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On the distribution of squares of integral quaternions II

by

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1. Introduction and statement of the main result. Let $\mathbb{J} = \mathbb{Z}^4 \cup \left(\frac{1}{2} + \mathbb{Z}\right)^4$ denote the Hurwitz ring of integral quaternions. In the first part [4] of the paper we developed the following two asymptotic formulas, generalizing a result of H. Müller and W. G. Nowak [6] on the distribution of squares of Gaussian integers.

As $X \to \infty$,

(i) $$\#\{q^2 \mid q \in \mathbb{J} \wedge q^2 \in [-X, X]^4\} = cX^2 - \frac{2\pi}{3} X^{3/2} + O(X^{96/73} (\log X)^{461/146}),$$

where $c = 7.674124\ldots$,

(ii) $$\#\{q^2 \mid q \in \mathbb{J} \wedge |\mathrm{Re}(q^2)|, |\mathrm{Im}(q^2)| \le X\} = 2\pi X^2 - \frac{2\pi}{3} X^{3/2} + O(X^{7/6} (\log X)^{19/4}),$$

where $\mathrm{Re}(a) = a_0$ *is the real part and* $\mathrm{Im}(a) := (a_1, a_2, a_3)$ *is the imaginary* (*or vector*) *part of the quaternion* $a = (a_0, a_1, a_2, a_3)$, *and* $|\cdot|$ *is the Euclidean norm.*

A natural variation of the two distribution questions investigated in [4] arises from introducing the quaternions as hypercomplex numbers. Define $\mathbb{H} = \mathbb{C} + \mathbb{C}j$, where j is the "hyper-imaginary" unit, and define addition and multiplication formally with respect to $j^2 = -1$ and $jz = \bar{z}j$ for $z \in \mathbb{C}$. Then $\mathbb{H}$ equals the division ring of Hamilton's quaternions, and the subring $\mathbb{Z}\frac{1+i+j+ij}{2} + \mathbb{Z}i + \mathbb{Z}[i]j$ equals the Hurwitz ring $\mathbb{J}$.

If $q \in \mathbb{H}$ then let us call $\mathrm{CP}(q) := \alpha$ the *complex part* and $\mathrm{HCP}(q) := \beta$ the *hypercomplex part* of the quaternion $q = \alpha + \beta j$ ($\alpha, \beta \in \mathbb{C}$).

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The topic of the present paper is to derive an asymptotic formula for the number of quaternions q^2 with $q \in \mathbb{J}$ and $|\mathrm{CP}(q^2)|, |\mathrm{HCP}(q^2)| \le X$, where X is a large positive parameter. Now, the main result of the present paper is the following theorem.

Theorem 1. *For positive real X let*

$$A(X) := \#\{q^2 \mid q \in \mathbb{J} \wedge |\mathrm{CP}(q^2)|, |\mathrm{HCP}(q^2)| \le X\}.$$

Then as $X \to \infty$,

$$A(X) = C_1 X^2 + C_2 X^{3/2} + O(X^{96/73}(\log X)^{461/146}),$$

where $C_1 = 6.393466\ldots$ is a numerical constant, and

$$C_2 := \pi\left(\frac{\sqrt{17\sqrt{2}+23}-8}{6}\right) = -0.597588\ldots$$

Remark. The main term $C_1 X^2$ equals the volume of the non-convex, four-dimensional domain $\{q \in \mathbb{H} \mid |\mathrm{CP}(q^2)|, |\mathrm{HCP}(q^2)| \le X\}$. As in [4], the second main term $C_2 X^{3/2}$ occurs because of the exceptional role of the *imaginary space* $\mathrm{Im}\,\mathbb{H} := \mathbb{R}i + \mathbb{C}j$.

2. Preparation for the proof. For $X > 0$, define the four-dimensional body

$$K(X) := \{(a_0, a_1, a_2, a_3) \in \mathbb{R}^4 \mid 4a_0^2(a_2^2 + a_3^2) \le X^2 \wedge \\ (a_0^2 - a_1^2 - (a_2^2 + a_3^2))^2 + 4a_0^2 a_1^2 \le X^2\}.$$

By adding the two inequalities we observe that

$$(a_0, a_1, a_2, a_3) \in K(X) \\ \Leftrightarrow\ (a_0^2 + a_1^2 + a_2^2 + a_3^2)^2 \le X^2 + 4a_0^2(a_2^2 + a_3^2) \le 2X^2,$$

so that $K(X)$ is contained in the four-dimensional ball with radius $\sqrt[4]{2}\sqrt{X}$ and center in the origin. In particular, $K(X)$ is compact and $K(X) \subset [-\sqrt[4]{2}\sqrt{X}, \sqrt[4]{2}\sqrt{X}]^4$. It is plain that $K(X)$ is not convex. Further, we obtain $K(X)$ by "blowing up" the basic domain $K(1)$ with factor $\sqrt{X}$, i.e. $K(X) = \sqrt{X} \cdot K(1)$.

