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The class number one problem for the non-abelian normal CM-fields of degree 24 and 40

by

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1. Introduction. We fix an algebraic closure of $\mathbb{Q}$. Assume that all the number fields are subfields of the field $\mathbb{C}$ of complex numbers. We let c denote complex conjugation and recall that if $\mathbf{N}$ is a normal CM-field, then c is in the center $Z(\mathbf{G})$ of its Galois group $\mathbf{G}$.

There are 11 possible Galois groups for non-abelian normal CM-fields $\mathbf{N}$ of degree 24: $C_2 \times \mathcal{A}_4$, $\mathrm{SL}_2(\mathbf{F}_3)$, $C_3 \times Q_8$, Q_{24}, $C_3 \rtimes C_8$, D_{24}, $C_2 \times Q_{12}$, $C_2 \times D_{12}$, $C_4 \times D_6$, $C_3 \times D_8$, and $V_{24} = C_3 \rtimes D_8$. For two of them, namely $C_2 \times \mathcal{A}_4$ and $\mathrm{SL}_2(\mathbf{F}_3)$, whose 3-Sylow subgroups are not normal, there are exactly 3 such CM-fields with class number one (see [LLO]). For the other 9 groups, $\mathbf{N}$ contains a normal octic CM-subfield $\mathbf{N}_8$ and the relative class number of $\mathbf{N}_8$ divides that of $\mathbf{N}$ (see [LOO, Thm. 5]). Hence, $\mathbf{N}$ with Galois group $C_3 \times Q_8$ or Q_{24} have even relative class numbers since the quaternion octic CM-fields $\mathbf{N}_8$ have even relative class numbers ([LO2]). Moreover, the relative class numbers of CM-fields $\mathbf{N}$ with Galois group $C_3 \rtimes C_8$ are greater than one (see [Lou4]) and there is only one dihedral CM-field of degree 24 with class number one (see [Lef]). Therefore, it remains to deal with the following 5 groups: $C_2 \times Q_{12}$, $C_2 \times D_{12}$, $C_3 \rtimes D_8$, $C_4 \times D_6$, and $C_3 \times D_8$. We will prove:

THEOREM 1. (1) *There is only one normal CM-field* $\mathbf{N}$ *with Galois group* $C_2 \times D_{12}$ *and class number one*: $\mathbf{N} = \mathbf{K}_3\mathbf{N}_8$ *where* $\mathbf{K}_3$ *is the non-normal cubic field defined by the polynomial* $x^3 - 6x - 2$ *and* $\mathbf{N}_8 = \mathbb{Q}(\sqrt{-3}, \sqrt{-4}, \sqrt{-7})$. *Notice that* $d_{\mathbf{K}_3} = 2^2 \cdot 3^3 \cdot 7$, $d_{\mathbf{N}_8} = 2^8 \cdot 3^4 \cdot 7^4$, *and* $d_{\mathbf{N}} = 2^{32} \cdot 3^{28} \cdot 7^{12}$.

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(2) *There is only one normal CM-field* $\mathbf{N}$ *with Galois group* $C_4 \times D_6$ *and relative class number one*: $\mathbf{K} = \mathbf{K}_3\mathbf{N}_8$ *where* $\mathbf{K}_3$ *is the non-normal totally real cubic field defined by the polynomial* $x^3 - 10x - 10$ *and* $\mathbf{N}_8 = \mathbb{Q}(\exp(2\pi i/5), \sqrt{13})$. *This* $\mathbf{N}$ *has class number one. Notice that* $d_{\mathbf{K}_3} = 2^2 \cdot 5^2 \cdot 13$, $d_{\mathbf{N}_8} = 5^6 \cdot 13^4$, *and* $d_{\mathbf{N}} = 2^{16} \cdot 5^{22} \cdot 13^{12}$.

(3) *There is only one normal CM-field* $\mathbf{N}$ *with Galois group* $C_3 \times D_8$ *and relative class number one*: $\mathbf{K} = \mathbf{N}_3\mathbf{N}_8$ *where* $\mathbf{N}_3$ *is the real cyclic cubic field defined by the polynomial* $x^3 - x^2 - 4x - 1$ *and*

$$\mathbf{N}_8 = \mathbb{Q}(\sqrt{13}, \sqrt{17}, \sqrt{-(9+\sqrt{13})/2})$$

is a dihedral octic CM-field. This $\mathbf{N}$ *has class number one. Notice that* $d_{\mathbf{N}_3} = 13^2$, $d_{\mathbf{N}_8} = 13^4 \cdot 17^4$, *and* $d_{\mathbf{N}} = 13^{20} \cdot 17^{12}$.

(4) *The relative class numbers of normal CM-fields* $\mathbf{N}$ *with Galois group* $C_2 \times Q_{12}$ *or* $C_3 \rtimes D_8$ *are greater than one.*

Let $\mathbf{N}$ be a non-abelian normal CM-fields of degree $8p$ with $p \geq 5$ a prime and let $\mathbf{G} = \mathrm{Gal}(\mathbf{N}/\mathbb{Q})$. Since the Sylow p-subgroup of $\mathbf{G}$ is normal $\mathbf{N}$ is a cyclic extension of degree p of a normal octic CM-field $\mathbf{N}_8$, and the relative class number $h^-_{\mathbf{N}_8}$ of $\mathbf{N}_8$ divides that of $\mathbf{N}$ (see [LOO, Thm. 5]). Let us consider the CM-fields $\mathbf{N}$ of degree 40. There are 11 non-abelian finite groups $\mathbf{G}$ of order 40: $C_5 \rtimes C_8$ (2 groups), $C_2 \times D_{20}$, $C_2 \times Q_{20}$, $C_4 \times D_{10}$, $C_2 \times F_{5,4}$, $C_5 \times D_8$, D_{40}, $V_{40} = C_5 \rtimes D_8$, $C_5 \times Q_8$, and Q_{40}. Moreover, since the center $Z(\mathbf{G})$ of each of these eleven $\mathbf{G}$'s contains an element of order 2, nothing prevents each of these groups from being the Galois group for some non-abelian normal CM-field $\mathbf{N}$ of degree 40. If $\mathbf{G} = D_{40}$ then $h^-_{\mathbf{N}} > 1$ (see [Lef]). If $\mathbf{G} = C_5 \times Q_8$ or Q_{40} then $h^-_{\mathbf{N}_8}$ is even (see [LO1]), hence $h^-_{\mathbf{N}}$ is even. Moreover, if $\mathbf{G} = C_5 \overset{1}{\rtimes} C_8 = \langle a, b : a^5 = b^8 = 1, b^{-1}ab = a^4\rangle = \langle \sigma, \tau : \sigma^{20} = 1, \tau^2 = \sigma^5, \tau^{-1}\sigma\tau = \sigma^9\rangle$ or $C_5 \overset{2}{\rtimes} C_8 = \langle a, b : a^5 = b^8 = 1, b^{-1}ab = a^2\rangle = \langle \sigma, \tau : \sigma^{10} = 1, \tau^4 = \sigma^5, \tau^{-1}\sigma\tau = \sigma^3\rangle$ then analysis similar to that in the proof of [Lou4, Thm. 5] shows $h^-_{\mathbf{N}} > 1$. Therefore, it remains to deal with the following 6 groups: $C_2 \times Q_{20}$, $C_2 \times D_{20}$, $C_5 \rtimes D_8$, $C_4 \times D_{10}$, $C_5 \times D_8$, $C_2 \times F_{5,4} = C_2 \times \langle a, b : a^5 = b^4 = 1, b^{-1}ab = a^2\rangle = \langle \sigma, \tau : \sigma^{10} = \tau^4 = 1, \tau^{-1}\sigma\tau = \sigma^7\rangle$. We will prove:

THEOREM 2. *There is only one normal CM-field* $\mathbf{N}$ *of degree* 40 *with class number one*: $\mathbf{N} = \mathbf{K}_5\mathbf{N}_8$, *where* $\mathbf{K}_5$ *is the non-normal totally real quintic field defined by the polynomial* $x^5 - 10x^3 + 20x + 10$ *and* $\mathbf{N}_8 = \mathbb{Q}(\exp(2\pi i/5), \sqrt{-7})$. *Notice that* $d_{\mathbf{K}_5} = 2^4 \cdot 5^5 \cdot 7^2$, $d_{\mathbf{N}_8} = 5^6 \cdot 7^4$, *and* $d_{\mathbf{N}} = 2^{32} \cdot 5^{46} \cdot 7^{20}$. *Its Galois group* $G(\mathbf{N}/\mathbb{Q})$ *is* $C_2 \times F_{5,4}$.

2. Lattices of subfields. According to the foregoing, it is natural to closely investigate the non-abelian normal CM-fields $\mathbf{N}$ of degree $8p$, $p \geq 3$

an odd prime, with Galois group **G** isomorphic to one of the following five groups:

$$C_2 \times Q_{4p} = \langle \sigma, \tau, u : \sigma^{2p} = \tau^2 = 1,\ u^2 = \sigma^p,\ u^{-1}\sigma u = \sigma^{2p-1},\ \sigma\tau = \tau\sigma,\ \tau u = u\tau \rangle,$$
$$C_2 \times D_{4p} = \langle \sigma, \tau, u : \sigma^{2p} = \tau^2 = u^2 = 1,\ u^{-1}\sigma u = \sigma^{2p-1},\ \sigma\tau = \tau\sigma,\ \tau u = u\tau \rangle,$$
$$C_p \rtimes D_8 = \langle \sigma, \tau, u : \sigma^{2p} = \tau^2 = 1,\ u^2 = \tau,\ u^{-1}\sigma u = \sigma^{2p-1},\ \sigma\tau = \tau\sigma,\ \tau u = u\tau \rangle,$$
$$C_4 \times D_{2p} = \langle \sigma, \tau : \sigma^{4p} = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^{2p-1} \rangle,$$
$$C_p \times D_8 = \langle \sigma, \tau : \sigma^{4p} = \tau^2 = 1,\ \tau^{-1}\sigma\tau = \sigma^{2p+1} \rangle.$$

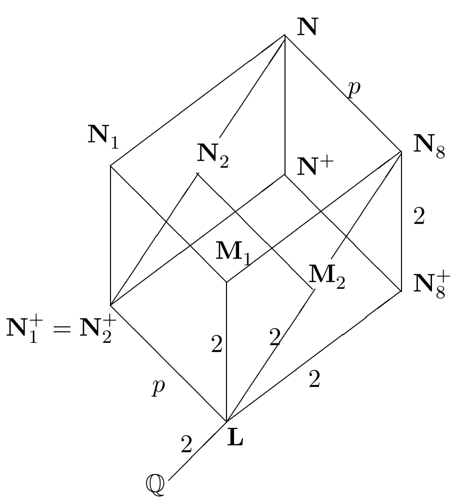
Then we have

Table 1

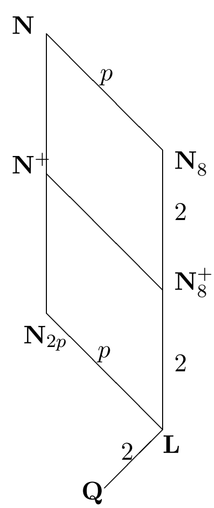
$\mathbf{G}$	$C_2 \times Q_{4p}$	$C_2 \times D_{4p}$	$C_p \rtimes D_8$	$C_4 \times D_{2p}$	$C_p \times D_8$
$Z(\mathbf{G})$	$\langle \sigma^p, \tau \rangle$	$\langle \sigma^p, \tau \rangle$	$\langle \tau \rangle$	$\langle \sigma^p \rangle$	$\langle \sigma^2 \rangle$
$D(\mathbf{G})$	$\langle \sigma^2 \rangle$	$\langle \sigma^2 \rangle$	$\langle \sigma^2\tau \rangle$	$\langle \sigma^4 \rangle$	$\langle \sigma^{2p} \rangle$
c	σ^p or τ	τ	τ	σ^{2p}	σ^{2p}

where $Z(\mathbf{G})$, $D(\mathbf{G})$, and $c \in Z(\mathbf{G})$ denote the center of **G**, the derived subgroup of **G** and the complex conjugation of **G**, respectively.

