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An arithmetic group associated with a Pisot unit, and its symbolic-dynamical representation

by

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1. The definition of the Pisot group and its basic properties. Let $\beta > 1$ be a *Pisot number*, i.e. an algebraic integer whose conjugates have the moduli strictly less than 1. Let the characteristic polynomial of β be $g(x) = x^m - k_1 x^{m-1} - k_2 x^{m-2} - \ldots - k_m$. We assume β to be a *unit*, i.e. $k_m = \pm 1$.

We recall that since β is a Pisot number, $\|\beta^n\| \to 0$ as $n \to +\infty$, where $\|y\| = \min\{|y-k| : k \in \mathbb{Z}\}$. Let

$$\mathcal{P}_\beta = \{\xi : \|\xi\beta^n\| \to 0,\ n \to +\infty\}.$$

Let us first establish some auxiliary facts. It is well known that $\mathcal{P}_\beta \subset \mathbb{Q}(\beta)$ (see, e.g., [Cas]). Let $\mathrm{Tr}(\xi)$ denote the trace of ξ, i.e. the sum of ξ and all its conjugates.

LEMMA 1.1. *The set $\mathcal{P}_\beta$ is a commutative group under addition containing $\mathbb{Z}[\beta]$. It can be characterized as follows*:

$$\mathcal{P}_\beta = \{\xi \in \mathbb{Q}(\beta) : \mathrm{Tr}(a\xi) \in \mathbb{Z} \textit{ for any } a \in \mathbb{Z}[\beta]\}. \tag{1.1}$$

Proof. The first claim is a consequence of the inequalities $\|\xi_1 \pm \xi_2\| \le \|\xi_1\| + \|\xi_2\|$ and of the fact that $\|\beta^n\| \to 0$. To prove (1.1), we observe that the classical theorem due to Pisot and Vijayaraghavan says that for any Pisot β, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{P}_\beta = \{\xi \in \mathbb{Q}(\beta) : \mathrm{Tr}(\beta^k\xi) \in \mathbb{Z},\ k \ge k_0\}$ (see, e.g., [Cas]). Hence (1.1) follows, because β is a unit, whence β^{-1} is an algebraic integer. ■

Thus, if we regard $\mathbb{Z}[\beta]$ as a lattice over $\mathbb{Z}$, then by (1.1) and the definition, $\mathcal{P}_\beta$ is the dual lattice for $\mathbb{Z}[\beta]$ (notation: $\mathcal{P}_\beta = (\mathbb{Z}[\beta])^*$). The following claim follows from some general statement about lattices [FrolTa, Chapter III, (2.20)].

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Proposition 1.2. *There exists* $\xi_0 \in \mathbb{Q}(\beta) \setminus \mathbb{Z}[\beta]$ *such that*

$$\mathcal{P}_\beta = \xi_0 \cdot \mathbb{Z}[\beta].$$

It can be given by the formula $\xi_0 = (g'(\beta))^{-1}$.

Definition. Let the *Pisot group* $\mathcal{A}_\beta$ be defined as the quotient group $\mathcal{P}_\beta/\mathbb{Z}[\beta]$.

Let $\beta_1 = \beta$, and $\beta_2, \beta_3, \ldots, \beta_m$ denote its conjugates. Finally, let $D = D(\beta) = \prod_{i \neq j}(\beta_i - \beta_j)^2$ be the discriminant of β.

Theorem 1.3. *The order of the Pisot group is* $|D(\beta)|$.

Proof. Again, we will use some facts from classical number theory. Let

$$M_\beta = \begin{pmatrix} \mathrm{Tr}(1) & \mathrm{Tr}(\beta) & \ldots & \mathrm{Tr}(\beta^{m-1}) \\ \mathrm{Tr}(\beta) & \mathrm{Tr}(\beta^2) & \ldots & \mathrm{Tr}(\beta^m) \\ \ldots & \ldots & \ldots & \ldots \\ \mathrm{Tr}(\beta^{m-1}) & \mathrm{Tr}(\beta^m) & \ldots & \mathrm{Tr}(\beta^{2m-1}) \end{pmatrix}.$$

For any lattice M over $\mathbb{Z}$,

$$M^* : M \cong \mathbb{Z}^m / M_\beta \mathbb{Z}^m, \tag{1.2}$$

whence the order of $M^* : M$ is $|D|$, where $D = \det M_\beta$ (see [FrolTa, Chapter III]).

It suffices to show that $\det M_\beta = D(\beta)$. Sometimes, this relation is taken as the definition of the discriminant. Observe that $M_\beta = V_\beta \cdot V_\beta^T$, where

$$V_\beta = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \beta_1 & \beta_2 & \ldots & \beta_m \\ \beta_1^2 & \beta_2^2 & \ldots & \beta_m^2 \\ \ldots & \ldots & \ldots & \ldots \\ \beta_1^{m-1} & \beta_2^{m-1} & \ldots & \beta_m^{m-1} \end{pmatrix},$$

whence $\det M_\beta = (\det V_\beta)^2 = \prod_{i \neq j}(\beta_i - \beta_j)^2 = D$. ∎

The following simple fact answers the question about the finer structure of the Pisot group. Let $d = \min\{l \geq 1 : lM_\beta^{-1} \in \mathbb{Z}^{m \times m}\}$.

Lemma 1.4. *The Pisot group* $\mathcal{A}_\beta$ *contains* $\mathbb{Z}/d\mathbb{Z}$ *as a subgroup. In particular, if* $d = |D|$, *then* $\mathcal{A}_\beta$ *is cyclic, and if* $|D|/d$ *is prime, then* $\mathcal{A}_\beta$ *is isomorphic to* $(\mathbb{Z}/d\mathbb{Z}) \times (\mathbb{Z}/(|D|/d)\mathbb{Z})$.

Proof. This is a direct consequence of (1.2). ∎

Corollary 1.5. *Assume the entries of* DM_β^{-1} *to be coprime. Then* $\mathcal{A}_\beta$ *is cyclic.*

Remark. All the above results are valid for *any* algebraic unit if one takes (1.1) for the definition of $\mathcal{P}_\beta$.

