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Uniformizing functions for certain Shimura curves, in the case $D = 6$

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Introduction. Let $\mathbb{H}_6$ be the rational quaternion algebra of discriminant $D = 6$, which we consider embedded into $\mathbf{M}(2, \mathbb{R})$. Let $\varGamma_6 \subseteq \mathbf{SL}(2, \mathbb{R})$ be the group of units of norm 1 in a maximal order $\mathcal{O}_6 \subseteq \mathbb{H}_6$ of discriminant $D = 6$. The quotient group $\overline{\varGamma}_6 = \varGamma_6/\{\pm 1\} \subseteq \mathbf{PSL}(2, \mathbb{R})$ is a co-compact Fuchsian group whose action on the upper half-plane $\mathcal{H}$ gives rise to a nonsingular complete curve X_6 of genus zero. According to Shimura, the curve X_6 has a canonical model defined over $\mathbb{Q}$. Our purpose is to compute a complex uniformizing function t_6 of X_6 relying on the fundamental domain for $\overline{\varGamma}_6$ obtained in [1], and to derive from it a canonical model, j_6.

The function j_6 is an analog of the elliptic modular function j, which is obtained from the split quaternion algebra $\mathbf{M}(2, \mathbb{Q})$, of discriminant $D = 1$. However, the functions j and j_6 present notable differences. The function j is automorphic under the modular group $\mathbf{PSL}(2, \mathbb{Z})$, which is a triangle Fuchsian group endowed with parabolic transformations. The function j_6 is automorphic for a quadrilateral Fuchsian group without parabolic transformations, $\overline{\varGamma}_6 \subseteq \mathbf{PSL}(2, \mathbb{R})$. The lack of cusps in this case prevents the use of Fourier series expansions.

The complex uniformizing function we aim to determine satisfies a nonlinear differential equation of the third order, which can be obtained from the fundamental domain. In principle, it depends on eight parameters. Six of these parameters can be specified uniquely by prescribing the values of t_6 at three points, but the remaining two need to be determined by other means. In order to find their value, we simultaneously undertake the uniformization of several quotients of X_6: the curves $X_6^{(2)}$, $X_6^{(3)}$, $X_6^{(6)}$, X_6^{+}, attached to a group $\overline{\varGamma}_6^{+}/\overline{\varGamma}_6$ of involutions of X_6.

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The Fuchsian groups uniformizing the curves $X_6^{(2)}$, $X_6^{(3)}$, X_6^{+} are triangle groups, but the Fuchsian group uniformizing $X_6^{(6)}$ is again a quadrilateral group. The associated differential equations in the triangle case are determined by particular choices of the uniformizing functions. But from the two quadrilateral cases, X_6 and $X_6^{(6)}$, four accessory parameters remain. Two of them will be determined from algebraic equations of the coverings, which we compute beforehand. Formal integration of the differential equations will provide the other two.

Once the five uniformizing equations are fully determined, we integrate them to obtain the uniformizing functions t_6, $t_6^{(2)}$, $t_6^{(3)}$, $t_6^{(6)}$, t_6^{+}. We expand these functions as power series in local uniformizing parameters q_P attached to the vertices P of fundamental half domains. Moreover, we make explicit the local development of t_6 at the special complex multiplication points of X_6. Knowledge of the functions q_P involves the local isotropy at P, the exact determination of the integration constants, and the right choice of local normalizing factors.

In the last part of the paper, we use the values of our functions at the elliptic points and the special complex multiplication points to derive the canonical models of all the curves involved.

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1. The Fuchsian group $\overline{\Gamma}_6$ and the Shimura curve X_6. Let $\mathbb{H}_6$ be the rational quaternion algebra with basis $\{1, \mathrm{I}, \mathrm{J}, \mathrm{K}\}$ and defined by $\mathrm{I}^2 = 3$, $\mathrm{J}^2 = -1$, $\mathrm{IJ} = -\mathrm{JI} = \mathrm{K}$. Its discriminant D, given by the product of the places v where $\mathbb{H}_6 \otimes \mathbb{Q}_v$ is a division algebra, is equal to 6. The algebra $\mathbb{H}_6$ is the rational non-split ($D \neq 1$) quaternion algebra of lowest discriminant which is unramified at ∞. In particular, it can be embedded into $\mathbf{M}(2,\mathbb{R})$.

Throughout the paper we fix the embedding $\Phi : \mathbb{H}_6 \to \mathbf{M}(2,\mathbb{R})$ given by

$$x + y\mathrm{I} + z\mathrm{J} + t\mathrm{K} \mapsto \begin{bmatrix} x + y\sqrt{3} & z + t\sqrt{3} \\ -(z - t\sqrt{3}) & x - y\sqrt{3} \end{bmatrix}.$$

The reduced trace and the reduced norm of a quaternion are given by $\mathrm{tr}(x + y\mathrm{I} + z\mathrm{J} + t\mathrm{K}) = 2x$ and $\mathrm{n}(x + y\mathrm{I} + z\mathrm{J} + t\mathrm{K}) = x^2 - 3y^2 + z^2 - 3t^2$. They agree with the trace and the determinant of the matrix $\Phi(x + y\mathrm{I} + z\mathrm{J} + t\mathrm{K})$.

All maximal orders in $\mathbb{H}_6$ are conjugate, and we fix the representative in this conjugacy class to be $\mathcal{O}_6 := \mathbb{Z}[1, \mathrm{I}, \mathrm{J}, (1 + \mathrm{I} + \mathrm{J} + \mathrm{K})/2]$. Let $(\mathcal{O}_6)_1^* = \{\gamma \in \mathcal{O}_6 : \mathrm{n}(\gamma) = 1\}$ be the group of its units of reduced norm equal to one. The group $(\mathcal{O}_6)_1^*$ can be identified with its image $\Gamma_6 \subseteq \mathbf{SL}(2,\mathbb{R})$ under Φ. This group admits the following description, given in [1]:

$$\Gamma_6 = \left\{ \gamma = \frac{1}{2} \begin{bmatrix} \alpha & \beta \\ -\beta' & \alpha' \end{bmatrix} : \alpha, \beta \in \mathbb{Z}[\sqrt{3}],\ \det \gamma = 1,\ \alpha \equiv \beta \equiv \alpha\sqrt{3} \pmod 2 \right\}.$$

Here α' denotes the non-trivial Galois conjugate of an element $\alpha \in \mathbb{Q}(\sqrt{3})$. The projection $\overline{\Gamma}_6 := \Gamma_6/\{\pm 1\}$ in $\mathbf{PSL}(2,\mathbb{R})$ is a Fuchsian group which does not have parabolic transformations.

Let $\mathcal{H} = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ be the complex upper half-plane. The group $\mathbf{PSL}(2,\mathbb{R}) = \mathbf{GL}^+(2,\mathbb{R})/\mathbb{R}^*$ acts freely on $\mathcal{H}$ by homographic transformations $\gamma(z) = (az+b)/(cz+d)$, defined by matrices $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{GL}^+(2,\mathbb{R})$.