Lattice I



Lattice II



If $\mathbf{G} = C_2 \times Q_{4p}$, $C_2 \times D_{4p}$, or $C_p \rtimes D_8$, we let $\mathbf{L}$, $\mathbf{N}_8, \mathbf{N}_1$, and $\mathbf{N}_2$ be the fixed subfields of the subgroups $\langle \sigma, \tau \rangle$, $\langle \sigma^2 \rangle, \langle \sigma^p\tau \rangle$ and $\langle c\sigma^p\tau \rangle$, respectively (Lattice I). Hence, **L** is a real quadratic subfield of **N**, the extension $\mathbf{N}/\mathbf{L}$ is abelian with Galois group $G(\mathbf{N}/\mathbf{L}) \simeq C_{2p} \times C_2$ and $\mathbf{N}_8$ is an octic CM-field containing $\mathbf{L}$. The field $\mathbf{N}$ is the compositum of two CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ of degree $4p$ with the same maximal real dihedral subfield $\mathbf{N}_1^+ = \mathbf{N}_2^+$ of degree $2p$ containing $\mathbf{L}$. Let $\mathbf{M}_i = \mathbf{N}_8 \cap \mathbf{N}_i$ for $i = 1, 2$. Then $\mathbf{N}_8$ is also the compositum of two CM-subfields $\mathbf{M}_1$ and $\mathbf{M}_2$ with the same maximal real quadratic subfield $\mathbf{L}$. We have

$$h^-_{\mathbf{N}}/h^-_{\mathbf{N}_8} = (h^-_{\mathbf{N}_1}/h^-_{\mathbf{M}_1})(h^-_{\mathbf{N}_2}/h^-_{\mathbf{M}_2}). \tag{1}$$

If $\mathbf{G} = C_4 \times D_{2p}$ or $C_p \times D_8$, we let $\mathbf{L}$, $\mathbf{N}_8$, and $\mathbf{N}_{2p}$ be the fixed subfields of the cyclic subgroup generated by σ, σ^4, and σ^p, respectively (see Lattice II). Hence, the extension $\mathbf{N}/\mathbf{L}$ is cyclic. Considering $D(\mathbf{G})$ we can easily verify that $w_{\mathbf{N}} = w_{\mathbf{N}_8}$.

Table 2

Lattice I

$G(\mathbf{N}/\mathbb{Q})$	$C_2 \times Q_{4p}$	$C_2 \times D_{4p}$	$C_p \rtimes D_8$
$G(\mathbf{N}_8/\mathbb{Q})$	$C_4 \times C_2$	$(C_2)^3$	D_8
Remarks	$\mathbf{N}_1$ or $\mathbf{N}_2$ dicyclic	$\mathbf{N}_1$ and $\mathbf{N}_2$ dihedral	$\mathbf{N}_1 \simeq \mathbf{N}_2$, $\mathbf{M}_1 \simeq \mathbf{M}_2$ non-normal CM-fields $\mathbf{N}_1/\mathbf{L}$ cyclic

Lattice II

$G(\mathbf{N}/\mathbb{Q})$	$C_4 \times D_{2p}$	$C_p \times D_8$
$G(\mathbf{N}_8/\mathbb{Q})$	$C_4 \times C_2$	D_8
Remarks	$\mathbf{N}_{2p}$ dihedral $\mathbf{N}/\mathbf{L}$ cyclic	$\mathbf{N}_{2p}$ cyclic $\mathbf{N}/\mathbf{L}$ cyclic

Let us set some notations we will use throughout this paper. If $\mathbf{N}$ is a number field, we let $d_{\mathbf{N}}$, $A_{\mathbf{N}}$, $w_{\mathbf{N}}$, $h_{\mathbf{N}}$, and $\zeta_{\mathbf{N}}$ denote the absolute value of its discriminant, its ring of integers, its number of complex roots of unity, its class number, and its Dedekind zeta function, respectively. If $\mathbf{N}$ is a CM-field, we let $\mathbf{N}^+$, $h^-_{\mathbf{N}}$ and $Q_{\mathbf{N}} \in \{1, 2\}$ denote its maximal real subfield, relative class number, and Hasse unit index, respectively (see [Wa]). If $\mathbf{L}$ is a quadratic number field, we let $\chi_{\mathbf{L}}$ denote the primitive quadratic Dirichlet character modulo $d_{\mathbf{L}}$ associated with $\mathbf{L}$. For any abelian extension $\mathbf{E}/\mathbf{F}$ let $\mathcal{F}_{\mathbf{E}/\mathbf{F}}$ be the finite part of its conductor and $f_{\mathbf{E}/\mathbf{F}} = N_{\mathbf{F}/\mathbb{Q}}(\mathcal{F}_{\mathbf{E}/\mathbf{F}})$ the norm of the finite part of this conductor. Finally, we recall:

PROPOSITION 3. (1) ([LOO, Th. 5]) *Let $\mathbf{k} \subseteq \mathbf{K}$ be two CM-fields. Assume that $[\mathbf{K} : \mathbf{k}]$ is odd. Then $Q_{\mathbf{K}} = Q_{\mathbf{k}}$ and $h^-_{\mathbf{k}}$ divides $h^-_{\mathbf{K}}$.*

(2) ([LOO, Prop. 8]) *Let p be any odd prime number. Let $\mathbf{K}/\mathbf{M}$ be a cyclic extension of degree p of CM-fields and let $\mathbf{K}^+/\mathbf{M}^+$ also be cyclic. Let t be the number of prime ideals of $\mathbf{M}^+$ which split $\mathbf{M}/\mathbf{M}^+$ and are ramified*

in $\mathbf{K}^+/\mathbf{M}^+$. *Then* $p^{t-1}h^-_{\mathbf{M}}$ *divides* $h^-_{\mathbf{K}}$, *and* $p^t h^-_{\mathbf{M}}$ *divides* $h^-_{\mathbf{K}}$ *if* p *does not divide* $w_{\mathbf{M}}$.

(3) ([LO1]) *Let* t *denote the number of prime ideals of* $\mathbf{K}$ *which are ramified in the quadratic extension* $\mathbf{K}/\mathbf{K}^+$. *Then* 2^{t-1} *divides* $h^-_{\mathbf{K}}$.

(4) *Let* p *be an odd prime number. Let* $\mathbf{K}$ *be a real dihedral field of degree* $2p$ *which is cyclic over a real quadratic field* $\mathbf{L}$.

(a) ([LPL]) *There exists a positive rational integer* $F_{\mathbf{K}/\mathbf{L}}$ *such that the conductor of* $\mathbf{K}/\mathbf{L}$ *is given by* $\mathcal{F}_{\mathbf{K}/\mathbf{L}} = (F_{\mathbf{K}/\mathbf{L}})$.

(b) ([Mar]) *Let* $\mathcal{Q}$ *be a prime ideal of* $\mathbf{L}$ *above a rational prime* q. *If* q *does not split in* $\mathbf{L}/\mathbb{Q}$, *then* $\mathcal{Q}$ *is not inert in* $\mathbf{K}/\mathbf{L}$. *Moreover*, *if* q *is totally ramified in* $\mathbf{K}/\mathbb{Q}$ *then* $q = p$. *If* $q \neq p$ *and* $\mathcal{Q}$ *is ramified in* $\mathbf{K}/\mathbf{L}$ *then* $q \equiv \chi_{\mathbf{L}}(q) \pmod{p}$.

2.1. *Numerical computation of relative class numbers.* We use the technique developed in [Lou5] and [Lou6] to compute efficiently relative class numbers of the CM-fields:

PROPOSITION 4 (use [Lou6, Theorem 9]). *Let* $\mathbf{E}$ *be a CM-field. Assume that there exists some totally real subfield* $\mathbf{L}$ *of* $\mathbf{E}^+$ *such that the extension* $\mathbf{E}/\mathbf{L}$ *is cyclic of degree* $2^r p$ *with* $r \geq 1$ *and* $p \geq 3$ *any odd prime. Let* $\mathbf{F}$ *be a CM-subfield of* $\mathbf{N}$ *such that* $\mathbf{L} \subseteq \mathbf{F} \subseteq \mathbf{E}$ *and the degree of the extension* $\mathbf{E}/\mathbf{F}$ *is* p. *Finally*, *let* χ *be any one of the characters of order* $2^r p$ *associated with the cyclic extension* $\mathbf{E}/\mathbf{L}$. *Then* $w_{\mathbf{E}} L(0,\chi)$ *is an algebraic integer of the cyclotomic field* $\mathbb{Q}(\zeta_{2^r p})$, $h^-_{\mathbf{F}}$ *divides* $h^-_{\mathbf{E}}$, $w_{\mathbf{F}}$ *divides* $w_{\mathbf{E}}$, *and*

$$h^-_{\mathbf{E}}/h^-_{\mathbf{F}} = (w_{\mathbf{E}}/w_{\mathbf{F}}) N_{\mathbb{Q}(\zeta_{2^r p})/\mathbb{Q}}\left(\frac{1}{2^m} L(0,\chi)\right). \tag{2}$$

We refer the reader to [Lou6] to see how to use [Lou5] to compute the exact value of $L(0,\chi)$ (i.e. the values of the rational integers which are the coordinates in a given $\mathbb{Z}$-basis of the algebraic integer $w_{\mathbf{E}} L(0,\chi)$), prior to using (2).