Recall that (having identified $\mathbb{H}$ with $\mathbb{R}^4$) for $q = (a_0, a_1, a_2, a_3) \in \mathbb{H}$,

$$q^2 = (a_0^2 - a_1^2 - a_2^2 - a_3^2, 2a_0a_1, 2a_0a_2, 2a_0a_3),$$

whence $|\mathrm{CP}(q^2)|, |\mathrm{HCP}(q^2)| \le X$ iff $q \in K(X)$. Referring to [4] we have

$$q^2 = p^2 \quad \text{iff} \quad q = p \text{ or } q = -p$$

for all $q, p \in \mathbb{H} \setminus \mathrm{Im}\,\mathbb{H} = \mathbb{R}^4 \setminus (\{0\} \times \mathbb{R}^3)$. Consequently, with $\mathbb{J} = \mathbb{Z}^4 \cup (1/2 + \mathbb{Z})^4$,

$$\#\{q^2 \mid q \in \mathbb{J} \setminus \mathrm{Im}\,\mathbb{H} \wedge |\mathrm{CP}(q^2)|, |\mathrm{HCP}(q^2)| \le X\} \\ = \#(K(X) \cap \mathbb{J} \cap (]0, \infty[\, \times \mathbb{R}^3)).$$

For $q, p \in \operatorname{Im}\mathbb{H} = \{0\} \times \mathbb{R}^3$ we have $q^2 = p^2$ iff $|q| = |p|$. Further, $\operatorname{HCP}(q^2) = 0$ and $\operatorname{CP}(q^2) = -|q|^2$ for every $q \in \operatorname{Im}\mathbb{H}$, whence

$$\#\{q^2 \mid q \in \mathbb{J} \cap \operatorname{Im}\mathbb{H} \wedge |\operatorname{CP}(q^2)|, |\operatorname{HCP}(q^2)| \leq X\} \leq X + 1.$$

Thus we have

$$A(X) = \#(K(X) \cap \mathbb{J} \cap (]0, \infty[\times \mathbb{R}^3)) + O(X).$$

By symmetry we can write

$$A(X) = \#(K_0(X) \cap \mathbb{J}) + 2 \cdot \#(K_+(X) \cap \mathbb{J}) + O(X), \tag{2.1}$$

where

$$K_0(X) := K(X) \cap (]0, \infty[\times \{0\} \times \mathbb{R} \times \mathbb{R}),$$
$$K_+(X) := K(X) \cap (]0, \infty[\times]0, \infty[\times \mathbb{R} \times \mathbb{R}).$$

Thus our problem is to count all integral lattice points in the three-dimensional domain $K_0(X)$ (note that $K_0(X) \cap \mathbb{J} = K_0(X) \cap \mathbb{Z}^4$) and all integral and half odd integral lattice points in the four-dimensional domain $K_+(X)$.

3. Counting the lattice points in $K_0(X)$. For abbreviation throughout the paper, define constants

$$c_1 := \sqrt{\frac{\sqrt{2} - 1}{2}}, \quad c_2 := \sqrt{\frac{1}{2}}, \quad c_3 := \sqrt[4]{\frac{1}{2}}, \quad c_4 := \sqrt{\frac{\sqrt{2} + 1}{2}},$$

so that $0 < c_1 < c_2 < c_3 < 1 < c_4 < \sqrt[4]{2}$.

For $X > 0$ and $a \neq 0$ define circular rings

$$D(X; a) := \{(x, y) \in \mathbb{R}^2 \mid -X + a^2 \leq x^2 + y^2 \leq \min\{X^2/(4a^2), X + a^2\}\},$$

so that $D(X; a)$ is a circular disk iff $|a| \leq \sqrt{X}$ and $D(X; a) = \emptyset$ iff $|a| > c_4\sqrt{X}$. Since

$$\begin{aligned}&\#(K_0(X) \cap \mathbb{J}) \\ &\quad = \#\{(a_0, a_2, a_3) \in \mathbb{N} \times \mathbb{Z} \times \mathbb{Z} \mid a_2^2 + a_3^2 \leq X^2/(4a_0^2) \wedge |a_0^2 - (a_2^2 + a_3^2)| \leq X\},\end{aligned}$$

we can write

$$\#(K_0(X) \cap \mathbb{J}) = \sum_{0 < a \leq c_4\sqrt{X}} \#(D(X; a) \cap \mathbb{Z}^2). \tag{3.1}$$

Referring to Huxley's deep result concerning the circle problem (cf. [2], Theorem 18.3.2), we note that uniformly in $\varrho \in [0, 1]$, as $T \to \infty$,

$$\#\{(x, y) \in (\varrho + \mathbb{Z})^2 \mid x^2 + y^2 \leq T\} = \pi T + O(T^{23/73} (\log T)^{315/146}). \tag{3.2}$$

As a consequence, since $r_2(n) \ll n^{\varepsilon}$, we also have

$$\#\{(x, y) \in (\varrho + \mathbb{Z})^2 \mid x^2 + y^2 < T\} = \pi T + O(T^{23/73} (\log T)^{315/146}) \tag{3.3}$$

for $\varrho = 0$ and $\varrho = 1/2$.

Now, applying (3.2) and (3.3) to (3.1) we can write

$$\frac{1}{\pi}\cdot\#(K_0(X)\cap\mathbb{J})=\sum_{0<a\le c_1\sqrt{X}}(X+a^2)+\sum_{c_1\sqrt{X}<a\le c_4\sqrt{X}}\frac{X^2}{4a^2}$$
$$-\sum_{\sqrt{X}<a\le c_4\sqrt{X}}(a^2-X)+O(X^{119/146+\varepsilon}).$$

By making use of a rough version of the Euler summation formula, i.e.

$$\sum_{\alpha<k\le\beta}f(k)=\int_\alpha^\beta f(t)\,dt+O(\max_{\alpha\le t\le\beta}|f(t)|)\quad\text{for monotonic } f:[\alpha,\beta]\to\mathbb{R},$$

we obtain

$$\#(K_0(X)\cap\mathbb{J})=C_3X^{3/2}+O(X),\tag{3.4}$$

where $C_3=(4c_4-2)\pi/3=2.507762\ldots$

4. Uniform estimates of certain rounding error sums. Let the rounding error function ψ be defined by

$$\psi(z)=z-[z]-1/2\quad(z\in\mathbb{R})$$

throughout the paper. ([ ] are the Gauss brackets.)