The action of $\overline{\Gamma}_6$ on $\mathcal{H}$ yields a compact Riemann surface $\overline{\Gamma}_6\backslash\mathcal{H}$ or, equivalently, a projective non-singular curve X_6, which is of genus zero. Note that the explicit identification of $\overline{\Gamma}_6\backslash\mathcal{H}$ with $\mathbf{P}^1(\mathbb{C})$ is by no means obvious, since it implies the construction of coordinate functions automorphic under $\overline{\Gamma}_6$. By a theorem due to Shimura, we know that the curve X_6 has a model defined over $\mathbb{Q}$ without rational points.

A fundamental domain for the action of $\overline{\Gamma}_6$ in $\mathcal{H}$, together with some distinguished complex multiplication points in it, called special complex multiplication points (SCM), was computed in [1]. We summarize some of its properties.

THEOREM 1.1. *Consider the hyperbolic hexagon* $[P_1, P_2, \ldots, P_6]$ *with vertices* P_i *given in Table* 1. *For each* i, *the isotropy group at* P_i *under* $\overline{\Gamma}_6$ *is generated by the class of the matrix* η_i *in Table* 2. *Under the identifications*

$$\text{(a) } \eta_2[P_3,P_2] = [P_1,P_2], \quad \text{(b) } \eta_4[P_3,P_4] = [P_5,P_4], \quad \text{(c) } \eta_6[P_5,P_6] = [P_1,P_6],$$

the hexagon yields a fundamental domain for the action of $\overline{\Gamma}_6$ *in* $\mathcal{H}$ (*see Figure* 1). *The vertices* $P_1 \equiv P_3 \equiv P_5 \pmod{\overline{\Gamma}_6}$ *and* P_6 *are elliptic of order* 2; *the remaining vertices* P_2, P_4 *are elliptic of order* 3. *The points* P_0, $P_7 \equiv P_8 \pmod{\overline{\Gamma}_6}$ *in Table* 1 *are a full set of representatives of the special complex multiplication* (*SCM*) *points of the genus zero curve* X_6. ∎

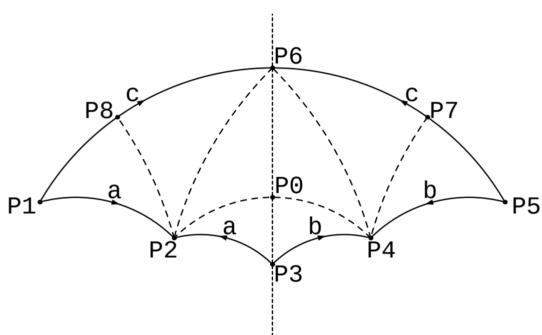

Fig. 1. Fundamental domain for X_6

Table 1. Vertices of a fundamental domain for X_6 and SCM points

P_1	$(-\sqrt{3}+i)/2$	P_2	$(-1+i)/(1+\sqrt{3})$	P_3	$(2-\sqrt{3})i$
P_4	$(1+i)/(1+\sqrt{3})$	P_5	$(\sqrt{3}+i)/2$	P_6	i
P_7	$(1+\sqrt{2}\,i)/\sqrt{3}$	P_8	$(-1+\sqrt{2}\,i)/\sqrt{3}$	P_0	$(\sqrt{6}-\sqrt{2})i/2$

Table 2. Matrices representing generators for the isotropy groups at the vertices of the hexagon

η_1	$\begin{bmatrix} \sqrt{3} & 2 \\ -2 & -\sqrt{3} \end{bmatrix}$	η_2	$\frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & 3-\sqrt{3} \\ -3-\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$	η_3	$\begin{bmatrix} 0 & -2+\sqrt{3} \\ 2+\sqrt{3} & 0 \end{bmatrix}$
η_4	$\frac{1}{2}\begin{bmatrix} 1+\sqrt{3} & -3+\sqrt{3} \\ 3+\sqrt{3} & 1-\sqrt{3} \end{bmatrix}$	η_5	$\begin{bmatrix} \sqrt{3} & -2 \\ 2 & -\sqrt{3} \end{bmatrix}$	η_6	$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

2. The Fuchsian group $\overline{\Gamma}_6^+$ and quotients of X_6. Let $N(\mathcal{O}_6)$ be the normalizer of $\mathcal{O}_6$ in $\mathbb{H}_6$. The elements of $N(\mathcal{O}_6)$ of positive reduced norm define a subgroup whose image in $\mathbf{GL}^+(2,\mathbb{R})$ will be denoted by Γ_6^+. The group Γ_6 is contained in Γ_6^+ as a normal subgroup and the quotient $\overline{\Gamma}_6^+/\overline{\Gamma}_6$ is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Its classes are represented by elements $w_d \in \mathcal{O}_6$ of norm d dividing $D = 6$. They give rise to involutions of the curve X_6, denoted ω_d (cf. [8]). We shall consider, together with the Shimura curve X_6, its quotients $X_6^{(d)} := X_6/\langle \omega_d \rangle$ and $X_6^+ := X_6/\langle \{\omega_d : d \,|\, 6\} \rangle$. All these Shimura curves fit in the following diagram of Galois covers of degree two:

$$\begin{tikzcd}
 & X_6 \arrow[dl] \arrow[d] \arrow[dr] & \\
X_6^{(2)} \arrow[dr] & X_6^{(3)} \arrow[d] & X_6^{(6)} \arrow[dl] \\
 & X_6^+ &
\end{tikzcd}$$

In what follows, we shall compute fundamental domains for the curves so defined.

First, consider the elements $w_2 := 1 + \mathrm{J}$, $w_3 := (-3 - \mathrm{I} - 3\mathrm{J} + \mathrm{K})/2$, $w_3' := (3 + \mathrm{I} - 3\mathrm{J} + \mathrm{K})/2$, and $w_6 := w_2 w_3 = -3\mathrm{J} + \mathrm{K}$ in $\mathcal{O}_6$. Their classes in $\overline{\Gamma}_6^+/\overline{\Gamma}_6$ define involutions ω_d, ω_d' of X_6. Observe that $\omega_3' = \omega_3$, because $w_3' = \eta_6 w_3$ and $\eta_6 \in \Gamma_6$; and $\omega_6 \eta_2^{-1} = \omega_6$, because $\eta_2 \in \Gamma_6$. Then we have the following polygon identifications: $\omega_2[P_1, P_2, P_6] = [P_3, P_4, P_6]$, $\omega_2[P_2, P_3, P_6] = [P_4, P_5, P_6]$, $\omega_3[P_2, P_3, P_6] = [P_2, P_6, P_1]$, $\omega_3'[P_3, P_4, P_6] = [P_6, P_4, P_5]$, $\omega_6[P_2, P_3, P_6] = [P_4, P_6, P_3]$, and $\omega_6 \eta_2^{-1}[P_1, P_2, P_6] = [P_6, P_4, P_5]$.

