ACTA ARITHMETICA
101.3 (2002)

On the Barban–Davenport–Halberstam theorem: XIV

by

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1. Introduction. We pursue further our researches on moments of the type

$$(1)\qquad H(x,k) = \sum_{0<a\le k} \{S(x;a,k) - f(a,k)x\}^2 = \sum_{0<a\le k} E^2(x;a,k), \quad \text{say}, \quad (k\le x)$$

and

$$G(x,Q) = \sum_{k\le Q} H(x,k) \qquad (Q\le x)$$

that appertain to given strictly increasing sequences of positive integers s obeying a condition of the type

$$(2)\qquad S(x;a,k) = \sum_{\substack{s\le x\\ s\equiv a,\,\mathrm{mod}\,k}} 1 = f(a,k)x + O\{\Delta_k(x)\}$$

for values of k that may be small compared with x. But, before revealing what we intend and having been reminded that previously

$$(3)\qquad \Delta_k(x) = x\log^{-A} x$$

by analogy with known properties of prime numbers, we should mention that our first discussion of this topic appeared in 1975 in the third article III of this series ([1]), in which the generalized Barban–Davenport–Halberstam inequality

$$(4)\qquad G(x,Q) = O(Qx) + O(x^2\log^{-A} x)$$

was derived under the additional hypothesis that

$$(5)\qquad f(a,k) = g\{k,(a,k)\}$$

2000 *Mathematics Subject Classification*: Primary 11B25.

([1]) We refer to these articles by the Roman numerals indicating their position in the series; their full particulars are given in the list of references at the end.

depended only on k and the highest common factor (a,k). Subsequently the subject of an alternative and improved treatment in our paper [3], this result was then augmented in IX by an asymptotic formula

$$(6)\quad G(x,Q)=\{D_1+o(1)\}Qx+O(x^2\log^{-A}x)\quad (o(1)\to 0 \text{ as } x/Q\to 1)$$

of Barban–Montgomery type under the extra supposition, often realized in familiar situations, that $g(k,k)$ was a fixed positive multiple of a multiplicative function of k. Then, in the following paper X, it was shewn that all previously imposed conditions on $f(a,k)$ were actually irrelevant to the truth of ([2]) (4) and (6) so that, in particular, the assumption (2) and the conclusion (4) are seen to be strictly equivalent when (3) is in place. Yet there remained the important question of when (6) fails to be a fully informative asymptotic formula because of the vanishing of D_1, the final discussion in IX having only uncovered some situations in which $D_1=0$ and another in which $D_1\neq 0$.

Parallel to the picture just painted, there is the one portrayed by Vaughan in two memoirs ([6] and [7]) published during the short interval between the appearances of IX and X. In the latter of these, slightly generalizing the underlying circumstances by attaching weights to the members s of the sequence but *retaining* the condition (5) that was removed in X, he employed the circle method to obtain Barban–Montgomery type formulae with accurate remainder terms that were expressed in terms of a function $\Delta(x)$ of x alone appearing in lieu of $\Delta_k(x)$ in a generalization of (2). Although we have formed the view that the two approaches are roughly equal in power in several situations, Vaughan's method and insight lead to a full understanding of the quiddity of D_1 that we failed to reach in IX and X.

The genesis of our present investigation springs from thoughts similar to those expressed by Professor Montgomery when he stated that the form of Croft's asymptotic formula [1]

$$G(x,Q)\sim D_2Q^{3/2}x^{1/2}\qquad (x^{2/3+\varepsilon}<Q\leq x)$$

for square-free numbers s could be foreseen from the likelihood of the truth of the asymptotic formula

$$S(x)=S(x;0,1)=\frac{6x}{\pi^2}+O(x^{1/4+\varepsilon}).$$

We are thus prompted to see what can be learnt about the behaviour of $H(x,k)$ and $G(x,Q)$ for large k and Q in the light of any known estimate $E(x;a,k)=O(\Delta_k(x))$ for values of k that are relatively small compared with x. Indeed, put in the crudest terms, our inquiry could be whether $G(x,Q)$

([2]) We mention (4) as well as (6) because the treatment of the former is often a precursor to that of the latter.

for large Q be essentially equal to the sum

$$\sum_{k\le Q} k\varDelta_k^2(x), \tag{7}$$

for which, if necessary, the given definition of $\varDelta_k(x)$ for small k would be extended for larger k in a natural way. Yet, that we may overreach ourselves in making such an ambitious assertion without qualification can be appreciated from our knowledge that the formally dominant term in $G(x,Q)$ stems only from the Dirichlet's series $\varPhi(s)$ in IX and X that is defined in terms of $f(a,k)$ alone. In fact, since the addition of an appropriately thick rogue sequence of zero density to a given sequence would increase the size of the allowable value $\varDelta_k(x)$ in (2) without affecting the value of $f(a,k)$ and hence of the main constituent in $G(x,Q)$, we must abate our speculations to the contemplation of essentially one-sided inequalities between $G(x,Q)$ and (7). We are also reinforced in this view by the realization that formula (4) implies that $E(x;a,k)$ is usually not more than about $\sqrt{x/k}$ for values of k close to x even when $\varDelta_k(x)$ is taken to be no better than $x\log^{-A}x$ for arbitrary positive values of A.

Enough has been said to indicate that we should study our problem in the case where there is still no explicitly stated restriction on $f(a,k)$ and where

$$\varDelta_k(x) = O\{(x/k)^\alpha\} \tag{8}$$

for

$$k \le x^{1/2} \tag{9}$$

when α is a given constant satisfying

$$0 < \alpha < 1/2. \tag{10}$$

In this situation, influenced by Vaughan's observations about the number D_1 in (6) that will be seen to be still valid without restriction (5), we use the circle method in an iterative and slightly novel form to come close to our expectation by shewing that

$$G(x,Q) = O(Q^{2-2\alpha}x^{2\alpha}\log^2(2x/Q)) \tag{11}$$

for largish values of Q up to x and hence that the bound

$$E(x;a,k) = O\{(x/k)^\alpha \log(2x/k)\}$$

comparable with (8) is true on average for values of k in some range bounded above by x; thus, in particular, $D_1 = 0$. As a by-product of the first part of the method, an asymptotic formula of the type

$$\nu_c(x) \sim A(c)(x-c) \tag{12}$$

is also produced for the number $\nu_c(x)$ of pairs of numbers s, s' not exceeding x that differ by a given number $c < x$. From a more accurate variant of this, we then increase our knowledge of $E(x;a,k)$ for some large *individual* values of k by deducing that in case (5)

$$H(x,k) = O\{k^{1-2\alpha}x^{2\alpha}(x/k)^{\varepsilon}\} \tag{13}$$

when k is a prime number that is fairly close to x (here (5) is probably necessary to ensure the truth of the result for each and every such k). Thence, by assuming in addition that the function $g(d) = g(d,d)$ of (5) above is multiplicative in the sense of criterion S of IX, we obtain in similar circumstances the keener relation

$$H(x,k) = O\{k^{1-2\alpha}x^{2\alpha}\log^3(2x/k)\} \tag{14}$$

which only falls short of what one might expect by a logarithmic factor. In addition, though straying somewhat from the theme of this series, we should mention that our proof of (12) is interesting methodologically in that an hypothesis of type (8) with (9) and (10) removes the obstacles that currently preclude a treatment of the prime twins conjecture by the circle method even when all reasonable attributes of $\theta(x;a,k)$ are assumed, albeit it has to be said that it is seldom in practice we are presented with a specific sequence that is known to satisfy such generous assumptions in full.

As yet, the bound found for $G(x,Q)$ may well be imperfect because of the presence of the factor $\log^2(2x/Q)$. While not able to effect any improvement in the estimate under the circumstances so far assumed, we go on to shew that the expected bound

$$G(x,Q) = O(Q^{2-2\alpha}x^{2\alpha}), \tag{15}$$

is certainly true for all Q up to x provided that the range of validity of our supposition (8) be extended to

$$k \le x^{2/3}\log^{4/3} x. \tag{16}$$

At least two questions remain. The first is about how much our conditions for smaller k could be loosened with jeopardizing (11) or (15). We might, for example, suspect that a smaller range of validity for (8) than (9) would still be enough to ensure the continuation of (15) for large Q but, at present, see no way to settle the matter. One also might hope that in most or all of our work we could dispense with criteria such as (8) in favour of consequential inequalities of type (15) for smallish Q. Yet we do not attempt to go down such an avenue on the present occasion because such a journey, if successful, would lengthen the exposition to an unacceptable extent and because the statements of our theorems are at their vividest when expressed in terms of (8). Instead we shall hold back any discussion of this possible advance until such time as an adequate method has been developed to our satisfaction.

The second question concerns the circumstances in which (11) or (15) can be converted into an asymptotic formula with (non-zero) main term when Q is large. Vaughan [7], indeed, has shewn there is such a formula when the function $f(a,k) = g\{k,(a,k)\}$ fulfils certain conditions, although it must be said it is neither clear just what sequences conform to these requirements nor how they can be characterized in terms of $E(x;a,k)$ for smaller k. Here we do two things. First, under the circumstances in which (15) was established for all $Q \le x$, we shew that the asymptotic formula

$$G(x,Q) \sim D_2 Q^{2-2\alpha} x^{2x} \quad (D_2 > 0)$$

holds for $x^{2/3}\log^{4/3} x < Q \le x$ whenever it is known to be true at the top of the complementary range of Q; thus again we have an example where the behaviour of $S(x;a,k)$ for small k induces a similar behaviour for larger k. Moreover, in this situation we find we can deduce an improvement in formula (14) for $H(x,k)$. On the other hand, however, we shall also construct a sequence satisfying (3) to which there does not answer an asymptotic formula and for which the true order of magnitude of $G(x,Q)$ is so variable that it fluctuates between values as large and as small as $Q^{1+\varepsilon}x^{1-\varepsilon}$ and $Q^{\varepsilon}x^{2-\varepsilon}$.

That there is at least a loose association between the functions $f(a,k)$ and the remainder term in (2) is manifest from what has already been said and from what follows. But further research will be needed to elucidate it and to refine any of our results that depend on it.

2. The interpretation of the constant D_1. The entry to most of our theorems is through the interpretation Vaughan would give to the constant D_1, which we shall shew to be still valid when it is only assumed in the first instance that (3)

$$S(x;a,k) = f(a,k)x + o(x) \tag{17}$$

without the extra condition (5) being necessarily imposed. First we express (17) for the special case $k = 1$ as

$$S(x) = S(x;0,1) = Cx + o(x) \tag{18}$$

and then must get our bearings by referring to X, drawing from it some relevant facts and extending the notation therein. Starting with X(27) and the left part of X(7) in a slightly different symbolism and setting

$$F(\theta) = F(\theta,x) = \sum_{s \le x} e^{2\pi i s\theta}, \tag{19}$$

(3) It is important to note that at present we are only discussing the value of D_1 as it has been defined in terms of $f(a,k)$. We are not asserting that (17) alone implies a Barban–Montgomery type theorem, which can be easily seen to be false in some situations when $o(x)$ divided by x tends very slowly to 0 as $x \to \infty$.

we have that

$$\sum_{\substack{0<h\le k\\(h,k)=1}} \left|F\left(\frac{h}{k}\right)\right|^2$$

is equal to both

$$\frac{1}{k}\sum_{0<a\le k}\left(\sum_{d|k}\mu\left(\frac{k}{d}\right)dS(x;a,d)\right)^2 \tag{20}$$

and

$$\sum_{d|k}\mu\left(\frac{k}{d}\right)d\sum_{0<a\le d}S^2(k;a,d)$$

provided we bear in mind the provenance of the inner sum in (20). Hence, if we write $F(h/k)$ as

$$\sum_{s\le x}e^{2\pi ihs/k}=\sum_{0<b\le k}e^{2\pi ihb/k}S(x;b,k) \tag{21}$$

and use condition (17), we deduce that

$$\begin{aligned}\sum_{\substack{0<h\le k\\(h,k)=1}}\Big|\sum_{0<b\le k}e^{2\pi ihb/k}f(b,k)\Big|^2 &= \frac{1}{k}\sum_{0<a\le k}\left(\sum_{d|k}\mu\left(\frac{k}{d}\right)df(a,d)\right)^2\\ &= \sum_{d|k}\mu\left(\frac{k}{d}\right)d\sum_{0<a\le d}f^2(a,d)\end{aligned} \tag{22}$$

for any given value of k after dividing by x^2 and letting $x\to\infty$, which equation we express as both

$$R(k)=\frac{1}{k}\sum_{0<a\le k}w^2(a,k)=ka_k=N(k) \tag{23}$$

and

$$R(k)=\sum_{\substack{0<h\le k\\(h,k)=1}}|P(h,k)|^2 \tag{24}$$

after setting

$$P(h,k)=\sum_{0<b\le k}e^{2\pi ihb/k}f(b,k) \tag{25}$$

and recalling the definitions on pp. 4, 5, and 9 of X. Thus some of the conclusions in the last page of X are tantamount to the convergence of the series

$$\sum_{k=1}^{\infty}R(k)=C_1,\quad\text{say,} \tag{26}$$

and to the determination of D_1 in Theorem 2 therein by

$$D_1 = C - C_1. \tag{27}$$

The obvious fact that $D_1 \geq 0$ can be otherwise and indirectly deduced by an appeal to that form of the large sieve inequality which asserts that the function $F(\theta)$ in (19) is subject to the inequality

$$\begin{aligned}(28)\quad \sum_{k\le Q}\ \sum_{\substack{0<h\le k\\(h,k)=1}} \left|F\left(\frac{h}{k}\right)\right|^2 &\le (x + A_1Q^2)\int_0^1 |F(\theta)|^2\,d\theta = (x+A_1Q^2)S(x)\\ &= (x+A_1Q^2)(Cx + o(x)),\end{aligned}$$

this being a return in more accurate form to a relation otherwise exploited in X. Hence, treating the inner sum on the left of this as in the derivation of (22), we deduce after letting $x \to \infty$ that

$$\sum_{k\le Q} R(k) \le C$$

for any Q and hence that $C_1 \le C$ by (26).