Note that for every z, $\psi(z+a)=\psi(z)$ if $a\in\mathbb{Z}$, and $\psi(z+a)=\psi(z+1/2)$ if $a\in 1/2+\mathbb{Z}$.

Further, define functions α, β, and σ depending on our parameter $X\to\infty$ by

$$\alpha(X;u):=\sqrt{X-u^2}\quad(0\le u\le\sqrt{X}),\qquad\beta(X;u):=\frac{X}{2u}\quad(u>0),$$

and

$$\sigma(X;u):=\sqrt{\sqrt{2}\,X-u^2-\frac{X^2}{4u^2}}\quad(c_1\sqrt{X}\le u\le c_4\sqrt{X}),$$

so that

$$\alpha(X;u)=\sqrt{X}\,a\Big(\frac{u}{\sqrt{X}}\Big),\qquad\beta(X;u)=\sqrt{X}\,b\Big(\frac{u}{\sqrt{X}}\Big),$$
$$\sigma(X;u)=\sqrt{X}s\Big(\frac{u}{\sqrt{X}}\Big),$$

with

$$a(t):=\sqrt{1-t^2}\quad(0\le t\le 1),\qquad b(t):=\frac{1}{2t}\quad(t>0),$$

and

$$s(t):=\sqrt{\sqrt{2}-t^2-\frac{1}{4t^2}}\quad(c_1\le t\le c_4).$$

Clearly, the graph of a is a quarter of a circle, and the graph of b is a branch of a hyperbola. Obviously, $a(t) \leq b(t)$ $(0 < t \leq 1)$ and $a(t) = b(t)$ iff $t = c_2$.

Concerning the graph of s we note that $s(c_1) = s(c_4) = 0$, and $\max\{s(t) \mid c_1 \leq t \leq c_4\} = s(c_2) = \sqrt{2}\,c_1$. The tangents at the endpoints $(c_1, 0)$ and $(c_4, 0)$ are vertical. For the second derivative we have $s''(t) < -4$ on $c_1 < t < c_4$, so that s is strictly concave.

Further, we always have $b(t) \geq s(t)$ (with equality iff $t = c_3$), and $a(t) = s(t)$ iff $t = c_2c_4 \in \,]c_2, c_3[$, $a(t) < s(t)$ when $c_2c_4 < t \leq 1$, and $a(t) > s(t)$ when $c_1 \leq t < c_2c_4$.

For the proof of Theorem 1 we will need estimates of ψ-sums involving the functions α, β, and σ. To obtain these estimates the Discrete Hardy–Littlewood Method is required. (See Huxley [2] for a profound presentation of the method and its various applications to important problems of geometry and analytic number theory.)

Notation. For abbreviation, for $f : [a, b] \to \mathbb{R}$ and $\varrho \in \mathbb{R}$ define

$$\sum_{a \leq n \leq b}^{(\varrho)} f(n) := \sum_{\substack{a \leq n \leq b \\ n \in \varrho + \mathbb{Z}}} f(n).$$

LEMMA 1. *As* $X \to \infty$,

$$\sum_{a \leq n \leq b}^{(\varrho)} \psi(\tau + \alpha(X; n)) \ll X^{23/73} (\log X)^{315/146}$$

uniformly in $0 \leq a \leq b \leq \sqrt{X}$ *and uniformly in* $\varrho, \tau \in \mathbb{R}$.

Proof. This has been proved in [5]. Note that it is not possible to adapt the proof of Lemma 2 in the first part [4] of the paper because $a'''(0) = 0$.

LEMMA 2. *As* $X \to \infty$,

$$\sum_{a \leq n \leq b}^{(\varrho)} \psi(\tau + \beta(X; n)) \ll X^{23/73} (\log X)^{461/146}$$

uniformly in $0 < a \leq b \leq 2\sqrt{X}$ *and uniformly in* $\varrho, \tau \in \mathbb{R}$.

Proof. Apply [2], Theorem 18.2.3, with $T = X$ and $F(x) = 1/(2x + 2\varrho/M) + M\tau/X$ to every part of a dyadic division of the sum.

LEMMA 3. *Let* $\tau \in \{0, 1/2\}$. *Then as* $X \to \infty$,

$$\sum_{a \leq n \leq b}^{(\tau)} \psi(\tau + \sigma(X; n)) \ll X^{23/73} (\log X)^{315/146}$$

uniformly in $c_1\sqrt{X} \leq a \leq b \leq c_4\sqrt{X}$.