EXAMPLES. 1. *The quadratic units.* Let $\beta^2 = k\beta \pm 1$ with $k \geq 1$ for $+1$ and $k \geq 3$ for -1. Then by direct inspection,

$$M_\beta = \begin{pmatrix} 2 & k \\ k & k^2 \pm 2 \end{pmatrix},$$

and $\xi_0 = 1/\sqrt{D}$. Then by Lemma 1.4, $\mathcal{A}_\beta$ is isomorphic to $\mathbb{Z}/D\mathbb{Z}$ if k is odd, and to $(\mathbb{Z}/(D/2)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$ otherwise. The Pisot group for this case was considered in [SV2].

2. Let β be the "tribonacci number", i.e. the positive root of $x^3 = x^2 + x + 1$. Here $D = -44$,

$$M_\beta = \begin{pmatrix} 3 & 1 & 3 \\ 1 & 3 & 7 \\ 3 & 7 & 11 \end{pmatrix}, \qquad M_\beta^{-1} = \frac{1}{22}\begin{pmatrix} 8 & -5 & 1 \\ -5 & -12 & 9 \\ 1 & 9 & -4 \end{pmatrix},$$

$\xi_0^{-1} = g'(\beta) = -1 - 2\beta + 3\beta^2$, and $\xi_0 = \frac{1}{22}(1 + 9\beta - 4\beta^2)$. By Lemma 1.4, $\mathcal{A}_\beta \cong (\mathbb{Z}/22\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$.

3. Finally, let β be the smallest Pisot number, i.e. the root of $\beta^3 = \beta + 1$ (see, e.g., [DuPi]). Here $D = -23$,

$$M_\beta = \begin{pmatrix} 3 & 0 & 2 \\ 0 & 2 & 3 \\ 2 & 3 & 2 \end{pmatrix}, \qquad M_\beta^{-1} = \frac{1}{23}\begin{pmatrix} 5 & -6 & 4 \\ -6 & -2 & 9 \\ 4 & 9 & -6 \end{pmatrix},$$

and since 23 is a prime, $\mathcal{A}_\beta \cong \mathbb{Z}/23\mathbb{Z}$.

To end this section, we establish a link between the groups $\mathcal{P}_\beta$ and $\mathcal{A}_\beta$ and certain groups of recurrent sequences, which will be used in the next section. Let

$$R_\beta = \Big\{ \{T_n\}_{n=1}^\infty \in \mathbb{Z}^\infty \,\Big|\, \exists j : T_{n+m} = \sum_{i=1}^m k_i T_{n+m-i},\ n \geq j \Big\}.$$

Obviously, R_β is a group under addition. Define $h : R_\beta \to \mathcal{P}_\beta$ by

$$h(\{T_n\}) := \lim_{n\to+\infty} \beta^{-n} T_n.$$

The image does belong to $\mathcal{P}_\beta$, as $\|\xi\beta^n\| = |\xi\beta^n - T_n| \to 0$ (in view of β being a Pisot number, whence $|\xi - \beta^{-n}T_n| = o(\beta^{-n})$).

LEMMA 1.6. *The mapping h is an epimorphism of groups.*

Proof. It is obvious from the definition that h is a homomorphism. Let $\xi \in \mathcal{P}_\beta$. We define T_n as the integer closest to $\xi\beta^n$. Since β is a Pisot number, $\{T_n\}$ will eventually satisfy the corresponding recurrence relation, whence $h^{-1}(\xi) \neq \emptyset$. ∎

Let now the equivalence relation $\sim$ on R_β be defined as follows: $\{T_n\} \sim \{T_n'\}$ iff $\lim_n \beta^{-n}(T_n - T_n') \in \mathbb{Z}[\beta]$. Then obviously, the quotient map h' is an epimorphism as well.

2. Symbolic realization of the Pisot group

Definition ([Pa]). A representation of an $x \in [0,1)$ of the form

$$x = \psi_\beta(\varepsilon_1, \varepsilon_2, \ldots) := \sum_{k=1}^{\infty} \varepsilon_k \beta^{-k} \tag{2.1}$$

is called the *β-expansion* of x if the "digits" $\{\varepsilon_k\}_{k=1}^{\infty}$ are obtained by means of the greedy algorithm (similarly to the decimal expansion), i.e., $\varepsilon_1 = \varepsilon_1(x) = [\beta x]$, $\varepsilon_2 = \varepsilon_2(x) = [\beta\{\beta x\}]$, etc. The set of all possible sequences $\{\{\varepsilon_k(x)\}_{k=1}^{\infty} : x \in [0,1)\}$ is called the (one-sided) *β-compactum* and will be denoted by X_β^+.

The β-compactum can be described more explicitly. Let $1 = \sum_{k=1}^{\infty} d_k \beta^{-k}$ be the expansion of 1 defined by the greedy algorithm, i.e., $d_1' = [\beta]$, $d_2' = [\beta\{\beta\}]$, etc. If the sequence $\{d_n'\}$ is not finite, we put $d_n \equiv d_n'$. Otherwise let $k = \max\{j : d_j' > 0\}$, and $(d_1, d_2, \ldots) := (\overline{d_1', \ldots, d_{k-1}', d_k' - 1})$, where the bar denotes a period.

We will write $\{x_n\}_{n=1}^{\infty} \prec \{y_n\}_{n=1}^{\infty}$ if $x_n < y_n$ for the smallest $n \geq 1$ such that $x_n \neq y_n$. Then

$$X_\beta^+ = \{\{\varepsilon_n\}_{n=1}^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{N}\}$$

(see [Pa]). Similarly, we define the *two-sided β-compactum* as

$$X_\beta = \{\{\varepsilon_n\}_{n=-\infty}^{\infty} : (\varepsilon_n, \varepsilon_{n+1}, \ldots) \prec (d_1, d_2, \ldots) \text{ for all } n \in \mathbb{Z}\}.$$

It is shown in [Pa] that $\psi_\beta : X_\beta^+ \to [0,1)$ is one-to-one everywhere with the exception of some countable set of points.

Both compacta are naturally endowed with the weak topology, i.e. with the topology of coordinatewise convergence. For β Pisot the properties of the β-compactum are well studied. Its main property is that it is *sofic*, i.e., the shift on X_β is a factor of a subshift of finite type (see, e.g., the review [Bl]). Let $\mathrm{Fin}(\beta)$ denote the set of x's whose β-expansions are *finite*, i.e. have the tail 0^∞. Obviously, $\mathrm{Fin}(\beta) \subset \mathbb{Z}[\beta] \cap [0,1)$, but the inverse inclusion does not hold for an arbitrary Pisot unit.