The ideas embodied in (28) can be strengthened to form Vaughan's principle on our been prompted to appreciate that the series

$$\sum_{k=1}^{\infty} R(k) = C_1$$

is actually the singular series associated with a formal application of the circle method to the evaluation of the sum

$$S(x) = \int_0^1 |F(\theta)|^2\,d\theta = Cx + o(x),$$

as will become totally clear when we press these concepts to new conclusions. Thus, without specifying precisely what they are, the major arcs used in the attempted evaluation of the integral with positive integrand give rise to a term that is asymptotically equivalent to C_1x, whence once again $D_1 \geq 0$ and D_1 is the formal contribution of the minor arcs. Necessarily somewhat nebulous as yet, the notions involved will become rigorous once we have prepared for our analysis by restating the chief criterion mentioned in the introduction.

3. The tail of the series $\sum R(k)$. We are ready to study the sum $G(x,Q)$ for sequences s that satisfy Criterion V_1 to the effect that

$$|E(x;a,k)| < A_2\{(x/k)^{\alpha}\} \tag{29}$$

for $k \le x^{1/2}$ where A_2 is some positive constant and $0 < \alpha < 1/2$.

Our first theorem stems from the circle method, which for technical reasons is applied for *all* $y \geq 1$ to the evaluation of

$$S_1(y) = S(2y) - S(y) = \sum_{y<s\leq 2y} 1 \tag{30}$$

instead of $S(y)$ ([4]). Setting up the usual machinery, we define $F_1(\theta, y)$ through (17) and the equation

$$F_1(\theta_1, y) = F(\theta, 2y) - F(\theta, y),$$

by means of which we first get

$$S_1(y) = \int_0^1 |F_1(\theta, y)|^2 \, d\theta. \tag{31}$$

Next, the integrand being non-negative, we choose

$$M = M(y) = y^{1/2} \tag{32}$$

and use Dirichlet's theorem to bound the integral from above by covering the range of integration, mod 1, with the set of all (possibly overlapping) arcs of the form $|\phi - h/k| \leq 1/(Mk)$ corresponding to the fractions answering to the conditions $0 < h \leq k$, $(h, k) = 1$, and $k \leq M$, wherefore we gain the inequality

$$S_1(y) \leq \sum_{k\leq M} \sum_{\substack{0<h\leq k \\ (h,k)=1}} \int_{-1/(Mk)}^{1/(Mk)} |F_1(h/k+\phi, y)|^2 \, d\phi = \sum_{k\leq M} T(y, k), \quad \text{say.} \tag{33}$$

We proceed as usual to the integrand in (33) and develop (21) to obtain the equation

$$\begin{aligned} F(h/k, u) &= u \sum_{0<b\leq k} e^{2\pi ibh/k} f(b, k) + \sum_{0<b\leq k} e^{2\pi ibh/k} E(u; b, k) \\ &= uP(h, k) + P_1(u; h, k), \quad \text{say,} \end{aligned} \tag{34}$$

in the language of (25), whence by partial summation

$$\begin{aligned} F_1(h/k + \phi, y) &= \int_y^{2y} e^{2\pi iu\phi} \, dF(h/k, u) \\ &= P(h, k) \int_y^{2y} e^{2\pi iu\phi} \, du + [P_1(u; h, k)e^{2\pi iu\phi}]_y^{2y} \\ &\quad - 2\pi i\phi \int_y^{2y} P_1(u; h, k)e^{2\pi iu\phi} \, du \end{aligned} \tag{35}$$

([4]) If we were to work with $S(y)$, we would need the rather stronger requirement that $|E(u; a, k)| \leq A_2\{(x/k)^\alpha\}$ for $u \leq x$ and $k \leq x^{1/2}$.

$$= P(h,k)J(\phi,y) + P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi), \quad \text{say.}$$

Therefore, if we apply Minkowski's inequality to the value of $T(y,k)$ in (33) supplied by this, we obtain

$$\begin{aligned}(36)\quad T(y,k) \le R(k) &\int_{-1/(Mk)}^{1/(Mk)} |J(\phi,y)|^2\,d\phi \\ &+ \sum_{\substack{0<h\le k\\(h,k)=1}} \int_{-1/(Mk)}^{1/(Mk)} |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|^2\,d\phi \\ &+ 2\Big(R(k)\int_{-1/(Mk)}^{1/(Mk)} |J(\phi,y)|^2\,d\phi\Big)^{1/2} \\ &\times \Big(\sum_{0<h\le k}\int_{-1/(Mk)}^{1/(Mk)} |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|^2\,d\phi\Big)^{1/2} \\ &= T_1(y,k) + T_2(y,k) + 2(T_1(y,k)T_2(y,k))^{1/2}, \quad \text{say.}\end{aligned}$$

In this, by (35) and the Parseval–Plancherel theorem,

$$(37)\qquad T_1(y,k) \le R(k)\int_{-\infty}^{\infty} |J(\phi,y)|^2\,d\phi = yR(k)$$

so that

$$(38)\qquad T(y,k) \le yR(k) + T_2(y,k) + 2y^{1/2}R^{1/2}(k)T_2^{1/2}(y,k),$$

with which estimation the first stage in the treatment of the integral $S_1(y)$ is complete.

To bound $T_2(y,k)$ the Criterion V_1 is brought into play in the exploitation of the inequality

$$\begin{aligned}(39)\qquad T_2(y,k) \le 2&\int_{-1/(Mk)}^{1/(Mk)} \sum_{\substack{0<h\le k\\(h,k)=1}} |P_2(y;h,k;\phi)|^2\,d\phi \\ &+ 8\pi^2\int_{-1/(Mk)}^{1/(Mk)} \phi^2 \sum_{\substack{0<h\le k\\(h,k)=1}} |P_3(y;h,k;\phi)|^2\,d\phi \\ &= 2T_3(y,k) + 8\pi^2 T_4(y,k), \quad \text{say,}\end{aligned}$$

where (35) and (34) shew that the integrand of $T_3(y,k)$ does not exceed

$$2 \sum_{\substack{0<h\le k\\(h,k)=1}} (|P_1(2y;h,k)|^2 + |P_1(y;h,k)|^2)$$
$$\le 2 \sum_{0<h\le k} \Big(\Big| \sum_{0<b\le k} e^{2\pi ibh/k} E(2y;b,k)\Big|^2 + \Big| \sum_{0<b\le k} e^{2\pi ibh/k} E(y;b,k)\Big|^2\Big)$$
$$= 2k \sum_{0<b\le k} (E^2(2y;b,k) + E^2(y;b,k))$$
$$\le \frac{6A_2^2k^2y^{2\alpha}}{k^{2\alpha}}$$

because $k \le M = y^{1/2} < (2y)^{1/2}$. Thus, integrating over a range of length $2/(Mk)$, we conclude that

(40) $$T_3(y,k) \le \frac{12A_2^2ky^{2\alpha}}{Mk^{2\alpha}}.$$

On the other hand, by (39), (35), and (34),

$$T_4(y,k) \le \frac{1}{M^2k^2} \int_{-1/(Mk)}^{1/(Mk)} \sum_{\substack{0<h\le k\\(h,k)=1}} \Big| \sum_{0<b\le k} e^{2\pi ibh/k} \int_y^{2y} E(u;b,k)e^{2\pi iu\phi}\,du\Big|^2 d\phi$$
$$\le \frac{1}{M^2k^2} \int_{-\infty}^{\infty} \sum_{0<h\le k} \Big| \sum_{0<b\le k} e^{2\pi ibh/k} \int_y^{2y} E(u;b,k)e^{2\pi iu\phi}\,du\Big|^2 d\phi$$
$$= \frac{1}{M^2k} \sum_{0<b\le k} \int_{-\infty}^{\infty} \Big| \int_y^{2y} E(u;b,k)e^{2\pi iu\phi}\,du\Big|^2 d\phi$$
$$= \frac{1}{M^2k} \sum_{0<b\le k} \int_y^{2y} E^2(u;b,k)\,du$$

from another application of the Parseval–Plancherel theorem. Hence, by Criterion V_1,

$$T_4(y,k) \le \frac{A_2^2}{M^2k^{2\alpha}} \int_y^{2y} u^{2\alpha}\,du < \frac{2A_2^2y^{2\alpha+1}}{M^2k^{2\alpha}},$$

which combined with (40) in (39) yields

(41) $$T_2(y,k) < \frac{24A_2^2ky^{2\alpha}}{Mk^{2\alpha}} + \frac{16\pi^2A_2^2y^{2\alpha+1}}{M^2k^{2\alpha}} < \frac{200A_2^2y^{2\alpha}}{k^{2\alpha}} = \frac{A_3y^{2\alpha}}{k^{2\alpha}}, \quad \text{say,}$$

on account of (32). Altogether, therefore, we deduce from this and (37) that

$$T(y,k) < yR(k) + \frac{A_3 y^{2\alpha}}{k^{2\alpha}} + \frac{2A_3^{1/2} y^{\alpha+1/2} R^{1/2}(k)}{k^\alpha}$$

and then conclude from (33) that

$$\begin{aligned} (42)\qquad S_1(y) &< y\sum_{k\le M} R(k) + A_3 y^{2\alpha}\sum_{k\le M}\frac{1}{k^{2\alpha}} + 2A_3^{1/2}y^{\alpha+1/2}\sum_{k\le M}\frac{R^{1/2}(k)}{k^\alpha} \\ &< y\sum_{k\le M} R(k) + \frac{A_3 y^{\alpha+1/2}}{1-2\alpha} + 2A_3^{1/2}y^{\alpha+1/2}\sum_{k\le M}\frac{R^{1/2}(k)}{k^\alpha} \\ &< y\sum_{k\le M} R(k) + A_4 y^{\alpha+1/2} + A_4^{1/2}y^{\alpha+1/2}\sum_{k\le M}\frac{R^{1/2}(k)}{k^\alpha}, \end{aligned}$$

where we must emphasize that A_4 is determined by A_2 and α and where we may clearly assume that

$$(43)\qquad A_4 > R(1), 2A_2.$$

This is the fundamental inequality for $S_1(y)$ upon which our final results depend, although to take advantage of it we shall need the following elementary

LEMMA 1. *Suppose the non-negative function $v(k)$ has the property that both $v(1) < A_5$ and*

$$V(u) = \sum_{k>u} v(k) < A_5 u^{-\beta}$$

for some β in the range $0 \le \beta < 1-2\alpha$ and for $u \ge 1$. Then

$$\sum_{k\le u}\frac{v^{1/2}(k)}{k^\alpha} < \frac{2A_5^{1/2}u^{1/2-\alpha-\beta/2}}{1/2-\alpha-\beta/2}$$

whenever $u \ge 1$.

By the Cauchy–Schwarz inequality

$$\begin{aligned} \Big(\sum_{k\le u}\frac{v^{1/2}(k)}{k^\alpha}\Big)^2 &= \Big(\sum_{k\le u}\frac{v^{1/2}(k)}{k^{\alpha/2-1/4-\beta/4}}\cdot\frac{1}{k^{\alpha/2+1/4+\beta/4}}\Big)^2 \\ &\le \Big(\sum_{k\le u}\frac{v(k)}{k^{\alpha-1/2-\beta/2}}\Big)\Big(\sum_{k\le u}\frac{1}{k^{\alpha+1/2+\beta/2}}\Big), \end{aligned}$$

the first factor in which is

$$v(1) - \int_1^u t^{-\alpha+1/2+\beta/2}\,dV(t)$$
$$= v(1) - u^{-\alpha+1/2+\beta/2}V(u) + V(1) + \left(\frac{1}{2} + \frac{1}{2}\beta - \alpha\right)\int_1^u t^{-\alpha-1/2+\beta/2}V(t)\,dt$$
$$\le 2A_5 + A_5\int_0^u t^{-\alpha-1/2-\beta/2}\,dt$$
$$= 2A_5 + \frac{A_5 u^{1/2-\alpha-\beta/2}}{1/2-\alpha-\beta/2} < \frac{3A_5 u^{1/2-\alpha-\beta/2}}{1/2-\alpha-\beta/2}$$

owing to the condition imposed on β. Also the second factor does not exceed

$$\int_0^u \frac{dt}{t^{\alpha+1/2+\beta/2}} = \frac{u^{1/2-\alpha-\beta/2}}{1/2-\alpha-\beta/2},$$

whence the left side of the stated inequality does not exceed

$$\frac{2A_5^{1/2}u^{1/2-\alpha-\beta/2}}{1/2-\alpha-\beta/2},$$

as proposed.

Our route to the first set of conclusions can now be traversed but is somewhat circuitous because it depends on the repeated use of (42) and Lemma 1, the first appeal to which will shew amongst other things that the constant D_1 in (27) is zero when Criterion V_1 is in place.

Let us suppose that for some β satisfying $0 \le \beta < 1-2\alpha$ the end part

$$\Gamma(u) = \sum_{k>u} R(k) \quad (u \ge 1) \tag{44}$$

of the convergent series representing C_1 in (26) is subject to the inequality

$$\Gamma(u) < A_5 u^{-\beta}, \tag{45}$$

where for convenience it may be assumed that

$$A_5 > A_4 \tag{46}$$

and where $R(1) < A_5$ by (43). Then, by Lemma 1 and (32), the sum

$$\sum_{k\le M} \frac{R^{1/2}(k)}{k^\alpha}$$

in the last component in (42) would not exceed

$$\frac{2A_5^{1/2}y^{1/4-\alpha/2-\beta/4}}{1/2-\alpha-\beta/2},$$

and we would therefore deduce that

$$(47)\quad S_1(y) < y\sum_{k\le M} R(k) + A_4 y^{\alpha+1/2} + \frac{2(A_4A_5)^{1/2}}{1/2-\alpha-\beta/2} y^{\alpha/2+3/4-\beta/4}$$
$$< y\sum_{k\le M} R(k) + \frac{3(A_4A_5)^{1/2}}{1/2-\alpha-\beta/2} y^{\alpha/2+3/4-\beta/4}$$

by (42) and the conditions on α and β. From this relationship we deduce that Criterion V_1 implies that $D_1 = 0$ on the strength of the truth of (45) for $\beta = 0$, which through (18) with remainder term $O(x^\alpha)$ leads to the inequality

$$Cy + O(y^\alpha) < y\sum_{k\le M} R(k) + O(y^{\alpha/2+3/4}) \le y\sum_{k=1}^{\infty} R(k) + O(y^{\alpha/2+3/4}).$$

Hence, dividing by y and letting $y \to \infty$, we infer that

$$C \le \sum_{k=1}^{\infty} R(k)$$

and hence that

$$(48)\qquad D_1 = C - C_1 = 0$$

because $D_1 \ge 0$.