PROPOSITION 2.1. *Figures 2 and 3 show fundamental domains for the genus zero curves $X_6^{(2)}$, $X_6^{(3)}$, and $X_6^{(6)}$. The identifications of the sides are given as follows*:

	$X_6^{(2)}$	$X_6^{(3)}$	$X_6^{(6)}$
(a)	$\eta_4^{-1}\omega_2[P_2,P_3]=[P_4,P_3]$	$\omega_3[P_2,P_3]=[P_2,P_6]$	$\omega_6[P_0,P_3]=[P_0,P_6]$
(b)	$\omega_2[P_2,P_6]=[P_4,P_6]$	$\omega_3'[P_4,P_3]=[P_4,P_6]$	$\eta_4\omega_6\eta_6[P_6,P_7]=[P_5,P_7]$
(c)	$*$	$*$	$\eta_4[P_3,P_4]=[P_5,P_4]$

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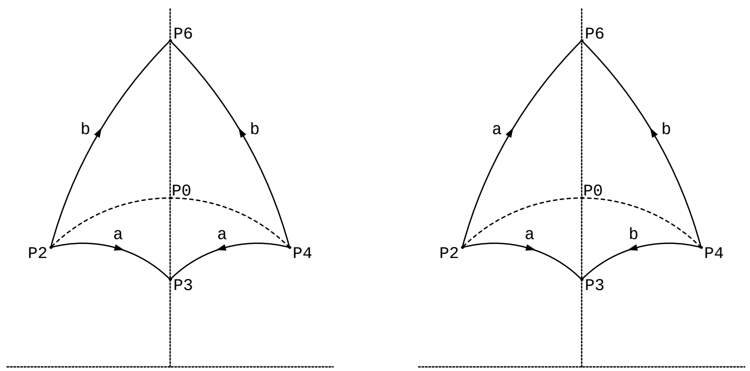

Fig. 2. Fundamental domains for $\Gamma_6^{(2)}$ and $\Gamma_6^{(3)}$

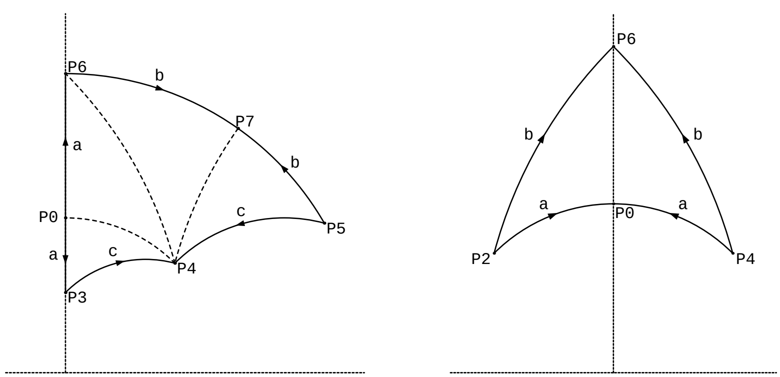

Fig. 3. Fundamental domains for $\Gamma_6^{(6)}$ and Γ_6^{+}

Now, since the curve X_6^+ can be obtained as a quotient of any of the curves $X_6^{(d)}$, $d\neq 1$, by any involution $\omega_{d'}$, $d'\neq d,1$, we may use the identification $\omega_6[P_2,P_3,P_4]=[P_4,P_6,P_2]$ to obtain a fundamental domain for the curve X_6^+ from the fundamental domain for $X_6^{(2)}$ or, alternatively, for $X_6^{(3)}$. We state the result in the next proposition.

PROPOSITION 2.2. *Figure* 3 *shows a fundamental domain for the genus zero curve* X_6^+. *The identifications of the sides are as follows*:

$$\text{(a) } \omega_6[P_2, P_0] = [P_4, P_0], \qquad \text{(b) } \omega_6\omega_3^{-1}[P_2, P_6] = [P_4, P_6]. \ \blacksquare$$

3. Fundamental half domains. The fundamental domains we have obtained up to now show particular symmetries. We will take advantage of this fact to compute the uniformizing functions we are looking for, that is, functions t such that $\mathbb{C}(X(\overline{\Gamma})) \simeq \mathbb{C}(t)$. For example, in Figure 1, besides the obvious reflection in the imaginary axis, the two closed hyperbolic segments joining P_2 with P_6 and P_6 with P_4 are a unique symmetry axis for X_6, denoted (P_2, P_6, P_4), the symmetry being the involution ω_2 composed with the reflection in the imaginary axis. Similarly, (P_8, P_2, P_4, P_7) is a symmetry axis for X_6, the symmetry being the involution ω_6 composed with the reflection in the imaginary axis (observe that $\omega_6 = \eta_4\omega_6 = \eta_2\omega_6$).

But not all the symmetries above are suitable for our purposes. In fact, we require them to cut out half domains containing all the vertices of a fundamental domain exactly once. These half domains will be called *fundamental*. Thus the reflection in the imaginary axis is not suitable, because it cuts out half domains containing twice the vertex $P_1 \equiv P_3 \equiv P_5 \pmod{\overline{\Gamma}_6}$, while one of the vertices P_2 or P_4 is missing. Similarly, the symmetry with axis $(P_8, P_2, P_0, P_4, P_7)$ is not suitable; for example, it identifies the vertices $P_1 \equiv P_3 \equiv P_5$ with P_6 but $P_7 \equiv P_8$ appears twice in each half domain. A convenient symmetry is the one with axis (P_2, P_6, P_4), since it cuts out the quadrilateral $[P_2, P_3, P_4, P_6]$ as a fundamental half domain for X_6.

PROPOSITION 3.1. *Table* 3 *lists axes of symmetries and polygons defining fundamental half domains for the curves* X_6, $X_6^{(2)}$, $X_6^{(3)}$, $X_6^{(6)}$, *and* X_6^+. *The fundamental half domains are cut out of the fundamental domains in* 1.1, 2.1 *and* 2.2. $\blacksquare$

Table 3. Fundamental half domains

Curve	Axis of symmetry	Polygons
X_6	(P_2, P_6, P_4)	$[P_2, P_3, P_4, P_6]$
$X_6^{(2)}$	(P_3, P_6)	$[P_3, P_4, P_6]$
$X_6^{(3)}$	(P_2, P_4)	$[P_2, P_4, P_6]$
$X_6^{(6)}$	(P_0, P_4, P_7)	$[P_0, P_4, P_7, P_6]$
X_6^+	(P_0, P_6)	$[P_0, P_4, P_6]$

At this point, we compute the angles at the different vertices of the polygons defining the corresponding fundamental half domains. Since the angle at any vertex in a fundamental half domain equals half the sum of the angles at the homologous vertices of the complete fundamental domain, we obtain the results displayed in Table 4.