Definition. A Pisot unit is called *finitary* if

$$\mathrm{Fin}(\beta) = \mathbb{Z}[\beta] \cap [0,1).$$

Examples. 1. The Pisot number which is the dominant root of the equation $x^m = k_1 x^{m-1} + \ldots + k_m$, $k_1 \geq k_2 \geq \ldots \geq k_m \geq 1$, is known to be finitary [FrSo].

2. A quadratic Pisot unit is finitary if and only if its norm is $+1$. Indeed, if $\beta^2 = k\beta + 1$, $k \geq 1$, then the claim follows from the previous one. If $\beta^2 = k\beta - 1$, $k \geq 3$, then in view of $(\beta-1)^2 = \beta^2 - 2\beta + 1 = (k-2)\beta$,

we have the β-expansion $\beta - 1 = (k-2)\beta^{-1} + (k-2)\beta^{-2} + \ldots$, whence $\mathbb{Z}[\beta] \cap [0,1) \not\subset \mathrm{Fin}(\beta)$.

3. For the cubic Pisot units there also exists a full criterion due to S. Akiyama [Ak1]. Namely, the norm of a finitary cubic β must be -1, i.e. $\beta^3 = k_1\beta^2 + k_2\beta + 1$. Furthermore, such a β is finitary if and only if $k_1 \ge 0$ and $-1 \le k_2 \le k_1 + 1$. Thus, the tribonacci number and the smallest Pisot number are both finitary, whereas, for example, the principal root of $x^3 = 3x^2 - 2x + 1$ is not.

4. Finally, the second Pisot number, i.e. the positive root of $x^4 = x^3 + 1$, is nonfinitary, as $\beta^{-2} + \beta^{-3} = \beta^{-1} + \beta^{-6} + \beta^{-11} + \ldots$

From now on we will assume β to be finitary.

Remark. The β-expansions can be easily extended from $[0,1)$ to $\mathbb{R}_+$ in the way analogous to the decimal expansions. This yields the possibility of adding in X_β two sequences finite to the left. Namely, if $\overline{\varepsilon}$ and $\overline{\varepsilon}'$ are both finite to the left, put $x := \sum_{k=-\infty}^{\infty} (\varepsilon_k + \varepsilon'_k)\beta^{-k}$; then by definition, $\overline{\varepsilon} + \overline{\varepsilon}'$ is the β-expansion of x. The same is true for subtraction, though $\overline{\varepsilon} - \overline{\varepsilon}'$ is well defined only for those sequences for which $\sum_k (\varepsilon_k - \varepsilon'_k)\beta^{-k} \ge 0$.

Theorem (A. Bertrand, K. Schmidt). *For a Pisot β, any $x \in \mathbb{Q}(\beta) \cap \mathbb{R}_+$ has an ultimately periodic β-expansion.*

Thus, all the elements of $\mathcal{P}_\beta \cap \mathbb{R}_+$ have ultimately periodic β-expansions.

Lemma 2.1. *For the nonnegative elements of $\mathcal{P}_\beta$ the set of their periods is finite.*

Proof. Let $q = |D|$; then all the denominators of the elements in question in the standard basis of $\mathbb{Q}(\beta)$ are bounded by q (see Proposition 1.2). Therefore, it suffices to show that the intersection of the set of numbers in $(0,1)$ having purely periodic β-expansions and $\frac{1}{q}\mathbb{Z}[\beta]$ is finite.

The proof is basically the same as that of Lemma 2 of [Ak2]. Namely, it was shown there that for each purely periodic element in $(0,1)$ the values of its conjugates are uniformly bounded. Therefore, after the natural embedding of $\mathbb{Q}(\beta)$ into $\mathbb{R}^m$ we will have a lattice lying in a ball, and it will hence be finite. ∎

Our goal is to find a realization of the Pisot group $\mathcal{A}_\beta$ in X_β. We will perform the following operation: to a nonnegative $\xi \in \mathcal{P}_\beta$ the mapping $\widetilde{g}$ assigns the purely periodic two-sided sequence in X_β which is the extension to the left of the periodic tail of the β-expansion of ξ. Now let g be the corresponding quotient map from $\mathcal{A}_\beta$. To any equivalence class $[\xi] \in \mathcal{A}_\beta$ it assigns the class of all purely periodic sequences in X_β whose tails can be obtained from the β-expansions of the elements of $[\xi]$.

By Lemma 2.1, $\#g([\xi]) < \infty$ for any $\xi \in \mathcal{P}_\beta \cap \mathbb{R}_+$. Let $\mathfrak{A}_\beta := g(\mathcal{A}_\beta)$.

LEMMA 2.2. *The mapping* $g : \mathcal{A}_\beta \to \mathfrak{A}_\beta$ *is one-to-one.*

Proof. We need to prove the injectivity of g only. Let $g([\xi]) = g([\xi'])$. This means that there exist $\xi_1 \in [\xi]$, $\xi_2 \in [\xi']$ whose tails of the β-expansions coincide. But this implies $\xi_1 - \xi_2 \in \mathbb{Z}[\beta]$, whence $[\xi] = [\xi']$. ∎

Thus, we have defined a certain finite arithmetic group of classes of sequences that may be regarded as a realization of the Pisot group in X_β. The arithmetic operations on $\mathfrak{A}_\beta$ inherited from $\mathcal{A}_\beta$ will be denoted by $\oplus$ and $\ominus$ respectively. That is,

$$[\overline{\varepsilon}] \oplus [\overline{\varepsilon}'] := g(g^{-1}([\overline{\varepsilon}]) + g^{-1}([\overline{\varepsilon}'])),$$

and similarly for "circular" subtraction. Note also that for $[\overline{\varepsilon}] \in \mathfrak{A}_\beta$,

$$g^{-1}([\overline{\varepsilon}]) = \Big[\sum_{j=-n}^{\infty} \varepsilon_j \beta^{-j}\Big] \quad \text{for any } n \in \mathbb{Z} \tag{2.2}$$

(the square brackets on the right-hand side denote the equivalence class in $\mathcal{A}_\beta$).