Our first conclusion having been reached, we note from (47) that assumption (29) now means that we would have

$$y\sum_{k=1}^{\infty} R(k) - (2^\alpha+1)A_2 y^\alpha \le y\sum_{k\le M} R(k) + \frac{3(A_4A_5)^{1/2}}{1/2-\alpha-\beta/2} y^{\alpha/2+3/4-\beta/2}$$

so that

$$\sum_{k>M} R(k) \le \frac{4(A_4A_5)^{1/2}}{1/2-\alpha-\beta/2} y^{\alpha/2-1/4-\beta/2}$$

in the light of (44) and (46). Thus, writing $M = u$ and $y = M^2 = u^2$, we would arrive at the inequality

$$\Gamma(u) < \frac{4(A_4A_5)^{1/2}}{1/2-\alpha-\beta/2} u^{\alpha-1/2-\beta/2}$$

for any $u \ge 1$. Consequently, in summary, (45) holds for $\beta = 0$, while its truth for constant A_5 and exponent $-\beta$ implies in fact an improved version with constant $A_5' = 4(A_4A_5)^{1/2}/(1/2-\alpha-\beta/2)$ and exponent $-\beta' = \alpha - 1/2 - \beta/2$; moreover, it is quickly confirmed that the initial constraints on A_5 in (46) and on β still apply to A_5' and β'.

Let us therefore build an inductive algorithm by means of which we produce the inequality

$$\Gamma(u) < B_n u^{-\beta_n} \qquad (u \geq 1)$$

for each non-negative integer n, where B_n and β_n are defined iteratively by $\beta_0 = 0$, $B_0 = A_5$,

$$\beta_n = \beta_{n-1}/2 + 1/2 - \alpha, \qquad B_n = \frac{4(A_4 B_{n-1})^{1/2}}{1/2 - \alpha - \beta_{n-1}/2}.$$

From the latter relations it follows that $(1 - 2\alpha - \beta_n) = \frac{1}{2}(1 - 2\alpha - \beta_{n-1})$ and hence that

$$\beta_n = 1 - 2\alpha - \frac{1}{2^n}(1 - 2\alpha)$$

and then

$$B_n = \frac{4A_4^{1/2} B_{n-1}^{1/2} 2^n}{1 - 2\alpha}.$$

Since therefore

$$B_n = \left(\frac{4A_4^{1/2}}{1 - 2\alpha}\right)^{1 + \frac{1}{2} + \ldots + \frac{1}{2^{n-1}}} \prod_{r=1}^{n} (2^r)^{1/2^{n-r}}$$

in which the exponent of 2 in the product does not exceed $2n$, we see first that

$$\Gamma(u) = O(2^{2n} u^{2\alpha - 1} u^{(1-2\alpha)/2^n})$$

and thus ascertain that

$$\Gamma(u) = O(u^{2\alpha - 1} \log^2 2u) \tag{49}$$

on choosing n so that $2^n = [\log 3u]$.

The bound (49) just obtained is the most important entity upon which the structures of the following treatments are based. In many instances it will be applied through the medium of an intermediate result, in whose statement we begin our practice of usually denoting the common value of $R(k)$ and $N(k)$ by the latter symbol because we no longer need its interpretation as an exponential sum. Derived by methods similar to those used in Lemma 1, this is enunciated without proof as

LEMMA 2. *Let α have the same meaning as in the statement of Criterion* V_1 *in* (29). *Then, for $u \geq 1$ and a given positive number η, we have*

$$\sum_{k \leq u} N(k) k^{1 - 2\alpha + \eta} = O(u^{\eta} \log^2 2u)$$

and

$$\sum_{k > u} N(k) k^{1 - 2\alpha - \eta} = O(u^{-\eta} \log^2 2u).$$

Also

$$\sum_{k\le u} N(k)k^{1-2\alpha} = O(\log^3 2u)$$

and

$$\sum_{k\le u} \frac{N^{1/2}(k)}{k^{\alpha}} = O(\log^2 2u).$$

4. The integral $I^*(u)$ and the first generalized Barban–Montgomery theorem. To identify the integral $I^*(u)$ that occupies an important place in the work, we must elaborate on the comparisons with X made in Section 2 and make more explicit a method therein that stemmed from the previous paper IX. First, the series $\Phi(s)$ on p. 10 of X being the same as

$$\sum_{k=1}^{\infty} \frac{N(k)}{k^{1+s}} \tag{50}$$

by (23), the value of $T^*(u)$ in X is seen to be

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \zeta(s+1)\Phi(s)\frac{u^s}{s(s+1)(s+2)}\,ds \qquad (c>0) \tag{51}$$

by the representation of $T^*(u)$ on p. 26 of IX after we take note of the removal of the factor C^2 in IX. Then, still interpreted with the symbolism of X, the equation

$$T^*(u) = \frac{1}{2}\Phi(0)\log u + \frac{1}{2}B + I(u)$$

of IX(3) introduces the integral $I(u)$ in the form

$$\frac{1}{2\pi i}\int_{-1/2-i\infty}^{-1/2+i\infty} \zeta(s+1)\Phi(s)\frac{u^s}{s(s+1)(s+2)}\,ds,$$

which for $0<\eta<1-2\alpha$ is equal to

$$\frac{\Phi(-1)}{2u} + \frac{1}{2\pi i}\int_{2\alpha-2+\eta-i\infty}^{2\alpha-2+\eta+i\infty} \zeta(s+1)\Phi(s)\frac{u^s}{s(s+1)(s+2)}\,ds \tag{52}$$

$$= \frac{\Phi(-1)}{2u} - \frac{1}{2}I^*(u), \quad \text{say,}$$

because the abscissa of (absolute) convergence of $\Phi(s)$ does not exceed $2\alpha-2$

by (50) and Lemma 2. Next, let us express $\Phi(s)$ in $I^*(u)$ as ([5])

$$\sum_{k\le u}\frac{N(k)}{k^{1+s}}+\sum_{k>u}\frac{N(k)}{k^{1+s}}=\Phi_1(s)+\Phi_2(s),\quad\text{say,}$$

denoting the contributions to $I^*(u)$ due to $\Phi_1(s)$ and $\Phi_2(s)$ as $I_1(u)$ and $I_2(u)$, respectively. Accordingly, since Lemma 2 implies that

$$|\Phi_2(s)|\le\sum_{k>u}\frac{N(k)}{k^{2\alpha-1+\eta}}=O(u^{-\eta}\log^2 2u)$$

on the line of integration, we first have that

$$I_2(u)=O(u^{2\alpha-2+\eta}u^{-\eta}\log^2 2u)=O(u^{2\alpha-2}\log^2 2u)\tag{53}$$

after heeding known bounds for $\zeta(s+1)$ in terms of t. As for $I_1(u)$, we move the line of integration to $\sigma=2\alpha-2-\eta>-2$ (where now η denotes a positive number less than 2α), on which

$$|\Phi_1(s)|\le\sum_{k\le u}\frac{N(k)}{k^{2\alpha-1-\eta}}=O(u^{\eta}\log^2 2u)$$

with the consequence that

$$I_1(u)=O(u^{2\alpha-2-\eta}u^{\eta}\log^2 2u)=O(u^{2\alpha-2}\log^2 2u)\tag{54}$$

by reasoning like that used before. Hence, in conclusion, we first have that

$$T^*(u)=\frac{1}{2}\Phi(0)\log u+\frac{1}{2}B+\frac{C_1}{2u}-\frac{1}{2}I^*(u)\tag{55}$$

by (52) and the value of $\Phi(-1)$ furnished by (26), while also

$$I^*(u)=O(u^{2\alpha-2}\log^2 2u)\tag{56}$$

by (53) and (54).

An improved version of Theorem 2 in X emerges at once in the present circumstances because Criterion V_1 is readily seen to be stronger than the previously used hypothesis of X(3). Indeed, leaving all work in X unaltered save that appertaining to the remainder term in the unstated analogue of IX(32), we find the constant $D_1=C-C_1$ is zero by (48) and then replace the main term in the formula by

$$x^2I^*(x/Q)=O(Q^{2-2\alpha}x^{2\alpha}\log^2(2x/Q))$$

in virtue of (56). Thus, in recognition of Criterion V_1, the first formula in Theorem 2 of X now reads as

$$G(x,Q)=O(Q^{2-2\alpha}x^{2\alpha}\log^2(2x/Q))+O(x^2\log^{-A}x)\tag{57}$$

([5]) It might seem more natural to interchange the meanings of $\Phi_1(s)$ and $\Phi_2(s)$ but in so doing we would violate the definition of the former in IX.

and hence as

$$G(x,Q) = O(Q^{2-2\alpha}x^{2\alpha}\log^2(2x/Q))$$

for ([6]) $x\log^{-A}x < Q \le x$, which result is not inconsistent with the expectation that at the very least $E(x;a,k)$ is usually not more than $O\{(x/k)^{\alpha}\log(2x/k)\}$ when $k > x\log^{-A}x$.

Alongside (56), there is another property of the function $I^*(u)$ that can be usefully drawn from the above analysis. For any given number $u \ge 1$, let $Q = x/u$ and suppose that $x \to \infty$. Then (57) can be expressed as

$$\frac{1}{x^2}G(x,x/u) = I^*(u) + O(\log^{-A}x)$$

so that

$$\lim_{x\to\infty}\frac{1}{x^2}G(x,x/u) = I^*(u).$$

Hence, since $G(x,x/u_1) \ge G(x,x/u_2) \ge 0$ for $1 \le u_1 < u_2$, we deduce that *$I^*(u)$ is a non-negative decreasing function of u for $u \ge 1$.*

But what has so far been accomplished falls short of what is attainable because we have only taken cognizance of the effect of Criterion V_1 on part of the analysis of IX and X. To take full account of what is available will be the task of the next section.

5. The stronger generalized Barban–Montgomery theorems. To take full advantage of the power of Criterion V_1 we must radically reappraise several other aspects of X in the light of equation (49), including the preparations for a revised Barban–Davenport–Halberstam theorem that will supplement Theorem I of X as a necessary portal to the treatment.

We pick up the analysis with the sums $\Psi_x(k)$ and

$$\sum_{k\le Q}\Psi_x(k) = \sum_{lm\le Q}\frac{W(x,l)}{lm}$$

in X(9) and X(10), where

$$W(x,l) = \sum_{0<a\le l} w(a,l)S(x;a,l)$$

as in X(15). Secondly, by Criterion V_1, we have in place of X(16) the equation

$$W(x,l) = xN(l) + O\biggl(\frac{x^{\alpha}}{l^{\alpha}}\sum_{0<a\le l}|w(a,l)|\biggr) \tag{58}$$

for $l \le x^{1/2}$, while always

$$W(x,l) = xN(l) + O\biggl(\frac{x}{l}\sum_{0<a\le l}|w(a,l)|\biggr) \tag{59}$$

([6]) Here we use (56) with $(2-2\alpha)A$ replacing A.

as before. Hence, if we follow the derivations of X(17) and X(19) (assuming as always that $Q \le x$), we see that for ([7]) $\xi_1 \le x^{1/2}$

$$(60)\quad \sum_{k\le Q}\Psi_x(k) = x\sum_{k\le Q}M(k) + O\Big(x^\alpha \log x \sum_{l\le\xi_1}\frac{1}{l^{1+\alpha}}\sum_{0<a\le l}|w(a,l)|\Big)$$
$$+ O\Big(x\log x\sum_{\xi_1<l\le Q}\frac{1}{l^2}\sum_{0<a\le l}|w(a,l)|\Big)$$
$$= x\sum_{k\le Q}M(k) + O\Big(x^\alpha\log x\sum\nolimits_1\Big) + O\Big(x\log x\sum\nolimits_2\Big),\quad \text{say,}$$

in which the sums $\sum_1$ and $\sum_2$ are estimated after we have been reminded of (23) and Lemma 2. Thence, by the Cauchy–Schwarz inequality,

$$\sum\nolimits_1 = \sum_{l\le\xi_1}\frac{1}{l}\cdot\frac{1}{l^\alpha}\sum_{0<\alpha\le l}|w(a,l)| \le \Big(\sum_{l\le\xi_1}\frac{1}{l}\Big)^{1/2}\Big(\sum_{l\le\xi_1}l^{1-2\alpha}N(l)\Big)^{1/2}$$
$$= O\{(\log 2\xi_1\cdot\log^3 2\xi_1)^{1/2}\} = O(\log^2 x)$$

and

$$\sum\nolimits_2 \le \sum_{l>\xi_1} l^{\alpha/2-3/2}l^{-\alpha/2-1/2}\sum_{0<a\le l}|w(a,l)| \le \Big(\sum_{l>\xi_1}l^{\alpha-2}\Big)^{1/2}\Big(\sum_{l>\xi_1}l^{-\alpha}N(l)\Big)^{1/2}$$
$$= O\{(\xi_1^{\alpha-1}\xi_1^{\alpha-1}\log^2 2\xi_1)^{1/2}\} = O(\xi_1^{\alpha-1}\log 2\xi_1),$$

from which and (60) we deduce the formula

$$(61)\qquad \sum_{k\le Q}\Psi_x(k) = x\sum_{k\le Q}M(k) + O(x^{1/2+\alpha/2}\log^2 x)$$

on choosing ξ_1 to be $x^{1/2}$. Consequently, under our present circumstances, the quantity $Z_x(k)$ in X(21) obeys the estimate

$$(62)\qquad \sum_{k\le Q}Z_x(k) = O(x^{\alpha/2+3/2}\log^2 x)$$

whenever $Q \le x$; also, before we go on to other elements of the analysis, we should note for future reference that the right side of this could be changed into

$$(63)\qquad O(x^{2\alpha/3+4/3}\log^{2-\beta(1-\alpha)} x)$$

if the range of applicability of (29) were stretched to $k \le x^{2/3}\log^\beta x$.