THEOREM 5. *Let* $\mathbf{K}$ *be a non-abelian normal CM-field of degree* $2n = 2^r p$ ($r \geq 2$) *which is cyclic over a real quadratic field* $\mathbf{L}$ *and cyclic over a CM-subfield* $\mathbf{K}_{2^r}$ *of degree* 2^r. *Assume also that* $w_{\mathbf{K}} = w_{\mathbf{K}_{2^r}}$. *Then* $h^-_{\mathbf{K}_{2^r}}$ *divides* $h^-_{\mathbf{K}}$ *and* $h^-_{\mathbf{K}}/h^-_{\mathbf{K}_{2^r}} = (h^-_{\mathbf{K}/\mathbf{K}_{2^r}})^2$ *is a perfect square.*

Proof. Let χ be any character of order n associated with $\mathbf{K}/\mathbf{L}$. Then

$$h^-_{\mathbf{K}}/h^-_{\mathbf{K}_{2^r}} = N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}\left(\frac{1}{4} L(0,\chi)\right)$$

(Proposition 4). Let τ be the non-trivial element in $G(\mathbf{L}/\mathbb{Q})$. Then for any ideal $\mathcal{I}$ of $\mathbf{L}$ we have $\chi(\tau(\mathcal{I})) = \chi(\mathcal{I})^k$ where $k^2 \equiv 1 \pmod{n}$, which yields $\chi \circ \tau = \chi^k$. Let σ_k denote the automorphism of $\mathbb{Q}(\zeta_n)$ which sends ζ_n to ζ_n^k. Then $\sigma_k(L(0,\chi)) = L(0,\chi^k) = L(0,\chi \circ \tau) = L(0,\chi)$ and $L(0,\chi)$ lies in the fixed subfield $\mathbf{F}$ of $\mathbb{Q}(\zeta_n)$ by σ_k. Therefore, we have $h^-_{\mathbf{K}}/h^-_{\mathbf{K}_{2^r}} = N_{\mathbb{Q}(\zeta_n)/\mathbb{Q}}\big(\frac{1}{4}L(0,\chi)\big) = \big(N_{\mathbf{F}/\mathbb{Q}}\big(\frac{1}{4}L(0,\chi)\big)\big)^2$, which completes the proof. ∎

3. The case $\mathbf{G} \simeq C_2 \times Q_{4p}$

LEMMA 6 (due to S. Louboutin). *Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be two distinct CM-fields with the same maximal totally real subfield. Set $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$. Assume that $Q_{\mathbf{N}_1} = 1$ and that 4 does not divide $w_{\mathbf{N}_1}$. Then $h^-_{\mathbf{N}_1}h^-_{\mathbf{N}_2}$ divides $2h^-_{\mathbf{N}}$. In particular, if $G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times Q_{4p}$ then $h^-_{\mathbf{N}_1}h^-_{\mathbf{N}_2}$ divides $2h^-_{\mathbf{N}}$.*

Proof. We have $h^-_{\mathbf{N}} = \eta_{\mathbf{N}} h^-_{\mathbf{N}_1}h^-_{\mathbf{N}_2}$, where $\eta_{\mathbf{N}} = Q_{\mathbf{N}}w_{\mathbf{N}}/(Q_{\mathbf{N}_1}w_{\mathbf{N}_1}Q_{\mathbf{N}_2}w_{\mathbf{N}_2}) = Q_{\mathbf{N}}w_{\mathbf{N}}/(Q_{\mathbf{N}_2}w_{\mathbf{N}_1}w_{\mathbf{N}_2})$ and $w_{\mathbf{N}_1}w_{\mathbf{N}_2}$ divides $2w_{\mathbf{N}}$ (see [LO2, Proof of Prop. 2, point (b)]). We must prove that $2\eta_{\mathbf{N}}$ is a positive rational integer. Clearly, we may assume that $w_{\mathbf{N}} = \frac{1}{2}w_{\mathbf{N}_1}w_{\mathbf{N}_2}$ and $Q_{\mathbf{N}_2} = 2$. Now, we must prove that $Q_{\mathbf{N}} = 2$. Since $Q_{\mathbf{N}_2} = 2$, we have $W_{\mathbf{N}_2} = \langle \varepsilon_2/\overline{\varepsilon}_2\rangle$ for some $\varepsilon_2 \in U_{\mathbf{N}_2}$, and since $Q_{\mathbf{N}_1} = 1$ we have $W^2_{\mathbf{N}_2} = \langle \varepsilon_1/\overline{\varepsilon}_1\rangle$ for some $\varepsilon_1 \in U_{\mathbf{N}_1}$. Now, $\phi : (\varepsilon, \varepsilon') \in W^2_{\mathbf{N}_1} \times W_{\mathbf{N}_2} \to \varepsilon\varepsilon' \in W_{\mathbf{N}}$ is injective (for $W^2_{\mathbf{N}_1} \cap W_{\mathbf{N}_2} \subset W_{\mathbf{N}_1\cap\mathbf{N}_2} = W_{\mathbf{N}_1^+} = \{\pm1\}$ and $-1 \notin W^2_{\mathbf{N}_1}$) and $\#\operatorname{Im}\phi = \frac{1}{2}w_{\mathbf{N}_1}w_{\mathbf{N}_2} = w_{\mathbf{N}}$. Thus, ϕ is surjective and $W_{\mathbf{N}} \subseteq U^{1-c}_{\mathbf{N}}$. Hence, $Q_{\mathbf{N}} = 2$. If $G(\mathbf{N}/\mathbb{Q}) \simeq Q_{4p} \times C_2$, then we may assume that $\mathbf{N}_1$ is dicyclic. We have $Q_{\mathbf{N}_1} = Q_{\mathbf{M}_1} = 1$ and $4 \nmid w_{\mathbf{N}_1} = w_{\mathbf{M}_1}$, where $\mathbf{M}_1$ is a cyclic quartic extension of $\mathbb{Q}$. ∎

PROPOSITION 7. *Let $\mathbf{N}$ be a normal CM-field of degree $8p$ with Galois group $G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times Q_{4p}$. If $p \equiv 3 \pmod 4$, then 2^{p-2} divides $h^-_{\mathbf{N}}$ which is therefore even. If $p \equiv 1 \pmod 4$, then $h^-_{\mathbf{N}} > 1$.*

Proof. We may assume that $\mathbf{N}_1$ is a dicyclic CM-field of degree $4p$. Then 2^{p-1} divides $h^-_{\mathbf{N}_1}$ if $p \equiv 3 \pmod 4$ (see [LOO, Thm. 6]) and $h^-_{\mathbf{N}_1} \geq 4$ in any case (see [LP]). Therefore, using Lemma 6, we get the desired results. ∎

4. The case $\mathbf{G} \simeq C_2 \times D_{4p}$.

Using the determination of all the dihedral CM-fields of degree $4p$ with relative class number one and the determination of some imaginary abelian octic fields with class number one, we determine all the normal CM-fields $\mathbf{N}$ of degree $8p$ with Galois group $G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times D_{4p}$ which have class number one. We use the following lemma whose proof is left to the reader (use Hilbert class fields):

LEMMA 8. *Let $\mathbf{N}_1$ and $\mathbf{N}_2$ be two distinct CM-fields with the same maximal totally real subfield. Set $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ and assume that $h_{\mathbf{N}} = 1$. If neither $h_{\mathbf{N}_1}$ nor $h_{\mathbf{N}_2}$ is one, then $h^-_{\mathbf{N}_1} = h^-_{\mathbf{N}_2} = 1$.*

THEOREM 9. *There is only one normal CM-field of degree $8p$ with Galois group $G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times D_{4p}$ of class number one: the one given in Theorem 1(1).*

Proof. Recall that $\mathbf{N} = \mathbf{N}_1\mathbf{N}_2$ is a compositum of two dihedral CM-fields $\mathbf{N}_1$ and $\mathbf{N}_2$ of degree $4p$ which are cyclic over the same real quadratic number field $\mathbf{L}$ and that $\mathbf{N}$ is a cyclic extension of degree p of an elementary imaginary abelian octic field $\mathbf{N}_8$ containing $\mathbf{L}$ (see Lattice I and Table 2). We let $\mathcal{N}_8$ denote the finite set of all the elementary imaginary abelian octic fields $\mathbf{N}_8$ which are equal to their own genus field and have relative class number one. There are 17 such $\mathbf{N}_8 \in \mathcal{N}_8$: the 14 last fields in [Lou2, Table 2] and the following three: $\mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-3})$, $\mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{-11})$ and $\mathbb{Q}(\sqrt{-1}, \sqrt{-2}, \sqrt{5})$ (see [CK]). Since $h_{\mathbf{N}} = 1$, the genus field of $\mathbf{N}_8$ is included in $\mathbf{N}$. Since $\mathbf{N}_8$ is the maximal abelian subfield of $\mathbf{N}$, $\mathbf{N}_8$ is its own genus field and $h^-_{\mathbf{N}_8} = 1$ (use Proposition 3(1)). Hence, $\mathbf{N}_8 \in \mathcal{N}_8$. Moreover, according to Lemma 8 and noticing that no two of the 18 dihedral CM-fields of degree $4p$ with relative class number one are cyclic extensions of the same real quadratic field (see [Lef, Th. 4.1]), we may assume that $\mathbf{N}_1$ is one of the 10 dihedral CM-fields of degree $4p$ with class number one. Hence, $p = 3$ or 5. Now, only 5 out of these 10 dihedral CM-fields $\mathbf{N}_1$ are such that the imaginary biquadratic bicyclic subfield $\mathbf{M}_1$ of $\mathbf{N}_1$ is a subfield of one of the 17 fields $\mathbf{N}_8 \in \mathcal{N}_8$. These five dihedral CM-fields are of degree 12 and are the ones of indices $i_{\mathbf{N}_1} = 1$, 2, 3, 4 and 6 in [Lef, Table 1, p. 85]. We have only the following 16 possible choices for the pair $(\mathbf{N}_1, \mathbf{N}_8)$:

Table 3

$i_{\mathbf{N}_1}$	$\mathbf{M}_1$	$d_{\mathbf{L}}$	$\mathbf{N}_8$	$\mathbf{M}_2$	$h^-_{\mathbf{M}_2}$	$h^-_{\mathbf{N}}$
1	$(-3,-15)$	5	$(-3,-4,5)$	$(-4,-20)$	1	3^2
1	$(-3,-15)$	5	$(-3,-7,5)$	$(-7,-35)$	1	6^2
1	$(-3,-15)$	5	$(-3,-8,5)$	$(-8,-40)$	1	3^2
2	$(-3,-4)$	12	$(-3,-4,5)$	$(-15,-20)$	2	3^2
2	$(-3,-4)$	12	$(-3,-4,-7)$	$(-7,-84)$	2	18^2
2	$(-3,-4)$	12	$(-3,-4,-11)$	$(-11,-33)$	2	9^2
2	$(-3,-4)$	12	$(-3,-4,-19)$	$(-19,-57)$	2	18^2
2	$(-3,-4)$	12	$(-1,-2,-3)$	$(-24,-8)$	2	6^2
3	$(-3,-7)$	21	$(-3,-7,5)$	$(-15,-35)$	2	3^2
3	$(\mathbf{-3},\mathbf{-7})$	21	$(\mathbf{-3},\mathbf{-7},\mathbf{-4})$	$(\mathbf{-4},\mathbf{-84})$	**2**	**1**
3	$(-3,-7)$	21	$(-3,-7,-8)$	$(-8,-168)$	2	3^2
4	$(-3,-8)$	24	$(-3,-8,5)$	$(-15,-40)$	2	11^2
4	$(-3,-8)$	24	$(-3,-7,-8)$	$(-7,-42)$	2	20^2
4	$(-3,-8)$	24	$(-1,-2,-3)$	$(-4,-24)$	2	2^2
6	$(-3,-19)$	57	$(-3,-19,-4)$	$(-4,-57)$	2	3^2
6	$(-3,-19)$	57	$(-3,-19,-11)$	$(-11,-627)$	2	24^2

To compute $h^-_{\mathbf{N}}$, we use Theorem 5. Note that according to (1) we have $h^-_{\mathbf{N}} = (h^-_{\mathbf{N}_1}/h^-_{\mathbf{M}_1})(h^-_{\mathbf{N}_2}/h^-_{\mathbf{M}_2}) = h^-_{\mathbf{N}_2}/h^-_{\mathbf{M}_2}$. According to Table 3, Theorem 9 is proved. ∎

5. The cases $\mathbf{G} \simeq C_p \rtimes D_8$, $C_4 \times D_{2p}$ or $C_p \times D_8$. We use of the same plan as in [LP].

5.1. *Lower bounds for relative class numbers*

THEOREM 10 (see the proof of [LP, Thm. 6]). *Let* $\mathbf{K}$ *be a CM-field of degree* $2n$ *which is cyclic of degree* $2m$ *over an abelian real field* $\mathbf{L}$. *Let* f_+ *denote the norm of the conductor of the extension* $\mathbf{K}^+/\mathbf{L}$. *Set* $\varepsilon_{\mathbf{K}} = \max\{\varepsilon'_{\mathbf{K}}, \varepsilon''_{\mathbf{K}}\}$ *where* $\varepsilon'_{\mathbf{K}} = \frac{2}{5}\exp(-2\pi n/d_{\mathbf{K}}^{1/(2n)})$, $\varepsilon''_{\mathbf{K}} = 1-(2\pi n e^{1/n}/d_{\mathbf{K}}^{1/(2n)})$. *Let* $\mathbf{M}/\mathbf{L}$ *be the only quadratic subextension of* $\mathbf{K}/\mathbf{L}$. *Assume that* $\zeta_{\mathbf{M}}(s) \le 0$ *for* $0 < s < 1$. *Then for some constant* $\mu_{\mathbf{L}}$ *depending on* $\mathbf{L}$ *only, we have*

$$h_{\mathbf{K}}^- \ge \varepsilon_{\mathbf{K}} \frac{2Q_{\mathbf{K}} w_{\mathbf{K}} \sqrt{d_{\mathbf{K}}/d_{\mathbf{K}^+}}}{e(2\pi)^n (\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}))^m \left(\frac{1}{2}\log f_+ + 2\mu_{\mathbf{L}}\right)^{m-1} \log d_{\mathbf{K}}}.$$

To compute the numerical approximations of $\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})$ and $\mu_{\mathbf{L}}$ we use [Lou3].

THEOREM 11. (1) ([CK]) *Let* $\mathbf{N}_8 = \mathbf{L}_4\mathbf{L}_2$ *be an imaginary abelian octic field with Galois group* $G(\mathbf{N}_8/\mathbb{Q}) \simeq C_4 \times C_2$, *where* $\mathbf{L}_4$ *is an imaginary cyclic quartic of conductor* f_4 *and* $\mathbf{L}_2$ *is a real quadratic of conductor* f_2. *Then* $h_{\mathbf{N}_8}^- = 1$ *if and only if* $(f_4, f_2) \in \{(5,8), (5,13), (5,17), (13,5), (13,8), (16,5)\}$. *For these six fields* $\mathbf{N}_8$ *we have* $\zeta_{\mathbf{N}_8^+}(s) \le 0$ *for* $0 < s < 1$.

(2) ([LO1]) *There are exactly* 19 *dihedral octic CM-fields* $\mathbf{N}_8$ *with relative class number one*: *the narrow Hilbert* 2*-class fields of the* 19 *real quadratic field* $\mathbf{L}$ *which appear in Table* 6. *For these* 19 *fields* $\mathbf{N}_8$ *we have* $\zeta_{\mathbf{N}_8^+}(s) \le 0$ *for* $0 < s < 1$.

(3) ([LO1]) *There are* 38 *non-normal quartic CM-fields* $\mathbf{M}$ *with relative class number one* (*they are pairwise isomorphic and their normal closures are the previous* 19 *dihedral octic CM-fields with relative class number one*). *For these* 38 *fields* $\mathbf{M}$ *we have* $\zeta_{\mathbf{M}}(s) \le 0$ *for* $0 < s < 1$.

COROLLARY 12. *Let* $p \ge 3$ *be a given odd prime. We can compute an explicit bound on the discriminants of the normal CM-fields of degree* $8p$ *with Galois groups* $\mathbf{G} \simeq C_p \rtimes D_8$, $C_4 \times D_{2p}$, *or* $C_p \times D_8$ *and relative class number one. More precisely*:

(1) *Assume that* $\mathbf{G} \simeq C_4 \times D_{2p}$ *or* $C_p \times D_8$ (*Lattice II*). *Then* $h_{\mathbf{N}_8}^- = 1$. *For each of these* 25 $(= 6 + 19)$ *CM-fields* $\mathbf{N}_8$ *of relative class number one we can compute a bound* $B_p(\mathbf{L})$ *on the norm* f_+ *of the conductor of the extension* $\mathbf{N}^+/\mathbf{L}$ *such that* $h_{\mathbf{N}}^- = 1$ *implies* $f_+ \le B_p(\mathbf{L})$.

(2) *Assume that* $\mathbf{G} \simeq C_p \rtimes D_8$ (*Lattice I*). *Then* $h_{\mathbf{N}}^- = 1$ *if and only if* $h_{\mathbf{N}_1}^- = 1$. *Now*, $h_{\mathbf{N}_1}^- = 1$ *implies* $h_{\mathbf{M}_1}^- = 1$. *For each of these* 38 *non-normal quartic CM-fields* $\mathbf{M}_1$ *of relative class number one we can compute a bound* $B_p(\mathbf{L})$ *on the norm* f_+ *of the conductor of the extension* $\mathbf{N}_1^+/\mathbf{L}$ *such that* $h_{\mathbf{N}_1}^- = 1$ *implies* $f_+ \le B_p(\mathbf{L})$.

Proof. First, assume that $\mathbf{G} \simeq C_4 \times D_{2p}$ or $C_p \times D_8$ (Lattice II). Then $\mathbf{N}/\mathbf{L}$ is a cyclic extension of degree $4p$. For s real we have

$$(\zeta_{\mathbf{N}}/\zeta_{\mathbf{N}_8^+})(s) = \prod_{\{\chi,\overline{\chi}\},\,\chi^2\neq 1} |L(s,\chi)|^2 \geq 0$$

(where χ ranges over the $4p-2$ non-quadratic characters associated with the extension $\mathbf{N}/\mathbf{L}$), and we conclude that $\zeta_{\mathbf{N}_8^+}(s) \leq 0$ for $0 < s < 1$ implies $\zeta_{\mathbf{N}}(s) \leq 0$ for $0 < s < 1$. Now, assume that $h_{\mathbf{N}}^- = 1$. Since $h_{\mathbf{N}_8}^-$ divides $h_{\mathbf{N}}^-$ (Proposition 3), we have $h_{\mathbf{N}_8}^- = 1$ and $\zeta_{\mathbf{N}_8^+}(s) \leq 0$ for $0 < s < 1$ (Theorem 11(1), (2)). On applying Theorem 10 with $\mathbf{K} = \mathbf{N}$ and $2m = 4p$, for each of the finitely many $\mathbf{N}_8$ with $h_{\mathbf{N}_8}^- = 1$ we obtain a good lower bound of $B_p(\mathbf{L})$. (See Tables 4, 6, and 7.)

Second, assume that $\mathbf{G} \simeq C_p \rtimes D_8$ (Lattice I). Then $\mathbf{M}_1$ is a non-normal quartic CM-field, hence $Q_{\mathbf{M}_1} = 1$ (see [Lou1, Lemma 1]), $Q_{\mathbf{N}_1} = Q_{\mathbf{M}_1} = 1$, by Proposition 3(1), and $h_{\mathbf{N}}^- = (Q_{\mathbf{N}}/2)(h_{\mathbf{N}_1}^-/Q_{\mathbf{N}_1})^2$ (see [LO2, Prop. 2]). Hence, $h_{\mathbf{N}}^- = 1$ implies $h_{\mathbf{N}_1}^- = 1$ (and $Q_{\mathbf{N}} = 2$) and $h_{\mathbf{M}_1}^- = 1$ (Proposition 3(1)). Conversely, $h_{\mathbf{N}_1}^- = 1$ implies $h_{\mathbf{N}}^- = 1$ (and $Q_{\mathbf{N}} = 2$). Since $\mathbf{N}_1/\mathbf{L}$ is cyclic of degree $2p$, as for the previous point we also obtain a good lower bound of $B_p(\mathbf{L})$. (See Table 9.) ∎

If we followed the same line of reasoning as in [Lef], we could determine all the normal CM-fields of degree $8p$, $p \geq 3$ any odd prime, with Galois group $\mathbf{G} \simeq C_p \rtimes D_8$, $C_4 \times D_{2p}$ or $C_p \times D_8$ with relative class number one. Instead of determining all the normal CM-fields of degree $8p$ we determine the fields of degree 24 and 40 in the following three sections. We will prove:

THEOREM 13. *The only normal CM-fields of degree* 24 *with Galois group* $G(\mathbf{N}/\mathbb{Q}) \simeq C_3 \rtimes D_8$, $C_4 \times D_6$, *or* $C_3 \times D_8$ *of relative class number one are the two fields given in Theorem* 1(2), (3). *If* $\mathbf{N}$ *is a normal CM-field of degree* 40 *with Galois group* $G(\mathbf{N}/\mathbb{Q}) \simeq C_5 \rtimes D_8$, $C_4 \times D_{10}$, *or* $C_5 \times D_8$, *then* $h_{\mathbf{N}}^- > 1$.