Table 4. Angles at the vertices of fundamental half domains

	P_0	P_2	P_3	P_4	P_6	P_7
X_6	$*$	$\pi/3$	$\pi/2$	$\pi/3$	$\pi/2$	$*$
$X_6^{(2)}$	$*$	$*$	$\pi/4$	$\pi/3$	$\pi/4$	$*$
$X_6^{(3)}$	π	$\pi/6$	$*$	$\pi/6$	$\pi/2$	$*$
$X_6^{(6)}$	$\pi/2$	$*$	$*$	$\pi/3$	$\pi/2$	$\pi/2$
X_6^+	$\pi/2$	$*$	$*$	$\pi/6$	$\pi/4$	$*$

4. Equations for the covers. The next step in our computation is to determine the equations of the covers. We shall begin by controlling their ramification. The ramification points can be easily seen from the pictures of the fundamental domains, and we have the following result.

PROPOSITION 4.1. *Table* 6 *shows the ramification points of all the covers of degree two that we have. The different ways in which they are identified depend on the curve*, *and the identifications are shown in Table* 5. ∎

Table 5. Identification of points

X_6	P_0	$P_1 \equiv P_3 \equiv P_5$	P_2	P_4	P_6	$P_7 \equiv P_8$
$X_6^{(2)}$	$P_0 \equiv P_7$	P_3	$P_2 \equiv P_4$	$P_2 \equiv P_4$	P_6	$P_0 \equiv P_7$
$X_6^{(3)}$	$P_0 \equiv P_7$	$P_3 \equiv P_6$	P_2	P_4	$P_3 \equiv P_6$	$P_0 \equiv P_7$
$X_6^{(6)}$	P_0	$P_3 \equiv P_6$	$P_2 \equiv P_4$	$P_2 \equiv P_4$	$P_3 \equiv P_6$	P_7
X_6^+	$P_0 \equiv P_7$	$P_3 \equiv P_6$	$P_2 \equiv P_4$	$P_2 \equiv P_4$	$P_3 \equiv P_6$	$P_0 \equiv P_7$

Table 6. All ramification and some splitting points of the covers

Cover	P_0	P_2	P_3	P_4	P_6	P_7
$X_6 \to X_6^{(2)}$	P_0P_7	$*$	P_3^2	P_2P_4	P_6^2	$*$
$X_6 \to X_6^{(3)}$	P_0P_7	P_2^2	$*$	P_4^2	P_3P_6	$*$
$X_6 \to X_6^{(6)}$	P_0^2	$*$	$*$	P_2P_4	P_3P_6	P_7^2
$X_6^{(2)} \to X_6^+$	P_0^2	$*$	$*$	P_4^2	P_3P_6	$*$
$X_6^{(3)} \to X_6^+$	P_0^2	$*$	$*$	P_2P_4	P_6^2	$*$
$X_6^{(6)} \to X_6^+$	P_0P_7	$*$	$*$	P_4^2	P_6^2	$*$

Since the curves we are dealing with are all of genus zero, their function field can be generated by a single automorphic function in each case. Due to the fact that the group of automorphisms of $\mathbf{P}^1(\mathbb{C})$ is isomorphic to $\mathbf{PSL}(2,\mathbb{C})$, we may isolate a generator by requiring that it assumes three given different values at three given different points.

PROPOSITION 4.2. *For each curve* X_6, $X_6^{(2)}$, $X_6^{(3)}$, $X_6^{(6)}$, X_6^+, *there exists a uniformizing function, uniquely determined by three values at three vertices of the fundamental half domains in Table* 3.

Proof. Since our curves are of genus zero and the Fuchsian groups attached to them admit fundamental half domains, the Riemann mapping theorem and the Schwarz reflection principle provide the existence of uniformizing functions (cf. [9]). Since $\mathbf{PSL}(2,\mathbb{C})$ is the automorphism group of $\mathbf{P}^1(\mathbb{C})$, the uniformizing functions are determined by their values at three points. ∎

Table 7. Values of the uniformizing functions at the vertices

t	P_0	P_2	P_3	P_4	P_6	P_7
t_6	$*$	a	0	1	∞	$*$
$t_6^{(2)}$	$*$	$*$	0	1	∞	$*$
$t_6^{(3)}$	$*$	0	$*$	1	∞	$*$
$t_6^{(6)}$	0	$*$	$*$	1	∞	b
t_6^+	0	$*$	$*$	1	∞	$*$

We choose or name some initial values for the functions we are aiming at in accordance with Table 7. Then we have the following theorem.

THEOREM 4.3. *The conditions in Table* 7 *define uniformizing functions* t_6, $t_6^{(2)}$, $t_6^{(3)}$, $t_6^{(6)}$, t_6^+ *which fulfil the following algebraic relations*:

(a) $4t_6^+t_6^{(2)} = (t_6^{(2)}+1)^2$, (b) $t_6^+ = (2t_6^{(3)}-1)^2$,
(c) $4t_6^{(2)}(2t_6^{(3)}-1)^2 = (t_6^{(2)}+1)^2$, (d) $t_6^2 = t_6^{(2)}$,
(e) $4t_6t_6^{(3)} = (t_6+1)^2$, (f) $t_6^+ + t_6^{(6)}(t_6^{(6)}-2) = 0$,
(g) $2t_6t_6^{(6)} = i(t_6-i)^2$, (h) $4t_6^2t_6^+ = (t_6^2+1)^2$,
(i) $(t_6^{(2)}+1)^2 + 4t_6^{(2)}t_6^{(6)}(t_6^{(6)}-2) = 0$, (j) $(2t_6^{(3)}-1)^2 + t_6^{(6)}(t_6^{(6)}-2) = 0$.

All these functions take real values on the boundaries of their respective half domains. Moreover, we have the following particular values:

(k) $t_6^{(2)}(P_0) = -1$, (l) $t_6^{(3)}(P_0) = 1/2$, (m) $b = t_6^{(6)}(P_7) = 2$,
(n) $a = t_6(P_2) = -1$, (o) $t_6(P_0) = i$, (p) $t_6(P_7) = -i$.

Proof. We write for the moment $c := t_6^{(2)}(P_0)$ and compute some divisors of the functions, taking into account the ramification given in Proposition 4.1:

$$\operatorname{div}(t_6^{(2)} - c)\left(1 - \frac{c}{t_6^{(2)}}\right) = \operatorname{div}(t_6^+), \quad \operatorname{div}(t_6^{(2)} - 1)\left(1 - \frac{1}{t_6^{(2)}}\right) = \operatorname{div}(t_6^+ - 1).$$

Since for genus 0 curves each zero degree divisor determines a function up to a non-zero constant factor, there exist non-zero constants $A, B \in \mathbb{C}$ such that

$$A(t_6^{(2)} - c)\left(1 - \frac{c}{t_6^{(2)}}\right) = t_6^+, \qquad B(t_6^{(2)} - 1)\left(1 - \frac{1}{t_6^{(2)}}\right) = t_6^+ - 1.$$

Solving the system, we obtain (a) and the value $c = -1$.

