p^2}\left(1 - \frac{1}{2p}\right)\right) \\ &= \varphi^*(c) \frac{\varphi^\dagger(ab)}{\varphi^\dagger(\gcd(ab, \gcd(a,b)c))} \mathcal{P}\varphi'(abc). \ \blacksquare\end{aligned}$$

Lemma 6. *Let $0 < \sigma \le 1$ be fixed. Let $0 \le t_1 < t_2$ and $I = [t_1, t_2]$. Let also $g : \mathbb{R}_{>0} \to \mathbb{R}$ be a function having a piecewise continuous derivative on I whose sign changes at most $R_g(I)$ times on I. Then*

$$\sum_{n \in I \cap \mathbb{Z}_{>0}} \psi_{a,b,c}(n) g(n) = \mathcal{P}\Psi(a,b,c) \int_I g(t)\,dt + O_\sigma(\varphi_\sigma(c) t_2^\sigma M_I(g)),$$

where $M_I(g) = (1 + R_g(I)) \sup_{t \in I \cap \mathbb{R}_{>0}} |g(t)|$.

Proof. We only treat the case where $t_1 > 0$ since the statement for $t_1 = 0$ easily follows from it. Let S be the function defined for $t > 0$ by

$$S(t) = \sum_{n \le t} \psi_{a,b,c}(n).$$

Splitting I into several ranges, we can assume that g has a continuous derivative. Partial summation gives

$$\sum_{n \in]t_1, t_2] \cap \mathbb{Z}_{>0}} \psi_{a,b,c}(n) g(n) = S(t_2) g(t_2) - S(t_1) g(t_1) - \int_{t_1}^{t_2} S(t) g'(t)\,dt.$$

Lemma 5 implies that $S(t) = \mathcal{P}\Psi(a,b,c) t + O(\varphi_\sigma(c) t^\sigma)$. An integration by parts reveals that the sum to be estimated is equal to

$$\mathcal{P}\Psi(a,b,c) \int_I g(t)\,dt + O\Big(\varphi_\sigma(c) t_2^\sigma \Big(|g(t_2)| + |g(t_1)| + \int_I |g'(t)|\,dt\Big)\Big).$$

It only remains to split I into the $R_g(I)+1$ ranges where g' has constant sign. ∎

2.3. Lemma for the final summation. Let $r\geq 1$ and $\mathbf{n}=(n_1,\dots,n_r)\in\mathbb{Z}_{>0}^r$ (and by analogy, $\mathbf{d}$, $\mathbf{k}$). Let $\mathcal{V}$ be the set of $\mathbf{n}\in\mathbb{Z}_{>0}^r$ satisfying the following conditions, indexed by $1\leq j\leq N$:

$$\prod_{i=1}^r n_i^{\beta_{i,j}}\leq X^{\varepsilon_j},$$

where $X\geq 1$ is a quantity independent of j and $\varepsilon_j\in\{-1,0,1\}$, and $\beta_{i,j}\in\mathbb{Q}$ are bounded by an absolute constant and such that the polytope $\mathcal{C}$ defined by $t_1,\dots,t_r\geq 0$ and the N inequalities

$$\sum_{i=1}^r \beta_{i,j}t_i\leq\varepsilon_j$$

satisfies $\mathcal{C}\subset[0,1]^r$. We are concerned with sums of the form

$$\sum_{\mathbf{n}\in\mathcal{V}}\frac{\Psi(\mathbf{n})}{n_1\cdots n_r},$$

where Ψ is an arithmetic function of r variables.

Lemma 7. *Let f be the characteristic function of a polytope $\mathcal{D}\subset[0,1]^r$. Then*

$$\sum_{n_1,\dots,n_r\leq X}\frac{f\big(\frac{\log(n_1)}{\log(X)},\dots,\frac{\log(n_r)}{\log(X)}\big)}{n_1\cdots n_r}=\mathrm{vol}(\mathcal{D})\log(X)^r+O(\log(X)^{r-1}).$$

Proof. Let us reason by induction on r. Let f be the characteristic function of $[a,b]\subset[0,1]$. We have

$$\sum_{n\leq X}\frac{f\big(\frac{\log(n)}{\log(X)}\big)}{n}=\sum_{X^a\leq n\leq X^b}\frac{1}{n}=(b-a)\log(X)+O(1),$$

as wished. Assume that the result is true for an integer $r-1\geq 1$ and let us prove it for r. The result for $r=1$ applied to n_r shows that the sum to be estimated is equal to

$$\sum_{n_1,\dots,n_{r-1}\leq X}\frac{1}{n_1\cdots n_{r-1}}\bigg(\log(X)\int_0^1 f\bigg(\frac{\log(n_1)}{\log(X)},\dots,\frac{\log(n_{r-1})}{\log(X)},t_r\bigg)\,dt_r+O(1)\bigg).$$

This quantity is plainly equal to

$$\log(X)\int_0^1\bigg(\sum_{n_1,\dots,n_{r-1}\leq X}\frac{f\big(\frac{\log(n_1)}{\log(X)},\dots,\frac{\log(n_{r-1})}{\log(X)},t_r\big)}{n_1\cdots n_{r-1}}\bigg)\,dt_r+O(\log(X)^{r-1}),$$

so the induction assumption applied to the $r-1$ remaining variables immediately completes the proof. ■

LEMMA 8. *Let Ψ be an arithmetic function of r variables satisfying*

$$\sum_{\mathbf{n}\in\mathbb{Z}_{>0}^r} \frac{|(\Psi*\boldsymbol{\mu})(\mathbf{n})|}{n_1\cdots n_r}\log\Big(2\prod_{i=1}^r n_i\Big) < +\infty, \tag{2.40}$$

where $\boldsymbol{\mu}$ is the generalized Möbius function defined by

$$\boldsymbol{\mu}(n_1,\dots,n_r) = \mu(n_1)\cdots\mu(n_r).$$

Then

$$\sum_{\mathbf{n}\in\mathcal{V}} \frac{\Psi(\mathbf{n})}{n_1\cdots n_r} = \mathrm{vol}(\mathcal{C})\Big(\sum_{\mathbf{n}\in\mathbb{Z}_{>0}^r}\frac{(\Psi*\boldsymbol{\mu})(\mathbf{n})}{n_1\cdots n_r}\Big)\log(X)^r + O(\log(X)^{r-1}).$$

Proof. Writing $\Psi = (\Psi*\boldsymbol{\mu})*\mathbf{1}$, we get

$$\sum_{\mathbf{n}\in\mathcal{V}} \frac{\Psi(\mathbf{n})}{n_1\cdots n_r} = \sum_{\mathbf{n}\in\mathcal{V}}\sum_{d_1|n_1,\dots,d_r|n_r}\frac{(\Psi*\boldsymbol{\mu})(\mathbf{d})}{n_1\cdots n_r} = \sum_{\mathbf{d}\in\mathbb{Z}_{>0}^r}\frac{(\Psi*\boldsymbol{\mu})(\mathbf{d})}{d_1\cdots d_r}\sum_{\mathbf{k}}\frac{1}{k_1\cdots k_r},$$

where the latter sum is over $\mathbf{k}$ such that

$$\Big(\prod_{i=1}^r k_i^{\beta_{i,j}}\Big)\Big(\prod_{i=1}^r d_i^{\beta_{i,j}}\Big) \le X^{\varepsilon_j}. \tag{2.41}$$

Let us estimate the difference between this sum and the sum over $\mathbf{k}$ satisfying

$$\prod_{i=1}^r k_i^{\beta_{i,j}} \le X^{\varepsilon_j}.$$

For a certain j_0, we have

$$X^{\varepsilon_{j_0}}\Big(\prod_{i=1}^r d_i^{\beta_{i,j_0}}\Big)^{-1} \le \prod_{i=1}^r k_i^{\beta_{i,j_0}} \le X^{\varepsilon_{j_0}}.$$

Summing first over k_{i_0} for which $\beta_{i_0,j_0}\neq 0$, we see that since the $\beta_{i,j}$ are bounded by an absolute constant, the above difference is bounded by

$$\log\Big(\prod_{i=1}^r d_i\Big)\sum_{k_1,\dots,\widehat{k}_{i_0},\dots,k_r}\frac{1}{k_1\cdots\widehat{k}_{i_0}\cdots k_r} \ll \log\Big(\prod_{i=1}^r d_i\Big)\log(X)^{r-1}.$$

Thus, Lemma 7 yields

$$\sum_{\mathbf{k},\,(2.41)}\frac{1}{k_1\cdots k_r} = \mathrm{vol}(\mathcal{C})\log(X)^r + O\Big(\log\Big(2\prod_{i=1}^r d_i\Big)\log(X)^{r-1}\Big).$$

The assumption (2.40) plainly implies the result. ■

3. Proof for the $3\mathbf{A}_1$ surface

3.1. The universal torsor. Using elementary techniques, Browning [Bro07] has made explicit a bijection between the set of points to be counted on U_1 and a certain set of integral points on the hypersurface defined by (1.4). A little thought reveals that the result proved by Browning [Bro07, Lemma 1] is equivalent to the following. We adopt the notation of Derenthal [Der06]. Let $\mathcal{T}_1(B)$ be the set of $(\eta_1,\eta_2,\eta_3,\eta_4,\eta_5,\eta_6,\eta_7,\eta_8,\eta_9) \in \mathbb{Z}_{\neq 0}^9$ such that $\eta_1,\eta_2,\eta_3,\eta_6,\eta_7 > 0$ and

$$\eta_4\eta_5 + \eta_1\eta_6\eta_7 + \eta_8\eta_9 = 0, \tag{3.1}$$

and satisfying the coprimality conditions

$$\gcd(\eta_8, \eta_1\eta_2\eta_4\eta_5\eta_6\eta_7) = 1, \tag{3.2}$$
$$\gcd(\eta_4, \eta_1\eta_2\eta_6\eta_7\eta_9) = 1, \tag{3.3}$$
$$\gcd(\eta_5, \eta_1\eta_3\eta_6\eta_7\eta_9) = 1, \tag{3.4}$$
$$\gcd(\eta_6, \eta_2\eta_7\eta_9) = 1, \tag{3.5}$$
$$\gcd(\eta_3, \eta_1\eta_2\eta_7\eta_9) = 1, \tag{3.6}$$
$$\gcd(\eta_1, \eta_2\eta_9) = 1, \tag{3.7}$$
$$\gcd(\eta_9, \eta_7) = 1 \tag{3.8}$$

and the height conditions

$$\eta_2\eta_3\eta_4^2\eta_5^2 \leq B, \tag{3.9}$$
$$\eta_1^2\eta_2\eta_3\eta_6^2\eta_7^2 \leq B, \tag{3.10}$$
$$\eta_1\eta_3^2|\eta_4|\eta_6^2|\eta_8| \leq B, \tag{3.11}$$
$$\eta_1\eta_2^2|\eta_5|\eta_7^2|\eta_9| \leq B. \tag{3.12}$$

Lemma 9. *We have the equality*

$$N_{U_1,H}(B) = \frac{1}{2}\#\mathcal{T}_1(B).$$

Browning [Bro07, Theorem 3] has used this description of the problem to prove the bound (1.6).

It is important to notice here that the contribution to $N_{U_1,H}(B)$ coming from the $(\eta_1,\ldots,\eta_9) \in \mathcal{T}_1(B)$ such that all the variables appearing in the torsor equation are bounded by an absolute constant is $\gg B$ since $\eta_2,\eta_3 \leq B^{1/2}$. That is why a result similar to (1.3) seems out of reach.

3.2. Calculation of Peyre's constant. The constant $c_{V_1,H}$ predicted by Peyre is

$$c_{V_1,H} = \alpha(\widetilde{V_1})\beta(\widetilde{V_1})\omega_H(\widetilde{V_1}),$$

where $\alpha(\widetilde{V_1}) \in \mathbb{Q}$ is the volume of a certain polytope in the dual of the effective cone of $\widetilde{V_1}$ with respect to the intersection form and where $\beta(\widetilde{V_1}) =$

$\#H^1(\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}), \mathrm{Pic}_{\overline{\mathbb{Q}}}(\widetilde{V_1}))$ and

$$\omega_H(\widetilde{V_1}) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^6 \omega_p,$$

with ω_∞ and ω_p being respectively the archimedean and p-adic densities. The work of Derenthal [Der07] reveals that

$$\alpha(\widetilde{V_1}) = \frac{1}{1440}.$$

Moreover, $\beta(\widetilde{V}) = 1$ for any del Pezzo surface V split over $\mathbb{Q}$ and finally, using a result of Loughran [Lou10, Lemma 2.3], we get

$$\omega_p = 1 + \frac{6}{p} + \frac{1}{p^2}.$$

Let us calculate ω_∞. Set $f_1(x) = x_0x_1 - x_2^2$ and $f_2(x) = x_2^2 + x_1x_2 + x_3x_4$. We parametrize the points of V_1 by x_0, x_2 and x_4. We have

$$\det \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_3} \end{pmatrix} = \begin{vmatrix} x_0 & 0 \\ x_2 & x_4 \end{vmatrix} = x_0x_4.$$

Moreover, $x_1 = x_2^2/x_0$ and $x_3 = -x_2^2(x_2+x_0)/(x_0x_4)$. Since $\mathbf{x} = -\mathbf{x}$ in $\mathbb{P}^4$,

$$\omega_\infty = 2 \iiint\limits_{x_0,x_4>0,\, x_0, x_2^2/x_0,\, x_2^2|x_2+x_0|/|x_0x_4|,\, x_4 \leq 1} \frac{dx_0\, dx_2\, dx_4}{x_0x_4}.$$

Define the function

$$h : (t_4,t_5,t_6) \mapsto \max\{t_4^2t_5^2, t_6^2, |t_4|t_6^2|t_4t_5+t_6|, |t_5|\}. \tag{3.13}$$

The change of variables given by $x_0 = t_4^2t_5^2$, $x_2 = t_4t_5t_6$ and $x_4 = t_5$ yields

$$\begin{aligned} \omega_\infty &= 4 \iiint\limits_{t_5,t_6>0,\, h(t_4,t_5,t_6)\leq 1} dt_4\, dt_5\, dt_6 \\ &= 2 \iiint\limits_{t_6>0,\, h(t_4,t_5,t_6)\leq 1} dt_4\, dt_5\, dt_6. \end{aligned} \tag{3.14}$$

3.3. Restriction of the domain. In order to be able to control the error terms showing up in our estimations, we need to assume that certain variables are greater in absolute value than a fixed power of $\log(B)$. The following result shows that this assumption does not affect the main term predicted by Manin's conjecture.

LEMMA 10. *Let $\mathcal{M}_1(B)$ be the overall contribution to $N_{U_1,H}(B)$ coming from the $(\eta_1,\dots,\eta_9) \in \mathcal{T}_1(B)$ such that $|\eta_i| \leq \log(B)^A$ for a certain $i \neq 2,3$, where $A > 0$ is any fixed constant. Then*

$$\mathcal{M}_1(B) \ll_A B \log(B)^4 \log(\log(B)).$$

LEMMA 11. *Let $K_1, K_4, \dots, K_9 \geq 1/2$ and define $M_1 = M_1(K_1, K_4, \dots, K_9)$ as the number of $(m_1, m_4, \dots, m_9) \in \mathbb{Z}^7$ such that $K_i < |m_i| \leq 2K_i$ for $i = 1$ and $4 \leq i \leq 9$, $\gcd(m_4 m_5, m_1 m_6 m_7) = 1$ and*

$$m_4 m_5 + m_1 m_6 m_7 + m_8 m_9 = 0. \tag{3.15}$$

Then

$$M_1 \ll K_1 K_6 K_7 \min(K_4 K_5, K_8 K_9).$$

Proof. We can assume by symmetry that $K_4 K_5 \leq K_8 K_9$. Let us first deal with the case where $K_1 K_6 K_7 \leq K_4 K_5$. Then (3.15) gives $K_8 K_9 \ll K_4 K_5$. Let M_1' be the number of $(m_1, m_4, \dots, m_9) \in \mathbb{Z}^7$ to be counted in this case. We can assume by symmetry that $K_4 \leq K_5$. The idea is to view (3.15) as a congruence modulo m_4. Since $|m_4| \ll (K_8 K_9)^{1/2}$, the number of m_5, m_8 and m_9 to be counted in M_1' is at most

$$\#\left\{ (m_8, m_9) \in \mathbb{Z}^2 : \begin{array}{l} K_i < |m_i| \leq 2K_i,\ i \in \{8, 9\} \\ \gcd(m_8 m_9, m_1 m_6 m_7) = 1 \\ m_8 m_9 \equiv -m_1 m_6 m_7 \pmod{m_4} \end{array} \right\} \ll \frac{K_8 K_9}{m_4}.$$

Summing over m_1, m_4, m_6 and m_7, we get

$$M_1' \ll K_1 K_6 K_7 K_8 K_9 \sum_{K_4 < |m_4| \leq 2K_4} \frac{1}{m_4} \ll K_1 K_6 K_7 K_4 K_5,$$

since $K_8 K_9 \ll K_4 K_5$.

We now treat the case where $K_1 K_6 K_7 > K_4 K_5$. Then (3.15) gives $K_8 K_9 \ll K_1 K_6 K_7$. Let M_1'' be the number of $(m_1, m_4, \dots, m_9) \in \mathbb{Z}^7$ to be counted under this assumption. We assume by symmetry that $K_8 \leq K_9$, which yields $|m_8| \ll (K_1 K_6 K_7)^{1/2}$. We can therefore use [HB03, Lemma 5] to deduce that the number of m_1, m_6, m_7 and m_9 to be counted in M_1'' is at most

$$\#\left\{ (m_1, m_6, m_7) \in \mathbb{Z}^3 : \begin{array}{l} K_i < |m_i| \leq 2K_i,\ i \in \{1, 6, 7\} \\ \gcd(m_1 m_6 m_7, m_4 m_5) = 1 \\ m_1 m_6 m_7 \equiv -m_4 m_5 \pmod{m_8} \end{array} \right\} \ll \frac{K_1 K_6 K_7}{\varphi(m_8)}.$$

We obtain

$$M_1'' \ll K_1 K_6 K_7 K_4 K_5 \sum_{K_8 < |m_8| \leq 2K_8} \frac{1}{\varphi(m_8)} \ll K_1 K_6 K_7 K_4 K_5,$$

as wished. ∎

We are now in a position to prove Lemma 10. Note that the following proof is largely inspired by Browning's proof of [Bro07, Theorem 3].

Proof of Lemma 10. Let $Y_i \geq 1/2$ for $i = 1, \dots, 9$ and define $\mathcal{N}_1 = \mathcal{N}_1(Y_1, \dots, Y_9)$ as the contribution of the $(\eta_1, \dots, \eta_9) \in \mathcal{T}_1(B)$ satisfying

$Y_i < |\eta_i| \leq 2Y_i$ for $i = 1, \dots, 9$. The height conditions imply that either $\mathcal{N}_1 = 0$ or we have the inequalities

$$Y_2 Y_3 Y_4^2 Y_5^2 \leq B, \tag{3.16}$$

$$Y_1^2 Y_2 Y_3 Y_6^2 Y_7^2 \leq B, \tag{3.17}$$

$$Y_1 Y_3^2 Y_4 Y_6^2 Y_8 \leq B, \tag{3.18}$$

$$Y_1 Y_2^2 Y_5 Y_7^2 Y_9 \leq B. \tag{3.19}$$

Using Lemma 11 and summing over η_2 and η_3, we get

$$\mathcal{N}_1 \ll Y_1 Y_2 Y_3 Y_6 Y_7 \min(Y_4 Y_5, Y_8 Y_9).$$

Let us recall the following basic estimates. Assume that we have to sum over all the ranges $Y < |y| \leq 2Y$ for all $|y| \leq \mathcal{Y}$; then

$$\sum_{Y \leq \mathcal{Y}} Y^{\delta} \ll_{\delta} \begin{cases} 1 & \text{if } \delta < 0, \\ \log(\mathcal{Y}) & \text{if } \delta = 0, \\ \mathcal{Y}^{\delta} & \text{if } \delta > 0. \end{cases}$$

In the following, the notation $\sum_{\widehat{Y}}$ means that the summation is over all the $Y_i \neq Y$. We only treat the case where $Y_4 Y_5 \leq Y_8 Y_9$ (the case $Y_4 Y_5 > Y_8 Y_9$ is identical).