5.2. *The cases* $\mathbf{G} \simeq C_4 \times D_6$ *and* $\mathbf{G} \simeq C_4 \times D_{10}$ (*Lattice II*)

PROPOSITION 14. *Let* $\mathbf{N}$ *be a normal CM-field of degree* $8p$ *with* $G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_{2p}$. *Assume that* $h_{\mathbf{N}}^-$ *is odd. If a rational prime* q *is ramified in* $\mathbf{N}_8/\mathbf{L}$, *then* q *divides* $f_{\mathbf{N}_{2p}/\mathbf{L}}$.

Proof. Suppose q splits in $\mathbf{L}/\mathbb{Q}$. Then $t_{\mathbf{N}_8/\mathbf{N}_8^+} \geq 2$ and $2 \mid h_{\mathbf{N}_8}^- \mid h_{\mathbf{N}}^-$ by Proposition 3(1), (3), which is a contradiction. Hence, q does not split in $\mathbf{L}/\mathbb{Q}$. Let $\mathcal{Q}_{\mathbf{L}}$ denote the prime ideal of $\mathbf{L}$ above q. According to Proposition 3(4), $\mathcal{Q}_{\mathbf{L}}$ is not inert in $\mathbf{N}_{2p}/\mathbf{L}$. Suppose $\mathcal{Q}_{\mathbf{L}}$ were not ramified in $\mathbf{N}_{2p}/\mathbf{L}$. Then $\mathcal{Q}_{\mathbf{L}}$ would split in $\mathbf{N}_{2p}/\mathbf{L}$. Since $\mathcal{Q}_{\mathbf{L}}$ is ramified in $\mathbf{N}_8/\mathbf{L}$ and since $\mathbf{N}/\mathbf{L}$ is cyclic, the p prime ideals $\mathcal{Q}_1, \ldots, \mathcal{Q}_p$ of $\mathbf{N}_{2p}$ above $\mathcal{Q}_{\mathbf{L}}$ would be ramified in the cyclic quartic extension $\mathbf{N}/\mathbf{N}_{2p}$, hence the prime ideals of $\mathbf{N}^+$ above $\mathcal{Q}_i$ would be ramified in the quadratic extension $\mathbf{N}/\mathbf{N}^+$, we would have $t_{\mathbf{N}/\mathbf{N}^+} \geq p$ and 2^{p-1} would divide $h_{\mathbf{N}}^-$ (by Proposition 3(2)). A contradiction. Hence, $\mathcal{Q}_{\mathbf{L}}$ is ramified in $\mathbf{N}_{2p}/\mathbf{L}$ and q divides $f_{\mathbf{N}_{2p}/\mathbf{L}}$. ■

First, assume that $G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_6$ and $h_{\mathbf{N}}^- = 1$. Then $h_{\mathbf{N}_8}^- = 1$ and $\mathbf{N}_8$ is known (Theorem 11(1)). In fact, we can get rid of two of the six possible fields $\mathbf{N}_8$ and we can decide which one of the three real quadratic subfields of a given $\mathbf{N}_8$ must be equal to $\mathbf{L}$:

LEMMA 15. *If* $h_{\mathbf{N}}^- = 1$ *then* $(f_4, f_2, d_{\mathbf{L}}) \in \{(16,5,5),(5,8,8),(5,13,13),(5,17,17)\}$.

Proof. If a rational prime q divides $f_{\mathbf{N}_8/\mathbf{L}}$, then $q \in \{2,5,13\}$ and $q \mid f_{\mathbf{N}_6/\mathbf{L}}$ by Proposition 14. Note that if q is ramified in $\mathbf{L}/\mathbb{Q}$, then q is totally ramified in $\mathbf{N}_6/\mathbb{Q}$ and $q = 3$. This implies that q is inert in $\mathbf{L}/\mathbb{Q}$ and $3 \mid (q+1)$ by Proposition 3(4), which yields the desired result. ■

For the four fields $\mathbf{N}_8$ we compute $B_3(\mathbf{L})$ such that $h_{\mathbf{N}}^- > 1$ if $f_{\mathbf{N}_6/\mathbf{L}} > B_3(\mathbf{L})$. Let n_f be the number of conductors of $\mathbf{N}/\mathbf{L}$ satisfying $f_{\mathbf{N}_6/\mathbf{L}} \leq B_3(\mathbf{L})$ and N_f the number of conductors of $\mathbf{N}/\mathbf{L}$ satisfying $f_{\mathbf{N}_6/\mathbf{L}} \leq B_3(\mathbf{L})$ and Proposition 14. We refer the reader to Table 4 for the result of our computation. Notice that N_f is much small than n_f, which clearly shows how useful Proposition 14 is for alleviating the amount of computation required. Finally, in Table 5 we give the results of our relative class number computations. According to Table 5, there is only one such CM-field with $h_{\mathbf{N}}^- = 1$. Notice that there are two fields $\mathbf{N}_6$ for which $\mathcal{F}_{\mathbf{N}_6/\mathbf{L}} = (5 \cdot 22)$. In the same way, for the case $G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_{10}$ we computed Table 6 according to which there is no such $\mathbf{N}$ with $h_{\mathbf{N}}^- = 1$.

Table 4 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_6$)

	$d_{\mathbf{L}}$	$h_{\mathbf{L}}$	$\mathrm{R}_{\mathbf{L}} \leq$	$\mu_{\mathbf{L}} \leq$	$f_{\mathbf{N}_8/\mathbf{L}}$	$B_3(\mathbf{L})$	n_f	N_f
1	5	1	0.431	0.1014	2^8	30^2	1	1
2	8	1	0.624	0.1409	5^2	310^2	20	5
3	13	1	0.663	0.2215	5^2	230^2	21	5
4	17	1	1.017	0.2167	5^2	390^2	27	9

Table 5 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_6$)

	$d_{\mathbf{L}}$	$\mathcal{F}_{\mathbf{N}_6/\mathbf{L}}$	$h_{\mathbf{N}}^-$		$d_{\mathbf{L}}$	$\mathcal{F}_{\mathbf{N}_6/\mathbf{L}}$	$h_{\mathbf{N}}^-$		$d_{\mathbf{L}}$	$\mathcal{F}_{\mathbf{N}_6/\mathbf{L}}$	$h_{\mathbf{N}}^-$
1	5	$(2 \cdot 3^2)$	65^2	8	13	$(5 \cdot 3^2)$	13^2	14	17	$(5 \cdot 19)$	52^2
2	8	$(5 \cdot 7)$	4^2	9	13	$(5 \cdot 18)$	61^2	15	17	$(5 \cdot 23)$	100^2
3	8	$(5 \cdot 3^2)$	10^2	10	13	$(5 \cdot 22)$	90^2	16	17	$(5 \cdot 29)$	261^2
4	8	$(5 \cdot 11)$	9^2		13	$(5 \cdot 22)$	90^2	17	17	$(5 \cdot 41)$	369^2
5	8	$(5 \cdot 31)$	81^2	11	13	$(5 \cdot 43)$	205^2	18	17	$(5 \cdot 43)$	541^2
6	8	$(5 \cdot 53)$	241^2	12	17	$(5 \cdot 3^2)$	25^2	19	17	$(5 \cdot 67)$	976^2
7	**13**	$(\mathbf{5 \cdot 2})$	**1**	13	17	$(5 \cdot 13)$	52^2	20	17	$(5 \cdot 71)$	1476^2

Table 6 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_4 \times D_{10}$)

	$d_{\mathbf{L}}$	$h_{\mathbf{L}}$	$\mathrm{R}_{\mathbf{L}} \le$	$\mu_{\mathbf{L}} \le$	$f_{\mathbf{N}_8}$	$B_5(\mathbf{L})$	n_f	N_f	$\mathcal{F}_{\mathbf{N}_{10}/\mathbf{L}}$	$h^-_{\mathbf{K}}$
1	40	2	1.151	0.3719	5	110^2	1	0	–	–
2	65	2	1.378	0.4718	5	105^2	1	1	95	145305^2
3	85	2	0.959	0.6116	5	55^2	0	0	–	–

5.3. *The cases* $\mathbf{G} \simeq C_3 \times D_8$ *and* $\mathbf{G} \simeq C_5 \times D_8$ (*Lattice II*). In these cases we use the following proposition similar to Proposition 14:

PROPOSITION 16. *Let* $\mathbf{N}$ *be a normal CM-field of degree* $8p$ *with* $G(\mathbf{K}/\mathbb{Q}) \simeq C_p \times D_8$. *Assume that* $h^-_{\mathbf{N}} = 1$. *If a rational prime* q *is inert in* $\mathbf{L}/\mathbb{Q}$, *then* q *does not divide* $f_{\mathbf{N}_{2p}/\mathbf{L}}$, *the norm of the conductor* $\mathcal{F}_{\mathbf{N}_{2p}/\mathbf{L}}$ *of* $\mathbf{N}_{2p}/\mathbf{L}$.

Proof. If $h^-_{\mathbf{N}} = 1$, then $h^-_{\mathbf{N}_8} = 1$ and $\mathbf{N}_8$ is the narrow Hilbert 2-class field of some real quadratic field $\mathbf{L}$ in Theorem 11(2). If q is ramified in $\mathbf{N}_{2p}/\mathbf{L}$, then, since q splits completely in $\mathbf{N}_8/\mathbf{L}$ and is ramified in $\mathbf{N}/\mathbf{N}^+$, we have $p \mid h^-_{\mathbf{N}}$ by Proposition 3(2). ■

We obtain Table 7 in the same way as Table 4. In Table 7, to provide the reader with an excerpt of our relative class number computations, for each of the 19 dihedral octic CM-fields $\mathbf{N}_8$ of relative class number one, we give the value of the relative class number of the $\mathbf{N}$ with $G(\mathbf{N}/\mathbb{Q}) \simeq C_3 \times D_8$ and containing $\mathbf{N}_8$ of least $f_{\mathbf{N}_6/\mathbf{L}}$. Table 8 provides the same data for the case $G(\mathbf{N}/\mathbb{Q}) \simeq C_5 \times D_8$. According to these results there is no $\mathbf{N}$ of relative class number one with $\mathbf{G} \simeq C_3 \times D_8$ and $\mathbf{G} \simeq C_5 \times D_8$. In Tables 7 and 8, $\mathcal{P}_q$ denotes the prime ideal of $\mathbf{L}$ above a prime q ramified in $\mathbf{L}/\mathbb{Q}$.