The next constituent that requires re-examination is the first sum on the right side of (60). Abandoning the methods suggested in IX and X as being

([7]) We should note that in the last term of the first line of X(17) we failed to divide by the l in the denominator on the right of (59) above. This did not, of course, invalidate the succeeding estimations, although such a division is essential in the present context.

now unsuitable, we start with the first equation on p. 10 of X to obtain

$$(64)\qquad \sum_{k\le u} M(k) = \sum_{lm\le u} \frac{N(l)}{lm} = \sum_{l\le u} \frac{N(l)}{l} \sum_{m\le u/l} \frac{1}{m}$$
$$= \sum_{l\le u} \frac{N(l)}{l}\Big(\log\frac{u}{l}+\gamma\Big) + O\Big(\frac{1}{u}\sum_{l\le u} N(l)\Big)$$
$$= \log u \sum_{l=1}^{\infty} \frac{N(l)}{l} - \sum_{l=1}^{\infty}\frac{N(l)}{l}(\log l - \gamma)$$
$$+ O\Big(\sum_{l\ge u}\frac{N(l)\log 2l}{l}\Big) + O\Big(\frac{1}{u}\sum_{l=1}^{\infty} N(l)\Big)$$
$$= \Phi(0)\log u + B_1 + O(u^{2\alpha-2}\log^3 2u) + O(1/u)$$
$$= \Phi(0)\log u + B_1 + O(1/u)$$

by (50) and a variant of Lemma 2. Furthermore, for the function

$$(65)\qquad M_\xi(k) = \frac{1}{k}\sum_{\substack{l|k\\ l>\xi}} N(l),$$

we shall require the estimate

$$(66)\qquad \sum_{k\le u} M_\xi(k) = O(\xi^{2\alpha-2}\log^3 2\xi)$$

that is proved in like manner.

An analogue of $\Psi_x(k)$ that arose in IX and X could be handled without trouble there because it was involved in a summation over a range as short as an arbitrary power of $\log x$. More delicacy, however, is now requisite and the sum

$$(67)\qquad {\sum}_3 = \sum_{k<v}\sum_{s\le x-kx/v}(x-kx/v-s)f(s,k) \qquad (v\le x)$$

in question must be treated by an appropriate modification of our previous method that will bring in the integral

$$W_1(v_1,l) = \sum_{s\le v_1}(v_1-s)w(s,l)$$

of the sum $W(u,l)$ appearing above and in X(14). But, ere we sketch the analysis employed, let us note that a potential source of difficulty is that Criterion V_1 of itself does not imply a similar one for the Césaro means of the sums $S(x;a,k)$.

To take care of this point and, especially, of the possibility that v_1 and u in the integral

$$W_1(v_1, l) = \int_0^{v_1} W(u, l)\, du$$

may be less than l, we must where necessary modify the foundations laid down in X by constructing them from the inequality

$$S(x; a, k) = O\left(\frac{x}{k}\right) + O(1) \tag{68}$$

in place of X(8). Then, as the integrand in $W_1(v_1, l)$ is estimated by (58) and the appropriate analogue of (59), we have

$$\begin{aligned} W_1(v_1, l) &= \frac{1}{2}v_1^2 N(l) + O\biggl(\int_{l^2}^{\max(v_1, l^2)} \frac{u^\alpha}{l^\alpha} \sum_{0<a\le l} |w(a, l)|\, du\biggr) \\ &\quad + O\biggl\{\int_0^{\min(v_1, l^2)} \Bigl(\frac{u}{l} + 1\Bigr) \sum_{0<a\le l} |w(a, l)|\, du\biggr\} \\ &= \frac{1}{2}v_1^2 N(l) + O\biggl(\frac{v_1^{1+\alpha}}{l^\alpha} \sum_{0<a\le l} |w(a, l)|\biggr) + O\Bigl(l^3 \sum_{0<a\le l} |w(a, l)|\Bigr) \end{aligned}$$

by a slightly eccentric procedure that caters for both cases $v_1 \ge l^2$ and $v_1 < l^2$ simultaneously. Consequently, starting at (67) and imitating the way (60) was inferred, we discover that

$$\begin{aligned} \sum\nolimits_3 &= \sum_{k<v} \sum_{s\le x-kx/v} (x - kx/v - s)\frac{1}{k} \sum_{lm=k} w(s, l) \\ &= \sum_{lm<v} \frac{1}{lm} \sum_{s\le x-lmx/v} (x - lmx/v - s) w(s, l) \\ &= \frac{1}{2}\sum_{k<v} (x - kx/v)^2 M(k) + O\biggl(x^{1+\alpha} \sum_{lm\le v} \frac{1}{l^{1+\alpha}} \sum_{0<a\le l} |w(a, l)|\biggr) \\ &\quad + O\biggl(\sum_{lm\le v} \frac{l^2}{m} \sum_{0<a\le l} |w(a, l)|\biggr) \\ &= x^2 T^*(v) + O\biggl(x^{1+\alpha} \log x \sum_{l\le v} \frac{1}{l^{1+\alpha}} \sum_{0<a\le l} |w(a, l)|\biggr) \\ &\quad + O\Bigl(\log x \sum_{l\le v} l^2 \sum_{0<a\le l} |w(a, l)|\Bigr) \end{aligned}$$

in the notation of IX(17) as modified on p. 10 of X. In this the inner sums are $O(\log^2 x)$ and $O(v^{3+\alpha} \log^2 x)$ by the method for estimating $\sum_1$ and $\sum_2$, whence we decide that

$$\sum\nolimits_3 = x^2T^*(v) + O(x^{1+\alpha}\log^3 x) + O(v^{3+\alpha}\log^3 x). \tag{69}$$

We are now primed to obtain the revised intermediate generalized Barban–Davenport–Halberstam theorem and then to deduce via this our strengthened Barban–Montgomery type theorem. Let us then take the special case

$$H(x,k) = \frac{1}{k}\sum_{\substack{q\mid k\\ q\le\xi}}\{V_x(q) - x^2N(q)\} + \frac{1}{k}\sum_{\substack{q\mid k\\ q>\xi}} V_x(q) - x^2M_\xi(k) + Z_x(k) \tag{70}$$

of X(25), supposing that $\xi = x^{1/2}$ and deducing first that

$$\sum_{q\mid k}\{V_x(q) - x^2N(q)\} = kH(x,k) - kZ_x(k) = O(k^{2-2\alpha}x^{2\alpha}) - kZ_x(k)$$

for $k \le x^{1/2}$ by (65) and Criterion V_1. This, by the Möbius inversion formula, implies that

$$\begin{aligned} V_x(q) - x^2N(q) &= O\Big(x^{2\alpha}\sum_{d\mid q} d^{2-2\alpha}\Big) - \sum_{d\mid q}\mu\Big(\frac{q}{d}\Big)dZ_x(d) \\ &= O(x^{2\alpha}q^{2-2\alpha}) - \sum_{d\mid q}\mu\Big(\frac{d}{q}\Big)dZ_x(d) \end{aligned}$$

for $q \le x^{1/2}$, wherefore the first term on the right of (70) transforms into

$$\begin{aligned} O\Big(\frac{x^{2\alpha}}{k}\sum_{\substack{q\mid k\\ q\le\xi}} q^{2-2\alpha}\Big) - \frac{1}{k}\sum_{\substack{dd_1d_2=k\\ dd_1\le\xi}} dZ_x(d)\mu(d_1) \\ = O\Big(\frac{x^{1+\alpha}d(k)}{k}\Big) - \sum_{\substack{dd_1d_2=k\\ dd_1\le\xi}} \frac{Z_x(d)\mu(d_1)}{d_1d_2}. \end{aligned}$$

Thence, deploying the consequential expression for $H(x,k)$, summing over k, and then using (66) and (62), we arrive at the equation

$$\begin{aligned} G(x,Q) &= O(x^{1+\alpha}\log^2 x) - \sum_{\substack{dd_1d_2\le Q\\ dd_1\le\xi}} \frac{Z_x(d)\mu(d_1)}{d_1d_2} + \sum_{k\le Q}\frac{1}{k}\sum_{\substack{q\mid k\\ q>\xi}} V_x(q) \\ &\quad - x^2\sum_{k\le Q} M_\xi(k) + \sum_{k\le Q} Z_x(k) \\ &= -\sum_{\substack{dd_1d_2\le Q\\ dd_1\le\xi}} \frac{Z_x(d)\mu(d_1)}{d_1d_2} + O\Big(\sum_{\xi<q\le Q}\frac{1}{q}\log\frac{2Q}{q}V_x(q)\Big) \\ &\quad + O(x^{1+\alpha}\log^2 x) + O(x^{1+\alpha}\log^3 x) + O(x^{\alpha/2+3/2}\log^2 x) \end{aligned}$$

$$= -\sum_{\substack{dd_1d_2\le Q\\ dd_1\le\xi}} \frac{Z_x(d)\mu(d_1)}{d_1d_2} + O\biggl(\sum_{\xi<q\le Q}\frac{1}{q}\log\frac{2Q}{q}V_x(q)\biggr)$$
$$+ O(x^{\alpha/2+3/2}\log^2 x),$$

in which the second term is

$$O\biggl\{\biggl(Q+\frac{x}{\xi}\log Q\biggr)x\biggr\} = O(Qx) + O(x^{3/2}\log x)$$

by the large sieve inequality and in which the first term is

$$-\sum_{d_1d_2\le Q}\frac{\mu(d_1)}{d_1d_2}\sum_{d\le\min(Q/d_1d_2,\xi/d_1)} Z_x(d) = O\biggl(x^{\alpha/2+3/2}\log^2 x\sum_{d_1,d_2\le Q}\frac{1}{d_1d_2}\biggr)$$
$$= O(x^{\alpha/2+3/2}\log^4 x)$$

by another application of (62). Hence, under Criterion V_1, we have the generalized Barban–Davenport–Halberstam theorem of X in the improved form

$$G(x,Q) = O(Qx) + O(x^{\alpha/2+3/2}\log^4 x). \tag{71}$$

In particular, we observe from this that

$$G(x,Q) = O(Qx)$$

whenever $x^{3/4}\le Q\le x$.

To complete the proof we now shun (70) and go back to its antecedent (21) of X, it being appropriate as in X to use the notation $T(u)$ for the sum

$$\sum_{k\le u} M(k).$$

Then, summing this equation over the range $Q_1<k\le Q_2$ for choices of Q_1,Q_2 to be so specified later that

$$x^{1/2} < Q_1 < Q_2 \le x, \tag{72}$$

we obtain through (62) the equality

$$G(x;Q_1,Q_2) = \sum_{Q_1<k\le Q_2}\sum_{0<a\le k} S^2(x;a,k) - x^2\{T(Q_2)-T(Q_1)\}$$
$$+ \sum_{Q_1<k\le Q_2} Z_x(k)$$
$$= \mathit{\Gamma}(x;Q_1,Q_2) - x^2\{T(Q_2)-T(Q_1)\} + O(x^{\alpha/2+3/2}\log^2 x)$$

as a stronger version of an equation on p. 9 of X, whence IX(11) takes the improved form

$$G(x;Q_1,Q_2) = C(Q_2-Q_1)x + 2J(x;Q_1,Q_2) + O(Q_2x^\alpha) - x^2\Phi(0)\log\frac{Q_2}{Q_1} + O\left(\frac{x^2}{Q_1}\right) + O(x^{\alpha/2+3/2}\log^2 x) = C(Q_2-Q_1)x + 2J(x;Q_1,Q_2) - x^2\Phi(0)\log\frac{Q_2}{Q_1} + O\left(\frac{x^2}{Q_1}\right) + O(x^{\alpha/2+3/2}\log^2 x) \tag{73}$$

in the light of (64) and IX(12). Next, if $J(x,Q)$ be defined as in IX where Q is either Q_1 or Q_2, then by analogy with IX

$$J(x,Q) = \sum_{l<x/Q}\ \sum_{s'<x-lQ}\ \sum_{\substack{s'+lQ<s\le x\\ s\equiv s',\,\mathrm{mod}\,l}} 1,$$

wherein the innermost sum equals

$$(x-lQ-s')f(s',l) + O\{(x/l)^\alpha\}$$

by Criterion V_1 because the lower limit $s'+lQ$ of summation exceeds $lQ > lx^{1/2} > l^2$. The remainder term in this being responsible for a contribution

$$O\left(x^{1+\alpha}\sum_{l<x/Q}\frac{1}{l^\alpha}\right) = O\left(\frac{x^2}{Q^{1-\alpha}}\right) \tag{74}$$

to $J(x,Q)$, we thus reach the sum

$$\sum_{l<x/Q}\ \sum_{s'<x-lQ}(x-lQ-s')f(s',l) = x^2T^*(x/Q) + O(x^{1+\alpha}\log^3 x) + O\left(\frac{x^{3+\alpha}\log^3 x}{Q^{3+\alpha}}\right) \tag{75}$$

that (67) and (69) shew to be the other part of $J(x,Q)$. Then, in summation of what we have so far accomplished in this paragraph, we deduce that

$$\begin{aligned} G(x;Q_1,Q_2) &= C(Q_2-Q_1)x + 2x^2(T^*(x/Q_1) - T^*(x/Q_2)) \\ &\quad - x^2\Phi(0)\log(Q_2/Q_1) \\ &\quad + O\left(\frac{x^2}{Q_1^{1-\alpha}}\right) + O(x^{\alpha/2+3/2}\log^2 x) + O\left(\frac{x^{3+\alpha}\log^3 x}{Q_1^{3+\alpha}}\right) \\ &= (C-C_1)(Q_2-Q_1)x + x^2(I^*(x/Q_2) - I^*(x/Q_1)) \\ &\quad + O(x^{\alpha/2+3/2}\log^2 x) \\ &= x^2(I^*(x/Q_2) - I^*(x/Q_1)) + O(x^{\alpha/2+3/2}\log^2 x) \end{aligned} \tag{76}$$

by using in succession (73)–(75), (55), (48), and (71).

If we combine the above result with the Barban–Davenport–Halberstam inequality (71), we get

$$G(x,Q_2)=O(Q_2^{2-2\alpha}x^{2\alpha}\log^2(2x/Q_2))+O(Q_1x)+O(x^{\alpha/2+3/2}\log^4 x)$$

because of (56). All that remains is to seek the range of Q_2 for which the first term on the right of this estimate formally dominates. Here the second term can be absorbed in the first if

$$Q_1x=A_6Q_2^{2-2\alpha}x^{2\alpha}\log^2(2x/Q_2),$$

which certainly entails the requirement $Q_1<Q_2$ for a suitably small A_6. Also the value of Q_1 must exceed $x^{1/2}$ by (72) so that the second term can be forgotten when $Q_2^{2-2\alpha}x^{2\alpha-1}>A_7x^{1/2}$ and surely therefore when $Q_2>x^{3/4}\log^2 x$, in which event it is easily confirmed that the first term also swallows up the third. Hence we have established

THEOREM 1. *Under Criterion* V_1, *we have*

$$G(x,Q)=O(Q^{2-2\alpha}x^{2\alpha}\log^2(2x/Q))$$

for $Q>x^{3/4}\log^2 x$.