First assume that $Y_1 Y_6 Y_7 \leq Y_4 Y_5$. We start by summing over

$$Y_6 \leq \min\left(\frac{Y_4 Y_5}{Y_1 Y_7}, \frac{B^{1/2}}{Y_1^{1/2} Y_3 Y_4^{1/2} Y_8^{1/2}}\right) \leq \frac{Y_4^{1/4} Y_5^{1/2} B^{1/4}}{Y_1^{3/4} Y_3^{1/2} Y_7^{1/2} Y_8^{1/4}},$$

and over Y_3 using (3.16). We get in this case

$$\begin{aligned} \sum_{Y_i} \mathcal{N}_1 &\ll \sum_{Y_i} Y_1 Y_2 Y_3 Y_4 Y_5 Y_6 Y_7 \\ &\ll B^{1/4} \sum_{\widehat{Y}_6} Y_1^{1/4} Y_2 Y_3^{1/2} Y_4^{5/4} Y_5^{3/2} Y_7^{1/2} Y_8^{-1/4} \\ &\ll B^{3/4} \sum_{\widehat{Y}_3, \widehat{Y}_6} Y_1^{1/4} Y_2^{1/2} Y_4^{1/4} Y_5^{1/2} Y_7^{1/2} Y_8^{-1/4}. \end{aligned}$$

Now sum over Y_2 using (3.19) and over $Y_4 \leq Y_5^{-1} Y_8 Y_9$ to obtain

$$\sum_{Y_i} \mathcal{N}_1 \ll B \sum_{\widehat{Y}_2, \widehat{Y}_3, \widehat{Y}_6} Y_4^{1/4} Y_5^{1/4} Y_8^{-1/4} Y_9^{-1/4} \ll B \sum_{\widehat{Y}_2, \widehat{Y}_3, \widehat{Y}_4, \widehat{Y}_6} 1.$$

We could have summed over Y_5 instead of Y_4 and over Y_7 instead of Y_6, so if we assume that $|\eta_i| \leq \log(B)^A$ for a certain $i \neq 2, 3$, where $A > 0$ is any fixed constant, we get an overall contribution $\ll_A B \log(B)^4 \log(\log(B))$.

Let us now assume $Y_1 Y_6 Y_7 > Y_4 Y_5$. Since $Y_4 Y_5 \leq Y_8 Y_9$, we deduce from (3.1) that $Y_1 Y_6 Y_7 \ll Y_8 Y_9$. Summing over Y_3 using (3.18) and over Y_2 using

(3.19) yields

$$\sum_{Y_i} \mathcal{N}_1 \ll B \sum_{\widehat{Y}_2, \widehat{Y}_3} Y_4^{1/2} Y_5^{1/2} Y_8^{-1/2} Y_9^{-1/2}$$
$$\ll B \sum_{\widehat{Y}_2, \widehat{Y}_3, \widehat{Y}_4} Y_1^{1/2} Y_6^{1/2} Y_7^{1/2} Y_8^{-1/2} Y_9^{-1/2} \ll B \sum_{\widehat{Y}_2, \widehat{Y}_3, \widehat{Y}_4, \widehat{Y}_6} 1,$$

where we have summed over $Y_4 < Y_1 Y_5^{-1} Y_6 Y_7$ and $Y_6 \ll Y_1^{-1} Y_7^{-1} Y_8 Y_9$. We can now conclude exactly as in the first case. ■

3.4. Setting up. To be able to apply Lemma 2, we need to assume

$$|\eta_9| \leq |\eta_8|.$$

Note that this assumption together with (3.1) and the height conditions (3.9) and (3.10) yield the following condition which plays a crucial role in the proof:

$$\eta_9^2 \leq 2 \frac{B^{1/2}}{\eta_2^{1/2} \eta_3^{1/2}}. \tag{3.20}$$

The symmetry given by $(\eta_3, \eta_4, \eta_6, \eta_8) \mapsto (\eta_2, \eta_5, \eta_7, \eta_9)$ and the following lemma prove that it suffices to multiply our main term by 2 to take into account this new assumption.

LEMMA 12. *Let $N_0(B)$ be the overall contribution from the $(\eta_1, \ldots, \eta_9) \in \mathcal{T}_1(B)$ such that $|\eta_8| = |\eta_9|$. Then*

$$N_0(B) \ll B \log(B).$$

Proof. Note that we have the inequality (3.20) here too. Define

$$\mathcal{X} = \frac{B^{1/2}}{\eta_2^{1/2} \eta_3^{1/2}}.$$

The number of η_1, η_4 and η_5 to be counted is

$$\ll \# \left\{ (\eta_1, \eta_4, \eta_5) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{\neq 0}^2 \colon \begin{array}{l} \eta_4 \eta_5 = \pm \eta_9^2 + \eta_1 \eta_6 \eta_7 \\ |\eta_4 \eta_5| \leq \mathcal{X} \end{array} \right\}$$
$$\ll \# \left\{ (\eta_4, \eta_5) \in \mathbb{Z}_{\neq 0}^2 \colon \begin{array}{l} \eta_4 \eta_5 \equiv \pm \eta_9^2 \pmod{\eta_6 \eta_7} \\ |\eta_4 \eta_5| \leq \mathcal{X} \end{array} \right\}$$
$$\ll \sum_{\substack{1 \leq |n| \leq \mathcal{X} \\ n \equiv \pm \eta_9^2 \,(\mathrm{mod}\, \eta_6 \eta_7)}} \tau(|n|) \ll \mathcal{X}^{\varepsilon} \left(\frac{\mathcal{X}}{\eta_6 \eta_7} + 1 \right),$$

for all $\varepsilon > 0$. Taking $\varepsilon = 1/4$ and summing over η_9 using (3.20), we get

$$N_0(B) \ll \sum_{\eta_2,\eta_3,\eta_6,\eta_7} \left(\frac{B^{7/8}}{\eta_2^{7/8}\eta_3^{7/8}\eta_6\eta_7} + \frac{B^{3/8}}{\eta_2^{3/8}\eta_3^{3/8}} \right)$$
$$\ll \sum_{\eta_2,\eta_6,\eta_7} \frac{B}{\eta_2\eta_6^{5/4}\eta_7^{5/4}} \ll B\log(B),$$

where we have summed over η_3 using (3.10). ■

Since $(\eta_8,\eta_9) \mapsto (-\eta_8,-\eta_9)$ is a bijection between the set of solutions with $\eta_9 > 0$ and the set of solutions with $\eta_9 < 0$, we can assume that $\eta_9 > 0$ if we multiply our main term by 2 once again. Furthermore, we need to assume that η_4 and η_5 are greater in absolute value than a power of $\log(B)$. To sum up, denote by $N(A,B)$ the contribution to $N_{U_1,H}(B)$ from the $(\eta_1,\ldots,\eta_9) \in \mathcal{T}_1(B)$ satisfying

$$0 < \eta_9 \le |\eta_8|, \tag{3.21}$$
$$\log(B)^A \le |\eta_4|, \tag{3.22}$$
$$\log(B)^A \le |\eta_5|, \tag{3.23}$$

where $A > 0$ is a constant to be chosen later. Note that combining (3.9) and (3.22), we get

$$\log(B)^{2A}\eta_2\eta_3\eta_5^2 \le B. \tag{3.24}$$

This inequality is crucial in the estimation of our error terms. Lemmas 9, 10 and 12 yield the following result.

Lemma 13. *For any fixed $A > 0$,*

$$N_{U_1,H}(B) = 2N(A,B) + O(B\log(B)^4\log(\log(B))).$$

Our goal is now to estimate $N(A,B)$ and for this, we start by investigating the contribution of the variables η_4, η_5 and η_8. The idea is to view the torsor equation (3.1) as a congruence modulo η_9. For this, we replace the height conditions (3.11) and (3.21) by the following (we keep denoting them by (3.11) and (3.21)), obtained using (3.1):

$$\eta_1\eta_3^2|\eta_4|\eta_6^2|\eta_4\eta_5+\eta_1\eta_6\eta_7|\eta_9^{-1} \le B, \quad \eta_9^2 \le |\eta_4\eta_5+\eta_1\eta_6\eta_7|.$$

Set $\boldsymbol{\eta}' = (\eta_1,\eta_2,\eta_3,\eta_6,\eta_7,\eta_9) \in \mathbb{Z}_{>0}^6$. Assume that $\boldsymbol{\eta}'$ is fixed and subject to the height conditions (3.10) and (3.20) and the coprimality conditions (3.5)–(3.8). Let $N(\boldsymbol{\eta}',B)$ be the number of η_4, η_5, η_8 satisfying the torsor equation (3.1), the height conditions (3.9), (3.11) and (3.12), the conditions (3.21)–(3.23) and the coprimality conditions (3.2)–(3.4). Recalling the definition (2.35) of φ^*, we have the following result.

LEMMA 14. *For any fixed $A \geq 7$,*

$$N(\boldsymbol{\eta}', B) = \frac{1}{\eta_9} \sum_{\substack{k_8 \mid \eta_2 \\ \gcd(k_8, \eta_7)=1}} \frac{\mu(k_8)}{k_8 \varphi^*(k_8 \eta_9)} \sum_{\substack{k_4 \mid \eta_1\eta_2\eta_6\eta_7 \\ \gcd(k_4, k_8\eta_9)=1}} \mu(k_4) \sum_{\substack{k_5 \mid \eta_1\eta_3\eta_6\eta_7 \\ \gcd(k_5, k_8\eta_9)=1}} \mu(k_5)$$
$$\times \sum_{\substack{\ell_4 \mid k_8\eta_9 \\ \ell_5 \mid k_8\eta_9}} \mu(\ell_4)\mu(\ell_5) C(\boldsymbol{\eta}', B) + R(\boldsymbol{\eta}', B),$$

where, with the notations $\eta_4 = k_4 \ell_4 \eta_4''$ and $\eta_5 = k_5 \ell_5 \eta_5''$,

$$C(\boldsymbol{\eta}', B) = \#\left\{ (\eta_4'', \eta_5'') \in \mathbb{Z}^2_{\neq 0} : \begin{array}{l} (3.9), (3.11), (3.12) \\ (3.21)\text{–}(3.23) \end{array} \right\},$$

and $\sum_{\boldsymbol{\eta}'} R(\boldsymbol{\eta}', B) \ll B \log(B)^2$.

The thrust of Lemma 14 is that the summation over η_8 has been carried out, which explains the absence of the torsor equation in $C(\boldsymbol{\eta}', B)$. The remainder of this section is devoted to proving Lemma 14.

Let us remove the coprimality condition (3.2) using a Möbius inversion. We get

$$N(\boldsymbol{\eta}', B) = \sum_{k_8 \mid \eta_1\eta_2\eta_4\eta_5\eta_6\eta_7} \mu(k_8) S_{k_8}(\boldsymbol{\eta}', B),$$

where

$$S_{k_8}(\boldsymbol{\eta}', B) = \#\left\{ (\eta_4, \eta_5, \eta_8') \in \mathbb{Z}^3_{\neq 0} : \begin{array}{l} \eta_4\eta_5 + k_8\eta_8'\eta_9 = -\eta_1\eta_6\eta_7 \\ (3.9), (3.11), (3.12) \\ (3.21)\text{–}(3.23) \\ (3.3), (3.4) \end{array} \right\}.$$

Clearly, if $\gcd(k_8, \eta_1\eta_6\eta_7) \neq 1$ or $\gcd(k_8, \eta_4\eta_5) \neq 1$ then $\gcd(\eta_4\eta_5, \eta_1\eta_6\eta_7) \neq 1$ and thus $S_{k_8}(\boldsymbol{\eta}', B) = 0$. We can therefore assume that $\gcd(k_8, \eta_1\eta_4\eta_5\eta_6\eta_7) = 1$. We have

$$S_{k_8}(\boldsymbol{\eta}', B) = \#\left\{ (\eta_4, \eta_5) \in \mathbb{Z}^2_{\neq 0} : \begin{array}{l} \eta_4\eta_5 \equiv -\eta_1\eta_6\eta_7 \pmod{k_8\eta_9} \\ (3.9), (3.11), (3.12) \\ (3.21)\text{–}(3.23) \\ (3.3), (3.4) \end{array} \right\} + R_0(\boldsymbol{\eta}', B),$$

where the error term $R_0(\boldsymbol{\eta}', B)$ comes from the fact that η_8' has to be nonzero. Otherwise, we would have $\eta_4\eta_5 = -\eta_1\eta_6\eta_7$ and so the coprimality condition $\gcd(\eta_4\eta_5, \eta_1\eta_6\eta_7) = 1$ would give $|\eta_4| = |\eta_5| = \eta_1 = \eta_6 = \eta_7 = 1$.

Summing over η_9 using (3.20), we obtain

$$\sum_{k_8,\boldsymbol{\eta}'} |\mu(k_8)| R_0(\boldsymbol{\eta}', B) \ll \sum_{\eta_2,\eta_3,\eta_9} 2^{\omega(\eta_2)} \ll \sum_{\eta_2,\eta_3} 2^{\omega(\eta_2)} \frac{B^{1/4}}{\eta_2^{1/4}\eta_3^{1/4}} \ll B\log(B)^2.$$

Let us remove the coprimality conditions (3.3) and (3.4). The main term of $N(\boldsymbol{\eta}', B)$ is equal to

$$\sum_{\substack{k_8 \mid \eta_2 \\ \gcd(k_8,\eta_1\eta_6\eta_7)=1}} \mu(k_8) \sum_{\substack{k_4 \mid \eta_1\eta_2\eta_6\eta_7\eta_9 \\ \gcd(k_4,k_8\eta_9)=1}} \mu(k_4) \sum_{\substack{k_5 \mid \eta_1\eta_3\eta_6\eta_7\eta_9 \\ \gcd(k_5,k_8\eta_9)=1}} \mu(k_5) S(\boldsymbol{\eta}', B),$$

where, with the notations $\eta_4 = k_4\eta_4'$ and $\eta_5 = k_5\eta_5'$,

$$S(\boldsymbol{\eta}', B) = \# \left\{ (\eta_4', \eta_5') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} \eta_4'\eta_5' \equiv -(k_4k_5)^{-1}\eta_1\eta_6\eta_7 \pmod{k_8\eta_9} \\ (3.9), (3.11), (3.12) \\ (3.21)\text{–}(3.23) \end{array} \right\}.$$

Indeed, k_4 and k_5 are invertible modulo $k_8\eta_9$ since $\gcd(k_8\eta_9, \eta_1\eta_6\eta_7) = 1$. We can therefore remove η_9 from the conditions on k_4 and k_5. Having in mind that our aim is to apply Lemma 2, we define

$$X = \frac{B^{1/2}}{k_4k_5\eta_2^{1/2}\eta_3^{1/2}}.$$

Let us prove that we can assume that $k_8 \leq (2k_4k_5)^{-1/2}X^{1/6}$, the contribution coming from the condition $k_8 > (2k_4k_5)^{-1/2}X^{1/6}$ being negligible. Indeed, let $N'(\boldsymbol{\eta}', B)$ be this contribution and define $a = -(k_4k_5)^{-1}\eta_1\eta_6\eta_7$. We have

$$\begin{aligned} S(\boldsymbol{\eta}', B) &\leq \# \left\{ (\eta_4', \eta_5') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} \eta_4'\eta_5' \equiv a \pmod{k_8\eta_9} \\ |\eta_4'\eta_5'| \leq X \end{array} \right\} \\ &= 2 \sum_{\substack{1 \leq |n| \leq X \\ n \equiv a \,(\mathrm{mod}\, k_8\eta_9)}} \tau(|n|). \end{aligned}$$

Thus, for all $\varepsilon > 0$,

$$S(\boldsymbol{\eta}', B) \ll X^{\varepsilon}\left(\frac{X}{k_8\eta_9} + 1\right) \ll (k_4k_5)^{1/4} \frac{X^{1+\varepsilon-1/12}}{k_8^{1/2}\eta_9} + X^{\varepsilon},$$

since $k_8 > k_8^{1/2}(2k_4k_5)^{-1/4}X^{1/12}$. Note that if k_4, k_5 or k_8 appears in the denominator then the arithmetic function involved by the corresponding Möbius inversion has average order $O(1)$ and therefore does not play any role in the estimation of the contribution of the error term. Thus we have

$$N'(\boldsymbol{\eta}', B) \ll \frac{1}{\eta_9}\left(\frac{B^{1/2}}{\eta_2^{1/2}\eta_3^{1/2}}\right)^{1+\varepsilon-1/12} + 2^{\omega(\eta_2)}\left(\frac{B^{1/2}}{\eta_2^{1/2}\eta_3^{1/2}}\right)^{\varepsilon}.$$

Let us estimate the overall contribution of the right-hand side summing over $\boldsymbol{\eta}'$. Using (3.20) to sum over η_9, we get

$$\sum_{\eta_9} N'(\boldsymbol{\eta}', B) \ll \left(\frac{B^{1/2}}{\eta_2^{1/2}\eta_3^{1/2}}\right)^{1+2\varepsilon-1/12} + 2^{\omega(\eta_2)}\left(\frac{B^{1/2}}{\eta_2^{1/2}\eta_3^{1/2}}\right)^{1/2+\varepsilon}.$$

Taking $\varepsilon = 1/48$ and summing over η_3 using (3.10), we obtain

$$\begin{aligned}\sum_{\boldsymbol{\eta}'} N'(\boldsymbol{\eta}', B) &\ll \sum_{\eta_1,\eta_2,\eta_6,\eta_7} \left(\frac{B}{\eta_1^{25/24}\eta_2\eta_6^{25/24}\eta_7^{25/24}} + 2^{\omega(\eta_2)}\frac{B}{\eta_1^{71/48}\eta_2\eta_6^{71/48}\eta_7^{71/48}}\right)\\ &\ll B\log(B)^2.\end{aligned}$$