Table 7 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_3 \times D_8$)

	$d_{\mathbf{L}}$	$h_{\mathbf{L}}$	$\mathrm{R}_{\mathbf{L}} \le$	$\mu_{\mathbf{L}} \le$	$B_3(\mathbf{L})$	n_f	N_f	$\mathcal{F}_{\mathbf{N}_6/\mathbf{L}}$	$h^-_{\mathbf{N}}$
1	136	2	1.458	0.6285	127000	45	16	(9)	4^2
2	205	2	1.051	0.8512	22300	20	11	(7)	4^2
3	**221**	2	0.728	1.0622	6300	15	7	$\boldsymbol{\mathcal{P}_{13}}$	**1**
4	305	2	1.578	0.9137	49700	33	13	(7)	7^2
5	377	2	1.266	1.1927	19300	22	10	(19)	91^2
6	545	2	1.418	1.2455	15600	19	7	(13)	52^2
7	584	2	0.939	1.4452	3700	9	3	(13)	64^2
8	712	2	1.210	1.4269	6300	12	8	(9)	52^2
9	745	2	2.500	0.9936	56300	31	14	(9)	67^2
10	1345	6	3.004	1.0595	40000	27	10	(7)	97^2
11	1537	2	2.626	1.2130	21800	20	13	(7)	52^2
12	1864	2	1.979	1.3345	6100	11	7	(7)	109^2
13	1945	2	2.657	1.3607	16100	17	6	(9)	157^2
14	2041	2	3.362	1.1322	30200	31	15	(7)	172^2
15	2248	2	1.680	1.5128	2800	7	6	(7)	112^2
16	2329	2	2.926	1.3022	16500	18	6	(9)	196^2
17	2353	2	2.612	1.3896	11500	20	15	$\mathcal{P}_{13}$	52^2
18	4369	2	3.573	1.3589	11500	15	8	(7)	217^2
19	7081	2	3.737	1.4961	6400	16	11	(7)	724^2

Table 8 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_5 \times D_8$)

$d_{\mathbf{L}}$	$B_5(\mathbf{L})$	n_f	N_f	$\mathcal{F}_{\mathbf{N}_{10}/\mathbf{L}}$	$h^-_{\mathbf{N}}$
136	9500	6	3	(11)	71^2
205	2400	4	3	$\mathcal{P}_{41}$	11^2
221	900	2	2	(11)	181^2
305	4500	5	3	$\mathcal{P}_{61}$	121^2
377	2200	4	3	(11)	1361^2
545	1800	4	2	$\mathcal{P}_5^3$	1991^2
584	600	1	1	(11)	3001^2
712	900	2	0	(61)	4250131^2
745	5000	5	3	$\mathcal{P}_5^3$	4366^2
1345	3800	5	4	(11)	18481^2
1537	2400	4	2	(31)	2620621^2
1864	900	2	2	(11)	26081^2
1945	1900	4	3	(11)	39581^2
2041	3000	4	2	(25)	1522576^2
2248	500	2	2	(11)	83171^2
2329	1900	4	3	(25)	2573371^2
2353	1400	4	2	$\mathcal{P}_{181}$	122305^2
4369	1400	3	1	(25)	8527696^2
7081	900	2	1	(25)	29035651^2

5.4. *The cases* $\mathbf{G} \simeq C_3 \rtimes D_8$ *and* $\mathbf{G} \simeq C_5 \rtimes D_8$ (*Lattice I*). Assume that

$$G(\mathbf{N}/\mathbb{Q}) \simeq C_3 \rtimes D_8 \quad \text{and} \quad h^-_{\mathbf{N}} = 1.$$

For each of the 38 non-normal quartic CM-fields $\mathbf{M}_1$ of relative class number one we have computed an upper bound $B_3(\mathbf{L})$ such that $f_{\mathbf{N}_1^+/\mathbf{L}} > B_3(\mathbf{L})$ implies $h^-_{\mathbf{N}_1} > 1$. For each possible $\mathbf{N}_1$, we have computed $h^-_{\mathbf{N}_1}/h^-_{\mathbf{M}_1}$ for the non-normal CM-field $\mathbf{N}_1$ which is cyclic of degree 3 over a non-normal CM-field $\mathbf{M}_1$ by using (2). Finally, our computation shows that in all the cases considered we have $h^-_{\mathbf{N}_1} > 1$, which implies $h^-_{\mathbf{N}} > 1$. In Table 9, we also give the value of $h^-_{\mathbf{N}}$ for least $f_{\mathbf{N}_1^+/\mathbf{L}}$ and we let n_f denote the number of conductors of $\mathbf{N}/\mathbf{L}$ satisfying $f_{\mathbf{N}_1^+/\mathbf{L}} \leq B_3(\mathbf{L})$.

For the case $G(\mathbf{N}/\mathbb{Q}) \simeq C_5 \rtimes D_8$ we obtain Table 10 in the same way. In Table 10, we give all possible 12 non-normal CM-fields $\mathbf{N}$ with $\mathbf{G} \simeq C_5 \rtimes D_8$ and $f_{\mathbf{N}_1^+/\mathbf{L}} \leq B_5(\mathbf{L})$.

According to these results there is no $\mathbf{N}$ of relative class number one with $\mathbf{G} \simeq C_3 \rtimes D_8$ or $\mathbf{G} \simeq C_5 \rtimes D_8$. In Tables 9 and 10, $\mathcal{P}_q$ denotes a prime ideal of $\mathbf{L}$ above a split prime q. Note that there are two possible prime ideals $\mathcal{P}_q$. If we choose the other, then we get exactly the other isomorphic non-normal CM-fields $\mathbf{N}_2$.

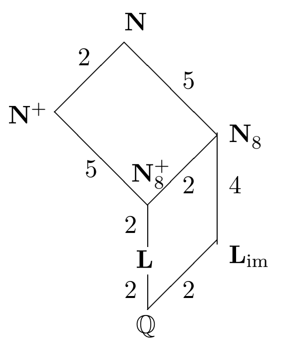
Table 9 $(G(\mathbf{N}/\mathbb{Q}) \simeq V_{24} = C_3 \rtimes D_8)$

	$d_{\mathbf{L}}$	$f_{\mathbf{M}_1/\mathbf{L}}$	$B_3(\mathbf{L})$	n_f	$\mathcal{F}_{\mathbf{N}_1/\mathbf{L}}$	$h_{\mathbf{N}}^- = (h_{\mathbf{N}_1}^-)^2$
1	8	17	120^2	10	$(29)\mathcal{P}_{17}$	208^2
2	8	73	40^2	2	$(29)\mathcal{P}_{73}$	148^2
3	8	89	30^2	1	$(29)\mathcal{P}_{89}$	124^2
4	8	233	20^2	0	–	–
5	8	281	20^2	0	–	–
6	5	41	50^2	5	$(18)\mathcal{P}_{41}$	57^2
7	5	61	40^2	3	$(18)\mathcal{P}_{61}$	84^2
8	5	109	30^2	1	$(18)\mathcal{P}_{109}$	63^2
9	5	149	20^2	1	$(18)\mathcal{P}_{149}$	100^2
10	5	269	20^2	1	$(18)\mathcal{P}_{269}$	211^2
11	5	389	10^2	0	–	–
12	13	17	90^2	8	$(10)\mathcal{P}_{17}$	12^2
13	13	29	60^2	5	$(10)\mathcal{P}_{29}$	27^2
14	13	157	20^2	3	$(10)\mathcal{P}_{157}$	196^2
15	13	181	20^2	3	$(10)\mathcal{P}_{181}$	228^2
16	17	137	30^2	1	$(11)\mathcal{P}_{137}$	324^2
17	17	257	20^2	1	$(11)\mathcal{P}_{257}$	444^2
18	29	53	20^2	2	$(9)\mathcal{P}_{53}$	52^2
19	73	97	30^2	1	$(5)\mathcal{P}_{97}$	292^2
20	17	8	670^2	47	$(11)\mathcal{P}_2^3$	4^2
21	73	8	550^2	35	$(5)\mathcal{P}_2^3$	16^2
22	89	8	330^2	5	$(29)\mathcal{P}_2^3$	400^2
23	233	8	150^2	14	$(17)\mathcal{P}_2^3$	516^2
24	281	8	190^2	7	$(9)\mathcal{P}_2^3$	208^2
25	41	5	860^2	43	$(17)\mathcal{P}_5$	19^2
26	61	5	360^2	34	$(22)\mathcal{P}_5$	57^2
27	109	5	280^2	27	$(11)\mathcal{P}_5$	36^2
28	149	5	110^2	16	$(18)\mathcal{P}_5$	133^2
29	269	5	70^2	3	$(2)\mathcal{P}_5$	4^2
30	389	5	70^2	4	$(2)\mathcal{P}_5$	4^2
31	17	13	190^2	13	$(11)\mathcal{P}_{13}$	16^2
32	29	13	60^2	6	$(9)\mathcal{P}_{13}$	12^2
33	157	13	30^2	4	$(10)\mathcal{P}_{13}$	43^2
34	181	13	30^2	3	$(17)\mathcal{P}_{13}$	516^2
35	137	17	50^2	3	$(9)\mathcal{P}_{17}$	268^2
36	257	17	30^2	1	$\mathcal{P}_{17}$	4^2
37	53	29	20^2	3	$(10)\mathcal{P}_{29}$	84^2
38	97	73	30^2	2	$(23)\mathcal{P}_{73}$	756^2

Table 10 ($G(\mathbf{N}/\mathbb{Q}) \simeq V_{40} = C_5 \rtimes D_8$)

	$d_{\mathbf{L}}$	$\mathcal{F}_{\mathbf{N}_1/\mathbf{L}}$	$h^-_{\mathbf{N}} = (h^-_{\mathbf{N}_1})^2$		$d_{\mathbf{L}}$	$\mathcal{F}_{\mathbf{N}_1/\mathbf{L}}$	$h^-_{\mathbf{N}} = (h^-_{\mathbf{N}_1})^2$
1	17	$(79)\cdot\mathcal{P}_2^3$	73205^2	7	89	$(5^2)\cdot\mathcal{P}_2^3$	86525^2
2	41	$(109)\cdot\mathcal{P}_5$	3688955^2	8	89	$(59)\cdot\mathcal{P}_2^3$	2732816^2
3	41	$(179)\cdot\mathcal{P}_5$	5263280^2	9	109	$(79)\cdot\mathcal{P}_5$	2044655^2
4	41	$(199)\cdot\mathcal{P}_5$	9782005^2	10	181	$(19)\cdot\mathcal{P}_{13}$	194011^2
5	61	$(59)\cdot\mathcal{P}_5$	101680^2	11	257	$(19)\cdot\mathcal{P}_{17}$	1030480^2
6	73	$(5^2)\cdot\mathcal{P}_2^3$	9136^2	12	389	$(29)\cdot\mathcal{P}_5$	228005^2