DEFINITION. We define the *natural arithmetic* on $\mathfrak{A}_\beta$ as follows: $\overline{\varepsilon} + \overline{\varepsilon}'$ is by definition, the class of all sequences $\overline{\varepsilon}'' \in X_\beta$ which can be obtained as weak limits of the sequence $\{\varepsilon^{(N)} + \varepsilon'^{(N)}\}_{N\ge 1}$, where $\varepsilon^{(N)} = (\ldots, 0, 0, \ldots, 0, \varepsilon_{-N}, \varepsilon_{-N+1}, \ldots)$. The difference $\overline{\varepsilon} - \overline{\varepsilon}'$ is defined in a similar way, namely, if $\sum_{j=-N}^{\infty} (\varepsilon_j - \varepsilon'_j)\beta^{-j} < 0$, then we take $(\ldots, 0, 0, \ldots, 0, 1, \varepsilon_{-N}, \varepsilon_{-N+1}, \ldots)$ instead of $\varepsilon^{(N)}$.

PROPOSITION 2.3. *The operations* $+$ *and* $\oplus$ (*respectively* $-$ *and* $\ominus$) *on* $\mathfrak{A}_\beta$ *coincide.*

Proof. Let $[\overline{\varepsilon}], [\overline{\varepsilon}'] \in \mathfrak{A}_\beta$ and $S_1 := [\overline{\varepsilon}] + [\overline{\varepsilon}']$, $S_2 := [\overline{\varepsilon}] \oplus [\overline{\varepsilon}']$. We first show that $S_1 \subset S_2$. Let $\overline{\varepsilon}'' \in [\overline{\varepsilon}] + [\overline{\varepsilon}']$. This means by definition that there exists a subsequence $\{N_k\}_{k=1}^{\infty}$ such that $\overline{\varepsilon}'' = \lim_k (\varepsilon^{(N_k)} + \varepsilon'^{(N_k)})$. We need to show that $g^{-1}([\overline{\varepsilon}'']) = g^{-1}([\overline{\varepsilon}]) + g^{-1}([\overline{\varepsilon}'])$, which follows from (2.2) with $n = N_k$ with k large enough to ensure that $\varepsilon^{(N_k)} + \varepsilon'^{(N_k)}$ will coincide with $\overline{\varepsilon}''$ at, say, all positive indices.

To prove the reverse inclusion, let $\overline{\varepsilon}'' \in S_2$. To show that $\overline{\varepsilon}'' \in S_1$, we put $N_k = kr$, where r is the length of a common period of $\overline{\varepsilon}, \overline{\varepsilon}'$ and $\overline{\varepsilon}''$. By definition,

$$\eta := \Big|\sum_{j=0}^{\infty} \varepsilon''_j \beta^{-j} - \sum_{j=0}^{\infty} (\varepsilon_j + \varepsilon'_j)\beta^{-j}\Big| \in \mathbb{Z}[\beta],$$

whence

$$\Big|\sum_{j=-N_k}^{\infty} \varepsilon''_j \beta^{-j} - \sum_{j=-N_k}^{\infty} (\varepsilon_j + \varepsilon'_j)\beta^{-j}\Big| = \beta^{kr}\eta, \tag{2.3}$$

and by our assumption that β is finitary, η has a finite β-expansion, whence by (2.3) and the definition of convergence in the weak topology,

$$\overline{\varepsilon}'' = \lim_k (\varepsilon^{(N_k)} + \varepsilon'^{(N_k)}).$$

The proof for subtraction is the same. ■

Thus, $\mathfrak{A}_\beta$ is an additive group of classes of sequences in X_β in the sense of its natural arithmetic. Its obvious property is that it is shift-invariant, i.e. contains any sequence together with all its shifts, because addition commutes with the shift—recall that the shift of a sequence finite to the left is just the multiplication of its value by β. Pursuing the idea from Section 1 (see Lemma 1.6), we will give another way of obtaining the sequences from $\mathfrak{A}_\beta$, which in a sense looks more traditional (see [FrSa1] for the case of the golden mean).

LEMMA 2.4. *For any element* $[\overline{\varepsilon}] \in \mathfrak{A}_\beta$ *there exists an eventually nonnegative sequence* $\{T_n\}_{n=1}^\infty \in R_\beta$ *such that the set of partial limits for the sequence of its* β*-expansions* $\{\overline{\varepsilon}_n\}_{n=1}^\infty$ *is exactly the class* $[\overline{\varepsilon}]$.

Proof. Take $[\xi] = g^{-1}([\overline{\varepsilon}])$, and, as above, define T_n as the integer closest to $\xi'\beta^n$ for some nonnegative $\xi' \sim \xi$ (for example, take ξ' equal to the fractional part of ξ). Then $T_n = \xi'\beta^n + o(1)$, and the T_n are eventually nonnegative, as $\xi' > 0$. Hence taking the set of partial limits of its β-expansions is the same as "pulling out" the periodic tail for a $\xi \in \mathcal{P}_\beta \cap \mathbb{R}_+$, because $T_n - \xi'\beta^n \to 0$, $n \to +\infty$. ■

EXAMPLES. 1. For the quadratic Pisot units $\beta^2 = k\beta \pm 1$ we proved in Section 1 that $\mathcal{P}_\beta = (1/\sqrt{D}) \cdot \mathbb{Z}[\beta]$. Since $1/\sqrt{D} = k\beta^2/(\beta^4 - 1)$, all sequences in $\mathfrak{A}_\beta$ are of period 4, 2 or 1. Let us consider some subcases. We will use the convention to write just the period instead of the whole periodic sequence.

1.1. $\beta^2 = \beta + 1$. Here $\mathfrak{A}_\beta = \{0000, 1000, 0100, 0010, 0001\}$. Recall that by definition, X_β does not contain sequences ending with $101010\ldots$; that is why there is no need to identify 0000 with 1010 and 0101. Thus, every class in $\mathfrak{A}_\beta$ consists of just one element. If $F_1 = 1, F_2 = 2, \ldots$ is the Fibonacci sequence, then by Lemma 2.4, the set of partial limits for the β-expansions of $(F_n)_{n=1}^\infty$ in X_β will give us just the four nonzero sequences in $\mathfrak{A}_\beta$ (1). However, this may be checked directly as well:

$$F_k = \begin{cases} \beta^{k-1} + \beta^{k-5} + \beta^{k-9} + \ldots + \beta^{-k+3} + \beta^{-k}, & k \text{ even}, \\ \beta^{k-1} + \beta^{k-5} + \beta^{k-9} + \ldots + \beta^{-k+5} + \beta^{-k+1}, & k \text{ odd}, \end{cases}$$

(1) It is easy to see that the Fibonacci sequence is the basis of the module R_β, i.e. for any sequence $\{T_n\} \in R_\beta$, $T_n = \sum_{k=0}^s \varepsilon_k F_{n+k}$, $\varepsilon_k \in \mathbb{Z}$, $n \geq n_0$.

whence the limits of $(F_{4k+j})_{k=1}^{\infty}$ in X_β do exist for $j = 0, 1, 2, 3$ and yield the four nonzero sequences in $\mathfrak{A}_\beta$. For more details see [FrSa1], [SV1].