Therefore, $N(\boldsymbol{\eta}', B)$ is the sum of the main term

$$\sum_{\substack{k_4 \mid \eta_1\eta_2\eta_6\eta_7\\ \gcd(k_4,\eta_9)=1}} \mu(k_4) \sum_{\substack{k_5 \mid \eta_1\eta_3\eta_6\eta_7\\ \gcd(k_5,\eta_9)=1}} \mu(k_5) \sum_{\substack{k_8 \mid \eta_2,\ k_8 \le (2k_4k_5)^{-1/2}X^{1/6}\\ \gcd(k_8,k_4k_5\eta_1\eta_6\eta_7)=1}} \mu(k_8) S(\boldsymbol{\eta}', B)$$

and an error term whose overall contribution is $\ll B\log(B)^2$. Note that thanks to (3.20), we now have $k_8\eta_9 \le X^{2/3}$. We want to apply Lemma 2 with $L_1 = \log(B)^A/k_4$, $L_2 = \log(B)^A/k_5$ and $T = \eta_1\eta_6\eta_7/(k_4k_5)$. Since $T \le X$ by (3.10) and $k_8\eta_9 \le X^{2/3}$, Lemma 2 proves that

$$S(\boldsymbol{\eta}', B) = S^*(\boldsymbol{\eta}', B) + O\left(\frac{X^{2/3+\varepsilon}}{(k_8\eta_9)^{1/2}} + \frac{X}{\varphi(k_8\eta_9)}\left(\frac{k_4}{\log(B)^A} + \frac{k_5}{\log(B)^A}\right)\right)$$

for all $\varepsilon > 0$, with

$$S^*(\boldsymbol{\eta}', B) = \frac{1}{\varphi(k_8\eta_9)}\#\left\{(\eta_4', \eta_5') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l}\gcd(\eta_4'\eta_5', k_8\eta_9) = 1\\ (3.9), (3.11), (3.12)\\ (3.21)\text{–}(3.23)\end{array}\right\}.$$

As explained above, k_4, k_5 and k_8 do not play any role in the estimation of the contribution of the first error term. Using (3.20) to sum over η_9, we find that the contribution of the first error term is

$$\begin{aligned}\sum_{\boldsymbol{\eta}'} \frac{B^{1/3+\varepsilon}}{\eta_2^{1/3+\varepsilon}\eta_3^{1/3+\varepsilon}\eta_9^{1/2}} &\ll \sum_{\eta_1,\eta_2,\eta_3,\eta_6,\eta_7} \frac{B^{11/24+\varepsilon}}{\eta_2^{11/24+\varepsilon}\eta_3^{11/24+\varepsilon}}\\ &\ll \sum_{\eta_1,\eta_2,\eta_6,\eta_7} \frac{B}{\eta_1^{13/12-2\varepsilon}\eta_2\eta_6^{13/12-2\varepsilon}\eta_7^{13/12-2\varepsilon}} \ll B\log(B)\end{aligned}$$

for $\varepsilon = 1/48$ and where we have summed over η_3 using (3.10). Furthermore, the contribution of the second error term is

$$\begin{aligned}\sum_{\boldsymbol{\eta}'} 2^{\omega(\eta_1\eta_2\eta_6\eta_7)}\frac{B^{1/2}\log(B)^{-A}}{\eta_2^{1/2}\eta_3^{1/2}\eta_9} &\ll \sum_{\eta_1,\eta_2,\eta_6,\eta_7,\eta_9} 2^{\omega(\eta_1\eta_2\eta_6\eta_7)}\frac{B\log(B)^{-A}}{\eta_1\eta_2\eta_6\eta_7\eta_9}\\ &\ll B\log(B)^{9-A},\end{aligned}$$

which is satisfactory if $A \geq 7$. The contribution of the third error term is easily seen to be also $\ll B \log(B)^{9-A}$. Furthermore,

$$S^*(\boldsymbol{\eta}', B) = \frac{1}{\varphi(k_8\eta_9)} \sum_{\ell_4 | k_8\eta_9} \mu(\ell_4) \sum_{\ell_5 | k_8\eta_9} \mu(\ell_5) C(\boldsymbol{\eta}', B),$$

where we have set $\eta_4' = \ell_4\eta_4''$ and $\eta_5' = \ell_5\eta_5''$. We now prove that we can remove the condition $k_8 \leq (2k_4k_5)^{-1/2}X^{1/6}$ from the sum over k_8. The height condition (3.9) plainly gives

$$C(\boldsymbol{\eta}', B) \ll \left(\frac{X}{\ell_4\ell_5}\right)^{1+\varepsilon}.$$

Let us bound the overall contribution corresponding to $k_8 > (2k_4k_5)^{-1/2}X^{1/6}$. Note that $\varphi(k_8\eta_9) = k_8\eta_9\varphi^*(k_8\eta_9)$ and write $k_8 > k_8^{1/2}(2k_4k_5)^{-1/4}X^{1/12}$. Once again, the Möbius inversions do not play any part in the estimation of the contribution of this error term, which we find to be less than

$$\begin{aligned}\sum_{\boldsymbol{\eta}'} \frac{1}{\eta_9}\left(\frac{B^{1/2}}{\eta_2^{1/2}\eta_3^{1/2}}\right)^{1+\varepsilon-1/12} &\ll \sum_{\eta_1,\eta_2,\eta_6,\eta_7,\eta_9} \frac{B}{\eta_1^{25/24}\eta_2\eta_6^{25/24}\eta_7^{25/24}\eta_9} \\ &\ll B\log(B)^2,\end{aligned}$$

where we have set $\varepsilon = 1/24$. Finally, the condition $\gcd(k_8, \eta_1\eta_6) = 1$ can be removed from the sum over k_8 since $k_8 \mid \eta_2$ and $\gcd(\eta_1\eta_6, \eta_2) = 1$, which completes the proof of Lemma 14.

3.5. Summing over η_4'', η_5'' and η_6. We intend to sum also over η_6 and thus we set $\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3, \eta_7, \eta_9) \in \mathbb{Z}_{>0}^5$. For $(r_1, r_2, r_3, r_7, r_9) \in \mathbb{Q}^5$, we introduce the useful notation

$$\boldsymbol{\eta}^{(r_1,r_2,r_3,r_7,r_9)} = \eta_1^{r_1}\eta_2^{r_2}\eta_3^{r_3}\eta_7^{r_7}\eta_9^{r_9}.$$

Setting

$$\begin{aligned} Y_4 &= \frac{\boldsymbol{\eta}^{(1,3/2,-1/2,2,1)}}{B^{1/2}}, & Y_4'' &= \frac{Y_4}{k_4\ell_4}, \\ Y_5 &= \frac{B}{\boldsymbol{\eta}^{(1,2,0,2,1)}}, & Y_5'' &= \frac{Y_5}{k_5\ell_5}, \\ Y_6 &= \frac{B^{1/2}}{\boldsymbol{\eta}^{(1,1/2,1/2,1,0)}}, \end{aligned}$$

and recalling the definition (3.13) of the function h, the height conditions (3.9)–(3.12) can be rewritten as

$$h(\eta_4''/Y_4'', \eta_5''/Y_5'', \eta_6/Y_6) \leq 1.$$

We also define the real-valued functions

$$g_1 : (t_5, t_6, t; \boldsymbol{\eta}, B) \mapsto \int\limits_{h(t_4,t_5,t_6)\le 1,\, t\le |t_4t_5+t_6|,\, |t_4|Y_4 \ge \log(B)^A} dt_4,$$

$$g_2 : (t_6, t; \boldsymbol{\eta}, B) \mapsto \int\limits_{|t_5|Y_5 \ge \log(B)^A} g_1(t_5, t_6, t; \boldsymbol{\eta}, B)\, dt_5,$$

$$g_3 : (t; \boldsymbol{\eta}, B) \mapsto \int\limits_{t_6 Y_6 \ge 1} g_2(t_6, t; \boldsymbol{\eta}, B)\, dt_6,$$

$$g_4 : t \mapsto \iiint\limits_{t_6>0,\, h(t_4,t_5,t_6)\le 1,\, t\le |t_4t_5+t_6|} dt_4\, dt_5\, dt_6.$$

The condition $t \le |t_4t_5 + t_6|$ corresponds to (3.21) which becomes, in our new notations,

$$\frac{\eta_9^2}{Y_4Y_5} \le \left| \frac{\eta_4''}{Y_4''} \frac{\eta_5''}{Y_5''} + \frac{\eta_6}{Y_6} \right|.$$

We denote by κ the left-hand side of this inequality. Note that (3.20) is exactly $\kappa \le 2$.

LEMMA 15. *We have the bounds*

$$g_1(t_5, t_6, t; \boldsymbol{\eta}, B) \ll |t_5|^{-2/3} t_6^{-2/3}, \quad g_2(t_6, t; \boldsymbol{\eta}, B) \ll t_6^{-2/3}.$$

Proof. Recall the definition (3.13) of h. The condition $|t_4|t_6^2|t_4t_5+t_6| \le 1$ shows that t_4 runs over a set whose measure is $\ll |t_5|^{-1/2}|t_6|^{-1}$. Since also $|t_4t_5| \le 1$, we derive the bound $g_1(t_5, t_6, t; \boldsymbol{\eta}, B) \ll \min(|t_5|^{-1/2}|t_6|^{-1}, |t_5|^{-1}) \le |t_5|^{-2/3} t_6^{-2/3}$. The bound for g_2 immediately follows since $|t_5| \le 1$. ∎

It is immediate to check that $\boldsymbol{\eta}$ is restricted to lie in the region

$$\mathcal{V} = \{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 : Y_5 \ge \log(B)^A,\ Y_6 \ge 1,\ 2Y_4Y_5 \ge \eta_9^2\}. \tag{3.25}$$

Assume that $\boldsymbol{\eta} \in \mathcal{V}$ and $\eta_6 \in \mathbb{Z}_{>0}$ are fixed and satisfy the height condition (3.10) and the coprimality conditions (3.5)–(3.8).

Our next task is to estimate $C(\boldsymbol{\eta}', B)$. Recall the condition (3.24) which can be rewritten as $|\eta_5''| \le Y_4 Y_5'' \log(B)^{-A}$. Let us sum over η_4'' using the basic estimate $\#\{n \in \mathbb{Z} : t_1 \le n \le t_2\} = t_2 - t_1 + O(1)$. The change of variable $t_4 \mapsto Y_4'' t_4$ shows that

$$C(\boldsymbol{\eta}', B) = \sum_{\eta_5'' \le Y_4 Y_5'' \log(B)^{-A}} (Y_4'' g_1(\eta_5''/Y_5'', \eta_6/Y_6, \kappa; \boldsymbol{\eta}, B) + O(1)).$$

The overall contribution of the error term is

$$\sum_{\boldsymbol{\eta}'} 2^{\omega(\eta_1\eta_2\eta_6\eta_7)} 2^{\omega(\eta_2\eta_9)} \frac{B^{1/2}\log(B)^{-A}}{\boldsymbol{\eta}^{(0,1/2,1/2,0,1)}} \ll \sum_{\boldsymbol{\eta}} 2^{\omega(\eta_1\eta_2\eta_7)} 2^{\omega(\eta_2\eta_9)} \frac{B\log(B)^{1-A}}{\boldsymbol{\eta}^{(1,1,1,1,1)}}$$
$$\ll B\log(B)^{12-A},$$

where we have summed over η_6 using (3.10). Let us now sum over η_5''. Partial summation and the change of variable $t_5 \mapsto Y_5'' t_5$ yield

$$C(\boldsymbol{\eta}', B) = Y_4'' Y_5'' g_2(\eta_6/Y_6, \kappa; \boldsymbol{\eta}, B) + O(Y_4'' \sup_{|t_5|Y_5 \geq \log(B)^A} g_1(t_5, \eta_6/Y_6, \kappa; \boldsymbol{\eta}, B)).$$

Since $h(t_4, t_5, t_6) \leq 1$ implies $|t_4 t_5| \leq 1$, we have $g_1(t_5, t_6, t; \boldsymbol{\eta}, B) \ll |t_5|^{-1}$ and thus

$$\sup_{|t_5|Y_5 \geq \log(B)^A} g_1(t_5, \eta_6/Y_6, \kappa; \boldsymbol{\eta}, B) \ll Y_5 \log(B)^{-A}.$$

Summing over η_6 using (3.10), the overall contribution of this error term is

$$\begin{aligned}\sum_{\boldsymbol{\eta}'} 2^{\omega(\eta_1\eta_3\eta_6\eta_7)} 2^{\omega(\eta_2\eta_9)} \frac{B^{1/2}\log(B)^{-A}}{\boldsymbol{\eta}^{(0,1/2,1/2,0,1)}} &\ll \sum_{\boldsymbol{\eta}} 2^{\omega(\eta_1\eta_3\eta_7)} 2^{\omega(\eta_2\eta_9)} \frac{B\log(B)^{1-A}}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \\ &\ll B\log(B)^{11-A}.\end{aligned}$$

Recalling Lemma 14, for any fixed $A \geq 10$, we have obtained

$$\begin{aligned}N(\boldsymbol{\eta}', B) = \frac{1}{\eta_9} g_2\left(\frac{\eta_6}{Y_6}, \kappa; \boldsymbol{\eta}, B\right) Y_4 Y_5 &\sum_{\substack{k_8 \mid \eta_2 \\ \gcd(k_8, \eta_7)=1}} \frac{\mu(k_8)}{k_8 \varphi^*(k_8\eta_9)} \sum_{\substack{k_4 \mid \eta_1\eta_2\eta_6\eta_7 \\ \gcd(k_4, k_8\eta_9)=1}} \frac{\mu(k_4)}{k_4} \\ &\times \sum_{\substack{k_5 \mid \eta_1\eta_3\eta_6\eta_7 \\ \gcd(k_5, k_8\eta_9)=1}} \frac{\mu(k_5)}{k_5} \sum_{\ell_4 \mid k_8\eta_9} \frac{\mu(\ell_4)}{\ell_4} \sum_{\ell_5 \mid k_8\eta_9} \frac{\mu(\ell_5)}{\ell_5} + R_1(\boldsymbol{\eta}', B),\end{aligned}$$

where $\sum_{\boldsymbol{\eta}'} R_1(\boldsymbol{\eta}', B) \ll B\log(B)^2$. A straightforward calculation reveals that the main term of $N(\boldsymbol{\eta}', B)$ is equal to

$$\theta(\boldsymbol{\eta}) \frac{\varphi^*(\eta_6)}{\varphi^*(\gcd(\eta_6, \eta_1\eta_2\eta_7))} \frac{\varphi^*(\eta_6)}{\varphi^*(\gcd(\eta_6, \eta_1\eta_3\eta_7))} g_2\left(\frac{\eta_6}{Y_6}, \kappa; \boldsymbol{\eta}, B\right) \frac{Y_4 Y_5}{\eta_9},$$

where

$$\theta(\boldsymbol{\eta}) = \varphi^*(\eta_1\eta_2\eta_7)\varphi^*(\eta_1\eta_3\eta_7) \frac{\varphi^*(\eta_2\eta_9)}{\varphi^*(\gcd(\eta_2, \eta_7))}.$$

For fixed $\boldsymbol{\eta} \in \mathcal{V}$ satisfying the coprimality conditions (3.6)–(3.8), let $\mathbf{N}(\boldsymbol{\eta}, B)$ be the sum over η_6 of the main term of $N(\boldsymbol{\eta}', B)$, with η_6 satisfying the height condition (3.10) and the coprimality condition (3.5). Let us use Lemma 6 to sum over η_6. We find that for any fixed $A \geq 10$ and $0 < \sigma \leq 1$,

$$\begin{aligned}\mathbf{N}(\boldsymbol{\eta}, B) = &\frac{1}{\eta_9} \mathcal{P} \Theta(\boldsymbol{\eta}) g_3(\kappa; \boldsymbol{\eta}, B) Y_4 Y_5 Y_6 \\ &+ O\left(\frac{Y_4 Y_5}{\eta_9} \varphi_\sigma(\eta_2\eta_7\eta_9) Y_6^\sigma \sup_{t_6 Y_6 \geq 1} g_2(t_6, \kappa; \boldsymbol{\eta}, B)\right),\end{aligned} \tag{3.26}$$

where

$$\Theta(\boldsymbol{\eta}) = \theta(\boldsymbol{\eta}) \varphi^*(\eta_2\eta_7\eta_9) \varphi^\dagger(\eta_3) \varphi'(\eta_1\eta_2\eta_3\eta_7\eta_9),$$

and where $\varphi^\dagger$, φ', φ_σ and $\mathcal{P}$ are respectively introduced in (2.36)–(2.39). Using the bound of Lemma 15 for g_2 and choosing $\sigma = 1/4$, we see that the overall contribution of the error term is

$$\sum_{\boldsymbol{\eta}} \varphi_\sigma(\eta_2\eta_7\eta_9)\frac{Y_4Y_5}{\eta_9}Y_6^{11/12} \ll \sum_{\eta_2,\eta_3,\eta_7,\eta_9} \varphi_\sigma(\eta_2\eta_7\eta_9)\frac{B}{\boldsymbol{\eta}^{(0,1,1,1,1)}} \ll B\log(B)^4,$$

since φ_σ has average order $O(1)$ and we have summed over η_1 using $Y_6 \geq 1$. Note that

$$\frac{Y_4Y_5Y_6}{\eta_9} = \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}}.$$

The aim now is to remove the conditions $|t_4|Y_4, |t_5|Y_5 \geq \log(B)^A$ from the integral defining g_3 in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.26) and to replace $t_6Y_6 \geq 1$ by $t_6 > 0$. This will replace $g_3(\kappa; \boldsymbol{\eta}, B)$ by $g_4(\kappa)$ in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.26). This is more subtle for t_4 than for t_5 and t_6. Indeed, since $Y_5 \geq \log(B)^A$ and $Y_6 \geq 1$, we can prove that the conditions $|t_5| < \log(B)^A/Y_5$ and $t_6 < 1/Y_6$ in the integral both yield a negligible contribution. However, we do not have $Y_4 \geq \log(B)^A$ so our reasoning consists in this case in proving that the contribution corresponding to $Y_4 < \log(B)^A$ is negligible, which will allow us to assume that $Y_4 \geq \log(B)^A$ and therefore conclude as for t_5 and t_6. For brevity, we set

$$D_h = \{(t_4, t_5, t_6) \in \mathbb{R}^3 : t_6 > 0,\ h(t_4, t_5, t_6) \leq 1\}.$$

Lemma 16. *For $Z_4, Z_5, Z_6 > 0$,*

$$\operatorname{meas}\{(t_4, t_5, t_6) \in D_h \colon |t_4|Z_4 \geq 1\} \ll Z_4^{1/4}, \tag{3.27}$$

$$\operatorname{meas}\{(t_4, t_5, t_6) \in D_h \colon |t_4|Z_4 < 1\} \ll Z_4^{-1}, \tag{3.28}$$

$$\operatorname{meas}\{(t_4, t_5, t_6) \in D_h \colon |t_5|Z_5 < 1\} \ll Z_5^{-1/3}, \tag{3.29}$$

$$\operatorname{meas}\{(t_4, t_5, t_6) \in D_h \colon t_6Z_6 < 1\} \ll Z_6^{-1/3}. \tag{3.30}$$

Proof. The conditions $|t_4|t_6^2|t_4t_5 + t_6| \leq 1$ and $|t_4t_5| \leq 1$ show that t_5 runs over a set whose measure is $\ll \min(t_4^{-2}t_6^{-2}, |t_4|^{-1}) \leq |t_4|^{-5/4}t_6^{-1/2}$, which proves (3.27) since $t_6 \leq 1$. The bound (3.28) is clear since $|t_5|, t_6 \leq 1$. The bound (3.29) follows from the bound of Lemma 15 for g_1 and $t_6 \leq 1$. In a similar way, (3.30) is a consequence of the bound of Lemma 15 for g_1 and $|t_5| \leq 1$. ∎

Using (3.29), we see that removing the condition $|t_5|Y_5 \geq \log(B)^A$ from the integral defining g_3 in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.26) yields an error term whose overall contribution is

$$\sum_{\boldsymbol{\eta}} Y_4Y_5^{2/3}Y_6\log(B)^{A/3} \ll \sum_{\eta_2,\eta_3,\eta_7,\eta_9} \frac{B}{\boldsymbol{\eta}^{(0,1,1,1,1)}} \ll B\log(B)^4,$$

where we have summed over η_1 using $Y_5 \geq \log(B)^A$. In a similar fashion, (3.30) shows that replacing $t_6 Y_6 \geq 1$ by $t_6 > 0$ in the integral defining g_3 in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.26) also creates an error term whose overall contribution is $\ll B \log(B)^4$.