Similarly, considering $d := t_6^{(3)}(P_0)$ and computing $\operatorname{div}(t_6^{(3)} - d)^2 = \operatorname{div}(t_6^+)$, $\operatorname{div}(t_6^{(3)} - 1)t_6^{(3)} = \operatorname{div}(t_6^+ - 1)$, we obtain (b) and the value $d = 1/2$.

(c) is an immediate consequence of (a) and (b). The remaining algebraic relations are obtained analogously.

The defining conditions of the automorphic triangle functions $t_6^{(2)}, t_6^{(3)}, t_6^+$ uniquely determine the conformal representations of the half domains in Table 3 in $\mathcal{H}$. Thus, these functions are real on the boundary of their half domains. The algebraic relations satisfied by the quadrangular functions $t_6^{(6)}$ and t_6 imply that these functions are real on the boundary of their respective half domains. In order to decide which of the two values i or $-i$ equals $t_6(P_0)$, we note that, since the function t_6 respects the orientation, the interior of the quadrilateral $[P_2, P_3, P_4, P_6]$ has to be mapped into $\mathcal{H}$. ∎

5. Uniformizing differential equations. We review the necessary definitions and facts regarding Schwarzian derivatives, drawn mostly from [2], [3] and [5].

Definition 5.1. Let $f(z)$ be a non-constant smooth function and let $D(f, z)$ be the usual derivative. The *Schwarzian derivative* of f is defined as

$$Ds(f, z) = \frac{2D(f, z)D^3(f, z) - 3D^2(f, z)^2}{D(f, z)^2};$$

and the *automorphic derivative* of f is defined as

$$Da(f, z) = \frac{Ds(f, z)}{D(f, z)^2}.$$

Neither the Schwarzian derivative nor the automorphic derivative are derivations in the usual sense, but these differential operators have some properties similar to those of the standard derivation. We now state some of these properties.

Proposition 5.2.

(a) *Let $f(z)$, $g(z)$ be non-constant smooth functions whose composition $g \circ f$ is defined. Then the automorphic derivative satisfies the following* chain rule:

$$Da(g \circ f, z) = Da(g, f(z)) + \frac{Da(f,z)}{D(g, f(z))^2}.$$

(b) *Suppose that $f(z) = w$ is a smooth function whose inverse function is $\mathbf{PGL}(2,\mathbb{C})$-multivalued. Then the Schwarzian derivative $Ds(f^{-1}, w)$ is single valued and $Ds(f^{-1}, w) = -Da(f, z)$, where $f^{-1}(w) = z$.*

(c) *For a homographic transformation $\gamma(z) = (az+b)/(cz+d)$, $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{GL}(2,\mathbb{C})$, we have $Da(\gamma, z) = 0$. In particular, $Da(f \circ \gamma, z) = Da(f, \gamma(z))$ for any function $f(z)$.*

(d) *Conversely, $Da(\gamma, z) = 0$ implies that $\gamma(z) = (az+b)/(cz+d)$ for some $\gamma = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathbf{M}(2,\mathbb{C})$.* ∎

For a Fuchsian group $\overline{\Gamma} \subseteq \mathbf{PSL}(2,\mathbb{R})$, and a $\overline{\Gamma}$-automorphic function f, we deduce the $\overline{\Gamma}$-invariance of $Da(f,z)$.

Corollary 5.3. *The automorphic derivative $Da(f,z)$ of a $\overline{\Gamma}$-automorphic function, $f(z)$, is again a $\overline{\Gamma}$-automorphic function. That is to say, the following equality holds*:

$$Da(f, \gamma(z)) = Da(f, z) \qquad \textit{for any } \gamma \in \overline{\Gamma}. \ ∎$$

The main tool in our approach to the differential treatment of the uniformizing functions will be the following theorem.

Theorem 5.4. *Let $\overline{\Gamma}$ be a Fuchsian group of the first kind such that the associated curve $X(\overline{\Gamma})$ is of genus 0. Assume that we know a fundamental half domain for the action of $\overline{\Gamma}$ in $\mathcal{H}$. Suppose that t is a generator of the field of $\overline{\Gamma}$-automorphic functions such that its values at the vertices of the fundamental half domain belong to $\mathbf{P}^1(\mathbb{R})$. Then there exists a rational function $R(t)$ such that $Da(t,z) + R(t) = 0$. If $\alpha_i\pi$ are the internal angles at the vertices of the fundamental half domain, then*

$$R(t) = \sum \frac{1-\alpha_i^2}{(t-a_i)^2} + \sum \frac{B_i}{t-a_i},$$

where the B_i are constants and the summation extends over all vertices of the fundamental half domain where the function t takes finite values a_i. Moreover, if the values of t at all vertices are finite, then

(a) $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum (1-\alpha_i^2) = 0$,

(c) $\sum a_i^2 B_i + \sum a_i(1-\alpha_i^2) = 0$.

But if ∞ is the value of t at a vertex with internal angle $\alpha\pi$, then

(a) $\sum B_i = 0$,

(b) $\sum a_i B_i + \sum(1-\alpha_i^2) - (1-\alpha^2) = 0$. ∎

In general, the above relations between the constants B_i, the angles α_i, and the values a_i do not suffice to determine all the constants. In our case, the relations between all our functions given in Theorem 4.3 fully determine the constants B_i only for three of the five functions $R(t)$, namely for those associated to $t_6^{(2)}$, $t_6^{(3)}$, and t_6^{+}. To determine the constants that remain, we formally integrate the differential equation computing some coefficients of the expansion in a neighbourhood of the point P_6 and compare the solutions using the equations for the covers we have obtained in Theorem 4.3. When comparing solutions for t_6 and $t_6^{(2)}$, we obtain the value of the constant term in the differential equation for t_6; and when comparing solutions for $t_6^{(6)}$ and t_6^{+}, we obtain the value of the constant term in the differential equation for $t_6^{(6)}$. This gives the values of the automorphic derivatives.