6. Formula for s-twins. As stated in the introduction, a feature of sequences s satisfying Criterion V_1 is that it is possible to establish an asymptotic formula for the number $\nu_c(x)$ of pairs of numbers s,s' that differ by a given positive number c. However, the form of Criterion V_1 adopted lends itself primarily to the study of the pairs s,s' satisfying

$$y<s,s'\le 2y,\qquad s-s'=c\qquad (0<c<y) \tag{77}$$

having cardinality $\nu_c^*(y)$, say, from whose asymptotic formula we can derive an expression for $\nu_c(x)$ by a simple argument. Being of interest in itself, this formula for $\nu_c(x)$ is stated in Theorem 2 below but is not suitable in its present form for our primary application to $H(x,k)$ because uniformity in c may be lost in the transition from [8] $\nu_c^*(y)$. We shall therefore proceed directly to $H(x,k)$ from our conclusions on $\nu_c^*(y)$.

Our procedure is to follow as far as possible the earlier analysis of Section 3, starting with the equation

$$\nu_c^*(y)=\int_0^1|F_1(\theta,y)|^2e^{-2\pi ic\theta}\,d\theta$$

and then using at the appropriate place the knowledge of $R(k)$ that was earlier gained. The integrand being no longer always positive, we use the same Farey series as before but now split up the range of integration, mod 1,

(8) The most obvious way to obtain a fully satisfactory formula for $\nu_c(x)$ is to adopt the hypothesis stated in footnote (4).

in the customary manner into the non-overlapping arcs

$$-\vartheta'_{h,k} \le \phi - h/k \le \vartheta_{h,k} \qquad (1/(2Mk) \le \vartheta_{h,k}, \vartheta'_{h,k} \le 1/(Mk))$$

so that we have

$$\begin{aligned}(78)\qquad \nu_c^*(y) &= \sum_{k\le M} \sum_{\substack{0<h\le k\\(h,k)=1}} e^{-2\pi ich/k} \int_{-\vartheta'_{h,k}}^{\vartheta_{h,k}} |F_1(h/k+\phi)|^2 e^{-2\pi ic\phi}\, d\phi \\ &= \sum_{k\le M} T^*(y,k), \quad \text{say,}\end{aligned}$$

as an analogue of (54). Next, since

$$\begin{aligned}|F_1(h/k+\phi,y)|^2 = |P(h,k)|^2 |J(x,\phi)|^2 &\\ + |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|^2 &\\ + O(|P(h,k)| \cdot |J(x,\phi)| &\\ \times |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|)&\end{aligned}$$

by formula (35) for $F_1(h/k+\phi,y)$, we deduce with the aid of the Cauchy–Schwarz inequality that [9]

$$\begin{aligned}(79)\qquad T^*(y,k) &\\ = \sum_{\substack{0<h\le k\\(h,k)=1}} |P(h,k)|^2 \int_{-\vartheta'_{h,k}}^{\vartheta_{h,k}} |J(\phi,y)|^2 e^{-2\pi ic\phi}\, d\phi &\\ + O\Big(\sum_{\substack{0<h\le k\\(h,k)=1}} \int_{-1/(Mk)}^{1/(Mk)} |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|^2\, d\phi\Big) &\\ + O\Big\{\Big(R(k) \int_{-1/(Mk)}^{1/(Mk)} |J(\phi,y)|^2\, d\phi\Big)^{1/2} &\\ \times \Big(\sum_{\substack{0<h\le k\\(h,k)=1}} \int_{-1/(Mk)}^{1/(Mk)} |P_2(y;h,k;\phi) - 2\pi i\phi P_3(y;h,k;\phi)|^2\, d\phi\Big)^{1/2}\Big\} &\\ = \sum_{\substack{0<h\le k\\(h,k)=1}} |P(h,k)|^2 \int_{-\vartheta'_{h,k}}^{\vartheta_{h,k}} |J(\phi,y)|^2 e^{-2\pi ic\phi}\, d\phi &\\ + O\{T_2(y,k)\} + O\{T_1^{1/2}(y,k) T_2^{1/2}(y,k)\}&\end{aligned}$$

([9]) The use of Minkowski's inequality is less appropriate here than it was in Section 3.

in the notation of (36). In the first term on the last line the limits on the integration may be expunged provided that we introduce a compensating term

$$O\Bigl(\sum_{\substack{0<h\le k\\(h,k)=1}}|P(h,k)|^2\int\limits_{1/(2MK)}^{\infty}\frac{d\phi}{\phi^2}\Bigr)=O(MkR(k)),$$

while estimates for $T_1(y,k)$ and $T_2(k,y)$ are supplied by (37) and (41). Consequently, exploiting the Fourier integral theorem, we first see that

$$T^*(y,k)=(y-c)\sum_{\substack{0<h\le k\\(h,k)=1}}|P(h,k)|^2e^{-2\pi ihc/k}+O(MkR(k))$$
$$+O\Bigl(\frac{y^{2\alpha}}{k^{2\alpha}}\Bigr)+O\Bigl(\frac{y^{\alpha+1/2}R^{1/2}(k)}{k^{\alpha}}\Bigr)$$

and then deduce from (78) that

$$\nu_c^*(y)=(y-c)\sum_{k\le M}\sum_{\substack{0<h\le k\\(h,k)=1}}|P(h,k)|^2e^{-2\pi ihc/k}+O(y^{2\alpha}M^{1-2\alpha})$$
$$+O\Bigl(M\sum_{k\le M}kR(k)\Bigr)+O\Bigl(y^{\alpha+1/2}\sum_{k\le M}\frac{R^{1/2}(k)}{k^{\alpha}}\Bigr).$$

Here, in contrast with the corresponding situation we reached in the previous analysis we are paralleling, we know the behaviour of the sums containing $R(k)=N(k)$ because of Lemma 2, and we therefore infer that

$$\begin{aligned}(80)\qquad \nu_c^*(y)&=(y-c)\sum_{k=1}^{\infty}\sum_{\substack{0<h\le k\\(h,k)=1}}|P(h,k)|^2e^{-2\pi ihc/k}+O\Bigl(y\sum_{k>M}N(k)\Bigr)\\&\quad+O(y^{2\alpha}M^{1-2\alpha})+O\Bigl(M\sum_{k\le M}kN(k)\Bigr)\\&\quad+O\Bigl(y^{\alpha+1/2}\sum_{k\le M}\frac{N^{1/2}(k)}{k^{\alpha}}\Bigr)\\&=(y-c)\mathfrak{S}(c)+O(yM^{2\alpha-1})+O(y^{2\alpha}M^{1-2\alpha}\log^2 2y)\\&\quad+O(M^{1+2\alpha}\log^2 2y)+O(y^{\alpha+1/2}\log^2 2y)\\&=(y-c)\mathfrak{S}(c)+O(y^{\alpha+1/2}\log^2 y),\quad\text{say},\end{aligned}$$

which equation with its derivation implies

LEMMA 3. *Let $\nu_c^*(y)$ be the number of s-twins associated with condition* (77). *Then, on Hypothesis* V_1, *we have*

$$\nu_c^*(y) = (y-c)\mathfrak{S}(c) + O(y^{1/2+\alpha}\log^2 2y)$$

uniformly for $0 < c < y$, *where*

$$\mathfrak{S}(c) = \sum_{k=1}^{\infty} \sum_{\substack{0<h\le k\\(h,k)=1}} |P(h,k)|^2 e^{-2\pi i hc/k}.$$

An equally valid formula for $\nu_c^*(y)$ is obtained if $\mathfrak{S}(c)$ be replaced by any of its partial sums consisting of all terms corresponding to values of k up to a limit not less than $y^{1/2}$.

The above lemma constitutes a satisfactory preparation for the next section but, as an aside, we mention that it implies the following

THEOREM 2. *If Hypothesis* V_1 *be assumed*, *then*

$$\nu_c(x) = (x-c)\mathfrak{S}(c) + O(x^{1/2+\alpha}\log^2 x) \qquad (c>0)$$

as $x\to\infty$, where the constant implied by the O-notation may depend on c.

7. The behaviour of $H(x,k)$ for primes k. Still adopting Criterion V_1 but also now assuming that condition (5) is in place, we examine the sum $H(x,k)$ defined in (1) for largish individual values of k with emphasis on the case where k is a prime number. That the secondary stipulation (5) or something similar should be imposed seems essential because in its absence there is little likelihood of there being a satisfactory asymptotic formula for $\Psi_k(x)$ in X(9) for large individual k. Yet, before we embark on this part of the work, we should make plain that we do not insist that k be a prime until necessary.

It is not convenient to study $H(x,k)$ directly for reasons associated with our previous remarks about formulae for $\nu_c(x)$ and $\nu_c^*(y)$. We therefore introduce the sum

$$H_1(y,k) = \sum_{0<a\le k} [S(2y;a,k) - S(y;a,k) - g\{k,(a,k)\}y]^2 \tag{81}$$

for $y > k$ and, letting $y_r = 2^{-r-1}x$ and $R = R(k,y) = O(\log y)$ be the greatest value of R for which $y_r \ge k$, employ it and the Cauchy–Schwarz inequality to gain the relation

$$H(x,k) = \sum_{0<a\le k}\Big(\sum_{0\le r\le R}[S(2y_r;a,k) - S(y_r;a,k) - g\{k,(a,k)\}y_r] \\ + S(y_R;a,k) - g\{k,(a,k)\}y_R\Big)^2$$

$$= O\Big\{\Big(\sum_{0\le r\le R}\frac{1}{2^{\alpha r}}\Big)\Big(\sum_{0\le r\le R}2^{\alpha r}H_1(y_r,k)\Big)\Big\} + O(H(y_R,k))$$

$$= O\Big(\sum_{0\le r\le R}2^{\alpha r}H_1(y_r,k)\Big) + O(H(y_R,k)).$$

In the summand of $H(y_R;a,k)$ both $S(y_R,k)$ and $g\{k,(a,k)\}y_R$ are $O(1)$ because of (68) and the given inequality $k \le y_R < 2k$; hence

$$H(y_R,k) = O(k)$$

and

$$H(x,k) = O\Big(\sum_{0\le r\le R}2^{\alpha r}H_1(y_r,k)\Big) + O(k), \tag{82}$$

it therefore being enough to derive a suitable bound for $H_1(y,k)$ when

$$k \le y \le x/2. \tag{83}$$

We do not tarry long over the earlier stages of the treatment of $H_1(y,k)$ because they resemble what has already occurred in [3], IX, and previous parts of this paper. First we decompose $H_1(y,k)$ in the same way as $H(x,k)$ in [3] to yield the equation

$$\begin{aligned} H_1(y,k) = {} & \sum_{0<a\le k}\{S(2y;a,k) - S(y;a,k)\}^2 \\ & - 2y\sum_{\delta|k}g(k,\delta)\sum_{\substack{0<a\le k\\(a,k)=\delta}}\{S(2y;a,k)-S(y;a,k)\} \\ & + y^2\sum_{\delta|k}\phi(k/\delta)g^2(k,\delta) \\ = {} & H_2(y,k) - 2yH_3(y,k) + y^2H_4(k), \quad \text{say}, \end{aligned} \tag{84}$$

and in the first place treat the constituents therein without further restrictions on k. Secondly, with appropriate interpretations of empty sums, the inner sum in $H_3(y,k)$ is shewn to be

$$\sum_{d|k/\delta}\mu(d)\sum_{\substack{s\le x\\ s\equiv 0,\,\mathrm{mod}\,d\delta}}1 = y\sum_{\substack{d\delta|k\\ d\delta\le y^{1/2}}}\mu(d)g(d\delta) + O\Big(y^\alpha\sum_{\substack{d|k/\delta\\ d\delta\le y^{1/2}}}\frac{1}{(d\delta)^\alpha}\Big)$$
$$+ O\Big(y\sum_{\substack{d|k/\delta\\ d\delta>y^{1/2}}}\frac{1}{d\delta}\Big)$$

$$= y \sum_{d\delta|k} \mu(d) g(d\delta) + O\left(\frac{y^{\alpha} d(k)}{\delta^{\alpha}}\right) + O\left(y \sum_{\substack{d|k/\delta \\ d\delta > y^{1/2}}} \frac{1}{d\delta}\right)$$

$$= y \sum_{d\delta|k} \mu(d) g(d\delta) + O(y^{1/2} d(k))$$

by Criterion V_1 if we adopt the notation $g(d\delta)$ of Criterion U of IX and use the relation $g(d\delta) = O(1/(d\delta))$. Then by IX(6) this expression can be replaced by

$$y\phi(k/\delta) g(k, \delta) + O(y^{1/2+\varepsilon})$$

with the implication that

$$H_3(y,k) = y \sum_{\delta|k} \phi(k/\delta) g^2(k,\delta) + O\left(\frac{y^{1/2+\varepsilon}}{k} \sum_{\delta|k} 1\right)$$

$$= yH_4(k) + O\left(\frac{y^{1/2+\varepsilon}}{k}\right),$$

whence

$$H_1(y,k) = H_2(y,k) - y^2 H_4(k) + O\left(\frac{y^{3/2+\varepsilon}}{k}\right) \tag{85}$$

from (84). Also, by what is now a familiar argument, we infer the equation

$$\begin{aligned} H_2(y,k) &= \sum_{0<a\le k} \Big(\sum_{\substack{y<s\le 2y \\ s\equiv a, \bmod k}} 1 \Big)^2 = \sum_{\substack{s-s'=lk \\ y<s,s'\le 2y}} 1 \\ &= S(2y) - S(y) + 2 \sum_{0<l<y/k} \sum_{\substack{y<s,s'\le 2y \\ s-s'=lk}} 1 \\ &= Cy + O(y^{\alpha}) + 2 \sum_{0<l<y/k} \nu^*_{lk}(y) \\ &= Cy + O(y^{\alpha}) + 2H_5(y,k), \qquad \text{say,} \end{aligned} \tag{86}$$

from (18) and the definition of $\nu^*_c(y)$ in the previous section. From this we then go on to refine through (5) the asymptotic formula in Lemma 3, it first being requisite to change the notation in the definition of $\mathfrak{S}(c)$ in (80) in recognition of our having now earmarked the letter k to denote the fixed modulus inherent in $H(x,k)$.