We now assume that $Y_4 < \log(B)^A$ and we bound the contribution of the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ under this assumption. The bound (3.27) shows that this contribution is

$$\sum_{\boldsymbol{\eta}} Y_4^{5/4} Y_5 Y_6 \log(B)^{-A/4} \ll \sum_{\eta_2,\eta_3,\eta_7,\eta_9} \frac{B}{\boldsymbol{\eta}^{(0,1,1,1,1)}} \ll B \log(B)^4.$$

We can therefore assume from now on that

$$Y_4 \geq \log(B)^A. \tag{3.31}$$

Under this assumption, exactly as for t_5 and t_6, the bound (3.28) shows that the overall contribution of the error term created by removing the condition $|t_4| Y_4 \geq \log(B)^A$ from the integral defining g_3 in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.26) is $\ll B \log(B)^4$. We have proved that for any fixed $A \geq 9$,

$$\mathbf{N}(\boldsymbol{\eta}, B) = \mathcal{P} g_4(\kappa) \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \Theta(\boldsymbol{\eta}) + R_2(\boldsymbol{\eta}, B), \tag{3.32}$$

where $\sum_{\boldsymbol{\eta}} R_2(\boldsymbol{\eta}, B) \ll B \log(B)^4$. The goal of the following lemma is to replace $g_4(\kappa)$ by $g_4(0)$ in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.32). By (3.14), $g_4(0)$ is equal to

$$\iiint_{t_6>0,\, h(t_4,t_5,t_6)\leq 1} dt_4\, dt_5\, dt_6 = \frac{\omega_\infty}{2}.$$

Lemma 17. *For $t > 0$,*

$$\operatorname{meas}\{(t_4,t_5,t_6) \in D_h \colon |t_4 t_5 + t_6| < t\} \ll t^{1/2}. \tag{3.33}$$

Proof. The conditions $|t_4| t_6^2 |t_4 t_5 + t_6| \leq 1$ and $|t_4 t_5 + t_6| < t$ imply that t_4 runs over a set whose measure is $\ll \min(|t_5|^{-1/2} t_6^{-1}, t|t_5|^{-1}) \leq t^{1/2} |t_5|^{-3/4} t_6^{-1/2}$, which suffices since $|t_5|, t_6 \leq 1$. ∎

Let us estimate the overall contribution of the error term which appears if we replace $g_4(\kappa)$ by $g_4(0)$ in the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ in (3.32). Using (3.33) and summing over η_9 using (3.20), we find that this contribution is

$$\sum_{\boldsymbol{\eta}} \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \kappa^{1/2} \ll \sum_{\eta_1,\eta_2,\eta_3,\eta_7} \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,0)}} \ll B \log(B)^4.$$

We have thus obtained the following result.

Lemma 18. *For any fixed $A \geq 10$,*

$$\mathbf{N}(\boldsymbol{\eta}, B) = \mathcal{P} \frac{\omega_\infty}{2} \frac{B}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \Theta(\boldsymbol{\eta}) + R_3(\boldsymbol{\eta}, B),$$

where $\sum_{\boldsymbol{\eta}} R_3(\boldsymbol{\eta}, B) \ll B \log(B)^4$.

3.6. Conclusion. Recall the definition (3.25) of $\mathcal{V}$. It remains to sum the main term of $\mathbf{N}(\boldsymbol{\eta}, B)$ over the $\boldsymbol{\eta} \in \mathcal{V}$ satisfying (3.31) and the coprimality conditions (3.6)–(3.8). It is easy to see that replacing $\{\boldsymbol{\eta} \in \mathcal{V} \colon (3.31)\}$ by the region

$$\mathcal{V}' = \{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5 \colon Y_4 \geq 1,\ Y_5 \geq 1,\ Y_6 \geq 1,\ Y_4 Y_5 \geq \eta_9^2\},$$

produces an error term contributing $\ll B \log(B)^4 \log(\log(B))$. Let us redefine Θ as being equal to zero if the remaining coprimality conditions (3.6)–(3.8) are not satisfied. Fixing for example $A = 10$ and combining Lemmas 13 and 18, we obtain

$$N_{U_1,H}(B) = \mathcal{P}\omega_\infty B \sum_{\boldsymbol{\eta} \in \mathcal{V}'} \frac{\Theta(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} + O(B \log(B)^4 \log(\log(B))).$$

Set $\mathbf{k} = (k_1, k_2, k_3, k_7, k_9)$ and define, for $s \in \mathbb{C}$ such that $\Re(s) > 1$,

$$F(s) = \sum_{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5} \frac{|(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})|}{\eta_1^s \eta_2^s \eta_3^s \eta_7^s \eta_9^s} = \prod_p \biggl(\sum_{\mathbf{k} \in \mathbb{Z}_{\geq 0}^5} \frac{|(\Theta * \boldsymbol{\mu})(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_7}, p^{k_9})|}{p^{k_1 s} p^{k_2 s} p^{k_3 s} p^{k_7 s} p^{k_9 s}} \biggr).$$

If $\mathbf{k} \notin \{0,1\}^5$ then $(\Theta * \boldsymbol{\mu})(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_7}, p^{k_9}) = 0$ and furthermore if only one of the k_i is equal to 1, then $(\Theta * \boldsymbol{\mu})(p^{k_1}, p^{k_2}, p^{k_3}, p^{k_7}, p^{k_9}) \ll 1/p$, so the local factors F_p of F satisfy

$$F_p(s) = 1 + O\biggl(\frac{1}{p^{\min(\Re(s)+1, 2\Re(s))}} \biggr),$$

and thus F actually converges in the half-plane $\Re(s) > 1/2$. This proves that Θ satisfies the assumption (2.40) of Lemma 8. We therefore get

$$\begin{aligned} N_{U_1,H}(B) = \mathcal{P}\omega_\infty \alpha \biggl(\sum_{\boldsymbol{\eta} \in \mathbb{Z}_{>0}^5} \frac{(\Theta * \boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}} \biggr) B \log(B)^5 \\ + O(B \log(B)^4 \log(\log(B))), \end{aligned}$$

where α is the volume of the polytope defined in $\mathbb{R}^5$ by $t_1, t_2, t_3, t_7, t_9 \geq 0$ and

$$\begin{aligned} 2t_1 + 3t_2 - t_3 + 4t_7 + 2t_9 &\geq 1, \\ t_1 + 2t_2 + 2t_7 + t_9 &\leq 1, \\ 2t_1 + t_2 + t_3 + 2t_7 &\leq 1, \\ t_2 + t_3 + 4t_9 &\leq 1. \end{aligned}$$

A computation using Franz's additional *Maple* package [Fra09] gives $\alpha = 1/1440$, that is,

$$\alpha = \alpha(\widetilde{V_1}),$$

and moreover

$$\sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5}\frac{(\Theta*\boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}}=\prod_p\Bigg(\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{(\Theta*\boldsymbol{\mu})(p^{k_1},p^{k_2},p^{k_3},p^{k_7},p^{k_9})}{p^{k_1}p^{k_2}p^{k_3}p^{k_7}p^{k_9}}\Bigg)$$
$$=\prod_p\left(1-\frac{1}{p}\right)^5\Bigg(\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{\Theta(p^{k_1},p^{k_2},p^{k_3},p^{k_7},p^{k_9})}{p^{k_1}p^{k_2}p^{k_3}p^{k_7}p^{k_9}}\Bigg).$$

We omit the details of the calculation of the series on the right-hand side; let us just say that the remaining coprimality conditions greatly simplify the calculation. We obtain

$$\sum_{\mathbf{k}\in\mathbb{Z}_{\geq 0}^5}\frac{\Theta(p^{k_1},p^{k_2},p^{k_3},p^{k_7},p^{k_9})}{p^{k_1}p^{k_2}p^{k_3}p^{k_7}p^{k_9}}=\varphi'(p)\left(1-\frac{1}{p}\right)\left(1+\frac{6}{p}+\frac{1}{p^2}\right),$$

and thus

$$\sum_{\boldsymbol{\eta}\in\mathbb{Z}_{>0}^5}\frac{(\Theta*\boldsymbol{\mu})(\boldsymbol{\eta})}{\boldsymbol{\eta}^{(1,1,1,1,1)}}=\mathcal{P}^{-1}\prod_p\left(1-\frac{1}{p}\right)^6\omega_p,$$

which completes the proof.

4. Proof for the $\mathbf{A}_1+\mathbf{A}_2$ surface

4.1. The universal torsor. We now proceed to define a bijection between the set of points to be counted on U_2 and a certain set of integral points on the affine variety defined by (1.5). Our choice of notation might be surprising but our aim is simply to highlight the similarities with the case of the $3\mathbf{A}_1$ surface. Note that for a given $(x_0:x_1:x_2:x_3:x_4)\in V_2$, we have $(x_0:x_1:x_2:x_3:x_4)\in U_2$ if and only if $x_0x_1x_2x_3x_4\neq 0$. Let $(x_0,x_1,x_2,x_3,x_4)\in\mathbb{Z}_{\neq 0}^5$ be such that $\gcd(x_0,x_1,x_2,x_3,x_4)=1$ and

$$x_0x_1-x_2x_3=0,$$
$$x_1x_2+x_2x_4+x_3x_4=0,$$

and $\max\{|x_i|\colon 0\leq i\leq 4\}\leq B$. Define $\xi_6=\gcd(x_0,x_1,x_2,x_3)>0$ and write $x_i=\xi_6x_i'$ for $i=0,1,2,3$. We thus have $\gcd(\xi_6,x_4)=1$ and $\gcd(x_0',x_1',x_2',x_3')=1$. Now let $\xi_3=\gcd(x_0',x_2',x_3')>0$. Since $\gcd(\xi_3,x_1')=1$, it follows that $\xi_3^2\mid x_0'$ and we can write $x_j'=\xi_3x_j''$ for $j=2,3$ and $x_0'=\xi_3^2x_0''$. Moreover, $\gcd(\xi_3x_0'',x_2'',x_3'')=1$. Let $\xi_8=\gcd(x_0'',x_3'')>0$ and write $x_0''=\xi_8\xi_4$ and $x_3''=\xi_8y_3$ with $\gcd(\xi_4,y_3)=1$. The first equation can be rewritten as $\xi_4x_1'=x_2''y_3$. Since $\gcd(\xi_4,y_3)=1$, we have $\xi_4\mid x_2''$ and we can write $x_2''=\xi_4y_2$ and thus $x_1'=y_2y_3$. Let us sum up what we have done until now. We have been able to find $(\xi_6,\xi_3,\xi_8,\xi_4,y_3,y_2)\in\mathbb{Z}_{>0}^3\times\mathbb{Z}_{\neq 0}^3$ such that $\gcd(\xi_6,x_4)=1$, $\gcd(\xi_3,y_2y_3)=1$, $\gcd(\xi_8,\xi_4y_2)=1$, $\gcd(\xi_4,y_3)=1$ and $x_0=\xi_6\xi_3^2\xi_8\xi_4$,

$x_1 = \xi_6 y_2 y_3$, $x_2 = \xi_6\xi_3\xi_4 y_2$, $x_3 = \xi_6\xi_3\xi_8 y_3$. Simplifying by $\xi_3\xi_6$, the second equation gives

$$\xi_6 y_2^2 y_3 \xi_4 + x_4(\xi_4 y_2 + \xi_8 y_3) = 0.$$

Let $y_{23} = \gcd(y_2, y_3) > 0$ and write $y_2 = y_{23}\xi_5$, $y_3 = y_{23}\xi_9$ with $\gcd(\xi_5, \xi_9) = 1$. We obtain

$$\xi_6 y_{23}^2 \xi_5^2 \xi_9 \xi_4 + x_4(\xi_4\xi_5 + \xi_8\xi_9) = 0.$$

Since $\gcd(\xi_4\xi_5, \xi_8\xi_9) = 1$, it is obvious that $\xi_4\xi_5^2\xi_9 \mid x_4$. If we write $x_4 = \xi_4\xi_5^2\xi_9 x_4'$, the equation becomes

$$\xi_6 y_{23}^2 + x_4'(\xi_4\xi_5 + \xi_8\xi_9) = 0.$$

We now see that since $\gcd(\xi_6, x_4') = 1$, we have $x_4' \mid y_{23}^2$ and thus there is a unique way to write $y_{23} = \xi_1\xi_2\xi_7$ and $x_4' = \xi_2^2\xi_7$ with $\gcd(\xi_1, \xi_2) = 1$ and $\xi_2 > 0$, $\xi_1\xi_7 > 0$. This leads to

$$\xi_4\xi_5 + \xi_1^2\xi_6\xi_7 + \xi_8\xi_9 = 0, \tag{4.1}$$

and so finally $x_0 = \xi_3^2\xi_4\xi_6\xi_8$, $x_1 = \xi_1^2\xi_2^2\xi_5\xi_6\xi_7^2\xi_9$, $x_2 = \xi_1\xi_2\xi_3\xi_4\xi_5\xi_6\xi_7$, $x_3 = \xi_1\xi_2\xi_3\xi_6\xi_7\xi_8\xi_9$, $x_4 = \xi_2^2\xi_4\xi_5^2\xi_7\xi_9$; a little thought reveals that, given (4.1), the coprimality conditions can be rewritten as

$$\begin{aligned}
&\gcd(\xi_4\xi_5, \xi_1\xi_6\xi_7) = 1,\\
&\gcd(\xi_4\xi_5, \xi_8\xi_9) = 1,\\
&\gcd(\xi_1\xi_6\xi_7, \xi_8\xi_9) = 1,\\
&\gcd(\xi_2, \xi_1\xi_3\xi_4\xi_6\xi_8) = 1,\\
&\gcd(\xi_3, \xi_1\xi_5\xi_7\xi_9) = 1,\\
&\gcd(\xi_6, \xi_7) = 1.
\end{aligned}$$

Since $\xi_1 \mapsto -\xi_1$ is a bijection on the set of solutions, we can assume that $\xi_1 > 0$ and thus $\xi_7 > 0$, keeping in mind that we have to divide our result by 2. In a similar fashion, $(\xi_8, \xi_9) \mapsto (-\xi_8, -\xi_9)$ shows that we can remove the condition $\xi_8 > 0$ multiplying our result by 2. To sum up, let $\mathcal{T}_2(B)$ be the number of $(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6, \xi_7, \xi_8, \xi_9) \in \mathbb{Z}_{\neq 0}^9$ with $\xi_1, \xi_2, \xi_3, \xi_6, \xi_7 > 0$ and satisfying (4.1), the coprimality conditions above and the height conditions

$$\xi_3^2|\xi_4|\xi_6|\xi_8| \le B, \tag{4.2}$$

$$\xi_1^2\xi_2^2|\xi_5|\xi_6\xi_7^2|\xi_9| \le B, \tag{4.3}$$

$$\xi_1\xi_2\xi_3|\xi_4\xi_5|\xi_6\xi_7 \le B, \tag{4.4}$$

$$\xi_1\xi_2\xi_3\xi_6\xi_7|\xi_8\xi_9| \le B, \tag{4.5}$$

$$\xi_2^2|\xi_4|\xi_5^2\xi_7|\xi_9| \le B. \tag{4.6}$$

Since we have not taken into account that $\mathbf{x} = -\mathbf{x} \in \mathbb{P}^4$ yet, we have finally proved the following lemma.

LEMMA 19. *We have the equality*

$$N_{U_2,H}(B) = \frac{1}{2}\#\mathcal{T}_2(B).$$

4.2. Calculation of Peyre's constant. We have

$$c_{V_2,H} = \alpha(\widetilde{V_2})\beta(\widetilde{V_2})\omega_H(\widetilde{V_2}),$$

where (see [Der07])

$$\alpha(\widetilde{V_2}) = \frac{1}{2160},$$

$\beta(\widetilde{V_2}) = 1$ since V_2 is split over $\mathbb{Q}$, and

$$\omega_H(\widetilde{V_2}) = \tau_\infty \prod_p \left(1 - \frac{1}{p}\right)^6 \tau_p,$$

where, thanks to [Lou10, Lemma 2.3],

$$\tau_p = 1 + \frac{6}{p} + \frac{1}{p^2}.$$

Let us calculate τ_∞. Set $f_1(x) = x_0x_1 - x_2x_3$ and $f_2(x) = x_1x_2 + x_2x_4 + x_3x_4$. Let us parametrize the points of V_2 by x_0, x_2 and x_3. We have

$$\det\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_4} \end{pmatrix} = \begin{vmatrix} x_0 & 0 \\ x_2 & x_2 + x_3 \end{vmatrix} = x_0(x_2 + x_3).$$

Moreover, $x_1 = x_2x_3/x_0$ and $x_4 = -x_2^2x_3/(x_0(x_2+x_3))$. Since $\mathbf{x} = -\mathbf{x}$ in $\mathbb{P}^4$, we have

$$\tau_\infty = -2 \iiint_{x_0>0,\, x_2+x_3<0, x_0,\, |x_2x_3/x_0|, |x_2|, |x_3|, |x_2^2x_3/(x_0(x_2+x_3))| \leq 1} \frac{dx_0\, dx_2\, dx_3}{x_0(x_2 + x_3)}.$$

We introduce the functions

$$h^a : (t_4, t_5, t_1) \mapsto \max\left\{ \begin{array}{l} |t_4|\,|t_4t_5 + t_1^2|, t_1^2|t_5|, t_1|t_4t_5|, \\ t_1|t_4t_5 + t_1^2|, |t_4|t_5^2 \end{array} \right\}, \tag{4.7}$$

$$h^b : (t_4, t_5, t_1) \mapsto \max\left\{ \begin{array}{l} |t_4|, t_1^2|t_5|\,|t_4t_5 + t_1^2|, t_1|t_4t_5|, \\ t_1|t_4t_5 + t_1^2|, |t_4|t_5^2|t_4t_5 + t_1^2| \end{array} \right\}. \tag{4.8}$$

The change of variables given by $x_0 = t_4(t_4t_5 + t_1^2)$, $x_2 = t_1t_4t_5$ and $x_3 = -t_1(t_4t_5 + t_1^2)$ yields

$$\begin{aligned} \tau_\infty &= 6 \iiint_{t_4(t_4t_5+t_1^2)>0,\, t_1>0,\, h^a(t_4,t_5,t_1)\leq 1} dt_4\, dt_5\, dt_1 \\ &= 3 \iiint_{t_1>0,\, h^a(t_4,t_5,t_1)\leq 1} dt_4\, dt_5\, dt_1. \end{aligned} \tag{4.9}$$

Moreover, the change of variables $x_0 = t_4$, $x_2 = t_1 t_4 t_5$ and $x_3 = -t_1(t_4 t_5 + t_1^2)$ gives the alternative expression

$$\tau_\infty = 6 \iiint_{t_4>0,\, t_1>0,\, h^b(t_4,t_5,t_1)\le 1} dt_4\, dt_5\, dt_1 \tag{4.10}$$
$$= 3 \iiint_{t_1>0,\, h^b(t_4,t_5,t_1)\le 1} dt_4\, dt_5\, dt_1.$$

Let us repeat here that the proof below is very similar to the one before, so we will sometimes allow ourselves to be concise.