1.2. $\beta^2 = 2\beta + 1$. Here $\#\mathfrak{A}_\beta = 8$, and $\mathfrak{A}_\beta = \{0000, 1010, 0101, 2000, 0200, 0020, 0002, 1111\}$. The elements 1010, 0101, 1111 are of order 2, whereas all other nonzero elements are of order 4. A similar pattern holds for $\beta^2 = k\beta + 1$, k even.

Remark. In [FrSa2] the authors study two similar groups for arbitrary quadratic Pisot units. One of them, H_β, is the group of *all* sequences of period 4; its order is $k^2 D$, and it is in fact isomorphic to $\frac{\mathbb{Z}[\beta]}{k\sqrt{D}}/\mathbb{Z}[\beta]$. Another one, G_β, is, on the contrary, smaller than $\mathfrak{A}_\beta$; it is related to a certain finite automaton. For example, in the case $\beta^2 = 2\beta + 1$, $G_\beta = \{0000, 0101, 1010, 1111\} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. In the general quadratic case one can show that $G_\beta \subset \mathfrak{A}_\beta \subset H_\beta$, both inclusions being, generally speaking, proper. However, in the case of the golden mean (and only in this case), $G_\beta = \mathfrak{A}_\beta = H_\beta \cong \mathbb{Z}/5\mathbb{Z}$.

2. For the tribonacci number the situation with the periods of the sequences in $\mathfrak{A}_\beta$ appears to be more complicated. Expanding the elements of $\mathcal{P}_\beta$, we see that these periods are 1, 2, 3 and 10. More precisely, besides 0, $\mathfrak{A}_\beta$ consists of exactly 40 sequences of period 10, namely: $\{1000110000, 1010000110, 1001011000, 1001101100\}$ together with all their shifts, two sequences of period two: $\{01, 10\}$ and *one* sequence of period 3: 100. One may ask: where are all its shifts, shouldn't they belong to $\mathfrak{A}_\beta$ as well? The answer is simple: *arithmetically* they are all equivalent, because

$$(\beta + \beta^{-2} + \beta^{-5} + \ldots) - (1 + \beta^{-3} + \beta^{-6} + \ldots) = \frac{\beta - 1}{1 - \beta^{-3}} = 1 \in \mathbb{Z}[\beta].$$

This example shows that even for a finitary β not necessarily any $\xi \in \mathcal{P}_\beta$ and $\xi + l$, $l \in \mathbb{Z}[\beta]$ will have one and the same tail. Thus, the definition of $\mathfrak{A}_\beta$ as a quotient set is essential.

3. $\beta^3 = \beta + 1$. Here $D = -23$. Let us prove a simple claim about the structure of the group $\mathfrak{A}_\beta$ in the case of prime $|D|$.

Lemma 2.5. *If $|D|$ is a prime number, then $|D|$ consists of $|D| - 1$ sequences of period $|D| - 1$ which are shifts of each other, and* 0.

Proof. Since $\mathfrak{A}_\beta$ is shift-invariant and cyclic, any $\overline{\varepsilon} \in \mathfrak{A}_\beta \setminus \{0\}$ belongs to it together with all its shifts, and they are arithmetically nonequivalent. ■

Returning to the example, the period of ξ_0 is 1000010000000100000000, and $\mathfrak{A}_\beta$ consists of this sequence together with its 21 shifts and 0.

3. Symbolic dynamics associated with the Pisot group. In this section we are going to show that a certain natural arithmetic coding of the companion toral automorphism is not one-to-one a.e. and that its “kernel” in fact coincides with the group $\mathfrak{A}_\beta$. We still assume β to be a finitary Pisot unit.

We need to recall some basic notions and facts from hyperbolic dynamics. Let T be a hyperbolic automorphism of the torus $\mathbb{T}^m = \mathbb{R}^m/\mathbb{Z}^m$, L_{s} and L_{u} denote respectively the leaves of the stable and unstable foliations passing through $\mathbf{0}$. Recall that a point homoclinic to $\mathbf{0}$ (or simply a *homoclinic point*) is a point belonging to $L_{\mathrm{s}} \cap L_{\mathrm{u}}$. In other words, $\mathbf{t}$ is homoclinic iff $T^n\mathbf{t} \to \mathbf{0}$ as $n \to \pm\infty$. The homoclinic points are a group under addition, isomorphic to $\mathbb{Z}^m$, and we will denote it by $\mathcal{H}(T)$. Each homoclinic point $\mathbf{t}$ can be obtained as follows: take some $\mathbf{n} \in \mathbb{Z}^m$ and project it onto L_{u} along L_{s} and then onto $\mathbb{T}^m$ by taking the fractional parts of all coordinates of the vector (see [Ver]).

Let $T = T(\beta)$ be the group automorphism of $\mathbb{T}^m$ given by the *companion matrix*, i.e. by

$$M = M(\beta) = \begin{pmatrix} k_1 & k_2 & k_3 & \dots & k_{m-1} & k_m \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}.$$

Then $(1, \beta^{-1}, \dots, \beta^{-m+1})$ is an eigenvector corresponding to the eigenvalue β.

LEMMA 3.1. *There exists a one-to-one correspondence between the homoclinic points and the elements of* $\mathcal{P}_\beta$. *Namely*, $\mathbf{t} \in \mathcal{H}(T)$ *if and only if* $\mathbf{t} = (\xi, \xi\beta^{-1}, \dots, \xi\beta^{-m+1}) \bmod \mathbb{Z}^m$ *for some* $\xi \in \mathcal{P}_\beta$.