Let us therefore write

$$\mathfrak{S}(c) = \sum_{m=1}^{\infty} \sum_{\substack{0<h\le m \\ (h,m)=1}} |P(h,m)|^2 e^{-2\pi i hc/m} \qquad (c > 0)$$

and size up the impact of the condition (5) on its terms with the help of the Ramanujan sum

$$c_q(n) = \sum_{\substack{0<b\le q\\(b,q)=1}} e^{2\pi inb/q} = \sum_{d|q;\, d|n} \mu\left(\frac{q}{d}\right) d. \tag{87}$$

If $(h,m)=1$, then (25) implies that

$$\begin{aligned} P(h,m) &= \sum_{\delta|m} g(m,\delta) \sum_{\substack{0<b\le m\\(b,m)=\delta}} e^{2\pi ihb/m} = \sum_{\delta|m} g(m,\delta) c_{m/\delta}(h) \\ &= \sum_{\delta|m} g(m,\delta) c_{m/\delta}(1) = P(1,m) \end{aligned}$$

and thus also that

$$P(1,m) = \sum_{\delta|m} \mu\left(\frac{m}{\delta}\right) g(m,\delta) = O\left(\frac{d(m)}{m}\right);$$

hence definition (24) translates into

$$R(m) = \phi(m)P^2(1,m) = O\left(\frac{m^\varepsilon}{m}\right), \tag{88}$$

while the inner sum in the singular series equals

$$P^2(1,m)c_m(c).$$

Consequently the formula in Lemma 3 can be expressed as

$$\nu_c^*(y) = (y-c) \sum_{m\le y^{1/2}} P^2(1,m)c_m(c) + O(y^{\alpha+1/2}\log^2 2y), \tag{89}$$

in which form it is used to evaluate the last term in (86).

To prepare for this evaluation we need an estimate for the sum

$$\sum_{l\le u}(u-l)c_m(lk)$$

for any positive integers k and m. This equals

$$\begin{aligned} &\sum_{l\le u}(u-l) \sum_{d|m;\, d|lk} \mu\left(\frac{m}{d}\right) d \\ &= \sum_{d|m} \mu\left(\frac{m}{d}\right) d \sum_{\substack{l\le u\\ l\equiv 0,\, \mathrm{mod}\, d/(d,k)}} (u-l) \\ &= \frac{1}{2}u^2 \sum_{d|m} \mu\left(\frac{m}{d}\right)(d,k) - \frac{1}{2}u \sum_{d|m} \mu\left(\frac{m}{d}\right) d + O\Big(\sum_{d|m} d^2\Big) \\ &= \frac{1}{2}u^2 \sum_{d|m} \mu\left(\frac{m}{d}\right)(d,k) - \frac{1}{2}u\phi(m) + O(m^2), \end{aligned} \tag{90}$$

in which

$$\sum_{d|m}\mu\left(\frac{m}{d}\right)(d,k)=\begin{cases}\phi(m) & \text{if } m\mid k,\\ 0 & \text{if } m\nmid k.\end{cases}\tag{91}$$

In turning these facts to initial advantage, we first assume merely that k is a given positive integer and $y\to\infty$ so that certainly $y/k>y^{2/3}$. Then, in this framework, (85), (88)–(90) shew that

$$\begin{aligned}H_5(y,k)&=k\sum_{m\le y^{1/2}}P^2(1,m)\sum_{l<y/k}\left(\frac{y}{k}-l\right)c_m(lk)+O\left(\frac{y^{\alpha+3/2}\log^2 y}{k}\right)\\&=\frac{y^2}{2k}\sum_{m|k}P^2(1,m)\phi(m)+O\Big(y\sum_{m\le y^{1/2}}P^2(1,m)\phi(m)\Big)\\&\quad+O\left(\frac{y^{\alpha+3/2}\log^2 y}{k}\right)\\&=\frac{y^2}{2k}\sum_{m|k}P^2(1,m)\phi(m)+O\Big(y\sum_{m=1}^{\infty}R(m)\Big)+O\left(\frac{y^{\alpha+3/2}\log^2 y}{k}\right)\\&=\frac{y^2}{2k}\sum_{m|k}P^2(1,m)\phi(m)+O\left(\frac{y^{\alpha+3/2}\log^2 y}{k}\right)\end{aligned}\tag{92}$$

because of (26) and (88). But Criterion V_1, (81), and (85) imply that $H_2(y,k)\sim y^2H_4(k)$ as $y\to\infty$ so that (91) leads to

$$H_4(k)=\frac{1}{k}\sum_{m|k}P^2(1,m)\phi(m),$$

an identity in k that is not altogether easy to verify directly.

Let us now approach more closely the conditions under which the theorem will be established, supposing that k is a prime number exceeding $x^{1/2}$ and therefore $y^{1/2}$. Then, by (85) and (86), we gain the equation

$$H_1(y,k)=Cy+2H_5(y,k)-\frac{y^2}{k}\sum_{m|k}P^2(1,m)\phi(m)+O(y^{3/2+\varepsilon}/k),\tag{93}$$

to progress from which we estimate $H_5(y,k)$ by modifying its previous treatment. First, we can write the series

$$\sum_{m\le y^{1/2}}P^2(1,m)c_m(lk)\tag{94}$$

in the formula (89) for $\nu^*_{lk}(y)$ as

$$\sum_{m\le y/k}P^2(1,m)c_m(lk)+O\Big(\sum_{y/k<m\le y^{1/2}}P^2(1,m)(m,l)\Big)\tag{95}$$

because $|c_m(lk)| \leq (m, lk)$ and k is a prime greater than the variable m of summation. Next the contribution due to the second portion of this is

$$(96)\quad O\Big(y \sum_{y/k<m\leq y^{1/2}} P^2(1,m) \sum_{l\leq y/k} (m,l)\Big)$$
$$= O\bigg(\frac{y^2}{k} \sum_{y/k<m\leq y^{1/2}} \frac{md(m)}{\phi(m)m} P^2(1,m)\phi(m)\bigg) = O\bigg(\frac{y^2}{k} \sum_{m>y/k} \frac{R(m)}{m^{1-\varepsilon}}\bigg),$$

which by (88) and Lemma 2 equals

$$(97)\quad O\bigg\{\frac{y^2}{k}\Big(\frac{y}{k}\Big)^{2\alpha-2+\varepsilon}\bigg\} = O\{k^{1-2\alpha}y^{2\alpha}(y/k)^{\varepsilon}\},$$

while the remainder term in (89) is responsible for a fourth gift of

$$(98)\quad O\bigg(\frac{y^{\alpha+3/2}\log^2 2y}{k}\bigg)$$

to $H_5(y,k)$. So far as the first term in (95) is concerned, we see by altering the analysis in (92) that it creates an amount

$$(99)\quad \frac{1}{2}\frac{y^2}{k} \sum_{\substack{m\leq y/k \\ m|k}} P^2(1,m)\phi(m) - \frac{1}{2}y \sum_{m\leq y/k} P^2(1,m)\phi(m)$$
$$+ O\Big(k \sum_{m\leq y/k} m^2 P^2(1,m)\Big)$$
$$\leq \frac{1}{2}\frac{y^2}{k} \sum_{m|k} P^2(1,m)\phi(m) - \frac{1}{2}y \sum_{m=1}^{\infty} R(m) + O\Big(y \sum_{m>y/k} R(m)\Big)$$
$$+ O\Big(k \log\log(10y/k) \sum_{m\leq y/k} mR(m)\Big)$$
$$= \frac{1}{2}\frac{y^2}{k} \sum_{m|k} P^2(1,m)\phi(m)$$
$$- \frac{1}{2}y \sum_{m=1}^{\infty} R(m) + O(k^{1-2\alpha}y^{2\alpha}\log^2(2y/k))$$
$$+ O(k^{1-2\alpha}y^{2\alpha}\log^2(2y/k)\log\log(10y/k))$$

because of Lemma 2. Thence, bearing in mind that

$$\sum_{m=1}^{\infty} R(m) = C_1 = C,$$

let us insert these results (97), (98), and (99) into (93) to infer that

$$\begin{aligned} H_1(y,k) &\le Cy - C_1 y + O(k^{1-2\alpha}y^{2\alpha}(y/k)^{\varepsilon}) + O\biggl(\frac{y^{\alpha+3/2+\varepsilon}}{k}\biggr) \\ &= O(k^{1-2\alpha}y^{2\alpha}(y/k)^{\varepsilon}) \end{aligned}$$

provided that $y^{\beta} < k \le y$, where β is any constant between $(3/2–\alpha)/(2(1-\alpha))$ and 1. This condition being stronger than what was previously assumed, we deduce that

$$H_1(y,k) = O(k^{1-2\alpha}y^{2\alpha}(y/k)^{\varepsilon})$$

because $H_1(y,k)$ is non-negative, whence via (82) we quickly obtain

THEOREM 3. *Let β be as above and suppose that k is a prime number between x^{β} and x. Then, for sequences satisfying Criterion* V_1 *and condition* (5), *we have*

$$H(x,k) = O(k^{1-2\alpha}x^{2\alpha}(x/k)^{\varepsilon}),$$

where $H(x,k)$ is defined in (1).

We examine the change in the above theorem that is brought about by our assuming that the sequence s satisfies Criterion S of IX to the effect that the function $g(d)/C = g(d,d)/C$ is multiplicative. It having been shewn in IX that the function $f(b,m)/C$ is then multiplicative in m for each integer b, it follows by standard methods that so is the sum

$$\frac{1}{C}P(1,m) = \frac{1}{C}\sum_{0<b\le m} e^{2\pi i b/m} f(b,m)$$

as defined in (25), whence we draw from (88) the multiplicativity of $R_1(m) = R(m)/C$ as the first implication of our additional supposition. From this we proceed to a new estimation of the tail of the series (94) but, for reasons to be explained, must begin by tightening up the initial calculations previously stemming from the relation $|c_m(lk)| = |c_m(l)| \le (m,l)$, which must be slightly refined by means of the familiar determination

$$c_{p^{\gamma}}(p^{\beta}) = \begin{cases} \phi(p^{\gamma}) & \text{if } \beta \ge \gamma, \\ p^{\gamma-1} & \text{if } \beta = \gamma - 1, \\ 0 & \text{otherwise,} \end{cases} \tag{100}$$

that is a corollary of (87). Let us now write

$$m = \prod_p p^{\gamma}$$

and then set

$$m_1 = \prod_p p^{\gamma-1}, \quad m_2 = \prod_p p$$

with an obvious interpretation for the third entity. Next, having established that we can certainly suppose that l is divisible by m_1 when estimating the sum

$$\sum\nolimits_{4,u} = \sum\nolimits_{4,u,m} = \sum_{l \le u} |c_m(l)|,$$

let us consider the influence of those values of l for which m_3 is the largest divisor of m_2 dividing l/m_1. Then, notwithstanding the possibility that $l/(m_1 m_3)$ and m_3 are not necessarily coprime, the summands in this case do not exceed $m_1 m_3$ by (100) and contribute in all

$$O\Big(m_1 m_3 \sum_{\substack{l \le u \\ l \equiv 0, \bmod m_1 m_3}} 1\Big) = O(u)$$

to $\sum_{4,u}$, the conclusion being that

$$\sum\nolimits_{4,u} = O\Big(u \sum_{m_3 | m_2} 1\Big) = O(2^{\omega(m)} u). \tag{101}$$

Thus $2^{\omega(m)}$ can replace $d(m)$ in the sum in the middle term of (96), an improvement ([10]) that in the prevailing situation is not as insubstantial as might appear.

To process the new sum we start with the associated sum

$$\sum_{m \le u} m 2^{\omega(m)} R_1(m),$$

which, by the multiplicitivity of $R_1(m)$ and then by Lemma 2, equals

$$\begin{aligned} \sum_{\substack{\lambda\mu \le u \\ (\lambda,\mu)=1}} \lambda\mu R_1(\lambda\mu) &= \sum_{\substack{\lambda\mu \le u \\ (\lambda,\mu)=1}} \lambda R_1(\lambda) \mu R_1(\mu) \le \sum_{\lambda \le \mu} \lambda R_1(\lambda) \sum_{\mu \le u/\lambda} \mu R_1(\mu) \\ &= O\Big(u^{2\alpha} \sum_{\lambda \le u} \lambda^{1-2\alpha} R_1(\lambda) \log^2 \frac{2u}{\lambda}\Big) \\ &= O(u^{2\alpha} \log^5 u). \end{aligned} \tag{102}$$

Then, progressing to the sum

$$\sum\nolimits_{5,u} = \sum_{m \le u} \frac{m^2}{\phi(m)} 2^{\omega(m)} R_1(m)$$

more closely connected with (95), we avail ourselves of the multiplicative

([10]) A further slight improvement is possible but does not permit any simplification in the ensuing calculations.

majorant

$$\frac{\pi^2}{6} \sum_{\substack{\lambda\mu=m \\ (\lambda,\mu)=1}} \frac{1}{\lambda_1}$$

of $m/\phi(m)$, where λ_1 is the square-free product of all the primes dividing λ. By means of this and (102), we get

$$\begin{aligned}\textstyle\sum_{5,u} &\leq \frac{\pi^2}{6} \sum_{\substack{\lambda\mu\leq u \\ (\lambda,\mu)=1}} \frac{\lambda\mu}{\lambda_1} 2^{\omega(\lambda)} 2^{\omega(\mu)} R_1(\lambda) R_1(\mu) \\ &\leq \frac{\pi^2}{6} \sum_{\lambda\leq u} \frac{\lambda 2^{\omega(\lambda)} R_1(\lambda)}{\lambda_1} \sum_{\mu\leq u/\lambda} \mu 2^{\omega(\mu)} R_1(\mu) \\ &= O\biggl(u^{2\alpha} \log^5 u \sum_{\lambda\leq u} \frac{\lambda^{1-2\alpha} 2^{\omega(\lambda)} R_1(\lambda)}{\lambda_1}\biggr)\end{aligned}$$

and hence, by (88),

$$\textstyle\sum_{5,u} = O\biggl(u^{2\alpha} \log^5 u \displaystyle\sum_{\lambda\leq u} \frac{1}{\lambda_1 \lambda^\alpha}\biggr) = O(u^{2\alpha} \log^5 u) \tag{103}$$

because $\alpha > 0$ and because the series $\sum_{\lambda=1}^{\infty} 1/(\lambda_1 \lambda^\alpha)$ is seen to be convergent by Euler's product formula. Being

$$\sum_{y/k<m\leq y^{1/2}} \frac{2^{\omega(m)} R_1(m)}{\phi(m)},$$

the replacement sum in (96) is estimated as

$$O\biggl\{\Bigl(\frac{y}{k}\Bigr)^{2\alpha-2} \log^5 \frac{2y}{k}\biggr\}$$

by (103) and partial summation, wherefore we obtain

$$H(x,k) = O\{k^{1-2\alpha} x^{2\alpha} \log^5(2x/k)\}$$

after improving (97) and retaining the estimations in (99) in their initial state. Accordingly, we have reached

Theorem 4. *To the data of Theorem* 3 *let us add the assumption that the sequence s satisfy Criterion* S *of* IX. *Then*

$$H(x,k) = O\{k^{1-2\alpha} x^{2\alpha} \log^5(2x/k)\}.$$

A brief observation on our procedures may be helpful. Without the assumption of Criterion S, the substitution of $2^{\omega(m)}$ for $d(m)$ is not helpful so that we preferred to go to Theorem 3 as quickly as possible without being unnecessarily diverted by the estimation of $\sum_{4,u}$ in (101). But, although normally in problems of multiplicative number theory the functions $2^{\omega(m)}$ and $d(m)$ are virtually indistinguishable in regard to their treatment, the multiplicativity of $R(m)$ can only be effectively exploited in the present context if the divisor function of m occurring can be interpreted through the factorization of m as a product of coprime integers. This is because problems about $R_1(m)$ must be structured so that they avoid too heavy an involvement with such quantities as $R_1(p^2)$, about which Lemma 2 provides little information.