Proof. Since it is not possible to apply [2], Theorem 18.2.2 or 18.2.3, because $s'''(t_0) = 0$ for $t_0 = 0.8129\ldots$, we are going to prove the result in a rather indirect way. Let us first consider the domain $\mathcal{H} := \{(x,y) \in \mathbb{R}^2 \mid c_1 \le x \le c_4 \wedge |y| \le s(x)\}$, which is obviously convex. The boundary of $\mathcal{H}$, which equals the union of the graphs of s and $-s$, is sufficiently smooth and its radius of curvature is bounded away from zero and infinity, so that the domain $\mathcal{H}$ satisfies the assumptions of Huxley [2], Theorem 18.3.2. Then the radius r of curvature attains its maximum $r_{\max} = 2c_4$ at the point $(c_1, 0)$ and its minimum $r_{\min} \approx 1/7$ near the point $(2/3, \pm s(2/3))$, whence always $0.1 < r < 2.2$.

Now we consider a family of circular disks of equal size given by

$$\mathcal{D}(u) := \{(x,y) \in \mathbb{R}^2 \mid (x - w(u))^2 + y^2 \le 9\} \quad (c_1 \le u \le c_4)$$

with $w(u) := u - \sqrt{9 - s(u)^2}$, so that $c_1 - 3 \le w(u) \le c_4 - 3 < 0$ and $w(u)$ is increasing in u. In this connection the number 9 is a "house number" that guarantees that the fixed radius of the circles is always greater than any radius r of curvature of the boundary $\partial\mathcal{H}$ of the domain $\mathcal{H}$.

Obviously, for every $(u,y) \in \partial\mathcal{H}$ we have

$$\partial\mathcal{D}(u) \cap \partial\mathcal{H} = \{(u,y),(u,-y)\} = \{(u,s(u)),(u,-s(u))\}$$

and $\mathcal{D}(u) \cap [u,\infty[\times \mathbb{R} \subset \mathcal{H}$, i.e. the whole circular arc on the right of the line $x = u$ is always lying within the domain $\mathcal{H}$.

By a straightforward adaptation of the proof of Huxley's main theorem [2], Theorem 18.3.2, we have for $\tau \in \{0, 1/2\}$ and as $X \to \infty$,

$$\begin{aligned} &(4.1)\quad \#((\sqrt{X} \cdot (\mathcal{D}(u) \cap \mathcal{H})) \cap (\tau + \mathbb{Z})^2) \\ &= X \cdot \operatorname{area}(\mathcal{D}(u) \cap \mathcal{H}) + O(X^{23/73}(\log X)^{315/146}) \quad \text{uniformly in } u \in [c_1, c_4]. \end{aligned}$$

Now let

$$\Psi_1(X,\tau;u) := \sum_{c_1\sqrt{X} \le n \le u\sqrt{X}}^{(\tau)} \psi(\tau + \sigma(X;n)) \quad (c_1 \le u \le c_4),$$

so that it suffices to prove, as $X \to \infty$,

$$(4.2)\qquad \Psi_1(X,\tau;u) \ll X^{23/73}(\log X)^{315/146} \quad \text{uniformly in } u \in [c_1, c_4].$$

Further, for $c_1 \le u \le c_4$, let

$$\Psi_2(X,\tau;u) := \sum_{u\sqrt{X} < n \le (w(u)+3)\sqrt{X}}^{(\tau)} \psi(\tau + \sqrt{9X - (n - w(u)\sqrt{X})^2}),$$

so that, with $\varrho = \tau - w(u)\sqrt{X}$, as $X \to \infty$ and uniformly in $u \in [c_1, c_4]$,

$$\tag{4.3} \begin{aligned} \Psi_2(X,\tau;u) &= \sum_{\sqrt{(9-s(u)^2)X}<n\le\sqrt{9X}}^{(\varrho)} \psi(\tau+\alpha(9X;n)) \\ &\ll X^{23/73}(\log X)^{315/146}, \end{aligned}$$

by applying Lemma 1 (with $9X$ instead of X).

Now, we count the lattice points in $\sqrt{X}\cdot(\mathcal{D}(u)\cap\mathcal{H})$ once again by writing

$$\begin{aligned} &\#((\sqrt{X}\cdot(\mathcal{D}(u)\cap\mathcal{H}))\cap(\tau+\mathbb{Z})^2) \\ &\quad = -2\Psi_1(X,\tau;u)-2\Psi_2(X,\tau;u)+2\sum_{c_1\sqrt{X}<n\le u\sqrt{X}}^{(\tau)}\sigma(X;n) \\ &\qquad +2\sum_{u\sqrt{X}<n\le(w(u)+3)\sqrt{X}}^{(\tau)}\sqrt{9X-(n-w(u)\sqrt{X})^2}. \end{aligned}$$

Hence, by applying the Euler summation formula (cf. [3], Theorem 1.3) to the last two sums we obtain

$$\tag{4.4} \begin{aligned} &\frac{1}{2}\cdot\#((\sqrt{X}\cdot(\mathcal{D}(u)\cap\mathcal{H}))\cap(\tau+\mathbb{Z})^2) \\ &\quad = -\Psi_1(X,\tau;u)-\Psi_2(X,\tau;u) \\ &\qquad + X\int_{c_1}^{u} s(t)\,dt + X\int_{\sqrt{9-s(u)^2}}^{3}\sqrt{9-t^2}\,dt \\ &\qquad + \int_{c_1\sqrt{X}}^{u\sqrt{X}} s'\Big(\frac{t}{\sqrt{X}}\Big)\psi(t-\tau)\,dt \\ &\qquad - \int_{\sqrt{(9-s(u)^2)X}}^{\sqrt{9X}} \frac{t}{\sqrt{9X-t^2}}\psi(t-\tau+w(u)\sqrt{X})\,dt. \end{aligned}$$

Obviously,

$$\tag{4.5} \int_{c_1}^{u} s(t)\,dt + \int_{\sqrt{9-s(u)^2}}^{3}\sqrt{9-t^2}\,dt = \frac{1}{2}\operatorname{area}(\mathcal{D}(u)\cap\mathcal{H}) \qquad (c_1\le u\le c_4).$$

Concerning the first ψ-integral we note that (as $X\ge 16$)

$$|s'(t/\sqrt{X})|\le 2X^{1/4} \qquad (c_1\sqrt{X}+1\le t\le c_4\sqrt{X}-1)$$

and $\sigma(X;t)\le 3X^{1/4}$ when $c_1\sqrt{X}\le t\le c_1\sqrt{X}+1$ or $c_4\sqrt{X}-1\le t\le c_4\sqrt{X}$.