Proof. Since β is a Pisot number, L_{u} is one-dimensional. Hence by Vershik's theorem cited above, any homoclinic point for T must be of the form $\mathbf{t} = (\xi, \xi\beta^{-1}, \dots, \xi\beta^{-m+1}) \bmod \mathbb{Z}^m$. To show that ξ belongs to $\mathcal{P}_\beta$, we observe that $T^n\mathbf{t} = (\xi\beta^n, \xi\beta^{n-1}, \dots, \xi\beta^{n-m+1}) \bmod \mathbb{Z}^m$. Since $T^n\mathbf{t}$ must tend to $\mathbf{0}$ as $n \to \pm\infty$, we are done. The converse is obvious. ■

Consider first two special homoclinic points. Let ξ_0 denote the generator of the Pisot group defined in Proposition 1.4, and $\mathbf{t}_0$ the fundamental homoclinic point given by the formula

$$\mathbf{t}_0 = (\xi_0, \xi_0\beta^{-1}, \dots, \xi_0\beta^{-m+1})$$

(we omit the natural projection of $\mathbb{R}^m$ onto the torus, i.e. write in the coordinates of $\mathbb{R}^m$), and

$$\mathbf{t}_1 := (1, \beta^{-1}, \dots, \beta^{-m+1}).$$

Consider the following two mappings from X_β onto $\mathbb{T}^m$:

$$\varphi_0(\varepsilon)=\sum_{k\in\mathbb{Z}}\varepsilon_k T^{-k}\mathbf{t}_0=\lim_{N\to+\infty}\Big(\sum_{k=-N}^{\infty}\varepsilon_k\beta^{-k}\Big)\begin{pmatrix}\xi_0\\ \xi_0\beta^{-1}\\ \vdots\\ \xi_0\beta^{-m+1}\end{pmatrix}\bmod\mathbb{Z}^m,$$

$$\varphi_1(\varepsilon)=\sum_{k\in\mathbb{Z}}\varepsilon_k T^{-k}\mathbf{t}_1=\lim_{N\to+\infty}\Big(\sum_{k=-N}^{\infty}\varepsilon_k\beta^{-k}\Big)\begin{pmatrix}1\\ \beta^{-1}\\ \vdots\\ \beta^{-m+1}\end{pmatrix}\bmod\mathbb{Z}^m.$$

Both mappings are well defined since $\|\beta^N\|\to 0$ and $\|\xi_0\beta^N\|\to 0$ as $N\to+\infty$ at an exponential rate. They are obviously continuous, and their important property is that they are bounded-to-one (see [Sch] for φ_0 and [Sid] for the general case of mappings of the form (4.1)—see below).

Their main value is that they both semiconjugate the shift τ_β on the compactum X_β and the automorphism T, i.e., $\varphi_0\tau_\beta=T\varphi_0$, $\varphi_1\tau_\beta=T\varphi_1$. Thus, both mappings may be regarded as *arithmetic codings* of T in the sense of [SV2], [Sid]. The mapping φ_0 was introduced in [SV2] for $m=2$, and in [Sch] for higher dimensions. At the same time, the mapping φ_1 had been historically the first attempt of arithmetic coding of an automorphism of the torus [Ber].

Theorem (K. Schmidt [Sch]). *The mapping φ_0 is one-to-one on the set of doubly transitive sequences of X_β, i.e. on the sequences $\overline{\varepsilon}$ such that the sets $\{\tau_\beta^n\overline{\varepsilon}:n\ge k\}$ and $\{\tau_\beta^n\overline{\varepsilon}:n\le -k\}$ are both dense in X_β for every $k\ge 0$. Therefore, φ_0 is bijective almost everywhere with respect to the measure of maximal entropy for τ_β (2).*

Remark. Recently the author proved that φ_0 is one-to-one a.e. for a wider class of Pisot units, namely, for those which we have called weakly finitary. More precisely, a Pisot unit β is called *weakly finitary* if for any $x\in\mathbb{Z}[\beta]$ and any $\delta>0$ there exists $f\in\mathrm{Fin}(\beta)\cap(0,\delta)$ such that $x+f\in\mathrm{Fin}(\beta)$ as well. There is a conjecture (shared by most experts in this area) that every Pisot unit is weakly finitary. For more details see [Sid].

Our goal in this section is to show that φ_1 is $|D|$-to-one a.e. and that $\varphi_1^{-1}(\mathbf{0})$, after some natural identification, will coincide with $\mathfrak{A}_\beta$.

Definition. We will say that $\overline{\varepsilon}\in X_\beta$ is *equivalent* to $\overline{\varepsilon}'\in X_\beta$ if $\varphi_0(\overline{\varepsilon})=\varphi_0(\overline{\varepsilon}')$.

---

(2) The nature of this measure is not important; in fact, one may take any shift-invariant measure that is strictly positive on all cylinders $[\varepsilon_1=i_1,\dots,\varepsilon_k=i_k]\subset X_\beta$.

We will denote this equivalence relation by $\sim$. Let $X'_\beta = X_\beta/{\sim}$; by Schmidt's theorem, $\#[\overline{\varepsilon}] = 1$ for a.e. sequence $\overline{\varepsilon}$. Let addition (subtraction) in X'_β be defined via the torus, i.e., $[\overline{\varepsilon}] \pm [\overline{\varepsilon}'] := \varphi_0^{-1}(\varphi_0(\overline{\varepsilon}) \pm \varphi_0(\overline{\varepsilon}'))$. Obviously, X'_β is a group under addition, isomorphic to $\mathbb{T}^m$.