6. The case $\mathbf{G} \simeq C_2 \times F_{5,4}$. Let $\mathbf{N}$ be a normal CM-field of degree 40 with Galois group $\mathbf{G} = G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times F_{5,4} = C_2 \times \langle a, b : a^5 = b^4 = 1,\ b^{-1}ab = a^2\rangle \simeq \langle \sigma, \tau : \sigma^{10} = \tau^4 = 1,\ \tau^{-1}\sigma\tau = \sigma^7\rangle$. Note that $\mathbf{N}^+$ is a normal real field with Galois group $G(\mathbf{N}^+/\mathbb{Q}) \simeq F_{5,4}$. Moreover, $D(\mathbf{G}) = \langle\sigma^2\rangle$ and $Z(\mathbf{G}) = \langle\sigma^5\rangle$. Hence, σ^5 is the complex conjugation in $\mathbf{G}$. Let $\mathbf{N}_8$ be the fixed subfield of the 5-Sylow normal subgroup $D(\mathbf{G})$ of $\mathbf{G}$. Then $\mathbf{N}_8$ is an imaginary abelian octic field whose maximal totally real subfield $\mathbf{N}_8^+$ is cyclic quartic, and we let $\mathbf{L}$ denote the quadratic subfield of $\mathbf{N}_8^+$ and $\mathbf{L}_{\rm im}$ be any one of the two imaginary quadratic subfield of $\mathbf{N}_8$. Notice that $w_{\mathbf{N}} = w_{\mathbf{N}_8}$. We have the following lattice of subfields:



$G(\mathbf{N}^+/\mathbb{Q}) \simeq F_{5,4}$

$G(\mathbf{N}_8/\mathbb{Q}) \simeq C_4 \times C_2$

$G(\mathbf{N}_8^+/\mathbb{Q}) \simeq C_4$

$G(\mathbf{N}/\mathbf{N}_8^+) \simeq C_{10}$

Proposition 17. *Let $\mathbf{K}/\mathbf{M}$ be a cyclic quintic extension of a real cyclic quartic field $\mathbf{M}$. Let χ be a character of order 5 associated with $\mathbf{K}/\mathbf{M}$. Fix a generator b of $G(\mathbf{M}/\mathbb{Q})$.*

(1) *$\mathbf{K}$ is a normal number field with Galois group $G(\mathbf{K}/\mathbb{Q}) \simeq F_{5,4}$ if and only if $\mathcal{F}_{\mathbf{K}/\mathbf{M}}$ is invariant under the action of $G(\mathbf{M}/\mathbb{Q})$ and for some $u \in \{2,3\}$ we have $\chi(b(\mathcal{P})) = \chi(\mathcal{P})^u$ for all prime ideals $\mathcal{P}$ of $\mathbf{M}$.*

(2) *Let $\mathbf{K}$ be a normal real field of degree 20 with $G(\mathbf{K}/\mathbb{Q}) \simeq F_{5,4}$. Let $\mathcal{P}_q$, e_q and f_q denote a prime ideal of $\mathbf{M}$ above a rational prime q, its ramification index, and its inertial degree, respectively.*

(a) *If q does not split completely in $\mathbf{M}/\mathbb{Q}$, then $\mathcal{P}_q$ is not inert in $\mathbf{K}/\mathbf{M}$. Moreover, if $\mathcal{P}_q$ is ramified in both $\mathbf{M}/\mathbb{Q}$ and $\mathbf{K}/\mathbf{M}$ then $q = 5$.*

(b) *Let* $\mathcal{I}_5$ *denote the ideal of* $\mathbf{M}$ *such that* $(5) = \mathcal{I}_5^{e_5}$. *Then*

$$\mathcal{F}_{\mathbf{K}/\mathbf{M}} = \mathcal{I}_5^e \Big(\prod q \Big)$$

where $\prod q$ *is a finite product of distinct rational primes* q*'s such that*

$$\begin{cases} q \equiv 1 \pmod 5 & \text{if } f_q = 1, \\ q \equiv \pm 1 \pmod 5 & \text{if } f_q = 2, \\ q \not\equiv 1 \pmod p & \text{if } f_q = 4, \end{cases}$$

and either $e = 0$ *or*

$$\begin{cases} e = 2 & \text{if } e_5 = 1, \\ e \in \{2,3\} & \text{if } e_5 = 2, \\ e \in \{2,3,4,6\} & \text{if } e_5 = 4. \end{cases}$$

Proof. (1) We first prove the necessity. Let $\Phi_{\mathbf{K}/\mathbf{M}}$ denote the Artin map associated with $\mathbf{K}/\mathbf{M}$. Note that

$$\chi(b(\mathcal{P})) = \chi(\mathcal{P})^u \ \Leftrightarrow\ \Phi_{\mathbf{K}/\mathbf{M}}(b(\mathcal{P})) = b^{-1}\Phi_{\mathbf{K}/\mathbf{M}}(\mathcal{P})b = \Phi_{\mathbf{K}/\mathbf{M}}(\mathcal{P})^u.$$

This shows that if $\mathcal{F}_{\mathbf{K}/\mathbf{M}}$ is invariant under b, so is the kernel $\mathrm{Ker}(\Phi_{\mathbf{K}/\mathbf{M}})$, which yields the normality of $\mathbf{K}$ (see [Cohn, Thm. 8.2.5]). Therefore, considering the Galois group $G(\mathbf{K}/\mathbb{Q})$, we get the desired result. The sufficiency is easily checked.

(2) First, if q does not split completely in $\mathbf{M}/\mathbb{Q}$ then there exists some $i_0 \in \{1,2,3\}$ such that $b^{i_0}(\mathcal{P}_q) = \mathcal{P}_q$. Hence, $\chi(b^{i_0}(\mathcal{P}_q)) = \chi(\mathcal{P}_q)^{u^{i_0}} = \chi(\mathcal{P}_q)$, which gives $\chi(\mathcal{P}_q) = 1$, and the first claim of (a) is proved. The last claim of (a) follows from ramification theory.

Second, assume that $q \neq 5$ and $\mathcal{P}_q$ divides $\mathcal{F}_{\mathbf{K}/\mathbf{M}}$. Then, since $\mathcal{F}_{\mathbf{K}/\mathbf{M}}$ is invariant under action of $G(\mathbf{M}/\mathbb{Q})$, (q) divides $\mathcal{F}_{\mathbf{K}/\mathbf{M}}$. By the method of [LPL, Lemma 5] we get $\nu_q(\mathcal{F}_{\mathbf{K}/\mathbf{M}}) = 1$, where ν_q denotes the q-adic valuation. Note that there exists a primitive modular character of order 5 on $(A_{\mathbf{M}}/(q))^*$ which is trivial on $\mathrm{Im}\,\mathbb{Z}$, the image of $\mathbb{Z}$. Hence, the order of $(A_{\mathbf{M}}/(q))^*/\mathrm{Im}\,\mathbb{Z}$ must be divisible by 5.

Third, assume that 5 is ramified in $\mathbf{K}/\mathbf{M}$. It is easily checked that $e > 1$. Assume that $e \geq 3$ for $e_5 = 1$, $e \geq 4$ for $e_5 = 2$, and $e \geq 7$ for $e_5 = 4$. Let $\alpha \equiv 1 \pmod{\mathcal{P}_5^{e-1}}$. Then there exists $\beta \in \mathcal{P}_5^{e-1-e_5}$ such that $\alpha = 1 + 5\beta$. By using $\nu_5(C_k^5) = 1 - \nu_5(k)$ for $1 \leq k \leq 5$, we obtain $\alpha \equiv (1+\beta)^5 \pmod{\mathcal{P}_5^e}$, which contradicts the existence of a primitive modular character of order 5 on $(A_{\mathbf{M}}/\mathcal{P}_5^e)^*$. Finally, by the same trick of [LPL, Lemma 5] we get $e \neq 5$ for $e_5 = 4$, which complete the proof of (b). ∎

THEOREM 18. *Let* χ *denote any primitive character of order* 10 *associated with* $\mathbf{N}/\mathbf{N}_8^+$ *and let* W_χ *denote the Artin root number associated with* χ. *Then* $W_\chi = \pm 1$ *and* $L(0,\chi) \in 16\mathbb{Z}$. *Moreover*, $h_{\mathbf{N}_8}^-$ *divides* $h_{\mathbf{N}}^-$, *and*

$h^-_{\mathbf{N}}/h^-_{\mathbf{N}_8} = (h^-_{\mathbf{N}/\mathbf{N}_8})^4$ *is the 4th power of the rational integer*:

$$h^-_{\mathbf{N}/\mathbf{N}_8} = \frac{1}{16} L(0,\chi).$$

Proof. The proof is similar to that of Theorem 5. Let σ_u denote a generator of the Galois group $G(\mathbb{Q}(\zeta_{10})/\mathbb{Q})$ such that $\sigma_u(\zeta_{10}) = \zeta_{10}^u$, where $\zeta_{10} = e^{2\pi i/10}$. Then, since for any ideal $\mathcal{I}$, $\sigma_u(\chi(\mathcal{I})) = \chi(\mathcal{I})^u = \chi(b(\mathcal{I}))$ for a generator $b \in G(\mathbf{N}_8^+/\mathbb{Q})$, we conclude that the algebraic number $L(0,\chi)$ which is invariant under the action of $G(\mathbb{Q}(\zeta_{10})/\mathbb{Q})$ is rational. ∎

To compute numerical approximations of $L(0,\chi)$ by using the technique developed in [Lou5] and [Lou6], we have to be able to compute the coefficients $a_n(\chi) := \sum_{N_{\mathbf{N}_8^+/\mathbb{Q}}(\mathcal{I})=n} \chi(\mathcal{I})$. For convenience, let us set some notations. Let $\mathcal{P}_q$ and f_q denote a prime ideal in $\mathbf{N}_8^+$ above a rational prime q and its inertial degree, respectively. We have:

Proposition 19. *Let χ_+ and χ_- denote the characters associated with the cyclic extensions $\mathbf{N}^+/\mathbf{N}_8^+$ and $\mathbf{N}_8/\mathbf{N}_8^+$, respectively, such that $\chi = \chi_+\chi_-$ is a character of order 10 associated with $\mathbf{N}/\mathbf{N}_8^+$. If either q divides $f_{\mathbf{N}/\mathbf{N}_8^+}$ or f_q does not divide k, then $a_{q^k}(\chi) = 0$. Otherwise, set $\varepsilon_q = \chi_-(\mathcal{P}_q) = \pm 1$ and $\eta_q = \chi_+(\mathcal{P}_q) = \zeta_5^n$, for some $n \in \mathbb{Z}$. Then*

$$a_{q^k}(\chi) = \begin{cases} \varepsilon_q^k & \text{if } e_q = 4 \text{ and } f_q = 1, \\ \varepsilon_q^{k/2} & \text{if } e_q = 2 \text{ and } f_q = 2, \\ \varepsilon_q^k(k+1) & \text{if } e_q = 2 \text{ and } f_q = 1, \\ 1 & \text{if } e_q = 1 \text{ and } f_q = 4, \\ k/2+1 & \text{if } e_q = 1 \text{ and } f_q = 2. \end{cases}$$

If $e_q = f_q = 1$, then

$$a_{q^k}(\chi) = \begin{cases} \dfrac{(k+1)(k+2)(k+3)}{6}\varepsilon_q^k & \text{if } \eta_q = 1, \\ \varepsilon_q^k & \text{if } k \equiv 0 \pmod 5 \text{ and if } \eta_q \neq 1, \\ -\varepsilon_q^k & \text{if } k \equiv 1 \pmod 5 \text{ and if } \eta_q \neq 1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Assume that $e_q = f_q = 1$. Then

$$a_{q^k}(\chi) = \varepsilon_q^k \sum_{\substack{r+s+t+u=k \\ r,s,t,u \geq 0}} \eta_q^{r+2s+3t+4u}.$$

If $\eta_q \neq 1$ then since $\sum_{r+s+t+u=k} \eta_q^{r+2s+3t+4u}$ is the coefficient of x^k in $(1-x)/(1-x^5) = (1-x)(\sum_{a\geq 0} x^{5a})$, we have the desired result. The others are immediate from the definition of $a_{q^k}(\chi)$ and ramification theory. ∎

Now, assume that $h^-_{\mathbf{N}} = 1$. Then $h^-_{\mathbf{N}_8} = 1$, and there are 18 such $\mathbf{N}_8$'s (see [CK]). An easy computation shows that $\zeta_{\mathbf{N}_8}(s) \leq 0$ in the range $0 < s < 1$

for these 18 fields $\mathbf{N}_8$. Therefore, by using Theorem 10, for each of these 18 fields $\mathbf{N}_8$ we can compute an upper bound $B(\mathbf{N}_8^+)$ such that $h_{\mathbf{N}}^- > 1$ if $f_{\mathbf{N}^+/\mathbf{N}_8^+} > B(\mathbf{N}_8^+)$. In Table 11, we let $f_{\mathbf{N}_8^+}$ and f_2 denote the conductor of $\mathbf{N}_8^+$ and that of an imaginary quadratic subfield of $\mathbf{N}_8$, respectively. In the factorization $f_{\mathbf{N}_8^+}$, we mark the conductor of a character of order 4 with the bold face. Let n_f denote the number of possible conductors of $\mathbf{N}^+/\mathbf{N}_8^+$ satisfying $f_{\mathbf{N}^+/\mathbf{N}_8^+} \leq B(\mathbf{N}_8^+)$ and let N_f denote the number of possible conductors of $\mathbf{N}/\mathbf{N}_8^+$ satisfying $f_{\mathbf{N}^+/\mathbf{N}_8^+} \leq B(\mathbf{N}_8^+)$ and being filtered by using either Proposition 3(2) or Propositions 3(3) and 17(2)(a). In Table 12, we list the relative class numbers $h_{\mathbf{N}}^- = (h_{\mathbf{N}/\mathbf{N}_8}^-)^4$ of the six CM-fields $\mathbf{K}$ which are obtained in the last column of Table 11. We should point out that in Tables 11 and 12, we used PARI-GP to construct primitive characters of order 5 on the ray class group $Cl_{\mathbf{N}_8^+}(\mathcal{F}_{\mathbf{N}^+/\mathbf{N}_8^+})$. According to our computations, the normal CM-field of degree 40 given in Theorem 2 is the only normal CM-field of degree 40 with $G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times F_{5,4}$ and relative class number one.

Table 11 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times F_{5,4}$)

	$f_{\mathbf{N}_8^+}$	f_2	$h_{\mathbf{N}_8^+}$	$Q_{\mathbf{N}}$	$w_{\mathbf{N}}$	$\mathrm{Reg}_{\mathbf{N}_8^+} \leq$	$\mu_{\mathbf{N}_8^+} \leq$	$B(\mathbf{N}_8^+)$	n_f	N_f
1	$\mathbf{5} \cdot 3$	3	1	2	30	0.2780	0.5089	23^4	2	2
2	$\mathbf{5} \cdot 4$	4	1	2	20	0.3315	0.6025	22^4	3	1
3	$\mathbf{5} \cdot 7$	7	1	2	10	0.3441	0.9326	17^4	3	2
4	$\mathbf{5} \cdot 8$	8	1	2	10	0.4028	0.9337	17^4	2	0
5	$\mathbf{13} \cdot 4$	4	1	2	4	0.3811	1.5474	12^4	0	0
6	$\mathbf{13} \cdot 7$	7	1	2	2	0.3238	2.1847	9^4	0	0
7	$\mathbf{16} \cdot 3$	3	1	2	6	0.6586	1.0155	19^4	0	0
8	$\mathbf{16}$	4	1	1	4	0.4317	0.5604	31^4	1	0
9	$\mathbf{16} \cdot 11$	11	1	2	2	0.4205	2.8114	9^4	0	0
10	$\mathbf{16} \cdot 5$	20	2	1	2	0.6950	1.3391	17^4	2	0
11	$\mathbf{37} \cdot 4$	4	1	2	4	1.4646	1.8139	10^4	0	0
12	$\mathbf{29} \cdot 8$	8	1	2	2	0.7201	2.5180	6^4	0	0
13	$\mathbf{16} \cdot 5$	4	2	1	4	0.6950	1.3391	15^4	2	0
14	$\mathbf{16}$	3	1	1	6	0.4317	0.5604	15^4	0	0
15	$\mathbf{16} \cdot 3 \cdot 11$	11	2	1	2	0.9479	2.4740	7^4	0	0
16	$\mathbf{17} \cdot 7 \cdot 3$	3	2	1	6	3.6084	1.9785	13^4	0	0
17	$\mathbf{17} \cdot 7 \cdot 11$	11	2	1	2	3.7957	3.0471	7^4	0	0
18	$\mathbf{61} \cdot 7$	7	5	2	2	2.2448	2.4298	6^4	2	1

Table 12 ($G(\mathbf{N}/\mathbb{Q}) \simeq C_2 \times F_{5,4}$, all $W_{\mathbf{N}} = 1$)

	$f_{\mathbf{N}_8^+}$	f_2	$\mathcal{F}_{\mathbf{N}^+/\mathbf{N}_8^+}$	$h_{\mathbf{N}/\mathbf{N}_8}^-$		$f_{\mathbf{N}_8^+}$	f_2	$\mathcal{F}_{\mathbf{K}^+/\mathbf{N}_8^+}$	$h_{\mathbf{N}/\mathbf{N}_8}^-$
1	$5 \cdot 3$	3	(10)	2	**4**	$\mathbf{5 \cdot 7}$	**7**	$(\mathbf{2}) \cdot \mathcal{P}_{\mathbf{5}}^{\mathbf{2}}$	**1**
2	$5 \cdot 3$	3	$(7) \cdot \mathcal{P}_5^2$	5	5	$5 \cdot 7$	7	(15)	12
3	$5 \cdot 4$	4	$\mathcal{P}_5^6$	3	6	$61 \cdot 7$	7	(1)	4

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References

[CK] K.-Y. Chang and S.-H. Kwon, *Class numbers of imaginary abelian number fields*, Proc. Amer. Math. Soc. 128 (2000), 2517–2528.

[Cohn] H. Cohn, *Introduction to the Construction of Class Fields*, Cambridge Univ. Press, Cambridge, 1985.

[Lef] Y. Lefeuvre, *Corps diédraux à multiplication complexe principaux*, Ann. Inst. Fourier (Grenoble) 50 (2000), 67–103.

[LLO] F. Lemmermeyer, S. Louboutin and R. Okazaki, *The class number one problem for some non-abelian normal CM-fields of degree 24*, J. Théor. Nombres Bordeaux 11 (1999), 387–406.

[Lou1] S. Louboutin, *On the class number one problem for non-normal quartic CM-fields*, Tôhoku Math. J. 46 (1994), 1–12.

[Lou2] —, *Corps quadratiques à corps de classes de Hilbert principaux et à multiplication complexe*, Acta Arith. 74 (1996), 121–140.

[Lou3] —, *Upper bounds on* $|L(1,\chi)|$ *and applications*, Canad. J. Math. 50 (1998), 794–815.

[Lou4] —, *The class number one problem for the dihedral and dicyclic CM-fields*, Colloq. Math. 80 (1999), 259–265.

[Lou5] —, *Computation of relative class numbers of CM-fields by using Hecke L-functions*, Math. Comp. 69 (2000), 371–393.

[Lou6] —, *Computation of* $L(0,\chi)$ *and of relative class numbers of CM-fields*, Nagoya Math. J. 161 (2001), 171–191.

[LO1] S. Louboutin and R. Okazaki, *Determination of all non-normal quartic CM-fields and of all non-abelian normal octic CM-fields with class number one*, Acta Arith. 67 (1994), 47–62.

[LO2] —, —, *The class number one problem for some non-abelian normal CM-fields of 2-power degrees*, Proc. London Math. Soc. 76 (1998), 523–548.

[LOO] S. Louboutin, R. Okazaki and M. Olivier, *The class number one problem for some non-abelian normal CM-fields*, Trans. Amer. Math. Soc. 349 (1997), 3657–3678.

[LP] S. Louboutin and Y.-H. Park, *Class number problems for dicyclic CM-fields*, Publ. Math. Debrecen 57/3-4 (2000), 283–295.

[LPL] S. Louboutin, Y.-H. Park and Y. Lefeuvre, *Construction of the real dihedral number fields of degree* $2p$*. Applications*, Acta Arith. 89 (1999), 201–215.

[Mar] J. Martinet, *Sur l'arithmétique des extensions galoisiennes à groupe de Galois diédral d'ordre* $2p$, Ann. Inst. Fourier (Grenoble) 19 (1969), no. 1, 1–80.

[Wa] L. C. Washington, *Introduction to Cyclotomic Fields*, Grad. Texts in Math. 83, Springer, 1997.

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