4.3. Restriction of the domain. The following two lemmas are the analogues of Lemmas 10 and 11 respectively.

LEMMA 20. *Let $\mathcal{M}_2(B)$ be the overall contribution to $N_{U_2,H}(B)$ coming from the $(\xi_1, \dots, \xi_9) \in \mathcal{T}_2(B)$ such that $|\xi_i| \le \log(B)^A$ for a certain $i \neq 1, 2, 3$, where $A > 0$ is any fixed constant. Then*

$$\mathcal{M}_2(B) \ll_A B \log(B)^4 \log(\log(B)).$$

LEMMA 21. *Let $L_1, L_4, \dots, L_9 \ge 1/2$ and define $M_2 = M_2(L_1, L_4, \dots, L_9)$ as the number of $(n_1, n_4, \dots, n_9) \in \mathbb{Z}^7$ such that $L_i < |n_i| \le 2L_i$ for $i = 1$ and $4 \le i \le 9$, $\gcd(n_4 n_5, n_1 n_6 n_7) = 1$ and*

$$n_4 n_5 + n_1^2 n_6 n_7 + n_8 n_9 = 0. \tag{4.11}$$

Then

$$M_2 \ll L_1(L_4 L_5 L_6 L_7 L_8 L_9)^{1/2} + L_1 L_6 L_7 \min(L_4 L_5, L_8 L_9).$$

Proof. We can assume by symmetry that $L_4 L_5 \le L_8 L_9$. Let us first deal with the case where $L_1^2 L_6 L_7 \le L_4 L_5$. Then (4.11) gives $L_8 L_9 \ll L_4 L_5$. Let M_2' be the number of $(n_1, n_4, \dots, n_9) \in \mathbb{Z}^7$ to be counted in this case. The first case of the proof of Lemma 11 shows that

$$M_2' \ll L_1 L_6 L_7 L_4 L_5.$$

In the other case where $L_1^2 L_6 L_7 > L_4 L_5$, (4.11) gives $L_8 L_9 \ll L_1^2 L_6 L_7$. Let M_2'' be the number of $(n_1, n_4, \dots, n_9) \in \mathbb{Z}^7$ to be counted here. Assume by symmetry that $L_4 \le L_5$, $L_6 \le L_7$ and $L_8 \le L_9$. Since $\gcd(n_4, n_1 n_6, n_8) = 1$, using [HB03, Lemma 6] we can deduce that

$$\# \left\{ (n_5, n_7, n_9) \in \mathbb{Z}^3 : \begin{array}{l} L_i < |n_i| \le 2L_i,\ i \in \{5,7,9\} \\ \gcd(n_5, n_7, n_9) = 1 \\ n_4 n_5 + n_1^2 n_6 n_7 + n_8 n_9 = 0 \end{array} \right\} \ll 1 + \frac{L_5 L_9}{n_1^2 n_6}.$$

Summing over n_1, n_4, n_6 and n_8, we get

$$M_2'' \ll L_1 L_4 L_6 L_8 + L_4 L_5 L_8 L_9 / L_1 \ll L_1(L_4 L_5 L_6 L_7 L_8 L_9)^{1/2} + L_4 L_5 L_1 L_6 L_7,$$

which ends the proof. ∎

Proof of Lemma 20. Let $Z_i \geq 1/2$ for $i = 1, \dots, 9$ and define $\mathcal{N}_2 = \mathcal{N}_2(Z_1, \dots, Z_9)$ as the contribution of the $(\xi_1, \dots, \xi_9) \in \mathcal{T}_2(B)$ satisfying $Z_i < |\xi_i| \leq 2Z_i$ for $i = 1, \dots, 9$. The height conditions imply that either $\mathcal{N}_2 = 0$, or

$$Z_3^2 Z_4 Z_6 Z_8 \leq B, \tag{4.12}$$

$$Z_1^2 Z_2^2 Z_5 Z_6 Z_7^2 Z_9 \leq B, \tag{4.13}$$

$$Z_1 Z_2 Z_3 Z_4 Z_5 Z_6 Z_7 \leq B, \tag{4.14}$$

$$Z_1 Z_2 Z_3 Z_6 Z_7 Z_8 Z_9 \leq B, \tag{4.15}$$

$$Z_2^2 Z_4 Z_5^2 Z_7 Z_9 \leq B. \tag{4.16}$$

Using Lemma 21 and summing over ξ_2 and ξ_3, we get

$$\mathcal{N}_2 \ll Z_1 Z_2 Z_3 (Z_4 Z_5 Z_6 Z_7 Z_8 Z_9)^{1/2} + Z_1 Z_2 Z_3 Z_6 Z_7 \min(Z_4 Z_5, Z_8 Z_9).$$

Assume that $Z_4 Z_5 \leq Z_8 Z_9$ (the case $Z_4 Z_5 > Z_8 Z_9$ is identical). Note that the torsor equation (4.1) then gives $Z_1^2 Z_6 Z_7 \ll Z_8 Z_9$. Denote by $\mathcal{N}_2'$ the first term of the right-hand side, and by $\mathcal{N}_2''$ the second term. We proceed to prove that

$$\sum_{Z_i} \mathcal{N}_2' \ll B \log(B)^4.$$

First assume that $Z_1^2 Z_6 Z_7 \leq Z_4 Z_5$. Summing over

$$Z_1 \leq \min\left(\frac{Z_4^{1/2} Z_5^{1/2}}{Z_6^{1/2} Z_7^{1/2}}, \frac{B^{1/2}}{Z_2 Z_5^{1/2} Z_6^{1/2} Z_7 Z_9^{1/2}} \right) \leq \frac{Z_4^{1/4} B^{1/4}}{Z_2^{1/2} Z_6^{1/2} Z_7^{3/4} Z_9^{1/4}}, \tag{4.17}$$

we get in this case

$$\begin{aligned}\sum_{Z_i} \mathcal{N}_2' &\ll B^{1/4} \sum_{\widehat{Z}_1} Z_2^{1/2} Z_3 Z_4^{3/4} Z_5^{1/2} Z_7^{-1/4} Z_8^{1/2} Z_9^{1/4} \\ &\ll B^{1/2} \sum_{\widehat{Z}_1, \widehat{Z}_2} Z_3 Z_4^{1/2} Z_7^{-1/2} Z_8^{1/2} \ll B \sum_{\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3} Z_6^{-1/2} Z_7^{-1/2} \\ &\ll B \log(B)^4,\end{aligned}$$

where we have summed over Z_2 and Z_3 using (4.16) and (4.12).

Let us treat the case where $Z_1^2 Z_6 Z_7 > Z_4 Z_5$. Summing over Z_2 using (4.15), we obtain

$$\begin{aligned}\sum_{Z_i} \mathcal{N}_2' &\ll B \sum_{\widehat{Z}_2} Z_4^{1/2} Z_5^{1/2} Z_6^{-1/2} Z_7^{-1/2} Z_8^{-1/2} Z_9^{-1/2} \\ &\ll B \sum_{\widehat{Z}_2, \widehat{Z}_4} Z_1 Z_8^{-1/2} Z_9^{-1/2} \ll B \sum_{\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_4} Z_6^{-1/2} Z_7^{-1/2} \\ &\ll B \log(B)^4,\end{aligned}$$

where we sum over $Z_4 < Z_1^2 Z_5^{-1} Z_6 Z_7$ and $Z_1 \ll Z_6^{-1/2} Z_7^{-1/2} Z_8^{1/2} Z_9^{1/2}$. Let us estimate the contribution of $\mathcal{N}_2''$ in the case $Z_1^2 Z_6 Z_7 \leq Z_4 Z_5$. Summing over Z_1 using (4.17), we get

$$\begin{aligned}\sum_{Z_i} \mathcal{N}_2'' &\ll B^{1/4} \sum_{\widehat{Z}_1} Z_2^{1/2} Z_3 Z_4^{5/4} Z_5 Z_6^{1/2} Z_7^{1/4} Z_9^{-1/4} \\ &\ll B \sum_{\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3} Z_4^{1/2} Z_5^{1/2} Z_8^{-1/2} Z_9^{-1/2} \ll B \sum_{\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4} 1,\end{aligned}$$

where we have summed over Z_2 and Z_3 using respectively (4.16) and (4.12). At the last step, we could have summed over Z_5 instead of Z_4, so if we assume that $|\xi_i| \leq \log(B)^A$ for a certain $i \neq 1, 2, 3$, where $A > 0$ is any fixed constant, we get an overall contribution $\ll_A B \log(B)^4 \log(\log(B))$.

We now deal with the case where $Z_1^2 Z_6 Z_7 > Z_4 Z_5$. Summing over Z_2 and Z_3 using (4.13) and (4.12), we get

$$\begin{aligned}\sum_{Z_i} \mathcal{N}_2'' &\ll B \sum_{\widehat{Z}_2, \widehat{Z}_3} Z_4^{1/2} Z_5^{1/2} Z_8^{-1/2} Z_9^{-1/2} \\ &\ll B \sum_{\widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4} Z_1 Z_6^{1/2} Z_7^{1/2} Z_8^{-1/2} Z_9^{-1/2} \ll B \sum_{\widehat{Z}_1, \widehat{Z}_2, \widehat{Z}_3, \widehat{Z}_4} 1,\end{aligned}$$

where we sum over $Z_4 < Z_1^2 Z_5^{-1} Z_6 Z_7$ and $Z_1 \ll Z_6^{-1/2} Z_7^{-1/2} Z_8^{1/2} Z_9^{1/2}$. We can plainly conclude just as above. ∎

4.4. Setting up. First, we note that the torsor equation (4.1) and the height conditions (4.4) and (4.5) give

$$\xi_1^3 \xi_2 \xi_3 \xi_6^2 \xi_7^2 \leq 2B. \tag{4.18}$$

Our goal is to tackle the equation (4.1) by viewing it as a congruence modulo ξ_9 if $|\xi_9| \leq |\xi_8|$ and modulo ξ_8 if $|\xi_9| > |\xi_8|$, so we split the proof into two parts. Let $N_a(A, B)$ be the contribution to $N_{U_2,H}(B)$ from the $(\xi_1, \ldots, \xi_9) \in \mathcal{T}_2(B)$ such that

$$0 < \xi_9 \leq |\xi_8|, \tag{4.19}$$

$$\log(B)^A \leq |\xi_4|, \tag{4.20}$$

$$\log(B)^A \leq |\xi_5|, \tag{4.21}$$

where $A > 0$ is a parameter at our disposal. Symmetrically, let $N_b(A, B)$ be the contribution to $N_{U_2,H}(B)$ from the $(\xi_1, \ldots, \xi_9) \in \mathcal{T}_2(B)$ such that

$$|\xi_9| > \xi_8 > 0, \tag{4.22}$$

$$\log(B)^A \leq |\xi_4|, \tag{4.23}$$

$$\log(B)^A \leq |\xi_5|. \tag{4.24}$$

Note that in both cases, combining (4.4) and $\log(B)^A \leq |\xi_4|$, we get

$$\log(B)^A \xi_1\xi_2\xi_3|\xi_5|\xi_6\xi_7 \leq B. \tag{4.25}$$

Since $(\xi_8, \xi_9) \mapsto (-\xi_8, -\xi_9)$ is a bijection on the set of solutions, assuming $\xi_9 > 0$ in the first case and $\xi_8 > 0$ in the second brings a factor 2. Thus, by Lemmas 19 and 20 we have the following.

LEMMA 22. *For any fixed $A > 0$,*

$$N_{U_2,H}(B) = N_a(A, B) + N_b(A, B) + O(B \log(B)^4 \log(\log(B))).$$

The next two sections are respectively devoted to the estimations of $N_a(A, B)$ and $N_b(A, B)$.

4.5. Estimating $N_a(A, B)$. We see that the assumption $\xi_9 \leq |\xi_8|$ and (4.5) give the following condition which is crucial in order to apply Lemma 2:

$$\xi_9^2 \leq \frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}. \tag{4.26}$$

We first estimate the contribution of the variables ξ_4, ξ_5 and ξ_8. We rewrite the coprimality conditions as

$$\gcd(\xi_8, \xi_1\xi_2\xi_4\xi_5\xi_6\xi_7) = 1, \tag{4.27}$$
$$\gcd(\xi_4, \xi_1\xi_2\xi_6\xi_7\xi_9) = 1, \tag{4.28}$$
$$\gcd(\xi_5, \xi_1\xi_3\xi_6\xi_7\xi_9) = 1, \tag{4.29}$$
$$\gcd(\xi_1, \xi_2\xi_3\xi_9) = 1, \tag{4.30}$$
$$\gcd(\xi_3, \xi_2\xi_7\xi_9) = 1, \tag{4.31}$$
$$\gcd(\xi_6, \xi_2\xi_7\xi_9) = 1, \tag{4.32}$$
$$\gcd(\xi_7, \xi_9) = 1. \tag{4.33}$$

We view the torsor equation (4.1) as a congruence modulo ξ_9. To do so, we replace the height conditions (4.2), (4.5) and (4.19) by the following (we keep denoting them by (4.2), (4.5) and (4.19) respectively), obtained using the torsor equation (4.1):

$$\xi_3^2|\xi_4|\xi_6|\xi_4\xi_5 + \xi_1^2\xi_6\xi_7|\xi_9^{-1} \leq B,$$
$$\xi_1\xi_2\xi_3\xi_6\xi_7|\xi_4\xi_5 + \xi_1^2\xi_6\xi_7| \leq B,$$
$$\xi_9^2 \leq |\xi_4\xi_5 + \xi_1^2\xi_6\xi_7|.$$

Set $\boldsymbol{\xi}'_a = (\xi_1, \xi_2, \xi_3, \xi_6, \xi_7, \xi_9) \in \mathbb{Z}^6_{>0}$. Assume that $\boldsymbol{\xi}'_a$ is fixed and subject to the height conditions (4.18) and (4.26) and to the coprimality conditions (4.30)–(4.33). Let $N_a(\boldsymbol{\xi}'_a, B)$ be the number of ξ_4, ξ_5 and ξ_8 satisfying the torsor equation (4.1), the height conditions (4.2)–(4.6), the conditions (4.19)–(4.21) and the coprimality conditions (4.27)–(4.29).

LEMMA 23. *For any fixed $A \geq 8$,*

$$N(\boldsymbol{\xi}'_a, B) = \frac{1}{\xi_9} \sum_{\substack{k_8 \mid \xi_2 \\ \gcd(k_8,\xi_7)=1}} \frac{\mu(k_8)}{k_8 \varphi^*(k_8\xi_9)} \sum_{\substack{k_4 \mid \xi_1\xi_2\xi_6\xi_7 \\ \gcd(k_4,k_8\xi_9)=1}} \mu(k_4) \sum_{\substack{k_5 \mid \xi_1\xi_3\xi_6\xi_7 \\ \gcd(k_5,k_8\xi_9)=1}} \mu(k_5) \times \sum_{\substack{\ell_4 \mid k_8\xi_9 \\ \ell_5 \mid k_8\xi_9}} \mu(\ell_4)\mu(\ell_5) C(\boldsymbol{\xi}'_a, B) + R(\boldsymbol{\xi}'_a, B),$$

where, setting $\xi_4 = k_4\ell_4\xi''_4$ and $\xi_5 = k_5\ell_5\xi''_5$,

$$C(\boldsymbol{\xi}'_a, B) = \#\left\{ (\xi''_4, \xi''_5) \in \mathbb{Z}^2_{\neq 0} : \begin{array}{l} (4.2)\text{–}(4.6) \\ (4.19)\text{–}(4.21) \end{array} \right\},$$

and $\sum_{\boldsymbol{\xi}'_a} R(\boldsymbol{\xi}'_a, B) \ll B \log(B)^2$.

Let us remove the coprimality condition (4.27) using a Möbius inversion. We get

$$N(\boldsymbol{\xi}'_a, B) = \sum_{k_8 \mid \xi_1\xi_2\xi_4\xi_5\xi_6\xi_7} \mu(k_8) S_{k_8}(\boldsymbol{\xi}'_a, B),$$

where

$$S_{k_8}(\boldsymbol{\xi}'_a, B) = \#\left\{ (\xi_4, \xi_5, \xi'_8) \in \mathbb{Z}^3_{\neq 0} : \begin{array}{l} \xi_4\xi_5 + k_8\xi'_8\xi_9 = -\xi_1^2\xi_6\xi_7 \\ (4.2)\text{–}(4.6) \\ (4.19)\text{–}(4.21) \\ (4.28), (4.29) \end{array} \right\}.$$

If $\gcd(k_8, \xi_1\xi_6\xi_7) \neq 1$ or $\gcd(k_8, \xi_4\xi_5) \neq 1$ then $\gcd(\xi_4\xi_5, \xi_1\xi_6\xi_7) \neq 1$ and thus $S_{k_8}(\boldsymbol{\xi}'_a, B) = 0$, so we can assume $\gcd(k_8, \xi_1\xi_4\xi_5\xi_6\xi_7) = 1$. We have

$$S_{k_8}(\boldsymbol{\xi}'_a, B) = \#\left\{ (\xi_4, \xi_5) \in \mathbb{Z}^2_{\neq 0} : \begin{array}{l} \xi_4\xi_5 \equiv -\xi_1^2\xi_6\xi_7 \pmod{k_8\xi_9} \\ (4.2)\text{–}(4.6) \\ (4.19)\text{–}(4.21) \\ (4.28), (4.29) \end{array} \right\} + R_0(\boldsymbol{\xi}'_a, B),$$

where the error term $R_0(\boldsymbol{\xi}'_a, B)$ comes from the fact that ξ'_8 has to be nonzero. Otherwise, we would have $\xi_4\xi_5 = -\xi_1^2\xi_6\xi_7$ and thus $|\xi_4| = |\xi_5| = \xi_1 = \xi_6 = \xi_7 = 1$. Summing over ξ_9 using (4.26), we easily obtain

$$\sum_{k_8, \boldsymbol{\xi}'_a} |\mu(k_8)| R_0(\boldsymbol{\xi}'_a, B) \ll B \log(B)^2.$$

Let us remove the coprimality conditions (4.28) and (4.29). The main term of $N(\boldsymbol{\xi}'_a, B)$ is equal to

$$\sum_{\substack{k_8 | \xi_2 \\ \gcd(k_8, \xi_1\xi_6\xi_7)=1}} \mu(k_8) \sum_{\substack{k_4 | \xi_1\xi_2\xi_6\xi_7\xi_9 \\ \gcd(k_4, k_8\xi_9)=1}} \mu(k_4) \sum_{\substack{k_5 | \xi_1\xi_3\xi_6\xi_7\xi_9 \\ \gcd(k_5, k_8\xi_9)=1}} \mu(k_5) S(\boldsymbol{\xi}'_a, B),$$

where, with the notations $\xi_4 = k_4\xi'_4$ and $\xi_5 = k_5\xi'_5$,

$$S(\boldsymbol{\xi}'_a, B) = \#\left\{ (\xi'_4, \xi'_5) \in \mathbb{Z}^2_{\neq 0} \colon \begin{array}{l} \xi'_4\xi'_5 \equiv -(k_4k_5)^{-1}\xi_1^2\xi_6\xi_7 \pmod{k_8\xi_9} \\ (4.2)\text{–}(4.6) \\ (4.19)\text{–}(4.21) \end{array} \right\}.$$