THEOREM 5.5. *The functions defined by Table* 7 *satisfy the differential equations $Da(t,z) + R(t) = 0$, where the rational functions $R(t)$ are listed in Table* 8. ∎

Table 8. Automorphic derivatives of the functions

Curve	Function t	Vertices / Angles	$-Da(t,z)$
X_6	t_6	$[P_2, P_3, P_4, P_6]$ $[\pi/3, \pi/2, \pi/3, \pi/2]$	$\dfrac{27t^4+74t^2+27}{36t^2(t^2-1)^2}$
$X_6^{(2)}$	$t_6^{(2)}$	$[P_3, P_4, P_6]$ $[\pi/4, \pi/3, \pi/4]$	$\dfrac{135t^2-142t+135}{144t^2(t-1)^2}$
$X_6^{(3)}$	$t_6^{(3)}$	$[P_2, P_4, P_6]$ $[\pi/6, \pi/6, \pi/2]$	$\dfrac{27t^2-27t+35}{36t^2(t-1)^2}$
$X_6^{(6)}$	$t_6^{(6)}$	$[P_0, P_4, P_7, P_6]$ $[\pi/2, \pi/3, \pi/2, \pi/2]$	$\dfrac{27t^4-108t^3+211t^2-206t+108}{36t^2(t^2-3t+2)^2}$
X_6^{+}	t_6^{+}	$[P_0, P_4, P_6]$ $[\pi/2, \pi/6, \pi/4]$	$\dfrac{135t^2-103t+108}{144t^2(t-1)^2}$

6. Local uniformizing parameters. Our goal is to obtain explicit expansions of the uniformizing functions around the elliptic points and around some CM points. The purpose of this section is to make a first choice of local uniformizing parameters adapted to our functions (see definition below).

Suppose that $P \in \mathcal{H}$ is any elliptic point of order e for the $\overline{\Gamma}$-action. By definition, the isotropy group at P, $\overline{\Gamma}_P$, will be generated by a transformation $g \in \mathbf{PSL}(2,\mathbb{R})$ of order $e > 1$. Let $G \in \Gamma \subseteq \mathbf{SL}(2,\mathbb{R})$ be a matrix defining g. Since in all our cases $-1 \in \Gamma$, we may take the matrix G of order $2e$, and since g is an elliptic transformation, the matrix G can be diagonalized. Let $H \in \mathbf{GL}(2,\mathbb{C})$ be such that $D := HGH^{-1} = \left[\begin{smallmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{smallmatrix}\right]$, where ζ is a $2e$th primitive root of unity. We denote by h and d the homographic transformations of $\mathbf{P}^1(\mathbb{C})$ defined by H and D, respectively. Then

$$(*) \qquad h(g(z)) = d(h(z)) = \zeta^2 h(z).$$

By evaluating $(*)$ at the points $z = P$ and $z = \overline{P}$, we obtain

$$h(P) = h(g(P)) = \zeta^2 h(P), \qquad h(\overline{P}) = h(g(\overline{P})) = \zeta^2 h(\overline{P}).$$

Since $e > 1$, we have $\zeta^2 \neq 1$, and since h is a bijective mapping of $\mathbf{P}^1(\mathbb{C})$, we must have $h(P) = 0$ and $h(\overline{P}) = \infty$ (or $h(P) = \infty$ and $h(\overline{P}) = 0$). Hence,

$$h(z) = k\,\frac{z - P}{z - \overline{P}} \qquad \left(\text{or } h(z) = k\,\frac{z - \overline{P}}{z - P}\right),$$

for some constant $k \in \mathbb{C} - \{0\}$ to be determined.

Now we can expand any $\overline{\Gamma}_P$-automorphic function t around the point P as a power series T in the variable $h(z) = k(z - P)/(z - \overline{P})$:

$$t(z) = T(h(z)) = \sum_{n=n_0}^{\infty} a_n h(z)^n.$$

We shall have $T(h(z)) = t(z) = t(g(z)) = T(h(g(z))) = T(\zeta^2 h(z))$. Thus $a_n = 0$ unless $n \equiv 0 \pmod{e}$.

We extend these considerations to the case $e = 1$ in the following definition.

Definition 6.1. A *local parameter* at a point $P \in \mathcal{H}$ for the $\overline{\Gamma}_P$-action is any function

$$q(z) := \left(k\,\frac{z - P}{z - \overline{P}}\right)^e,$$

where $e = \#\overline{\Gamma}_P$ is the order of the isotropy group at P and $k \in \mathbb{C} - \{0\}$ is a constant. The local parameter q is said to be *adapted* to a function $t = \sum_{n=m}^{\infty} a_{ne} q^n$ if, moreover, $a_{re} = 1$ in the case that $t - a_0$ has a zero of order er at $z = P$, or $a_{-re} = 1$ if t has a pole of order er at $z = P$.

In order to obtain local parameters adapted to our functions, we review some classical facts regarding the Schwarzian functions. For this purpose, let us consider the hypergeometric function defined by the series

$$F(a, b, c; w) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n}\,\frac{w^n}{n!}, \qquad (a)_n := a(a+1)\cdots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

which converges for $|w| < 1$ (cf. [7] and [11]).

THEOREM 6.2. *Assume that $c \neq 1$. The functions $F(a,b,c;w)$ and $w^{1-c}F(a-c+1,b-c+1,2-c;w)$ are two linearly independent solutions of the hypergeometric differential equation*

$$w(1-w)D^2(f,w)+(c-(1+a+b)w)D(f,w)-abf=0.$$

The function

$$z=s(a,b,c;w):=\frac{w^{1-c}F(a-c+1,b-c+1,2-c;w)}{F(a,b,c;w)}$$

provides a conformal representation of the upper half w-plane $\mathcal{H}$ onto the interior of a triangle $[A,B,C]$ in the z-plane and establishes a homeomorphism between $\mathbb{R}\cup\{\infty\}$ and the boundary of the triangle. The vertices A,B,C can be expressed in terms of Euler's gamma function as

$$\begin{aligned}
A&=s(a,b,c;0)=0,\\
B&=s(a,b,c;1)=\frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)},\\
C&=s(a,b,c;\infty)=e^{\pi i(1-c)}\,\frac{\Gamma(c-a)\Gamma(b)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)}.
\end{aligned}$$

The internal angles at these vertices are $\alpha\pi$, $\beta\pi$, $\gamma\pi$, where

$$\alpha=1-c\neq 0,\qquad \beta=c-a-b,\qquad \gamma=b-a. \ \blacksquare$$

In the next theorem we compare the triangle $[s(0),s(1),s(\infty)]$ with the triangles defining the functions t_6^+, $t_6^{(2)}$, and $t_6^{(3)}$. In each case, this will allow us to obtain the local constant k of the adapted local parameter in closed form.

THEOREM 6.3. *Let t be one of the three uniformizing functions defined in Table 7 for which a fundamental half domain is a triangle; that is, t_6^+, $t_6^{(2)}$, and $t_6^{(3)}$. Let $\alpha\pi$, $\beta\pi$, $\gamma\pi$ be the internal angles at the vertices A, B, C when t takes the values $t(A)=0$, $t(B)=1$, $t(C)=\infty$, or $t(A)=1$, $t(B)=\infty$, $t(C)=0$, or $t(A)=\infty$, $t(B)=0$, $t(C)=1$. Then the local constants k_A adapted to the functions t are listed in Table* 9.