8. The implications of Criterion V_2**.** We shall shew how to obtain a Barban–Montgomery theorem of expected quality for sequences s that satisfy Criterion V_2 to the effect that *inequality* (29) *in Criterion* V_1 *is valid for all* k *up to* $x^{2/3}\log^{\beta} x$, *where* β *will be chosen to be* $4/3$ *so that*

$$\beta < 2/(1-\alpha). \tag{104}$$

We begin with equation (73), in which the final remainder term $O(x^{\alpha/2+3/2}\log^2 x)$ can be replaced by $O(x^{4/3+2\alpha/3}\log^{2-\beta(1-\alpha)} x)$ because of our comments regarding (62) when Criterion V_2 replaces Criterion V_1; consequently a similar substitution is possible in the first line of (76) and thus in the second line also when $Q_1 \geq x^{2/3}\log^{(\frac{1}{1-\alpha})-\frac{1}{2}\beta} x$ because then it is easily verified that

$$x^2/Q_1^{1-\alpha},\, (x^{3+\alpha}\log^3 x)/Q_1^{3+\alpha} < x^{4/3+2\alpha/3}\log^{2-\beta(1-\alpha)} x$$

by (104). Therefore, setting

$$Q_1 = \frac{1}{3}x^{2/3}\log^{\beta} x > x^{2/3}\log^{(\frac{1}{1-\alpha})-\frac{1}{2}\beta} x, \tag{105}$$

we deduce after a short calculation that for $Q \geq Q_1$

$$\begin{aligned} G(x;Q_1,Q) &= x^2(I^*(x/Q) - I^*(x/Q_1)) + O(x^{4/3+2\alpha/3}\log^{2-\beta(1-\alpha)} x) \\ &= x^2(I^*(x/Q) - I^*(x/Q_1)) \\ &\qquad + O(Q^{2-2\alpha}x^{2\alpha}\log^{2-3\beta(1-\alpha)} x), \end{aligned} \tag{106}$$

which with another application of Criterion V_2 serves first as the foundation for estimating $I^*(v)$.

Let $Q = 2Q_1$ so that Q is less than the upper limit of validity for the operation of Criterion V_2. Then, since

$$G(x;Q/2,Q) \leq \sum_{k\leq Q}\sum_{0<a\leq k} E^2(x;a,k) = O(Q^{2-2\alpha}x^{2\alpha}), \tag{107}$$

it is deduced from (106) that

$$I^*(x/Q) - I^*(2x/Q) = O\{(x/Q)^{2\alpha-2}\},$$

which determines the behaviour of $I^*(v)$ as $v \to \infty$. In fact, simply determine x and Q uniquely by $v = \frac{3}{2}x^{1/3}\log^{-\beta} x$ and $Q = \frac{2}{3}x^{2/3}\log^{\beta} x$, where

$x, Q \to \infty$, and infer that

$$I^*(v) - I^*(2v) = O\left(\frac{1}{v^{2-2\alpha}}\right);$$

hence we arrive at the inequality

$$I^*(v) = O\left(\frac{1}{v^{2-2\alpha}} \sum_{\gamma=0}^{\infty} \frac{1}{(2^{1-\alpha})^{2\gamma}}\right) = O\left(\frac{1}{v^{2-2\alpha}}\right) \tag{108}$$

we sought, since the previous inferior inequality ([11]) (55) certainly implies that $I^*(v) \to 0$ as $v \to \infty$.

It is but a short step to the consequential Barban–Montgomery type theorem. If Q be not more that Q_1 as defined by (104), then

$$G(x, Q) = O(Q^{2-2\alpha} x^{2\alpha}) \tag{109}$$

by Criterion V_2. Alternatively, if $Q > Q_1$, we have that

$$G(x, Q) = G(x, Q_1) + G(x; Q_1, Q),$$

in which

$$G(x, Q_1) = O(Q_1^{2-2\alpha} x^{2\alpha}) = O(Q^{2-2\alpha} x^{2\alpha})$$

and

$$G(x; Q_1, Q) = O(Q^{2-2\alpha} x^{2\alpha})$$

by (106) and (108). Hence (109) holds for all Q up to x and we thus have proved

THEOREM 5. *If the sequence s meet the requirements of Criterion* V_2, *then*

$$G(x, Q) = O(Q^{2-2\alpha} x^{2\alpha})$$

for $Q \le x$.

Let us now suppose that we are presented with a sequence conforming to Criterion V_2 and having the additional property that there is an asymptotic formula ([12])

$$G(x, Q) \sim D_2 Q^{2-2\alpha} x^{2\alpha} \tag{110}$$

for a range of Q bounded above by $x^{2/3} \log^\beta x = 2Q_1$. Although typically (110) might be granted for all such Q bounded below by a function of x tending slowly to infinity, the only part of the range we actually need is the segment $Q_1 \le Q \le 2Q_1$. If first $Q = 2Q_1$ as in the deduction of (108), the inequality (107) should be superseded by

$$G(x; Q/2, Q) \sim D_2\left(1 - \frac{1}{2^{2-2\alpha}}\right) Q^{2-2\alpha} x^{2\alpha},$$

([11]) All that is needed here is that $I^*(u) = o(1/u)$, as proved in X.

([12]) Throughout the symbol of asymptotic equivalence and the o-symbol refer to a passage of x/Q to infinity; thus, as always, $x \to \infty$.

the insertion of which in (106) implies that

$$I^*(v) - I^*(2v) \sim D_2\left(1 - \frac{1}{2^{2-2\alpha}}\right)\frac{1}{v^{2-2\alpha}}$$

by the same choice as before of x and Q in terms of v. Thence, since now

$$I^*(v) \sim D_2\left(1 - \frac{1}{2^{2-2\alpha}}\right)\frac{1}{v^{2-2\alpha}}\sum_{\gamma=0}^{\infty}\frac{1}{2^{\gamma(1-\alpha)}} = \frac{D_2}{v^{2-2\alpha}} \tag{111}$$

rather than (108), we conclude from our hypothesis and (106) that

$$\begin{aligned} G(x,Q) &= G(x,Q_1) + G(x;Q_1,Q) \\ &= D_2Q_1^{2-2\alpha}x^{2\alpha} + D_2x^2\left(\frac{Q^{2-2\alpha}}{x^{2-2\alpha}} - \frac{Q_1^{2-2\alpha}}{x^{2-2\alpha}}\right) + o(Q^{2-2\alpha}x^{2\alpha}) \\ &= \{D_2 + o(1)\}Q^{2-2\alpha}x^{2\alpha} \end{aligned}$$

for all $Q > Q_1$. A Barban–Montgomery type theorem with explicit main term thus emerges in the circumstances described.

Other useful consequences follow from the assumption of Criterion V_2. To deduce the one that is perhaps the most interesting, we take the formula

$$\frac{\zeta(s+1)\Phi(s)}{s(s+1)(s+2)} = \int_1^\infty T^*(u)u^{s-1}\,du \qquad (\sigma > 0)$$

that is associated with formula (51) for $T^*(u)$ by means of the Mellin inversion theorem. Substituting in the integral the value of $T^*(u)$ furnished by (55), we have, for $\sigma > 0$ in the first place,

$$\begin{aligned} \frac{\zeta(s+1)\Phi(s)}{s(s+1)(s+2)} &= \frac{1}{2}\Phi(0)\int_1^\infty u^{-s-1}\log u\,du + \frac{1}{2}B\int_1^\infty u^{-s-1}\,du \\ &\quad + \frac{\Phi(-1)}{2}\int_1^\infty u^{-s-2}\,du - \frac{1}{2}\int_1^\infty I^*(u)u^{-s-1}\,du \\ &= \frac{\Phi(0)}{2s^2} + \frac{B}{2s} + \frac{\Phi(-1)}{2(s+1)} - \frac{1}{2}\int_1^\infty I^*(u)u^{-s-1}\,du \end{aligned}$$

and therefore find that the Dirichlet's integral

$$J(s) = -\frac{1}{2}\int_1^\infty I^*(u)u^{-s-1}\,du \tag{112}$$

produces the analytic continuation beyond the abscissa $\sigma = -1$ of the function ([13])

$$\frac{\zeta(s+1)\Phi(s)}{s(s+1)(s+2)} - \frac{\Phi(0)}{2s^2} - \frac{B}{2s} - \frac{\Phi(-1)}{2(s+1)}.$$

([13]) Although it is unnecessary, we can easily verify that the value of B is such that the effects of the poles at $s = 0$ annihilate themselves.

Also, by (108) $J(s)$ is regular and equal to

$$O\bigg(\int_1^\infty \frac{du}{u^{3-2\alpha+\sigma}}\bigg) = O\bigg(\frac{1}{\sigma-(2\alpha-2)}\bigg)$$

for $\sigma > 2\alpha - 2$. Therefore, for $0 < \eta < 1 - 2\alpha$,

$$\Phi(2\alpha - 2 + \eta) = O(1/\eta), \tag{113}$$

from which, on setting $\eta = 1/\log 2u$, we conclude that

$$\sum_{k\le u} N(k)k^{1-2\alpha} = O(\log 2u) \tag{114}$$

after we recall the definition of $\Phi(s)$ in (50). This represents an improvement in the third equation of Lemma 2, the exponent in the logarithm being reduced from 3 to 1; hence the exponents in the first two equations can also be reduced but only to 1.

A sharper estimate in Theorem 4 is a reward for the assumption of the condition V_2. Taking up the improved assessments stemming from (114), we get

THEOREM 6. *Let us replace Criterion* V_1 *by Criterion* V_2 *in the data of Theorem* 4. *Then*

$$H(x,k) = O\{k^{1-2\alpha}x^{2\alpha}\log^2(2x/k)\}.$$

We can even go a little further if once again we also assume the truth of (110), since from (111) and (112) a routine analytical argument would harvest the equation

$$\begin{aligned} 2J(s) &= -D_2\int_1^\infty \frac{du}{u^{s+3-2\alpha}} + \int_1^\infty o\bigg(\frac{1}{u^{s+3-2\alpha}}\bigg)\,du \\ &= \frac{-D_2}{s-(2\alpha-2)} + o\bigg(\frac{1}{s-(2\alpha-2)}\bigg) \end{aligned}$$

as $s \to 2\alpha - 2 + 0$. From this it then follows that

$$\Phi(2\alpha-2+\eta) \sim \frac{(2-2\alpha)(1-2\alpha)\alpha D_2}{-\zeta(2\alpha-1)}\cdot\frac{1}{\eta}$$

as $\eta \to 0$, whence

$$\sum_{k\le u} N(k)k^{1-2\alpha} \sim D_3\log 2u$$

as $u \to \infty$ by a familiar Tauberian theorem. The upshot of this being that the o-symbol can replace the O-symbol in the improved first two parts of Lemma 2 derived under Criterion V_2, we readily deduce a variant of Theorem 6 in which

$$H(x,k) = o\{k^{1-2\alpha}x^{2\alpha}\log^2(2x/k)\}.$$

9. A sequence for which the Barban–Montgomery type theorem has no main term of steady order of magnitude. The sequence s to be considered is constructed by removing from the natural numbers all multiples of primes p' belonging to a particular type of set for which

$$\sum_{p'} \frac{1}{p'} \tag{115}$$

is convergent and equal to a constant C_2 say. This, as will be shewn by a simple method that avoids any intricacies of sieve machinery, meets the conditions of Criteria U and S as originally stated in IX with remainder term $O(x \log^{-A} x)$.

Specializing in two stages the set of primes p' to be adopted, we first suppose that, for any positive number A, the tails of (115) are subject to the relation

$$P(u) = \sum_{p'>u} \frac{1}{p'} = O(\log^{-A} u) \qquad (u \geq 3/2), \tag{116}$$

from which, letting d' denote generally any square-free product (possibly 1) of the sieving primes, we easily deduce that

$$\sum_{d' \leq y} 1 = O(y \log^{-A} 2y) \quad \text{and} \quad \sum_{d'>y} 1/d' = O(\log^{-A} 2y) \qquad (y \geq 1) \tag{117}$$

as follows. Let $A > 1$ be fixed at will and let

$$\sigma_t = \frac{A \log\log 10t}{\log 10t},$$

noting that σ_t is decreasing for $t \geq 3/2$ and that (116) can be assumed with $2A$ instead of A. Then the first sum in (117) does not exceed

$$\begin{aligned} (10y)^{1-\sigma_y} \sum_{d' \leq y} \frac{1}{d'^{1-\sigma_y}} &\leq \frac{10y}{\log^A 10y} \prod_{p' \leq y} \left(1 + \frac{1}{p'^{1-\sigma_y}}\right) \\ &\leq \frac{10y}{\log^A 10y} \exp\left(\sum_{p' \leq y} \frac{p'^{\sigma_y}}{p'}\right), \end{aligned}$$

wherein

$$\begin{aligned} \sum_{p' \leq y} \frac{p'^{\sigma_y}}{p'} &= \int_{3/2}^{y} t^{\sigma_y}\, d\{C_2 - P(t)\} = -\int_{3/2}^{y} t^{\sigma_y}\, dP(t) \\ &\leq \left(\frac{3}{2}\right)^{\sigma_y} C_2 + O\left(\sigma_y \int_{3/2}^{y} \frac{t^{\sigma_t}}{t \log^{2A} t}\, dt\right) \\ &= O(1) + O\left(\int_{3/2}^{\infty} \frac{dt}{t \log^{A} t}\right) = O(1). \end{aligned}$$

Thus the first part of (117) is substantiated; the second part is deduced from this (with $A+1$ instead of A) by partial summation.