Indeed, we clearly have $\gcd(k_4k_5, k_8\xi_9) = 1$ since $\gcd(k_8\xi_9, \xi_1\xi_6\xi_7) = 1$. We can therefore remove ξ_9 from the conditions on k_4 and k_5. We now proceed to apply Lemma 3. To do so, define

$$X = \frac{B}{k_4k_5\xi_1\xi_2\xi_3\xi_6\xi_7}.$$

An argument identical to the one developed in the proof of Lemma 14 shows that assuming $k_8 \leq (k_4k_5)^{-1/2}X^{1/6}$ produces an error term $N'(\boldsymbol{\xi}'_a, B)$ with

$$\sum_{\xi_9} N'(\boldsymbol{\xi}'_a, B) \ll \left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{1+2\varepsilon-1/12} + 2^{\omega(\xi_2)}\left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{1/2+\varepsilon}.$$

Choosing $\varepsilon = 1/48$ and summing over ξ_3 using (4.18), we see that

$$\begin{aligned} \sum_{\boldsymbol{\xi}'_a} N'(\boldsymbol{\xi}'_a, B) &\ll \sum_{\xi_1,\xi_2,\xi_6,\xi_7} \left(\frac{B}{\xi_1^{13/12}\xi_2\xi_6^{25/24}\xi_7^{25/24}} + 2^{\omega(\xi_2)} \frac{B}{\xi_1^{47/24}\xi_2\xi_6^{71/48}\xi_7^{71/48}} \right) \\ &\ll B\log(B)^2. \end{aligned}$$

The assumption $k_8 \leq (k_4k_5)^{-1/2}X^{1/6}$ and (4.26) prove that now $k_8\xi_9 \leq X^{2/3}$. We apply the first estimate of Lemma 3 with $L_1 = \log(B)^A/k_4$, $L_2 = \log(B)^A/k_5$ and $T = \xi_1^2\xi_6\xi_7/(k_4k_5)$. We have $T \leq 2X$ by (4.18) and $k_8\xi_9 \leq X^{2/3}$, thus Lemma 3 shows that

$$S(\boldsymbol{\xi}'_a, B) = S^*(\boldsymbol{\xi}'_a, B) + O\left(\frac{X^{2/3+\varepsilon}}{(k_8\xi_9)^{1/2}} + \frac{X}{\varphi(k_8\xi_9)}\left(\frac{k_4}{\log(B)^A} + \frac{k_5}{\log(B)^A} \right)\right)$$

for all $\varepsilon > 0$, with

$$S^*(\boldsymbol{\xi}'_a, B) = \frac{1}{\varphi(k_8\xi_9)}\#\left\{ (\xi'_4, \xi'_5) \in \mathbb{Z}^2_{\neq 0} \colon \begin{array}{l} \gcd(\xi'_4\xi'_5, k_8\xi_9) = 1 \\ (4.2)\text{–}(4.6) \\ (4.19)\text{–}(4.21) \end{array} \right\}.$$

The Möbius inversions do not play any part in the estimation of the contribution of the first error term. Using (4.26) to sum over ξ_9, we find that this contribution is

$$\sum_{\xi_1,\xi_2,\xi_3,\xi_6,\xi_7} \left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{11/12+\varepsilon} \ll \sum_{\xi_1,\xi_2,\xi_6,\xi_7} \frac{B}{\xi_1^{7/6-2\varepsilon}\xi_2\xi_6^{13/12-\varepsilon}\xi_7^{13/12-\varepsilon}}$$
$$\ll B\log(B)$$

for $\varepsilon = 1/24$, where we have used (4.18) to sum over ξ_3. The contribution of the second error term is

$$\sum_{\boldsymbol{\xi}'_a} 2^{\omega(\xi_1\xi_2\xi_6\xi_7)} \frac{B\log(B)^{-A}}{\xi_1\xi_2\xi_3\xi_6\xi_7\xi_9} \ll B\log(B)^{10-A},$$

which is satisfactory provided that $A \geq 8$. The contribution of the third error term is also $\ll B\log(B)^{10-A}$. Furthermore,

$$S^*(\boldsymbol{\xi}'_a, B) = \frac{1}{\varphi(k_8\xi_9)} \sum_{\ell_4 | k_8\xi_9} \mu(\ell_4) \sum_{\ell_5 | k_8\xi_9} \mu(\ell_5) C(\boldsymbol{\xi}'_a, B),$$

with the notations $\xi'_4 = \ell_4\xi''_4$ and $\xi'_5 = \ell_5\xi''_5$. It is obvious that

$$C(\boldsymbol{\xi}'_a, B) \ll \left(\frac{X}{\ell_4\ell_5}\right)^{1+\varepsilon}.$$

Let us use this bound to estimate the overall contribution of the error term produced if we remove the condition $k_8 \leq (k_4k_5)^{-1/2}X^{1/6}$ from the sum over k_8. Writing $k_8 > k_8^{1/2}(k_4k_5)^{-1/4}X^{1/12}$ and choosing $\varepsilon = 1/24$, we infer that this contribution is

$$\sum_{\boldsymbol{\xi}'_a} \frac{1}{\xi_9}\left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{23/24} \ll \sum_{\xi_1,\xi_2,\xi_6,\xi_7,\xi_9} \frac{B}{\xi_1^{13/12}\xi_2\xi_6^{25/24}\xi_7^{25/24}\xi_9} \ll B\log(B)^2.$$

We can remove the condition $\gcd(k_8, \xi_1\xi_6) = 1$ from the sum over k_8 since it follows from $k_8 \mid \xi_2$ and $\gcd(\xi_1\xi_6, \xi_2) = 1$, which completes the proof of Lemma 23.

We intend to sum over ξ_1 also. For this, we set $\boldsymbol{\xi}_a = (\xi_2, \xi_3, \xi_6, \xi_7, \xi_9) \in \mathbb{Z}^5_{>0}$. We also define $\boldsymbol{\xi}_a^{(r_2,r_3,r_6,r_7,r_9)} = \xi_2^{r_2}\xi_3^{r_3}\xi_6^{r_6}\xi_7^{r_7}\xi_9^{r_9}$ for $(r_2, r_3, r_6, r_7, r_9) \in \mathbb{Q}^5$. Setting

$$Y_4 = \frac{B^{1/3}}{\boldsymbol{\xi}_a^{(-2/3,4/3,2/3,-1/3,-1)}}, \quad Y''_4 = \frac{Y_4}{k_4\ell_4},$$
$$Y_5 = \frac{B^{1/3}}{\boldsymbol{\xi}_a^{(4/3,-2/3,-1/3,2/3,1)}}, \quad Y''_5 = \frac{Y_5}{k_5\ell_5},$$
$$Y_1 = \frac{B^{1/3}}{\boldsymbol{\xi}_a^{(1/3,1/3,2/3,2/3,0)}},$$

and recalling the definition (4.7) of the function h^a, we can sum up the height conditions (4.2)–(4.6) as

$$h^a(\xi_4''/Y_4'', \xi_5''/Y_5'', \xi_1/Y_1) \leq 1.$$

Note also that (4.18) can be rewritten as

$$\xi_1/Y_1 \leq 2^{1/3}.$$

We also define the following real-valued functions:

$$\begin{aligned}
g_1^a &: (t_5, t_1, t; \boldsymbol{\xi}_a, B) \mapsto \int\limits_{h^a(t_4,t_5,t_1)\leq 1,\, t\leq |t_4t_5+t_1^2|,\, |t_4|Y_4\geq \log(B)^A} dt_4, \\
g_2^a &: (t_1, t; \boldsymbol{\xi}_a, B) \mapsto \int\limits_{|t_5|Y_5\geq \log(B)^A} g_1^a(t_5, t_1, t; \boldsymbol{\xi}_a, B)\, dt_5, \\
g_3^a &: (t; \boldsymbol{\xi}_a, B) \mapsto \int\limits_{t_1>0} g_2^a(t_1, t; \boldsymbol{\xi}_a, B) dt_1, \\
g_4^a &: t \mapsto \iiint\limits_{t_1>0,\, h^a(t_4,t_5,t_1)\leq 1,\, t\leq |t_4t_5+t_1^2|} dt_4\, dt_5\, dt_1.
\end{aligned}$$

The condition $t \leq |t_4t_5+t_1^2|$ corresponds to (4.19) which can now be rewritten as

$$\frac{\xi_9^2}{Y_4Y_5} \leq \left| \frac{\xi_4''}{Y_4''}\,\frac{\xi_5''}{Y_5''} + \left(\frac{\xi_1}{Y_1}\right)^2 \right|.$$

We denote by κ_a the left-hand side of this inequality.

Lemma 24. *We have the bounds*

$$g_1^a(t_5, t_1, t; \boldsymbol{\xi}_a, B) \ll t_1^{-1}|t_5|^{-1}, \quad g_2^a(t_1, t; \boldsymbol{\xi}_a, B) \ll 1.$$

Proof. Recall the definition (4.7) of the function h^a. The first bound is clear since $|t_4t_5|t_1 \leq 1$. Moreover, the conditions $|t_4|\,|t_4t_5 + t_1^2| \leq 1$ and $|t_4|t_5^2 \leq 1$ show that t_5 runs over a set whose measure is $\ll \min(t_4^{-2}, |t_4|^{-1/2})$. Splitting the integration of this minimum over t_4 depending on whether $|t_4|$ is greater or less than 1 completes the proof. ∎

It is easy to check that $\boldsymbol{\xi}_a$ is restricted to lie in the region

$$\mathcal{V}_a = \{\boldsymbol{\xi}_a \in \mathbb{Z}_{>0}^5 \colon Y_1 \geq 2^{-1/3},\ Y_1Y_4Y_5 \geq \xi_9^2\}. \tag{4.34}$$

Assume that $\boldsymbol{\xi}_a \in \mathcal{V}_a$ and $\xi_1 \in \mathbb{Z}_{>0}$ are fixed and satisfy the coprimality conditions (4.30)–(4.33).

We now proceed to estimate $C(\boldsymbol{\xi}_a', B)$. Recall the condition (4.25) which can be rewritten as $|\xi_5''| \leq Y_1\xi_1^{-1}Y_4Y_5''\log(B)^{-A}$. Let us sum over ξ_4'' using the basic estimate $\#\{n \in \mathbb{Z} \colon t_1 \leq n \leq t_2\} = t_2 - t_1 + O(1)$. The change of variable $t_4 \mapsto Y_4'' t_4$ shows that

$$C(\boldsymbol{\xi}_a', B) = \sum_{|\xi_5''| \le Y_1 \xi_1^{-1} Y_4 Y_5'' \log(B)^{-A}} (Y_4'' g_1^a(\xi_5''/Y_5'', \xi_1/Y_1, \kappa_a; \boldsymbol{\xi}_a, B) + O(1)).$$

We see that the overall contribution of the error term is

$$\sum_{\boldsymbol{\xi}_a'} 2^{\omega(\xi_1\xi_2\xi_6\xi_7)} 2^{\omega(\xi_2\xi_9)} \frac{B\log(B)^{-A}}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}\xi_1} \ll \log(B)^{13-A}.$$

Let us now sum over ξ_5''. Using partial summation and the change of variable $t_5 \mapsto Y_5'' t_5$, we obtain

$$\begin{aligned} C(\boldsymbol{\xi}_a', B) &= Y_4'' Y_5'' g_2^a(\xi_1/Y_1, \kappa_a; \boldsymbol{\xi}_a, B) \\ &\quad + O(Y_4'' \sup_{|t_5| Y_5 \ge \log(B)^A} g_1^a(t_5, \xi_1/Y_1, \kappa_a; \boldsymbol{\xi}_a, B)). \end{aligned}$$

Using the bound of Lemma 24 for g_1^a, we get

$$\sup_{|t_5| Y_5 \ge \log(B)^A} g_1^a(t_5, \xi_1/Y_1, \kappa; \boldsymbol{\xi}_a, B) \ll Y_5 \log(B)^{-A} Y_1/\xi_1.$$

The overall contribution coming from this error term is therefore

$$\sum_{\boldsymbol{\xi}_a'} 2^{\omega(\xi_1\xi_3\xi_6\xi_7)} 2^{\omega(\xi_2\xi_9)} \frac{B\log(B)^{-A}}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}\xi_1} \ll B\log(B)^{12-A}.$$

Recalling Lemma 23, we find that for any fixed $A \ge 9$,

$$\begin{aligned} N(\boldsymbol{\xi}_a', B) &= \frac{1}{\xi_9}\theta_a(\boldsymbol{\xi}_a) \frac{\varphi^*(\xi_1)}{\varphi^*(\gcd(\xi_1, \xi_2\xi_6\xi_7))} \frac{\varphi^*(\xi_1)}{\varphi^*(\gcd(\xi_1, \xi_3\xi_6\xi_7))} \\ &\quad \times g_2^a(\xi_1/Y_1, \kappa_a; \boldsymbol{\xi}_a, B) Y_4 Y_5 + R_1(\boldsymbol{\xi}_a', B), \end{aligned}$$

where

$$\theta_a(\boldsymbol{\xi}_a) = \varphi^*(\xi_2\xi_6\xi_7)\varphi^*(\xi_3\xi_6\xi_7)\frac{\varphi^*(\xi_2\xi_9)}{\varphi^*(\gcd(\xi_2, \xi_7))}$$

and $\sum_{\boldsymbol{\xi}_a'} R_1(\boldsymbol{\xi}_a', B) \ll B\log(B)^4$. For fixed $\boldsymbol{\xi}_a \in \mathcal{V}_a$ satisfying the coprimality conditions (4.31)–(4.33), let $\mathbf{N}(\boldsymbol{\xi}_a, B)$ be the sum over ξ_1 of the main term of $N(\boldsymbol{\xi}_a', B)$, with ξ_1 subject to the coprimality condition (4.30). Let us make use of Lemma 6 to sum over ξ_1. We find that for any fixed $A \ge 9$ and $0 < \sigma \le 1$,

$$\begin{aligned} \mathbf{N}(\boldsymbol{\xi}_a, B) &= \frac{1}{\xi_9}\mathcal{P}\Theta_a(\boldsymbol{\xi}_a) g_3^a(\kappa_a; \boldsymbol{\xi}_a, B) Y_4 Y_5 Y_1 \\ &\quad + O\left(\frac{Y_4 Y_5}{\xi_9}\varphi_\sigma(\xi_2\xi_3\xi_9) Y_1^\sigma \sup_{t_1 > 0} g_2^a(t_1, \kappa_a; \boldsymbol{\xi}_a, B)\right), \end{aligned} \tag{4.35}$$

where

$$\Theta_a(\boldsymbol{\xi}_a) = \theta_a(\boldsymbol{\xi}_a)\varphi^*(\xi_2\xi_3\xi_9)\varphi'(\xi_2\xi_3\xi_6\xi_7\xi_9).$$

Using the bound of Lemma 24 for g_2^a and choosing $\sigma = 1/2$, we see that the overall contribution of the error term is

$$\sum_{\boldsymbol{\xi}_a} \varphi_\sigma(\xi_2\xi_3\xi_9)\frac{Y_4Y_5}{\xi_9}Y_1^{1/2} \ll \sum_{\xi_2,\xi_3,\xi_6,\xi_9} \varphi_\sigma(\xi_2\xi_3\xi_9)\frac{B}{\boldsymbol{\xi}_a^{(1,1,1,0,1)}} \ll B\log(B)^4,$$

where we have summed over ξ_7 using $Y_1 \geq 2^{-1/3}$ and used the fact that φ_σ has average order $O(1)$. Note that

$$\frac{Y_4Y_5Y_1}{\xi_9} = \frac{B}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}}.$$

For brevity, we set

$$D_{h^a} = \{(t_4,t_5,t_1) \in \mathbb{R}^3 : t_1 > 0,\ h^a(t_4,t_5,t_1) \leq 1\}.$$

Lemma 25. *For $Z_4, Z_5 > 0$,*

$$\operatorname{meas}\{(t_4,t_5,t_1) \in D_{h^a} : |t_4|Z_4 \geq 1\} \ll Z_4, \tag{4.36}$$

$$\operatorname{meas}\{(t_4,t_5,t_1) \in D_{h^a} : |t_5|Z_5 \geq 1\} \ll Z_5, \tag{4.37}$$

$$\operatorname{meas}\{(t_4,t_5,t_1) \in D_{h^a} : |t_4|Z_4 < 1\} \ll Z_4^{-1/2}, \tag{4.38}$$

$$\operatorname{meas}\{(t_4,t_5,t_1) \in D_{h^a} : |t_5|Z_5 < 1\} \ll Z_5^{-1/2}. \tag{4.39}$$

Proof. First, the conditions $t_1|t_4t_5| \leq 1$ and $t_1|t_4t_5 + t_1^2| \leq 1$ show that we always have $t_1^3 \leq 2$. Using $|t_4|\,|t_4t_5 + t_1^2| \leq 1$, we see that t_5 runs over a set whose measure is $\ll |t_4|^{-2}$ and thus integrating over t_1 using $t_1^3 \leq 2$ gives (4.36). Since $|t_4|t_5^2 \leq 1$, we see that t_4 runs over a set whose measure is $\ll t_5^{-2}$. Integrating this over t_1 using $t_1^3 \leq 2$ leads to (4.37). Furthermore, integrating over t_5 using $|t_4|t_5^2 \leq 1$ and then over t_1 using $t_1^3 \leq 2$ leads to (4.38). Finally, the condition $|t_4|\,|t_4t_5 + t_1^2| \leq 1$ shows that t_4 runs over a set whose measure is $\ll |t_5|^{-1/2}$ and integrating this quantity over t_1 using $t_1^3 \leq 2$ proves (4.39). ∎

Exactly as in Section 3.5 for the case of the $3\mathbf{A}_1$ surface, the bounds (4.36) and (4.37) show that if we do not have $Y_4, Y_5 \geq \log(B)^A$, the contribution of the main term of $\mathbf{N}(\boldsymbol{\xi}_a, B)$ is $\ll B\log(B)^4$. Thus we can assume from now on that

$$Y_4 \geq \log(B)^A, \tag{4.40}$$

$$Y_5 \geq \log(B)^A, \tag{4.41}$$

and the two bounds (4.38), (4.39) therefore show that removing the conditions $|t_4|Y_4, |t_5|Y_5 \geq \log(B)^A$ from the integral defining g_3^a in the main term of $\mathbf{N}(\boldsymbol{\xi}_a, B)$ in (4.35) creates an error term whose overall contribution is $\ll B\log(B)^4$. We have thus proved that for any fixed $A \geq 9$,

$$\mathbf{N}(\boldsymbol{\xi}_a, B) = \mathcal{P} g_4^a(\kappa_a)\frac{B}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}}\Theta_a(\boldsymbol{\xi}_a) + R_2(\boldsymbol{\xi}_a, B), \tag{4.42}$$

where $\sum_{\boldsymbol{\xi}_a} R_2(\boldsymbol{\xi}_a, B) \ll B\log(B)^4$.