Proof. First we explain the results for the case $t(A)=0$. By formal integration of the differential equation of the third order in Theorem 5.5, and taking into account that $t(A)=0$, it follows that there exists a normalized power series in two variables

$$r(X,Y)=\sum_{n=1}^{\infty}a_{ne}X^{en}Y^{en},\quad a_e=1,$$

and a constant $\lambda\in\mathbb{C}$, such that

$$t(z)=r(\lambda;h_1(z))=\sum_{n=1}^{\infty}a_{ne}\lambda^{en}h_1^{en}(z)$$

Table 9. Local constants for the triangle functions

t	$[A,B,C]$	e_A	$t(A)$	$a,\ b,\ c$	ν_A	k_A
t_6^+	$[P_0,P_4,P_6]$	2	0	$\frac{1}{24},\ \frac{7}{24},\ \frac{1}{2}$	$2^3\cdot 3^2$	$i\,\frac{\sqrt{2}+\sqrt{3}}{2}\,\frac{\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
t_6^+	$[P_4,P_6,P_0]$	6	1	$\frac{1}{24},\ \frac{13}{24},\ \frac{5}{6}$	$2^{-1}\cdot 3^{-2}$	$\frac{2+\sqrt{3}-i}{12}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{7}{24})\Gamma(\frac{19}{24})}{\Gamma(\frac{5}{6})\Gamma(\frac{11}{24})\Gamma(\frac{23}{24})}$
t_6^+	$[P_6,P_0,P_4]$	4	∞	$\frac{1}{24},\ \frac{5}{24},\ \frac{3}{4}$	$2^5\cdot 3$	$-\frac{(\sqrt{2}+\sqrt{3})(1+i)}{4\sqrt{2}}\,\frac{\Gamma(\frac{1}{4})\Gamma(\frac{13}{24})\Gamma(\frac{17}{24})}{\Gamma(\frac{3}{4})\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
$t_6^{(2)}$	$[P_3,P_4,P_6]$	4	0	$\frac{1}{12},\ \frac{1}{3},\ \frac{3}{4}$	$3^2\cdot 2^{-4}$	$\frac{(1+\sqrt{3})(1+i)}{8}\,\frac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{12})}{\Gamma(\frac{3}{4})\Gamma(\frac{11}{12})}$
$t_6^{(2)}$	$[P_4,P_6,P_3]$	3	1	$\frac{1}{12},\ \frac{1}{3},\ \frac{2}{3}$	$2\cdot 3^{-1}$	$\frac{2+\sqrt{3}-i}{6}\,\frac{\Gamma(\frac{1}{3})^2\Gamma(\frac{7}{12})}{\Gamma(\frac{2}{3})^2\Gamma(\frac{11}{12})}$
$t_6^{(2)}$	$[P_6,P_3,P_4]$	4	∞	$\frac{1}{12},\ \frac{5}{12},\ \frac{3}{4}$	$2^8\cdot 3$	$-\frac{\sqrt{3}(1+i)}{4\sqrt{2}}\,\frac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}$
$t_6^{(3)}$	$[P_2,P_4,P_6]$	6	0	$\frac{1}{12},\ \frac{7}{12},\ \frac{5}{6}$	$2^{-3}\cdot 3^{-2}$	$\frac{(1+\sqrt{3})(1+i)}{12}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{7}{12})}{\Gamma(\frac{5}{6})\Gamma(\frac{11}{12})}$
$t_6^{(3)}$	$[P_4,P_6,P_2]$	6	1	$\frac{1}{12},\ \frac{1}{4},\ \frac{5}{6}$	$2^{-3}\cdot 3^{-2}$	$\frac{2+\sqrt{3}-i}{12}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{7}{12})}{\Gamma(\frac{5}{6})\Gamma(\frac{11}{12})}$
$t_6^{(3)}$	$[P_6,P_2,P_4]$	2	∞	$\frac{1}{12},\ \frac{1}{4},\ \frac{1}{2}$	2	$\frac{(1+\sqrt{3})(1-i)}{4}\,\frac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{12})}{\Gamma(\frac{3}{4})\Gamma(\frac{11}{12})}$

for any z in a neighbourhood of A. Here we take $h_1(z) := (z-A)/(z-\overline{A})$. Observe that λ is defined up to multiplication by an eth root of unity.

Consider the Schwarzian function $s(a,b,c;w)$ determined by the angles $\alpha\pi$, $\beta\pi$, $\gamma\pi$, that is to say, $a=(1-\alpha-\beta-\gamma)/2$, $b=(1-\alpha-\beta+\gamma)/2$, $c=1-\alpha$. Since r satisfies the conditions $r(\lambda;h_1(A))=0$, $r(\lambda;h_1(B))=1$, $r(\lambda;h_1(C))=\infty$, we can relate the inverse of the series defining $s(a,b,c;w)$ to the series defining $t(z)$. A direct computation of the first terms in both series suffices to establish the following lemma.

LEMMA 6.4. *Let $u(a,b,c;z)$ denote the inverse series of $s(a,b,c;w)$. Then*

$$r(\zeta_e;h_1(z)) = u(a,b,c;h_1(z))$$

for any $z\in\mathbb{C}$ in the convergence domain and any eth root of unity ζ_e. ∎

To continue the calculation of λ, we may use either the condition $t(B)=1$ or, alternatively, the condition $t(C)=\infty$. In the first case, we obtain $1 = t(B) = r(\lambda;h_1(B)) = r(1;\lambda h_1(B)) = r(\zeta_e;\zeta_e^{-1}\lambda h_1(B))$, and

$$\zeta_e^{-1}\lambda h_1(B) = s(a,b,c;1) = \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)}.$$

We can conclude that

$$\lambda = \zeta_e \, \frac{B-\overline{A}}{B-A} \, \frac{\Gamma(c-a)\Gamma(c-b)\Gamma(2-c)}{\Gamma(c)\Gamma(1-b)\Gamma(1-a)}.$$

In the second case, we obtain

$$\lambda = \zeta_e e^{\pi i(1-c)} \, \frac{C-\overline{A}}{C-A} \, \frac{\Gamma(c-a)\Gamma(b)\Gamma(2-c)}{\Gamma(c)\Gamma(b-c+1)\Gamma(1-a)}.$$

Both values of λ are equal and we may take k_A to be the product of λ by any eth root of unity. This is what we have done in Table 9, where we have taken $\zeta_e = 1$.

Now we explain the results for the cases $t(A) = 1$ or $t(A) = \infty$. When $t(A) = 1$, $t(B) = \infty$, $t(C) = 0$, we change our function t to $1 - 1/t$; this function has the properties of the function t in the preceding case, and we apply the same arguments. In the case $t(A) = \infty$, we change t to $1/(1-t)$ and proceed analogously. ∎

Now, we compute local parameters for the two remaining uniformizing functions. To begin with, we state a result that relates the local constants for two points in $\mathcal{H}$ in the same $\overline{\Gamma}$-orbit.