Consequently, by applying the second mean value theorem to a large integration interval of length $\asymp\sqrt{X}$ together with the estimate $|\int_a^b\psi(t)\,dt|\le 1/8$, and by applying the triangle inequality together with the estimate

$|\psi(\cdot)| \le 1/2$ to the remaining (one or two) short intervals of length ≤ 1, we obtain (as $X \ge 16$)

$$\Big|\int\limits_{c_1\sqrt{X}}^{u\sqrt{X}} s'(t/\sqrt{X})\psi(t-\tau)\,dt\Big| \le 4X^{1/4} \qquad (c_1 \le u \le c_4). \tag{4.6}$$

By a similar argument we obtain for every $\varrho \in \mathbb{R}$ (as $X \ge 1/9$)

$$\Big|\int\limits_{a}^{\sqrt{9X}} \frac{t}{\sqrt{9X-t^2}}\psi(t-\varrho)\,dt\Big| \le (9X)^{1/4} \qquad (0 \le a \le \sqrt{9X}). \tag{4.7}$$

Now we insert (4.5)–(4.7) into (4.4) and compare the result with (4.1); then (4.2) follows from (4.3).

5. Proof of Theorem 1. In order to calculate $\#(K_+(X)\cap\mathbb{J})$ we define circular rings $E(X;u,v)$ for $X \to \infty$ and $(u,v) \in {]0,\infty[}^2$ by

$$E(X;u,v) := \{(x,y)\in\mathbb{R}^2 \mid x^2+y^2 \le X^2/(4u^2)\wedge \\ (u^2-v^2-(x^2+y^2))^2+4u^2v^2 \le X^2\},$$

so that

$$\#(K_+(X)\cap\mathbb{J}) = \sum_{u\in\frac{1}{2}\mathbb{N}}\sum_{v>0}^{(u)} \#(E(X;u,v)\cap(u+\mathbb{Z})^2).$$

Naturally, the double sum is finite, because (as we will see below)

$$\#\{(u,v)\in\mathbb{N}^2\cup(1/2+\mathbb{N}_0)^2 \mid E(X;u,v)\ne\emptyset\} \le 3X \qquad (X>0). \tag{5.1}$$

In order to visualize the sets $E(X;u,v)$ we introduce functions F, G, and H depending on our parameter X by

$$G(X;u,v) := u^2-v^2-\sqrt{X^2-4u^2v^2} \qquad (u,v>0\wedge uv\le X/2),$$

$$H(X;u,v) := u^2-v^2+\sqrt{X^2-4u^2v^2} \qquad (u,v>0\wedge uv\le X/2),$$

and

$$F(X;u) := \frac{X^2}{4u^2} \qquad (u>0).$$

Then for $u,v>0$,

$$E(X;u,v) = \{(x,y)\in\mathbb{R}^2 \mid G(X;u,v)\le x^2+y^2 \\ \le \min\{F(X;u),H(X;u,v)\}\}$$

when $uv \le X/2$, and $E(X;u,v)=\emptyset$ otherwise. Further, for $uv\le X/2$ we also have $E(X;u,v)=\emptyset$ when $u<v$ and $u^2+v^2>X$ (since then $H(X;u,v)<0$), and also when $u>c_4\sqrt{X}$ (since then $G(X;u,v)>F(X;u)$). In particular, this yields (5.1).

Consequently, by (3.2) and (3.3) (and with area $\emptyset = 0$) we have

$$\#(K_+(X)\cap\mathbb{J}) = \sum_{\substack{0<u\le c_4\sqrt{X}\\ u\in\frac12\mathbb{Z}}}\ \sum_{0<v\le\eta(X;u)}^{(u)} \operatorname{area} E(X;u,v) + O(X^{96/73}(\log X)^{315/146}),$$

with

$$\eta(X;u) := \begin{cases} \alpha(X;u) & \text{if } 0<u\le c_2\sqrt{X},\\ \beta(X;u) & \text{if } u\ge c_2\sqrt{X}.\end{cases}$$

The O-term results from combining (5.1) with

$$\min\{F(X;u),H(X;u,v)\}\ll X\ (0<u\le c_4\sqrt{X},\ 0<v\le\eta(X;u)), \tag{5.2}$$

which is true as we will see below.