Lemma 26. *For $t > 0$,*

$$\operatorname{meas}\{(t_4, t_5, t_1) \in D_{h^a} : |t_4t_5 + t_1^2| \ge t\} \ll t^{-3/2}, \tag{4.43}$$

$$\operatorname{meas}\{(t_4, t_5, t_1) \in D_{h^a} : |t_4t_5 + t_1^2| < t\} \ll t^{1/2}. \tag{4.44}$$

Proof. First, the conditions $|t_4|\,|t_4t_5 + t_1^2| \le 1$, $t_1|t_4t_5 + t_1^2| \le 1$ and $|t_4t_5 + t_1^2| \ge t$ yield $t|t_4| \le 1$ and $tt_1 \le 1$. Therefore, integrating over t_5 using $|t_4|t_5^2 \le 1$ and then over $|t_4|, t_1 \le t^{-1}$ yields (4.43). In addition, the condition $|t_4t_5 + t_1^2| < t$ shows that t_4 runs over a set whose measure is $\ll \min(t|t_5|^{-1}, |t_5|^{-1/2}) \le t^{1/2}|t_5|^{-3/4}$. Integrating this quantity over t_5 using $t_1^2|t_5| \le 1$ and then over t_1 using $t_1^3 \le 2$ gives (4.44). ∎

The bound (4.43) shows that if $\kappa_a > 1$, the contribution of the main term of $\mathbf{N}(\boldsymbol{\xi}_a, B)$ is $\ll B\log(B)^4$, thus we assume from now on that $\kappa_a \le 1$, that is,

$$Y_4Y_5 \ge \xi_9^2. \tag{4.45}$$

Replacing $g_4^a(\kappa_a)$ by $g_4^a(0)$ in the main term of $\mathbf{N}(\boldsymbol{\xi}_a, B)$ in (4.42) therefore produces an error term whose overall contribution is $\ll B\log(B)^4$ thanks to (4.44). Since $g_4^a(0) = \tau_\infty/3$ by (4.9), we have obtained the following result.

Lemma 27. *For any fixed $A \ge 9$,*

$$\mathbf{N}(\boldsymbol{\xi}_a, B) = \mathcal{P}\frac{\tau_\infty}{3}\frac{B}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}}\Theta_a(\boldsymbol{\xi}_a) + R_3(\boldsymbol{\xi}_a, B),$$

where $\sum_{\boldsymbol{\xi}_a} R_3(\boldsymbol{\xi}_a, B) \ll B\log(B)^4$.

Recall the definition (4.34) of $\mathcal{V}_a$. It remains to sum the main term of $\mathbf{N}(\boldsymbol{\xi}_a, B)$ over the $\boldsymbol{\xi}_a \in \mathcal{V}_a$ satisfying (4.40), (4.41) and (4.45) and the coprimality conditions (4.31)–(4.33). It is easy to check that replacing $\{\boldsymbol{\xi}_a \in \mathcal{V}_a : (4.40), (4.41), (4.45)\}$ by the region

$$\mathcal{V}_a' = \{\boldsymbol{\xi}_a \in \mathbb{Z}_{>0}^5 : Y_4 \ge 1,\ Y_5 \ge 1,\ Y_1 \ge 1,\ Y_4Y_5 \ge \xi_9^2\}$$

produces an error term whose overall contribution is $\ll B\log(B)^4\log(\log(B))$. Let us redefine Θ_a as being equal to zero if the remaining coprimality conditions (4.31)–(4.33) are not satisfied. Lemma 27 proves that for any fixed $A \ge 9$,

$$N_a(A, B) = \mathcal{P}\frac{\tau_\infty}{3}B\sum_{\boldsymbol{\xi}_a \in \mathcal{V}_a'}\frac{\Theta_a(\boldsymbol{\xi}_a)}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}} + O(B\log(B)^4\log(\log(B))).$$

As in Section 3.6, Θ_a satisfies the assumption (2.40) of Lemma 8 and thus

$$N_a(A, B) = \mathcal{P}\frac{\tau_\infty}{3}\alpha_a\Bigg(\sum_{\boldsymbol{\xi}_a \in \mathbb{Z}_{>0}^5}\frac{(\Theta_a * \boldsymbol{\mu})(\boldsymbol{\xi}_a)}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}}\Bigg)B\log(B)^5 + O(B\log(B)^4\log(\log(B))),$$

where α_a is the volume of the polytope defined in $\mathbb{R}^5$ by $t_2, t_3, t_6, t_7, t_9 \geq 0$ and

$$\begin{aligned} -2t_2 + 4t_3 + 2t_6 - t_7 - 3t_9 &\leq 1, \\ 4t_2 - 2t_3 - t_6 + 2t_7 + 3t_9 &\leq 1, \\ t_2 + t_3 + 2t_6 + 2t_7 &\leq 1, \\ 2t_2 + 2t_3 + t_6 + t_7 + 6t_9 &\leq 2. \end{aligned}$$

A computation using [Fra09] gives

$$\alpha_a = \frac{1871}{2016000}, \tag{4.46}$$

and moreover, as in Section 3.6,

$$\sum_{\boldsymbol{\xi}_a \in \mathbb{Z}_{>0}^5} \frac{(\Theta_a * \boldsymbol{\mu})(\boldsymbol{\xi}_a)}{\boldsymbol{\xi}_a^{(1,1,1,1,1)}} = \mathcal{P}^{-1} \prod_p \left(1 - \frac{1}{p}\right)^6 \tau_p;$$

thus we have obtained the following lemma.

Lemma 28. *For any fixed $A \geq 9$,*

$$N_a(A, B) = \frac{1}{3} \alpha_a \omega_H(\widetilde{V_2}) B \log(B)^5 + O(B \log(B)^4 \log(\log(B))).$$

4.6. Estimating $N_b(A, B)$. Note that the assumption $|\xi_9| > \xi_8$ and (4.5) yield in this case

$$\xi_8^2 < \frac{B}{\xi_1 \xi_2 \xi_3 \xi_6 \xi_7}. \tag{4.47}$$

We estimate the contribution of the variables ξ_4, ξ_5 and ξ_9. To do so, we rewrite the coprimality conditions as

$$\gcd(\xi_9, \xi_1\xi_3\xi_4\xi_5\xi_6\xi_7) = 1, \tag{4.48}$$
$$\gcd(\xi_4, \xi_1\xi_2\xi_6\xi_7\xi_8) = 1, \tag{4.49}$$
$$\gcd(\xi_5, \xi_1\xi_3\xi_6\xi_7\xi_8) = 1, \tag{4.50}$$
$$\gcd(\xi_1, \xi_2\xi_3\xi_8) = 1, \tag{4.51}$$
$$\gcd(\xi_2, \xi_3\xi_6\xi_8) = 1, \tag{4.52}$$
$$\gcd(\xi_7, \xi_3\xi_6\xi_8) = 1, \tag{4.53}$$
$$\gcd(\xi_8, \xi_6) = 1, \tag{4.54}$$

This time, we want to view the torsor equation (4.1) as a congruence modulo ξ_8. To do so, we replace (4.3), (4.5), (4.6) and (4.22) by the following (we keep denoting them by (4.3), (4.5), (4.6) and (4.22)), obtained using (4.1):

$$\begin{aligned} \xi_1^2\xi_2^2|\xi_5|\xi_6\xi_7^2|\xi_4\xi_5+\xi_1^2\xi_6\xi_7|\xi_8^{-1} &\le B,\\ \xi_1\xi_2\xi_3\xi_6\xi_7|\xi_4\xi_5+\xi_1^2\xi_6\xi_7| &\le B,\\ \xi_2^2|\xi_4|\xi_5^2\xi_7|\xi_4\xi_5+\xi_1^2\xi_6\xi_7|\xi_8^{-1} &\le B,\\ \xi_8^2 &< |\xi_4\xi_5+\xi_1^2\xi_6\xi_7|. \end{aligned}$$

Set $\boldsymbol{\xi}_b' = (\xi_1, \xi_2, \xi_3, \xi_6, \xi_7, \xi_8) \in \mathbb{Z}_{>0}^6$. Assume that $\boldsymbol{\xi}_b'$ is fixed and satisfies the height conditions (4.18) and (4.47) and the coprimality conditions (4.51)–(4.54). Let $N_b(\boldsymbol{\xi}_b', B)$ be the number of ξ_4, ξ_5 and ξ_9 satisfying the torsor equation (4.1), the height conditions (4.2)–(4.6), the conditions (4.22)–(4.24) and the coprimality conditions (4.48)–(4.50).

Lemma 29. *For any fixed $A \ge 8$,*

$$N(\boldsymbol{\xi}_b', B) = \frac{1}{\xi_8} \sum_{\substack{k_9 | \xi_3\\ \gcd(k_9,\xi_6)=1}} \frac{\mu(k_9)}{k_9\varphi^*(k_9\xi_8)} \sum_{\substack{k_4|\xi_1\xi_2\xi_6\xi_7\\ \gcd(k_4,k_9\xi_8)=1}} \mu(k_4) \sum_{\substack{k_5|\xi_1\xi_3\xi_6\xi_7\\ \gcd(k_5,k_9\xi_8)=1}} \mu(k_5) \times \sum_{\substack{\ell_4|k_9\xi_8\\ \ell_5|k_9\xi_8}} \mu(\ell_4)\mu(\ell_5) C(\boldsymbol{\xi}_b', B) + R(\boldsymbol{\xi}_b', B),$$

where, setting $\xi_4 = k_4\ell_4\xi_4''$ and $\xi_5 = k_5\ell_5\xi_5''$,

$$C(\boldsymbol{\xi}_b', B) = \#\left\{(\xi_4'', \xi_5'') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} (4.2)\text{–}(4.6)\\ (4.22)\text{–}(4.24)\end{array}\right\},$$

and $\sum_{\boldsymbol{\xi}_b'} R(\boldsymbol{\xi}_b', B) \ll B\log(B)^2$.

Let us remove the coprimality condition (4.27) using a Möbius inversion. We get

$$N(\boldsymbol{\xi}_b', B) = \sum_{k_9|\xi_1\xi_3\xi_4\xi_5\xi_6\xi_7} \mu(k_9) S_{k_9}(\boldsymbol{\xi}_b', B),$$

where

$$S_{k_9}(\boldsymbol{\xi}_b', B) = \#\left\{(\xi_4, \xi_5, \xi_9') \in \mathbb{Z}_{\neq 0}^3 : \begin{array}{l} \xi_4\xi_5 + k_9\xi_8\xi_9' = -\xi_1^2\xi_6\xi_7\\ (4.2)\text{–}(4.6)\\ (4.22)\text{–}(4.24)\\ (4.49), (4.50)\end{array}\right\}.$$

If $\gcd(k_9, \xi_1\xi_6\xi_7) \neq 1$ or $\gcd(k_9, \xi_4\xi_5) \neq 1$ then $\gcd(\xi_4\xi_5, \xi_1\xi_6\xi_7) \neq 1$ and so $S_{k_9}(\boldsymbol{\xi}_b', B) = 0$; thus we can assume $\gcd(k_9, \xi_1\xi_4\xi_5\xi_6\xi_7) = 1$. We have

$$S_{k_9}(\boldsymbol{\xi}_b', B) = \#\left\{(\xi_4, \xi_5) \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} \xi_4\xi_5 \equiv -\xi_1^2\xi_6\xi_7 \pmod{k_9\xi_8}\\ (4.2)\text{–}(4.6)\\ (4.22)\text{–}(4.24)\\ (4.49), (4.50)\end{array}\right\} + R_0(\boldsymbol{\xi}_b', B),$$

where the error term $R_0(\boldsymbol{\xi}_b', B)$ comes from the fact that ξ_9' has to be nonzero. Otherwise, we would have $\xi_4\xi_5 = -\xi_1^2\xi_6\xi_7$ and thus $|\xi_4| = |\xi_5| = \xi_1 = \xi_6 = \xi_7 = 1$. Summing over ξ_8 using (4.47), we easily obtain

$$\sum_{k_9, \boldsymbol{\xi}_b'} |\mu(k_9)| R_0(\boldsymbol{\xi}_b', B) \ll B \log(B)^2.$$

We now remove the coprimality conditions (4.49) and (4.50). The main term of $N(\boldsymbol{\xi}_b', B)$ is equal to

$$\sum_{\substack{k_9 | \xi_3 \\ \gcd(k_9, \xi_1\xi_6\xi_7)=1}} \mu(k_9) \sum_{\substack{k_4 | \xi_1\xi_2\xi_6\xi_7\xi_8 \\ \gcd(k_4, k_9\xi_8)=1}} \mu(k_4) \sum_{\substack{k_5 | \xi_1\xi_3\xi_6\xi_7\xi_8 \\ \gcd(k_5, k_9\xi_8)=1}} \mu(k_5) S(\boldsymbol{\xi}_b', B),$$

where, setting $\xi_4 = k_4\xi_4'$ and $\xi_5 = k_5\xi_5'$,

$$S(\boldsymbol{\xi}_b', B) = \#\left\{ (\xi_4', \xi_5') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} \xi_4'\xi_5' \equiv -(k_4k_5)^{-1}\xi_1^2\xi_6\xi_7 \pmod{k_9\xi_8} \\ (4.2)\text{–}(4.6) \\ (4.22)\text{–}(4.24) \end{array} \right\}.$$

Indeed, since $\gcd(k_9\xi_8, \xi_1\xi_6\xi_7) = 1$, we have $\gcd(k_4k_5, k_9\xi_8) = 1$. We can therefore remove ξ_8 from the conditions on k_4 and k_5. Everything is now in place to apply Lemma 3. Set

$$X = \frac{B}{k_4k_5\xi_1\xi_2\xi_3\xi_6\xi_7}.$$

An argument identical to the one given in the proof of Lemma 14 shows that assuming $k_9 \le (k_4k_5)^{-1/2}X^{1/6}$ produces an error term $N'(\boldsymbol{\xi}_b', B)$ with

$$\sum_{\xi_8} N'(\boldsymbol{\xi}_b', B) \ll \left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{1+2\varepsilon-1/12} + 2^{\omega(\xi_3)}\left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7}\right)^{1/2+\varepsilon}.$$

Choosing $\varepsilon = 1/48$ and summing over ξ_2 using (4.18), we get

$$\begin{aligned}\sum_{\boldsymbol{\xi}_b'} N'(\boldsymbol{\xi}_b', B) &\ll \sum_{\xi_1,\xi_3,\xi_6,\xi_7} \left(\frac{B}{\xi_1^{13/12}\xi_3\xi_6^{25/24}\xi_7^{25/24}} + 2^{\omega(\xi_3)}\frac{B}{\xi_1^{47/24}\xi_3\xi_6^{71/48}\xi_7^{71/48}}\right) \\ &\ll B\log(B)^2.\end{aligned}$$

The assumption $k_9 \le (k_4k_5)^{-1/2}X^{1/6}$ and (4.47) give $k_9\xi_8 \le X^{2/3}$. We proceed to apply the second estimate of Lemma 3. Set as in the first case $L_1 = \log(B)^A/k_4$, $L_2 = \log(B)^A/k_5$ and $T = \xi_1^2\xi_6\xi_7/(k_4k_5)$. We have $T \le 2X$ by (4.18) and $k_9\xi_8 \le X^{2/3}$, thus Lemma 3 shows that

$$S(\boldsymbol{\xi}_b', B) = S^*(\boldsymbol{\xi}_b', B) + O\left(\frac{X^{4/5+\varepsilon}}{(k_9\xi_8)^{7/10}} + \frac{X}{\varphi(k_9\xi_8)}\left(\frac{k_4}{\log(B)^A} + \frac{k_5}{\log(B)^A}\right)\right)$$

for all $\varepsilon > 0$, with

$$S^*(\boldsymbol{\xi}_b', B) = \frac{1}{\varphi(k_9\xi_8)} \# \left\{ (\xi_4', \xi_5') \in \mathbb{Z}_{\neq 0}^2 : \begin{array}{l} \gcd(\xi_4'\xi_5', k_9\xi_8) = 1 \\ (4.2)\text{–}(4.6) \\ (4.22)\text{–}(4.24) \end{array} \right\}.$$

Using (4.47) to sum over ξ_8, we find that the contribution of the first error term is

$$\sum_{\xi_1,\xi_2,\xi_3,\xi_6,\xi_7} \left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7} \right)^{19/20+\varepsilon} \ll B\log(B)$$

for $\varepsilon = 1/40$, where we have summed over ξ_3 using (4.18). The contributions of the second and third error terms are easily seen to be both $\ll B\log(B)^{10-A}$, which is satisfactory if $A \geq 8$. Furthermore,

$$S^*(\boldsymbol{\xi}_b', B) = \frac{1}{\varphi(k_9\xi_8)} \sum_{\ell_4 | k_9\xi_8} \mu(\ell_4) \sum_{\ell_5 | k_9\xi_8} \mu(\ell_5) C(\boldsymbol{\xi}_b', B),$$

where we have set $\xi_4' = \ell_4\xi_4''$ and $\xi_5' = \ell_5\xi_5''$. It is plain that

$$C(\boldsymbol{\xi}_b', B) \ll \left(\frac{X}{\ell_4\ell_5} \right)^{1+\varepsilon}.$$

Let us use this bound to estimate the overall contribution of the error term produced by removing the condition $k_9 \leq (k_4k_5)^{-1/2}X^{1/6}$ from the sum over k_9. Writing $k_9 > k_9^{1/2}(k_4k_5)^{-1/4}X^{1/12}$ and choosing $\varepsilon = 1/24$, we see that this contribution is

$$\sum_{\boldsymbol{\xi}_a'} \frac{1}{\xi_8} \left(\frac{B}{\xi_1\xi_2\xi_3\xi_6\xi_7} \right)^{23/24} \ll B\log(B)^2,$$

as in Section 4.5. We can remove the condition $\gcd(k_9, \xi_1\xi_7) = 1$ from the sum over k_9 since it follows from $k_9 \mid \xi_3$ and $\gcd(\xi_1\xi_7, \xi_3) = 1$. This ends the proof of Lemma 29.

We intend to sum also over ξ_1 and we therefore set $\boldsymbol{\xi}_b = (\xi_2, \xi_3, \xi_6, \xi_7, \xi_8) \in \mathbb{Z}_{>0}^5$. We also set $\boldsymbol{\xi}_b^{(r_2,r_3,r_6,r_7,r_8)} = \xi_2^{r_2}\xi_3^{r_3}\xi_6^{r_6}\xi_7^{r_7}\xi_8^{r_8}$ for $(r_2, r_3, r_6, r_7, r_8) \in \mathbb{Q}^5$ and finally

$$Y_4 = \frac{B}{\boldsymbol{\xi}_b^{(0,2,1,0,1)}}, \qquad Y_4'' = \frac{Y_4}{k_4\ell_4},$$

$$Y_5 = \frac{\boldsymbol{\xi}_b^{(-2/3,4/3,2/3,-1/3,1)}}{B^{1/3}}, \qquad Y_5'' = \frac{Y_5}{k_5\ell_5},$$

$$Y_1 = \frac{B^{1/3}}{\boldsymbol{\xi}_b^{(1/3,1/3,2/3,2/3,0)}}.$$

Recalling the definition (4.8) of the function h^b, we can sum up the height conditions (4.2)–(4.6) as

$$h^b(\xi_4''/Y_4'', \xi_5''/Y_5'', \xi_1/Y_1) \leq 1.$$

Note that, as in the first case, (4.18) can be rewritten as

$$\xi_1/Y_1 \leq 2^{1/3}.$$

We also introduce the following real-valued functions:

$$g_1^b : (t_5, t_1, t; \boldsymbol{\xi}_b, B) \mapsto \int\limits_{h^b(t_4,t_5,t_1)\leq 1,\, t<|t_4t_5+t_1^2|,\, |t_4|Y_4\geq \log(B)^A} dt_4,$$

$$g_2^b : (t_1, t; \boldsymbol{\xi}_b, B) \mapsto \int\limits_{|t_5|Y_5\geq \log(B)^A} g_1^b(t_5, t_1, t; \boldsymbol{\xi}_b, B)\, dt_5,$$

$$g_3^b : (t; \boldsymbol{\xi}_b, B) \mapsto \int\limits_{t_1>0} g_2^b(t_1, t; \boldsymbol{\xi}_b, B)\, dt_1,$$

$$g_4^b : t \mapsto \iiint\limits_{t_1>0,\, h^b(t_4,t_5,t_1)\leq 1,\, t<|t_4t_5+t_1^2|} dt_4\, dt_5\, dt_1.$$

The condition $t < |t_4t_5+t_1^2|$ corresponds to (4.22) which can now be rewritten as

$$\frac{\xi_8^2}{Y_4Y_5} < \left| \frac{\xi_4''}{Y_4''} \frac{\xi_5''}{Y_5''} + \left(\frac{\xi_1}{Y_1}\right)^2 \right|.$$

We denote by κ_b the left-hand side of this inequality.