LEMMA 3.2. *The equivalence relation for $\mathfrak{A}_\beta$, described in the previous section, is precisely the restriction of the equivalence relation to X_β.*

Proof. We need to show that if $[\overline{\varepsilon}] = [\overline{\varepsilon}']$ in the sense of $\mathfrak{A}_\beta$, then $\varphi_0(\overline{\varepsilon}) = \varphi_0(\overline{\varepsilon}')$. Similarly to the proof of Proposition 2.3, let $r \in \mathbb{N}$ be a common period of the β-expansions of $\overline{\varepsilon}$ and $\overline{\varepsilon}'$ and $N_k := kr$. Then

$$\Big|\sum_{j=-N_k}^{\infty} (\varepsilon_j - \varepsilon'_j)\beta^{-j}\Big| = \beta^{kr}\eta, \quad \text{where} \quad \eta = \Big|\sum_{j=0}^{\infty}(\varepsilon_j - \varepsilon'_j)\beta^{-j}\Big| \in \mathbb{Z}[\beta].$$

Hence for $l = 0, 1, \ldots, m-1$,

$$\Big\|\sum_{j=-N_k}^{\infty} (\varepsilon_j - \varepsilon'_j)\beta^{-j}\xi_0\beta^{-l}\Big\| \to 0 \quad \text{as } k \to +\infty,$$

which implies $\varphi_0(\overline{\varepsilon}) = \varphi_0(\overline{\varepsilon}')$. ∎

THEOREM 3.3. (i) *The mapping φ_1 is well defined on the quotient compactum X'_β.*

(ii) $\varphi_1 : X'_\beta \to \mathbb{T}^m$ *is a group homomorphism with the kernel $\mathfrak{A}_\beta$.*

Proof. (i) We need to show that if $\varphi_0(\overline{\varepsilon}) = \varphi_0(\overline{\varepsilon}')$, then $\varphi_1(\overline{\varepsilon}) = \varphi_1(\overline{\varepsilon}')$. By the definition of φ_0, we have $\|\xi_0\beta^j u_N\| \to 0$ for any $j \in \mathbb{Z}$, where $u_N = \sum_{k=-N}^{\infty}(\varepsilon_k - \varepsilon'_k)\beta^{-k}$. By Proposition 1.2, $\xi_0^{-1} \in \mathbb{Z}[\beta]$, whence $\|\beta^j u_N\| \to 0$ as well. Hence by the definition of φ_1,

$$\|\varphi_1(\varepsilon^{(N)}) - \varphi_1(\varepsilon'^{(N)})\| \le \text{const} \cdot \|u_N\| \to 0.$$

(ii) As was mentioned above, the mapping φ_1 is bounded-to-one. Since the set $\mathcal{O} = \varphi_1^{-1}(\mathbf{0})$ is finite and shift-invariant, it should consist of purely periodic sequences only. Let $\overline{\varepsilon} \in \mathcal{O}$ have period r and $\alpha = \sum_{k=1}^{\infty} \varepsilon_k\beta^{-k}$. Then by the definition of φ_1, $\|\alpha\beta^{rN}\| \to 0$ as $N \to +\infty$. Considering the sequences $\tau_\beta\overline{\varepsilon}, \ \tau_\beta^2\overline{\varepsilon}, \ldots, \tau_\beta^{r-1}\overline{\varepsilon}$, we see that $\|\alpha\beta^{rN+j}\| \to 0$ for $j = 0, 1, \ldots, r-1$, whence $\|\alpha\beta^N\| \to 0$ as $N \to +\infty$, i.e., $\alpha \in \mathcal{P}_\beta$, and the claim follows from Lemma 2.2. Conversely, if $\alpha \in \mathcal{P}_\beta$, then $\|\alpha\beta^N\| \to 0$, whence $\varphi_1(\overline{\varepsilon}) = \mathbf{0}$. ∎

COROLLARY 3.4. *The mapping φ_1 is $|D|$-to-one a.e.*

Let $A = \varphi_1\varphi_0^{-1}$. This mapping is well defined a.e. on the torus, and we extend it to the whole $\mathbb{T}^m$ by continuity. The following claim is straightforward.

LEMMA 3.5. *The mapping $A : \mathbb{T}^m \to \mathbb{T}^m$ is an endomorphism of the torus. Its determinant equals $\pm D$.*

In practice, to find A, one might expand ξ_0^{-1} into the sum $\sum_{k=0}^{m-1} a_k\beta^k$ with $a_k \in \mathbb{Z}$. Then $A = \sum_{k=0}^{m-1} a_k T^k$.

EXAMPLES. 1. $\beta^2 = k\beta \pm 1$. Here, as we know, $\xi_0 = 1/\sqrt{D}$, $\xi_0^{-1} = \sqrt{D} = 2\beta - k$. Hence

$$A = 2T - kI = \begin{pmatrix} 2k & \pm 2 \\ 2 & 0 \end{pmatrix} - \begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix} = \begin{pmatrix} k & \pm 2 \\ 2 & -k \end{pmatrix},$$

and $\det A = -D(\beta)$.

2. $\beta^3 = \beta^2 + \beta + 1$. Here $\xi_0^{-1} = -1 - 2\beta + 3\beta^2$, whence

$$A = -I - 2\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 3 & 4 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & -1 \end{pmatrix},$$

and $\det A = 44 = -D(\beta)$.

4. Generalization of the Pisot group. It is natural to ask the same sort of questions for more general arithmetic codings of T, i.e. for those parameterized by arbitrary homoclinic points (= by arbitrary elements of the group $\mathcal{P}_\beta$). Let $\varphi_\xi : X_\beta \to \mathbb{T}^m$ be defined as follows:

$$\varphi_\xi(\overline{\varepsilon}) = \sum_{k\in\mathbb{Z}} \varepsilon_k T^{-k}\mathbf{t} \tag{4.1}$$

$$= \lim_{N\to+\infty} \Big(\sum_{k=-N}^{\infty} \varepsilon_k \beta^{-k} \Big) \begin{pmatrix} \xi \\ \xi\beta^{-1} \\ \vdots \\ \xi\beta^{-m+1} \end{pmatrix} \bmod \mathbb{Z}^m,$$

where $\xi = \xi(\mathbf{t}) \in \mathcal{P}_\beta$ (see Lemma 3.1). The questions are the following:

(1) What is the value of $\#\varphi_\xi^{-1}(x)$ for a typical $x \in \mathbb{T}^m$?
(2) How to describe the kernel of φ_ξ?

The next assertion answers the first question; it is a generalization of the corresponding result proven in [SV2] for $m = 2$.

THEOREM 4.1. *For a.e. $x \in \mathbb{T}^m$ with respect to the Haar measure,*

$$\#\varphi_\xi^{-1}(x) \equiv |DN(\xi)|,$$

where $N(\cdot)$ denotes the norm of an element of the extension $\mathbb{Q}(\beta)/\mathbb{Q}$.