Now we distinguish the several cases when $G>0$, $G\le 0$, $H<F$, $G\le F\le H$, and $F<G$, and note that for $0<u\le c_4\sqrt{X}$ and $0<v\le\eta(X;u)$,

$$G(X;u,v)\le 0\ \Leftrightarrow\ u^2+v^2\le X\ \Leftrightarrow\ v\le\alpha(X;u),$$

$$H(X;u,v)\le F(X;u)$$
$$\Leftrightarrow\ (u\le c_1\sqrt{X})\vee(c_1\sqrt{X}\le u\le c_3\sqrt{X}\wedge v\ge\sigma(X;u)),$$

which immediately implies (5.2), and

$$\begin{aligned}0\le G(X;u,v)\le F(X;u)&\\ \Leftrightarrow\ &(c_2\sqrt{X}\le u\le c_3\sqrt{X}\wedge\alpha(X;u)\le v\le\beta(X;u))\\ &\vee(c_3\sqrt{X}\le u\le\sqrt{X}\wedge\alpha(X;u)\le v\le\sigma(X;u))\\ &\vee(u\ge\sqrt{X}\wedge v\le\sigma(X;u)).\end{aligned}$$

Thus we derive, as $X\to\infty$,

$$\#(K_+(X)\cap\mathbb{J}) = \pi\sum_{i=1}^{3}S_i(X) - \pi\sum_{i=4}^{6}S_i(X) + \pi S_7(X) + O(X^{96/73}(\log X)^{315/146}), \tag{5.3}$$

where the terms $S_i(X)$ $(i=1,2,\dots,7)$ are double sums of the form

$$S_i(X) := \sum_{y_i<u\le z_i}^{*}\ \sum_{\gamma_i(u)<v\le\delta_i(u)}^{(u)} f_i(u,v), \tag{5.4}$$

with the summation limits $y_i, z_i, \gamma_i(u), \delta_i(u)$ and the functions f_i given by the following table:

i	y_i	z_i	$\gamma_i(u)$	$\delta_i(u)$	$f_i(u,v)$
1	0	$c_1\sqrt{X}$	0	$\alpha(X;u)$	$H(X;u,v)$
2	$c_1\sqrt{X}$	$c_2\sqrt{X}$	$\sigma(X;u)$	$\alpha(X;u)$	$H(X;u,v)$
3	$c_2\sqrt{X}$	$c_3\sqrt{X}$	$\sigma(X;u)$	$\beta(X;u)$	$H(X;u,v)$
4	$c_2\sqrt{X}$	$c_3\sqrt{X}$	$\alpha(X;u)$	$\beta(X;u)$	$G(X;u,v)$
5	$c_3\sqrt{X}$	$\sqrt{X}$	$\alpha(X;u)$	$\sigma(X;u)$	$G(X;u,v)$
6	$\sqrt{X}$	$c_4\sqrt{X}$	0	$\sigma(X;u)$	$G(X;u,v)$
7	$c_1\sqrt{X}$	$c_4\sqrt{X}$	0	$\sigma(X;u)$	$F(X;u)$

Moreover, the star symbol on the outer sum of (5.4) means that u runs through $\frac{1}{2}\mathbb{Z}$, i.e. we define for abbreviation

$$\sum_{y<u\le z}^{*} f(u) := \sum_{\substack{y<u\le z\\ u\in\frac{1}{2}\mathbb{Z}}} f(u) \qquad \text{for } f:[y,z]\to\mathbb{R}.$$

Let us first investigate S_7 which is only formally a double sum and actually is given by

$$S_7(X) = \sum_{c_1\sqrt{X}<u\le c_4\sqrt{X}}^{*} F(X;u)(\sigma(X;u)+\psi(u)) - \sum_{c_1\sqrt{X}<u\le c_4\sqrt{X}}^{*} \psi(u+\sigma(X;u))F(X;u).$$

The second sum, which is a weighted ψ-sum, is $\ll X^{96/73}(\log X)^{315/146}$ by applying Lemma 3 together with Abelian summation. By the Euler summation formula, the first sum equals

$$\begin{aligned} 2\int_{c_1\sqrt{X}}^{c_4\sqrt{X}} F(X;u)\sigma(X;u)\,du &- \frac{1}{2}\int_{c_1\sqrt{X}}^{c_4\sqrt{X}} F(X;u)\,du \\ &- \frac{1}{2}\psi(c_1\sqrt{X})F(X;c_1\sqrt{X}) + \frac{1}{2}\psi(c_4\sqrt{X})F(X;c_4\sqrt{X}) \\ &- \frac{1}{2}\int_{c_1\sqrt{X}}^{c_4\sqrt{X}} \left(\frac{d}{du}F(X;u)\right)\psi(u)\,du \\ &+ \int_{c_1\sqrt{X}}^{c_4\sqrt{X}} \left(\frac{d}{du}(F(X;u)\sigma(X;u))\right)\left(\psi(u)+\psi\left(u+\frac{1}{2}\right)\right)du. \end{aligned}$$

The ψ-terms and the first ψ-integral are trivially $\ll X$. With the help of the second mean value theorem and with regard to (4.6), we obtain

$$\int\limits_{c_1\sqrt{X}}^{c_4\sqrt{X}} \left(\frac{d}{du}(F(X;u)\sigma(X;u))\right)(\psi(u)+\psi(u+1/2))\,du \ll X^{5/4},$$

and hence we finally arrive at

$$S_7(X) = C_4X^2 - C_5X^{3/2} + O(X^{96/73}(\log X)^{315/146}), \tag{5.5}$$

where

$$C_5 := \frac{1}{2}\int\limits_{c_1}^{c_4} F(1;u)\,du = \frac{c_1}{\sqrt{8}},$$

and (with electronic support)