The formula for $S(x;a,k)$ is quickly derived. In the usual way, for

$$(a,k)=\delta, \tag{118}$$

we have first that

$$S(x;a,k)=\sum_{\substack{n\le x\\ n\equiv a,\,\mathrm{mod}\,k}}\ \sum_{d'|n}\mu(d')=\sum_{d'\le x}\mu(d')\sum_{\substack{n\le x\\ n\equiv a,\,\mathrm{mod}\,k\\ n\equiv 0,\,\mathrm{mod}\,d'}}1.$$

Next, the two congruences in the last inner sum are compatible if and only if $(k,d')\,|\,a$ and hence if and only if $(k,d')\,|\,\delta$, in which case the solutions in n belong to a single residue class, mod $[k,d']$. Hence

$$\begin{aligned}S(x;a,k)&=\sum_{\substack{d'\le x\\ (k,d')|\delta}}\left(\frac{x(k,d')\mu(d')}{kd'}+O(1)\right)\\ &=\frac{x}{k}\sum_{(k,d')|\delta}\frac{(k,d')\mu(d')}{d'}+O\Bigl(x\sum_{d'\ge x}\frac{1}{d'}\Bigr)+O\Bigl(\sum_{d'\le x}1\Bigr)\\ &=\frac{x}{k}\sum\nolimits_6+O\Bigl(\frac{x}{\log^A x}\Bigr),\qquad\text{say,}\end{aligned}$$

by (117). The terms in $\sum_6$ being multiplicative in d' in an obvious sense, we then have by absolute convergence that

$$\sum\nolimits_6=\prod_{p'|\delta}(1-1)\prod_{p'\nmid\delta;\,p'|k}1\prod_{p'\nmid k}\Bigl(1-\frac{1}{p'}\Bigr)=C\Psi(\delta)\prod_{p'|k}\Bigl(1-\frac{1}{p'}\Bigr)^{-1},$$

where

$$C=\prod_{p'}\Bigl(1-\frac{1}{p'}\Bigr)$$

and $\Psi(\delta)$ is the (trivially) multiplicative function that is 0 or 1 according as δ is or is not divisible by a sieving prime. Therefore, having identified $C\psi(k)=C\Psi(k)/k$ with $g(k,k)$ because of (118), we infer that

$$S(x;a,k)=\frac{Cx\Psi(\delta)}{k}\prod_{p'|k}\Bigl(1-\frac{1}{p'}\Bigr)^{-1}+O(x\log^{-A}x)$$

in accordance with Criteria U and S of IX.

By IX(27) we have in this case that

$$\begin{aligned}\Phi(s)&=\prod_p\left(1+\frac{1}{p-1}\sum_{m=1}^{\infty}\frac{\{p\psi(p^m)-\psi(p^{m-1})\}^2}{p^{ms+1}}\right)\\ &=\prod_{p'}\left(1+\frac{1}{(p'-1)p'^{s+1}}\right)=\sum_{d'}\frac{1}{\phi(d')d'^{s+1}}\end{aligned} \tag{119}$$

so that, in particular,

$$\Phi(-1) = \prod_{p'} \left(1 + \frac{1}{p'-1}\right) = \prod_{p'} \left(1 - \frac{1}{p'}\right)^{-1} = \frac{1}{C}.$$

Thus the sequences produced have the feature that the constant $D_1 = C - C^2\Phi(-1)$ in the formal main term of their Barban–Montgomery type theorem is zero.

Having satisfied a necessary condition for there to be a fluctuation in the true order of magnitude of $G(x,Q)$ for Q fairly close to x, we specialize the situation yet further by insisting that the sieving primes shall form a sequence of such a pronounced lacunary type that the numbers d' are also lacunary. Accordingly, letting ξ_r be a slowly increasing positive sequence tending to infinity, we construct a very rapidly increasing sequence of real numbers $u_1 > 1, u_2, \ldots, u_r, \ldots$ defined iteratively by the recurrence relation

$$u_{r+1} = e^{e^{u_r}}, \tag{120}$$

in terms of which the sieving primes p' are to be just those primes p that lie in intervals $\mathcal{I}_r$ of the type $[u_r, u_r + u_r \log^{-\xi_r} u_r)$. Since obviously

$$\frac{1}{2\log^{\xi_r+1} u_r} < \sum_{p\in\mathcal{I}_r} \frac{1}{p} < \frac{2}{\log^{\xi_r+1} u_r} \qquad (r > r_0)$$

provided ξ_r be chosen to increase sufficiently slowly, we first have the inequalities

$$\begin{aligned}\frac{1}{2\log^{\xi_r+1} u_r} &< \sum_{p'\ge u_r} \frac{1}{p'} < 2\sum_{s\ge r} \frac{1}{\log^{\xi_s+1} u_s} \\ &< 2\sum_{s\ge r} \frac{1}{\log^{\xi_r+1}(u_r^{2^{s-r}})} \\ &< \frac{2}{\log^{\xi_r+1} u_r} \sum_{\gamma=0}^{\infty} \frac{1}{2^\gamma} = \frac{4}{\log^{\xi_r+1} u_r}.\end{aligned} \tag{121}$$

Hence, as $\xi_r > A-1$ for sufficiently large r, we also have either that

$$\sum_{p'>u} \frac{1}{p'} \le \sum_{p'\ge u_r} \frac{1}{p'} < \frac{4}{\log^A u_r} < \frac{5}{\log^A u}$$

when $u > u_0(A)$ lies in an interval $\mathcal{I}_r$ or that

$$\sum_{p'>u} \frac{1}{p'} \le \sum_{p'\ge u_{r+1}} \frac{1}{p'} < \frac{4}{\log^A u_{r+1}} < \frac{4}{\log^A u}$$

when $u > u_0(A)$ lies between consecutive intervals $\mathcal{I}_r$ and $\mathcal{I}_{r+1}$. As a result, we have confirmed that the set of sieving primes p' conforms to the condition (116), on which the earlier work of the section was based.

Next, any square-free product of primes belonging to $\mathcal{I}_r$ does not exceed

$$(2u_r)^{u_r/\log u_r} < e^{2u_r}$$

with the implication that any product d'_r of numbers of type d' having prime factors belonging only to the first r intervals $\mathcal{I}_s$ does not exceed

$$\prod_{s \le r} e^{2u_r} = \exp\Bigl(2\sum_{s\le r} u_s\Bigr) < e^{3u_r} = v_r, \quad \text{say.} \tag{122}$$

In contrast, any number d' not of type d'_r cannot be less than the first prime in $\mathcal{I}_{r+1}$ and is therefore not less than u_{r+1}.

With this structure, we reinterpret conclusion (32) of IX for various values of u defined in terms of the sequence u_r. First, for any $\varepsilon > 0$, let us suppose that

$$v_r^{4/\varepsilon} \le u' < u_{r+1}^{1/2} \qquad (r > r_0(\varepsilon)). \tag{123}$$

Then, in the integral representations of $T^*(u)$ in IX (see also Section 4) for $u = u'$ and $u = u'^2$, we may replace $\Phi(s)$ by

$$\Phi_I(s) = \sum_{n \le v_r} \frac{a_n}{n^s}$$

because of the form of the Dirichlet's series exhibited in (119). Hence, following the pattern of previous analysis and letting B_{v_r} denote the analogue of B, we have

$$\begin{aligned} T^*(u) &= \frac{1}{2}\Phi_I(0)\log u + \frac{1}{2}B_{v_r} + \frac{\Phi_I(-1)}{2u} \\ &\qquad + \frac{1}{2}\int\limits_{\varepsilon/2-2-i\infty}^{\varepsilon/2-2+i\infty} \Phi_I(s)\zeta(s+1)\frac{u^s}{s(s+1)(s+2)}\,ds \\ &= \frac{1}{2}\Phi_I(0)\log u + \frac{1}{2}B_{v_r} + \frac{\Phi_I(-1)}{2u} + O\biggl(\frac{u^{\varepsilon/2}\Phi_1(\varepsilon/2-2)}{u^2}\biggr) \\ &= \frac{1}{2}\Phi_I(0)\log u + \frac{1}{2}B_{v_r} + \frac{\Phi_I(-1)}{2u} + O\Bigl(\frac{1}{u^{2-\varepsilon}}\Bigr), \end{aligned}$$

since

$$\Phi_I(\varepsilon/2-2) < v_r^{1-\varepsilon/2}\Phi_I(-1) < v_r\Phi(-1) = O(u^{\varepsilon/2})$$

by (123). If this be combined with the formula for $I^*(u)$ supplied by (52) and if we write

$$\Phi_{II}(s) = \sum_{n > v_r} \frac{a_n}{n^s} = \sum_{n \ge u_{r+1}} \frac{a_n}{n^s},$$

it follows that

$$\frac{1}{2}I^*(u') - \frac{1}{2}I^*(u'^2) = \frac{\Phi(-1)}{2}\left(\frac{1}{u'} - \frac{1}{u'^2}\right) - \frac{\Phi(0)}{2}\log u'$$
$$- \{T^*(u') - T^*(u'^2)\}$$
$$= \frac{\Phi_{II}(-1)}{2}\left(\frac{1}{u'} - \frac{1}{u'^2}\right) - \frac{\Phi_{II}(0)}{2}\log u' + O\left(\frac{1}{u'^{2-\varepsilon}}\right)$$
$$= \frac{\Phi_{II}(-1)}{2u'} - \frac{\Phi_{II}(0)}{2}\log u' + O\left(\frac{1}{u'^{2-\varepsilon}}\right)$$

and thence that

(124) $$I^*(u') = \frac{\Phi_{II}(-1)}{u'} - \Phi_{II}(0)\log u' + O\left(\frac{1}{u'^{2-\varepsilon}}\right)$$

because $I^*(u'^2) = O(u'^{-2})$ by (55) and IX(32). But

$$\Phi_{II}(0) < \frac{1}{u_{r+1}}\Phi_{II}(-1) < \frac{1}{u'^2}\Phi_{II}(-1),$$

and we therefore conclude that

(125) $$I^*(u') < \frac{\Phi_{II}(-1)}{u'} + O\left(\frac{1}{u'^{2-\varepsilon}}\right)$$

and

(126) $$I^*(u') > \frac{\Phi_{II}(-1)}{u'}\left(1 - \frac{\log u'}{u'}\right) + O\left(\frac{1}{u'^{2-\varepsilon}}\right)$$
$$> \frac{\Phi_{II}(-1)}{2u'} + O\left(\frac{1}{u'^{2-\varepsilon}}\right) \quad (r > r_0).$$

The required result is almost in sight. All numbers d' appearing in the Dirichlet development of $\Phi_{II}(s)$ having at least one prime factor not less than u_{r+1}, we have

$$\Phi_{II}(-1) = \prod_{p' < u_r + u_r \log^{-\xi_r} u_r}\left(1 + \frac{1}{p'}\right)\left\{\prod_{p' \geq u_{r+1}}\left(1 + \frac{1}{p'}\right) - 1\right\}$$

and infer that

(127) $$\Phi_{II}(-1) \simeq \frac{1}{\log^{\xi_{r+1}+1} u_{r+1}}$$

in the light of (121). To profit from this let A be any positive constant as before and first choose u' in (123) so that $u_{r+1}^{1/(2A)} \leq u' < u_{r+1}^{1/2}$. Then (126) and (127) shew that

$$\begin{aligned} I^*(u') &> \frac{A_1}{u' \log^{\xi_{r+1}+1} u_{r+1}} + O\biggl(\frac{1}{u'^{2-\varepsilon}}\biggr) \\ &\ge \frac{A_1}{u' \log^{\xi_{r+1}+1} u' \, (2A)^{\xi_{r+1}+1}} + O\biggl(\frac{1}{u'^{2-\varepsilon}}\biggr) \\ &> \frac{A_1}{u'^{1+\varepsilon}} + O\biggl(\frac{1}{u'^{2-\varepsilon}}\biggr) > \frac{1}{u'^{1+\varepsilon}} \end{aligned}$$

for $r > r_0(\varepsilon, A)$ provided that the rate of increase of ξ_r be sufficiently slow. In contrast, if $v_r^{4/\varepsilon} \le u' < v_r^{4A/\varepsilon}$, then the other inequality (125) leads to

$$I^*(u') < \frac{A_2}{u' \log^{\xi_{r+1}+1} u_{r+1}} + O\biggl(\frac{1}{u'^{2-\varepsilon}}\biggr) < \frac{1}{u'^{2-\varepsilon}}$$

for $r > r_0'(\varepsilon, A)$ because

$$\log^{\xi_{r+1}+1} u_{r+1} = e^{(\xi_{r+1}+1)u_r} = v_r^{\frac{1}{3}(\xi_{r+1}+1)} > u'^{\frac{\varepsilon}{12A}(\xi_{r+1}+1)} > u'$$

by (120) and (122). Thus there are indefinitely large values of v for which an inequality of the type

$$I^*(u) > 1/u^{1+\varepsilon} \tag{128}$$

holds for all u in a range of the type

$$v \le u < v^A, \tag{129}$$

whereas there are other such values of v for which

$$I^*(u) < 1/u^{2-\varepsilon} \tag{130}$$

in a similar range of u.

Finally, being guided by the analysis of Section 4 and remembering that here $D_1 = 0$, we have

$$G(x, Q) = x^2 I^*(x/Q) + O(x^2 \log^{-2A} x),$$

to apply which in the current situation we set $v = \log x$ and choose x so that either (128) or (130) is valid in the range (129). Since the consequential upper and lower bounds for $I^*(x/Q)$ are clearly not perturbed by the addition of the remainder term $O(x^2 \log^{-2A} x)$ when $x \log^{-A} x < Q \le x \log^{-1} x$, we deduce

THEOREM 7. *There are sequences s meeting the conditions of Criterion* U *which have the following features*:

(i) *there are indefinitely large values of x for which*

$$G(x, Q) > Q^{1+\varepsilon} x^{1-\varepsilon}$$

whenever $x \log^{-A} x < Q \le x \log^{-1} x$;

(ii) *there are also indefinitely large values of* x *for which*

$$G(x,Q) < Q^{2-\varepsilon}x^{\varepsilon}$$

whenever $x\log^{-A}x < Q \le x\log^{-1}x$.

The selection of $x\log^{-1}x$ as the upper bound for Q is not crucial, although we must make sure that x/Q does not become too small.

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Received on 12.2.2001 (3972)