Proof. Let $\ell := \xi/\xi_0 \in \mathbb{Z}[\beta]$. If $\ell = \sum_{i=0}^{m-1} c_i\beta^i$, then

$$\varphi_\xi = A_\xi \varphi_0, \tag{4.2}$$

where A_ξ is the endomorphism of $\mathbb{T}^m$ given by $A_\xi = \sum_{i=0}^{m-1} c_i T^i$. Indeed, for the basis sequence $e^{(0)} = (\ldots, 0, 0, \ldots, 0, 1, 0, \ldots, 0, 0, \ldots)$ with the unity at the first coordinate, we have

$$
\begin{aligned}
(A_\xi\varphi_0)(e^{(0)}) &= A_\xi(\xi_0,\xi_0\beta^{-1},\dots,\xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m\\
&= \sum_{i=0}^{m-1} c_i T^i(\xi_0,\xi_0\beta^{-1},\dots,\xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m\\
&= \sum_{i=0}^{m-1} c_i \beta^i(\xi_0,\xi_0\beta^{-1},\dots,\xi_0\beta^{-m+1}) \bmod \mathbb{Z}^m\\
&= (\xi,\xi\beta^{-1},\dots,\xi\beta^{-m+1}) \bmod \mathbb{Z}^m = \varphi_\xi(e^{(0)}).
\end{aligned}
$$

Since the relation (4.2) is shift-invariant, it holds for any $\overline{\varepsilon} = \tau_\beta^j(e^{(0)}), j \in \mathbb{Z}$, and therefore for any $\overline{\varepsilon} \in X_\beta$ by linearity and continuity.

As φ_0 is one-to-one a.e., φ_ξ will be K-to-one a.e. with $K = |\det A_\xi|$. By definition, $N(\ell)$ is the determinant of the matrix of the multiplication operator $x \mapsto \ell x$ in the standard basis of $\mathbb{Q}(\beta)$, whence $N(\ell) = \det A_\xi$, because T is given by the companion matrix. Finally, $N(\ell) = N(\xi)/N(\xi_0) = DN(\xi)$, as by the result of [Sam, Section 2.7], $N(\xi_0) = 1/D$ whenever ξ_0 is as in Proposition 1.2. ■

To answer the second question, let us recall that by definition of Section 3, X'_β is the quotient group $X_\beta/\sim$ with respect to the equivalence relation: $\overline{\varepsilon} \sim \overline{\varepsilon}'$ iff $\varphi_0(\overline{\varepsilon}) = \varphi_0(\overline{\varepsilon}')$.

Proposition 4.2. *The kernel of the mapping φ_ξ is a finite subgroup of X'_β whose size is $|DN(\xi)|$. Any element $\overline{\varepsilon} \in \operatorname{Ker}\varphi_\xi$ can be obtained as follows: take some $\varkappa \in \mathbb{Z}[\beta]$ such that $\alpha = (\xi_0/\xi)\varkappa > 0$ and let $\overline{\varepsilon}$ be the purely periodic sequence whose period coincides with the one of the β-expansion of α.*

Proof. The first claim is a direct consequence of Theorem 4.1. To prove the second one, suppose $\varphi_\xi(\overline{\varepsilon}) = \mathbf{0}$. By the same reasons as above, $\overline{\varepsilon}$ must be purely periodic. Let $\alpha = \sum_{k=1}^\infty \varepsilon_k\beta^{-k}$. By (4.1), $\|\xi\alpha\beta^n\| \to 0$ as $n \to +\infty$, whence by Proposition 1.2, $\xi\alpha = \xi_0\varkappa$, $\varkappa \in \mathbb{Z}[\beta]$. Hence $\alpha = (\xi_0/\xi)\varkappa$.

Conversely, if $\overline{\varepsilon}$ is the two-sided continuation of the period of $\varkappa\xi_0/\xi > 0$ for some $\varkappa \in \mathbb{Z}[\beta]$, and $\alpha = \sum_{k=1}^\infty \varepsilon_k\beta^{-k}$, then $\|\alpha\xi\beta^n\| = \|\xi_0\varkappa\beta^n\| + o(1) \to 0$ as $n \to +\infty$, whence by formula (4.1), $\varphi_\xi(\overline{\varepsilon}) = \mathbf{0}$. ■

Remark. In practice, to obtain $\operatorname{Ker}\varphi_\xi$, one may take $\varkappa = 1, \beta, \dots, \beta^{m-1}$ and consider the linear span of the corresponding set of sequences.

Corollary 4.3. *For any finitary Pisot β and any purely periodic sequence $\overline{\varepsilon} \in X_\beta$ there exists $\xi \in \mathcal{P}_\beta$ such that $\overline{\varepsilon} \in \operatorname{Ker}\varphi_\xi$.*

Proof. Let r denote the period of $\overline{\varepsilon}$. One may take $\xi = (\beta^r - 1)\xi_0$ and apply (4.1). ■

Example. Let $\beta = (1+\sqrt{5})/2$. Here, as we know, $\xi_0 = 1/\sqrt{5}$. Note first that if $\ell = \xi/\xi_0$ is not a unit of the ring $\mathbb{Z}[\beta]$ (i.e., it is not of the form $\pm\beta^k$ for some $k \in \mathbb{Z}$), then, as is well known, $|N(\ell)| \geq 4$, whence by Theorem 4.1, the number of pre-images for φ_ξ is greater than or equal to 4 as well. This corresponds to the observation that the smallest period of a nonzero sequence in X_β is three (as $\overline{01} \notin X_\beta$). Therefore, the set $\mathfrak{A} = \{0, 100, 010, 001\}$ is the smallest possible (in capacity) shift-invariant subset containing 0 and something else. As $1/2 = \beta^{-2} + \beta^{-5} + \beta^{-8} + \dots$, by Proposition 4.2, $\mathfrak{A}$ is indeed the kernel of the mapping φ_ξ with $\xi = 2/\sqrt{5}$. Obviously, in this case $A_\xi = \left(\begin{smallmatrix} 2 & 0 \\ 0 & 2 \end{smallmatrix}\right)$.

Thus, $\operatorname{Ker} \varphi_{2/\sqrt{5}}$ consists of *all* the sequences of period 3 and is in fact isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$. It is curious that $\operatorname{Ker} \varphi_{4/\sqrt{5}}$ is just the set of all sequences of period 6; it is isomorphic to $(\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/4\mathbb{Z})$.

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