$$C_4 := 2\int\limits_{c_1}^{c_4} F(1;u)s(u)\,du = 0.325322\ldots$$

In order to settle the other six terms $S_i(X)$ $(i = 1,\ldots,6)$ we apply the Euler summation formula once to the inner sum and twice to the outer sum of (5.4), which yields

$$S_i(X) = V_i(X) + R_i(X) + Q_i(X) + P_i(X) + T_i(X) \quad (i = 1,\ldots,6), \tag{5.6}$$

where for $i = 1,\ldots,6$ and $y := y_i$, $z := z_i$, $\gamma(u) := \gamma_i(u)$, $\delta(u) := \delta_i(u)$, and $f(u,v) := f_i(u,v)$,

$$V_i(X) := 2\int\limits_{y}^{z}\int\limits_{\gamma(u)}^{\delta(u)} f(u,v)\,dv\,du,$$

$$R_i(X) := (\psi(y)+\psi(y+1/2))\int\limits_{\gamma(y)}^{\delta(y)} f(y,v)\,dv$$
$$- (\psi(z)+\psi(z+1/2))\int\limits_{\gamma(z)}^{\delta(z)} f(z,v)\,dv,$$

$$Q_i(X) := \sum_{y<u\le z}{}^{*}\, \psi(\gamma(u)+u)f(u,\gamma(u)) - \sum_{y<u\le z}{}^{*}\, \psi(\delta(u)+u)f(u,\delta(u)),$$

$$P_i(X) := \sum_{y<u\le z}{}^{*} \int\limits_{\gamma(u)}^{\delta(u)} \frac{\partial f}{\partial v}(u,v)\psi(v+u)\,dv,$$

$$T_i(X) := \int\limits_{y}^{z}\left(\frac{\partial}{\partial u}\int\limits_{\gamma(u)}^{\delta(u)} f(u,v)\,dv\right)(\psi(u)+\psi(u+1/2))\,du.$$

Obviously, the terms $V_i(X)$ $(i = 1,\ldots,6)$ contribute to the main term in

Theorem 1. We compute

$$\sum_{i=1}^{3} V_i(X) - \sum_{i=4}^{6} V_i(X) \tag{5.7}$$
$$= 2X^2\Big(\Big(\int_0^{c_1}\int_0^{a(u)} + \int_{c_1}^{c_2}\int_{s(u)}^{a(u)} + \int_{c_2}^{c_3}\int_{s(u)}^{b(u)}\Big)H(1;u,v)\,dv\,du$$
$$- \Big(\int_{c_2}^{c_3}\int_{a(u)}^{b(u)} + \int_{c_3}^{1}\int_{a(u)}^{s(u)} + \int_{1}^{c_4}\int_{0}^{s(u)}\Big)G(1;u,v)\,dv\,du\Big) = C_6X^2,$$

where (with electronic support) $C_6 := 0.692229\ldots$, so that $\pi(C_4 + C_6) = C_1/2$.

As we sum up the terms $R_i(X)$ we are lucky that all but one of the twelve summands are annihilated, which is small wonder due to an obvious geometric argument, and compute

$$\sum_{i=1}^{3} R_i(X) - \sum_{i=4}^{6} R_i(X) = -\frac{1}{2}\int_0^{\sqrt{X}} H(X;0,v)\,dv = -\frac{1}{3}X^{3/2}, \tag{5.8}$$

which contributes to the second main term in Theorem 1 so that $C_3 - 2\pi C_5 - 2\pi/3 = C_2$.

In order to estimate the terms $Q_i(X)$ we note that $f(u,0) = u^2 \pm X$ and

$$f(u,\alpha(X;u)) = 2u^2 - X \pm |2u^2 - X|, \qquad f(u,\beta(X;u)) = u^2 - \frac{X^2}{4u^2},$$
$$f(u,\sigma(X;u)) = \frac{X^2}{4u^2} + 2u^2 - \sqrt{2}X \pm |2u^2 - \sqrt{2}X|.$$

Obviously, $f(u,0)$ is increasing in u and $\ll X$. Further, in all cases where $f(u,\alpha(X;u))$, $f(u,\beta(X;u))$, or $f(u,\sigma(X;u))$ appear, the first and the second function is increasing (or constant) in u and $\ll X$, whereas the third is always a sum of two functions which are monotonic in u and $\ll X$. Consequently, by applying Abelian summation together with Lemmas 1, 2, and 3, respectively, we obtain

$$\sum_{i=1}^{3} Q_i(X) - \sum_{i=4}^{6} Q_i(X) \ll X^{96/73}(\log X)^{461/146} \qquad (X \to \infty). \tag{5.9}$$

Now, in order to finish the proof it remains to estimate the terms $P_i(X)$ and $T_i(X)$.