LEMMA 30. *We have the bounds*

$$g_1^b(t_5, t_1, t; \boldsymbol{\xi}_b, B) \ll t_1^{-1}|t_5|^{-1}, \quad g_2^b(t_1, t; \boldsymbol{\xi}_b, B) \ll 1.$$

Proof. Recall the definition (4.8) of the function h^b. The first bound is clear since $t_1|t_4t_5| \leq 1$. For the other one, the conditions $|t_4| \leq 1$ and $|t_4|t_5^2|t_4t_5 + t_1^2| \leq 1$ show that t_4 runs over a set whose measure is $\ll \min(1, |t_5|^{-3/2})$. Splitting the integration of this minimum over t_5 depending on whether $|t_5|$ is greater or less than 1 provides the desired bound. ∎

It is immediate to check that $\boldsymbol{\xi}_b$ is restricted to lie in the region

$$\mathcal{V}_b = \{\boldsymbol{\xi}_b \in \mathbb{Z}_{>0}^5 \colon Y_4 \geq \log(B)^A,\ Y_1 \geq 2^{-1/3}\}. \tag{4.55}$$

Assume that $\boldsymbol{\xi}_b \in \mathcal{V}_b$ and $\xi_1 \in \mathbb{Z}_{>0}$ are fixed and satisfy the coprimality conditions (4.51)–(4.54).

We now turn to the estimation of $C(\boldsymbol{\xi}_b', B)$. Let us sum over ξ_4'' using the basic estimate $\#\{n \in \mathbb{Z} \colon t_1 \leq n \leq t_2\} = t_2 - t_1 + O(1)$. The change of variable $t_4 \mapsto Y_4''t_4$ shows that

$$C(\boldsymbol{\xi}_b', B) = \sum_{|\xi_5''|\leq Y_1\xi_1^{-1}Y_4Y_5''\log(B)^{-A}} (Y_4''g_1^b(\xi_5''/Y_5'', \xi_1/Y_1, \kappa_b; \boldsymbol{\xi}_b, B) + O(1)).$$

The overall contribution of the error term is

$$\sum_{\boldsymbol{\xi}'_b} 2^{\omega(\xi_1\xi_2\xi_6\xi_7)} 2^{\omega(\xi_3\xi_8)} \frac{B\log(B)^{-A}}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}\xi_1} \ll \log(B)^{12-A}.$$

Let us now sum over ξ''_5. Using partial summation and the change of variable $t_5 \mapsto Y''_5 t_5$, we obtain

$$\begin{aligned} C(\boldsymbol{\xi}'_b, B) &= Y''_4 Y''_5 g_2^b(\xi_1/Y_1, \kappa_b; \boldsymbol{\xi}_b, B) \\ &\quad + O(Y''_4 \sup_{|t_5|Y_5 \geq \log(B)^A} g_1^b(t_5, \xi_1/Y_1, \kappa_b; \boldsymbol{\xi}_b, B)). \end{aligned}$$

Using the bound of Lemma 30 for g_1^b, we get

$$\sup_{|t_5|Y_5 \geq \log(B)^A} g_1^b(t_5, \xi_1/Y_1, \kappa; \boldsymbol{\xi}_b, B) \ll Y_5 \log(B)^{-A} Y_1/\xi_1.$$

The overall contribution coming from this error term is therefore

$$\sum_{\boldsymbol{\xi}'_b} 2^{\omega(\xi_1\xi_3\xi_6\xi_7)} 2^{\omega(\xi_3\xi_8)} \frac{B\log(B)^{-A}}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}\xi_1} \ll B\log(B)^{13-A}.$$

Recalling Lemma 29, we find that for any fixed $A \geq 9$,

$$\begin{aligned} N(\boldsymbol{\xi}'_b, B) &= \frac{1}{\xi_8}\theta_b(\boldsymbol{\xi}_b) \frac{\varphi^*(\xi_1)}{\varphi^*(\gcd(\xi_1, \xi_2\xi_6\xi_7))} \frac{\varphi^*(\xi_1)}{\varphi^*(\gcd(\xi_1, \xi_3\xi_6\xi_7))} \\ &\quad \times g_2^b(\xi_1/Y_1, \kappa_b; \boldsymbol{\xi}_b, B) Y_4 Y_5 + R_1(\boldsymbol{\xi}'_b, B), \end{aligned}$$

where

$$\theta_b(\boldsymbol{\xi}_b) = \varphi^*(\xi_2\xi_6\xi_7)\varphi^*(\xi_3\xi_6\xi_7) \frac{\varphi^*(\xi_3\xi_8)}{\varphi^*(\gcd(\xi_3, \xi_6))}$$

and $\sum_{\boldsymbol{\xi}'_b} R_1(\boldsymbol{\xi}'_b, B) \ll B\log(B)^4$. For fixed $\boldsymbol{\xi}_b \in \mathcal{V}_b$ satisfying the coprimality conditions (4.52)–(4.54), let $\mathbf{N}(\boldsymbol{\xi}_b, B)$ be the sum over ξ_1 of the main term of $N(\boldsymbol{\xi}'_b, B)$, with ξ_1 subject to the coprimality condition (4.51). Let us make use of Lemma 6 to sum over ξ_1. We find that for any fixed $A \geq 9$ and $0 < \sigma \leq 1$,

$$\begin{aligned} \mathbf{N}(\boldsymbol{\xi}_b, B) &= \frac{1}{\xi_8}\mathcal{P}\Theta_b(\boldsymbol{\xi}_b) g_3^b(\kappa_b; \boldsymbol{\xi}_b, B) Y_4 Y_5 Y_1 \\ &\quad + O\bigg(\frac{Y_4 Y_5}{\xi_8}\varphi_\sigma(\xi_2\xi_3\xi_8) Y_1^\sigma \sup_{t_1>0} g_2^b(t_1, \kappa_b; \boldsymbol{\xi}_b, B)\bigg), \end{aligned} \tag{4.56}$$

where

$$\Theta_b(\boldsymbol{\xi}_b) = \theta_b(\boldsymbol{\xi}_b)\varphi^*(\xi_2\xi_3\xi_8)\varphi'(\xi_2\xi_3\xi_6\xi_7\xi_8).$$

Using the bound of Lemma 30 for g_2^b and choosing $\sigma = 1/2$, we see that the overall contribution of the error term is

$$\sum_{\boldsymbol{\xi}_b} \varphi_\sigma(\xi_2\xi_3\xi_8)\frac{Y_4Y_5}{\xi_8}Y_1^{1/2} \ll \sum_{\xi_2,\xi_3,\xi_6,\xi_8} \varphi_\sigma(\xi_2\xi_3\xi_8)\frac{B}{\boldsymbol{\xi}_b^{(1,1,1,0,1)}} \ll B\log(B)^4,$$

where we have summed over ξ_7 using $Y_1 \geq 2^{-1/3}$. Note that

$$\frac{Y_4Y_5Y_1}{\xi_8} = \frac{B}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}}.$$

For brevity, we set

$$D_{h^b} = \{(t_4,t_5,t_1) \in \mathbb{R}^3 \colon t_1 > 0,\ h^b(t_4,t_5,t_1) \leq 1\}.$$

Lemma 31. *For $Z_4, Z_5 > 0$,*

$$\text{meas}\{(t_4,t_5,t_1) \in D_{h^b} \colon |t_5|Z_5 \geq 1\} \ll Z_5^{1/2}, \tag{4.57}$$

$$\text{meas}\{(t_4,t_5,t_1) \in D_{h^b} \colon |t_4|Z_4 < 1\} \ll Z_4^{-1/3}, \tag{4.58}$$

$$\text{meas}\{(t_4,t_5,t_1) \in D_{h^b} \colon |t_5|Z_5 < 1\} \ll Z_5^{-1}. \tag{4.59}$$

Proof. As in Lemma 25, we have $t_1^3 \leq 2$. The condition $|t_4|t_5^2|t_4t_5+t_1^2| \leq 1$ shows that t_4 runs over a set whose measure is $\ll |t_5|^{-3/2}$ and integrating this over $t_1 \ll 1$ yields (4.57). Concerning (4.58), we split the proof into two cases depending on whether $|t_4t_5+t_1^2|$ is greater or less than $|t_4|^{1/3}$. If $|t_4t_5+t_1^2| \geq |t_4|^{1/3}$, the condition $|t_4|t_5^2|t_4t_5+t_1^2| \leq 1$ gives $|t_4|^{4/3}t_5^2 \leq 1$ and integrating over t_5 using this inequality and over $t_1 \ll 1$ gives the result. In the other case, t_5 runs over a set whose measure is $\ll |t_4|^{-2/3}$ and integrating this over $t_1 \ll 1$ also gives the result. Finally, (4.59) is an immediate consequence of $|t_4| \leq 1$ and $t_1 \ll 1$. ∎

As in Section 3.5, the bound (4.57) shows that if we do not have $Y_5 \geq \log(B)^A$, the contribution of the main term of $\mathbf{N}(\boldsymbol{\xi}_b, B)$ is $\ll B\log(B)^4$. Thus we can assume from now on that

$$Y_5 \geq \log(B)^A, \tag{4.60}$$

and since also $Y_4 \geq \log(B)^A$, the two bounds (4.58) and (4.59) prove that removing the conditions $|t_4|Y_4, |t_5|Y_5 \geq \log(B)^A$ from the integral defining g_3^b in the main term of $\mathbf{N}(\boldsymbol{\xi}_b, B)$ in (4.56) produces an error term whose overall contribution is $\ll B\log(B)^4$. We have thus proved that for any fixed $A \geq 9$,

$$\mathbf{N}(\boldsymbol{\xi}_b, B) = \mathcal{P}g_4^b(\kappa_b)\frac{B}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}}\Theta_b(\boldsymbol{\xi}_b) + R_2(\boldsymbol{\xi}_b, B), \tag{4.61}$$

where $\sum_{\boldsymbol{\xi}_b} R_2(\boldsymbol{\xi}_b, B) \ll B\log(B)^4$.

LEMMA 32. *For* $t > 0$,

$$\operatorname{meas}\{(t_4, t_5, t_1) \in D_{h^b} : |t_4 t_5 + t_1^2| > t\} \ll t^{-3/2}, \tag{4.62}$$

$$\operatorname{meas}\{(t_4, t_5, t_1) \in D_{h^b} : |t_4 t_5 + t_1^2| \le t\} \ll t^{1/2}. \tag{4.63}$$

Proof. For (4.62), the conditions $t_1 |t_4 t_5 + t_1^2| \le 1$, $|t_4| t_5^2 |t_4 t_5 + t_1^2| \le 1$ and $|t_4 t_5 + t_1^2| > t$ give $t_1 t \le 1$ and $|t_4| t_5^2 t \le 1$. Integrating over t_5 using this inequality, then over $|t_4| \le 1$ and over t_1 using $t_1 t \le 1$ yields (4.62). For (4.63), the conditions $t_1^2 |t_5|\, |t_4 t_5 + t_1^2| \le 1$ and $|t_4 t_5 + t_1^2| \le t$ show that t_5 runs over a set whose measure is $\ll \min(t_1^{-1} |t_4|^{-1/2}, t|t_4|^{-1}) \le t^{1/2} t_1^{-1/2} |t_4|^{-3/4}$. This concludes the proof since $t_1, |t_4| \ll 1$. ∎

The bound (4.62) shows that if $\kappa_b > 1$, the contribution of the main term of $\mathbf{N}(\boldsymbol{\xi}_b, B)$ is $\ll B \log(B)^4$, thus we assume from now on that $\kappa_b \le 1$, that is,

$$Y_4 Y_5 \ge \xi_8^2. \tag{4.64}$$

Replacing $g_4^b(\kappa_b)$ by $g_4^b(0)$ in the main term of $\mathbf{N}(\boldsymbol{\xi}_b, B)$ in (4.61) therefore creates an error term whose overall contribution is $\ll B \log(B)^4$. Since $g_4^b(0) = \tau_\infty/3$ by (4.10), we have obtained the following result.

LEMMA 33. *For any fixed* $A \ge 9$,

$$\mathbf{N}(\boldsymbol{\xi}_b, B) = \mathcal{P} \frac{\tau_\infty}{3} \frac{B}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}} \Theta_b(\boldsymbol{\xi}_b) + R_3(\boldsymbol{\xi}_b, B),$$

where $\sum_{\boldsymbol{\xi}_b} R_3(\boldsymbol{\xi}_b, B) \ll B \log(B)^4$.

Recall the definition (4.55) of $\mathcal{V}_b$. It remains to sum the main term of $\mathbf{N}(\boldsymbol{\xi}_b, B)$ over the $\boldsymbol{\xi}_b \in \mathcal{V}_b$ satisfying (4.60) and (4.64) and the coprimality conditions (4.52)–(4.54). It is easy to see that replacing $\{\boldsymbol{\xi}_b \in \mathcal{V}_b : (4.60), (4.64)\}$ by the region

$$\mathcal{V}_b' = \{\boldsymbol{\xi}_b \in \mathbb{Z}_{>0}^5 : Y_4 \ge 1,\ Y_5 \ge 1,\ Y_1 \ge 1,\ Y_4 Y_5 \ge \xi_8^2\}$$

produces an error term whose overall contribution is $\ll B \log(B)^4 \log(\log(B))$. Let us redefine Θ_b as being equal to zero if the remaining coprimality conditions (4.52)–(4.54) are not satisfied. Lemma 33 proves that for any fixed $A \ge 9$,

$$N_b(A, B) = \mathcal{P} \frac{\tau_\infty}{3} B \sum_{\boldsymbol{\xi}_b \in \mathcal{V}_b'} \frac{\Theta_b(\boldsymbol{\xi}_b)}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}} + O(B \log(B)^4 \log(\log(B))).$$

As in Section 3.6, Θ_b satisfies the assumption (2.40) of Lemma 8 and thus

$$N_b(A, B) = \mathcal{P} \frac{\tau_\infty}{3} \alpha_b \biggl(\sum_{\boldsymbol{\xi}_b \in \mathbb{Z}_{>0}^5} \frac{(\Theta_b * \boldsymbol{\mu})(\boldsymbol{\xi}_b)}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}} \biggr) B \log(B)^5 + O(B \log(B)^4 \log(\log(B))),$$

where α_b is the volume of the polytope defined in $\mathbb{R}^5$ by $t_2, t_3, t_6, t_7, t_8 \geq 0$ and

$$\begin{aligned} 2t_3 + t_6 + t_8 &\leq 1, \\ -2t_2 + 4t_3 + 2t_6 - t_7 + 3t_8 &\geq 1, \\ t_2 + t_3 + 2t_6 + 2t_7 &\leq 1, \\ 2t_2 + 2t_3 + t_6 + t_7 + 6t_8 &\leq 2. \end{aligned}$$

A computation using [Fra09] gives

$$\alpha_b = \frac{929}{2016000}, \tag{4.65}$$

and exactly as in the case of $N_a(A, B)$, a calculation provides

$$\sum_{\boldsymbol{\xi}_b \in \mathbb{Z}_{>0}^5} \frac{(\Theta_b * \boldsymbol{\mu})(\boldsymbol{\xi}_b)}{\boldsymbol{\xi}_b^{(1,1,1,1,1)}} = \mathcal{P}^{-1} \prod_p \left(1 - \frac{1}{p}\right)^6 \tau_p.$$

We have proved the following lemma.

Lemma 34. *For any fixed $A \geq 9$,*

$$N_b(A, B) = \frac{1}{3} \alpha_b \omega_H(\widetilde{V_2}) B \log(B)^5 + O(B \log(B)^4 \log(\log(B))).$$

We now fix $A = 9$ for example. The equalities (4.46) and (4.65) yield

$$\alpha_a + \alpha_b = 3\alpha(\widetilde{V_2}),$$

and we immediately complete the proof putting together Lemmas 22, 28 and 34.

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References

[BT98] V. V. Batyrev and Y. Tschinkel, *Manin's conjecture for toric varieties*, J. Algebraic Geom. 7 (1998), 15–53.

[BB07] R. de la Bretèche and T. D. Browning, *On Manin's conjecture for singular del Pezzo surfaces of degree 4. I*, Michigan Math. J. 55 (2007), 51–80.

[Bro07] T. D. Browning, *An overview of Manin's conjecture for del Pezzo surfaces*, in: Analytic Number Theory, Clay Math. Proc. 7, Amer. Math. Soc., Providence, RI, 2007, 39–55.

[BD09] T. D. Browning and U. Derenthal, *Manin's conjecture for a quartic del Pezzo surface with A_4 singularity*, Ann. Inst. Fourier (Grenoble) 59 (2009), 1231–1265.

[CLT02] A. Chambert-Loir and Y. Tschinkel, *On the distribution of points of bounded height on equivariant compactifications of vector groups*, Invent. Math. 148 (2002), 421–452.

[CT88] D. F. Coray and M. A. Tsfasman, *Arithmetic on singular Del Pezzo surfaces*, Proc. London Math. Soc. (3) 57 (1988), 25–87.

[Der06] U. Derenthal, *Singular del Pezzo surfaces whose universal torsors are hypersurfaces*, arXiv:math/0604194v1, 2006.

[Der07] —, *On a constant arising in Manin's conjecture for del Pezzo surfaces*, Math. Res. Lett. 14 (2007), 481–489.

[Der09] —, *Counting integral points on universal torsors*, Int. Math. Res. Notices 2009, no. 14, 2648–2699.

[DL10] U. Derenthal and D. Loughran, *Singular del Pezzo surfaces that are equivariant compactifications*, Zap. Nauchn. Sem. POMI 377 (2010), 26–43.

[DT07] U. Derenthal and Y. Tschinkel, *Universal torsors over del Pezzo surfaces and rational points*, in: Equidistribution in Number Theory, an Introduction, NATO Sci. Ser. II Math. Phys. Chem. 237, Springer, Dordrecht, 2007, 169–196.

[Est61] T. Estermann, *On Kloosterman's sum*, Mathematika 8 (1961), 83–86.

[FMT89] J. Franke, Y. I. Manin, and Y. Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. 95 (1989), 421–435.

[Fra09] M. Franz, *Convex—a Maple package for convex geometry, version 1.1*, http://www.math.uwo.ca/~mfranz/convex/, 2009.

[HB79] D. R. Heath-Brown, *The fourth power moment of the Riemann zeta function*, Proc. London Math. Soc. (3) 38 (1979), 385–422.

[HB86] —, *The divisor function $d_3(n)$ in arithmetic progressions*, Acta Arith. 47 (1986), 29–56.

[HB03] —, *The density of rational points on Cayley's cubic surface*, in: Proc. Session in Analytic Number Theory and Diophantine Equations, Bonner Math. Schriften 360, Univ. Bonn, Bonn, 2003, 33.

[Lou10] D. Loughran, *Manin's conjecture for a singular sextic del Pezzo surface*, J. Théor. Nombres Bordeaux 22 (2010), 675–701.

[Pey95] E. Peyre, *Hauteurs et mesures de Tamagawa sur les variétés de Fano*, Duke Math. J. 79 (1995), 101–218.

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