Concerning the terms $P_i(X)$ we note that $\frac{\partial f}{\partial v}(u,v) = -2v \pm 4\kappa(X;u,v)$ with $\kappa(X;u,v) := u^2v/\sqrt{X^2 - 4u^2v^2}$. Obviously, $\kappa(X;u,v)$ is increasing in v. Further, $\kappa(X;u,v) \le X^{3/4}$ when $0 \le v \le X/(2u) - 1$, and $\sqrt{X^2 - 4u^2v^2} \le$

$3X^{3/4}$ when $X/(2u)-1 \le v \le X/(2u)$, provided that $X \ge 5$ and $0 < u \le c_4\sqrt{X}$. Hence, by applying the same trick as in the derivation of (4.6), we obtain for $i = 1, \dots, 6$,

$$\int\limits_{\gamma_i(u)}^{\delta_i(u)} \kappa(X; u, v)\psi(v+u)\, dv \ll X^{3/4} \quad \text{uniformly in } y_i < u \le z_i.$$

Consequently, since $v \ll \sqrt{X}$, whence $\int_{\gamma(u)}^{\delta(u)} v\psi(v+u)\, dv \ll \sqrt{X}$, the trivial estimation yields

$$\sum_{i=1}^{3} P_i(X) - \sum_{i=4}^{6} P_i(X) \ll X^{5/4} \quad (X \to \infty). \tag{5.10}$$

Concerning the terms $T_i(X)$, for $y < u < z$ we have

$$\begin{aligned}\frac{\partial}{\partial u} \int\limits_{\gamma(u)}^{\delta(u)} f(u, v)\, dv& \\ &= \int\limits_{\gamma(u)}^{\delta(u)} \frac{\partial f(u,v)}{\partial u}\, dv - \gamma'(u) f(u, \gamma(u)) + \delta'(u) f(u, \delta(u)).\end{aligned} \tag{5.11}$$

We compute

$$\int \frac{\partial f(u,v)}{\partial u}\, dv = 2uv \pm \lambda(X; u, v) + C,$$

with

$$\lambda(X; u, v) := \frac{X^2}{4u^2} \arcsin\left(\frac{2uv}{X}\right) - \frac{v}{2u}\sqrt{X^2 - 4u^2v^2}.$$

We note that $\lambda(X; u, 0) = 0$ and

$$\lambda(X; u, \beta(X; u)) = \frac{\pi}{8} X^2/u^2 \le X \quad (c_2\sqrt{X} \le u \le c_3\sqrt{X}).$$

Further we have $0 \le \lambda(X; u, \sigma(X; u)) \le X$ $(c_1\sqrt{X} \le u \le c_4\sqrt{X})$, with $\lambda(X; u, \sigma(X; u))$ increasing on $c_1\sqrt{X} \le u \le c_3\sqrt{X}$ and decreasing on $c_3\sqrt{X} \le u \le c_4\sqrt{X}$, and finally we are lucky that $0 \le \lambda(X; u, \alpha(X; u)) \le X$ $(0 < u \le \sqrt{X})$, with $\lambda(X; u, \alpha(X; u))$ increasing on $0 < u \le c_2\sqrt{X}$ and decreasing on $c_2\sqrt{X} \le u \le \sqrt{X}$.

Hence, by applying the second mean value theorem, we get

$$\int\limits_{y_i}^{z_i} \int\limits_{\gamma_i(u)}^{\delta_i(u)} \frac{\partial f_i(u,v)}{\partial u} (\psi(u) + \psi(u + 1/2))\, dv\, du \ll X \quad (i = 1, \dots, 6). \tag{5.12}$$

Concerning the boundary terms in (5.11) recall that $f(u, \alpha(X; u))$ and $f(u, \beta(X; u))$ are always increasing in u and $\ll X$, and that $f(u, \sigma(X; u))$ is always a sum of two functions which are monotonic in u and $\ll X$. Further, $|\partial\beta(X;u)/\partial u| \le 1$ for $c_2\sqrt{X} \le u \le c_3\sqrt{X}$, and, by applying (4.7),

$$\int_{y_i}^{z_i} \frac{\partial\alpha(X;u)}{\partial u}\psi(u+\varrho)\,du \ll X^{1/4} \qquad (\varrho \in \{0,1/2\})$$

for $i = 1,2,4,5$ (actually, the integral is $\ll 1$ for $i = 1,2,4$), and, by (4.6),

$$\int_{y_i}^{z_i} \frac{\partial\sigma(X;u)}{\partial u}\psi(u+\varrho)\,du \ll X^{1/4} \qquad (\varrho \in \{0,1/2\})$$

for $i = 2,3,5,6$ (actually, the integral is $\ll 1$ for $i = 3,5$).

Consequently, by the second mean value theorem, for $1 \le i \le 6$,

$$\int_{y_i}^{z_i}\int_{\gamma_i(u)}^{\delta_i(u)} (-\gamma_i'(u)f_i(u,\gamma_i(u)) + \delta_i'(u)f_i(u,\delta_i(u)))\psi(u+\varrho)\,du \ll X^{5/4} \qquad (\varrho \in \{0,1/2\}). \tag{5.13}$$

Thus, via (5.11) and combining (5.12) and (5.13), we obtain

$$\sum_{i=1}^{3} T_i(X) - \sum_{i=4}^{6} T_i(X) \ll X^{5/4} \qquad (X \to \infty). \tag{5.14}$$

Now collect (5.7)–(5.10), and (5.14), then insert (5.5) and (5.6) into (5.3), and finally insert (5.3) and (3.4) into (2.1). This concludes the proof of Theorem 1. Additionally, from the execution of the proof it is clear that C_1X^2 equals the volume of the domain $K(X)$.

Final remark. In the meantime Huxley has announced a further improvement concerning the circle problem. Hence one may expect that the upper bound $X^{96/73}(\log X)^{461/146}$ in Theorem 1 (and also in Theorems 1 and 2 of the first part [4] of the present paper) can be sharpened to $X^{547/416}(\log X)^{26947/8320}$. A further improvement independent of the circle problem seems unrealistic. For instance, an application of the Poisson summation formula to the non-convex body $K(X)$ would lead to an error term no better than $O(X^{10/7})$.

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