Proposition 6.5. *Let $P \in \mathcal{H}$ be a point of order $e \geq 1$ for the $\overline{\Gamma}$-action. For any $w = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \Gamma \subseteq \mathbf{SL}(2,\mathbb{R})$, the local constants k_P and $k_{w(P)}$ adapted to a $\overline{\Gamma}$-automorphic function t, at P and $w(P)$, are related by*

$$k_{w(P)}^{er} = k_P^{er} \left(\frac{cP+d}{c\overline{P}+d} \right)^{er},$$

where the value of r is given in Definition 6.1. ∎

Theorem 6.6. *Table 10 lists the local constants k_P adapted to the quadrilateral uniformizing functions t_6 and $t_6^{(6)}$ in neighbourhoods of the vertices P of fundamental half domains and, also, in a neighbourhood of the SCM point P_0 for the function t_6.*

Proof. We begin by explaining the computation for the point P_3 and the curve X_6. For this, we shall take into account the equation $t_6^2 = t_6^{(2)}$ for the covering $X_6 \to X_6^{(2)}$, which we computed in Theorem 4.3. First, we integrate the differential equations in Theorem 5.5 for the curves X_6 and $X_6^{(2)}$ in a neighbourhood of the point P_3 in the form

$$t_6^{(2)}(z) = r_6^{(2)}(\lambda_6^{(2)}; h_1(z)) = \sum_{n=1}^{\infty} a_{4n}^{(2)} {\lambda_6^{(2)}}^{4n} h_1^{4n}(z), \quad a_4^{(2)} = 1,$$

$$t_6(z) = r_6(\lambda_6; h_1(z)) = \sum_{n=1}^{\infty} a_{2n} \lambda_6^{2n} h_1^{2n}(z), \qquad a_2 = 1,$$

Table 10. Local constants for the quadrilateral functions

t	P	e_P	$t(P)$	ν_P	k_P
$t_6^{(6)}$	P_0	2	0	$2^2\cdot 3^2$	$i\,\dfrac{\sqrt{2}+\sqrt{3}}{2\sqrt{2}}\,\dfrac{\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
$t_6^{(6)}$	P_4	3	1	3^{-1}	$\dfrac{2+\sqrt{3}+i}{12}\,\dfrac{\Gamma(\frac{1}{6})\Gamma(\frac{7}{24})\Gamma(\frac{19}{24})}{\Gamma(\frac{5}{6})\Gamma(\frac{11}{24})\Gamma(\frac{23}{24})}$
$t_6^{(6)}$	P_7	2	2	$2^2\cdot 3^2$	$\dfrac{(2\sqrt{3}+3\sqrt{2})(1-\sqrt{2}\,i)}{12}\,\dfrac{\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
$t_6^{(6)}$	P_6	2	∞	2^2	$i\,\dfrac{\sqrt{2}+\sqrt{3}}{4}\,\dfrac{\Gamma(\frac{1}{4})\Gamma(\frac{13}{24})\Gamma(\frac{17}{24})}{\Gamma(\frac{3}{4})\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
t_6	P_0	1	i	$2^2\cdot 3$	$i\,\dfrac{\sqrt{2}+\sqrt{3}}{2}\,\dfrac{\Gamma(\frac{7}{24})\Gamma(\frac{11}{24})}{\Gamma(\frac{19}{24})\Gamma(\frac{23}{24})}$
t_6	P_2	3	-1	3^{-1}	$\dfrac{1+(2+\sqrt{3})i}{6\sqrt[3]{2}}\,\dfrac{\Gamma(\frac{1}{3})^2\Gamma(\frac{7}{12})}{\Gamma(\frac{2}{3})^2\Gamma(\frac{11}{12})}$
t_6	P_3	2	0	$3\cdot 2^{-2}$	$\dfrac{(1+\sqrt{3})(1+i)}{8}\,\dfrac{\Gamma(\frac{1}{4})\Gamma(\frac{5}{12})}{\Gamma(\frac{3}{4})\Gamma(\frac{11}{12})}$
t_6	P_4	3	1	3^{-1}	$\dfrac{2+\sqrt{3}-i}{6\sqrt[3]{2}}\,\dfrac{\Gamma(\frac{1}{3})^2\Gamma(\frac{7}{12})}{\Gamma(\frac{2}{3})^2\Gamma(\frac{11}{12})}$
t_6	P_6	2	∞	2^4	$\dfrac{\sqrt{3}(1-i)}{4\sqrt{2}}\,\dfrac{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3})\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})\Gamma(\frac{7}{12})\Gamma(\frac{11}{12})}$

where $e_6^{(2)} = 4$, $e_6 = 2$ are the orders of the isotropy groups at P_3, and $h_1(z) = (z - P_3)/(z - \overline{P}_3)$.

Now, we impose the condition that these series expansions satisfy the equation $t_6^2 = t_6^{(2)}$; this gives us the relation $\lambda_6^4 = {\lambda_6^{(2)}}^4$, and so a local constant λ_6 adapted to t_6 at P_3 is, up to multiplication by a fourth root of unity, the local constant $\lambda_6^{(2)}$ adapted to $t_6^{(2)}$ at P_3.

To determine this fourth root of unity up to sign we note that the value of t_6 at P_0 may be obtained by estimating the series at the point $z = P_0$, because the series is absolutely convergent at this point. But, if for λ_6 we take $\pm i$ times the value of $\lambda_6^{(2)}$ listed in Table 9, then the value of the series at P_0 approximates $-i$ rather than i, as it should do. Thus, we must take ± 1 times the value of $\lambda_6^{(2)}$, as done in Table 10.

The local parameters adapted to t_6 at the points P_4, P_6 and P_0, and the ones adapted to $t_6^{(6)}$ at P_0, P_4, and P_6, are computed similarly. The relationship between the ones we want to compute and the ones we use to compute them are calculated taking into account the equations given in Theorem 4.3, and they are as follows:

$$2\lambda_6^{(6)}(P_0)^2 = \lambda_6^{+}(P_0)^2, \qquad \lambda_6^{(6)}(P_4)^6 = -\,\lambda_6^{+}(P_4)^6, \qquad \lambda_6^{(6)}(P_6)^4 = -\,\lambda_6^{+}(P_6)^4,$$
$$\lambda_6(P_0)^2 = 2\lambda_6^{(6)}(P_0)^2, \qquad 2\lambda_6(P_4)^3 = \lambda_6^{(2)}(P_4)^3, \qquad \lambda_6(P_6)^4 = \lambda_6^{(2)}(P_6)^4.$$

To determine the roots of unity that we need, we note that although we may estimate t_6 at P_0, because the series expansions of t_6 are absolutely convergent at this point, we cannot estimate $t_6^{(6)}$ at P_4 nor at P_6 with the development of $t_6^{(6)}$ at P_0, because this series is not absolutely convergent at any of these points. But we may estimate the value of $t_6^{(6)}$ along the segment joining P_0 to P_6, where $t_6^{(6)}$ must take negative real values, because in the open segment $[P_0, P_6)$ the development of $t_6^{(6)}$ at P_0 is absolutely convergent. This adjusts the root of unity for the local constant $\lambda_6(P_0)$.

For the other, we proceed in a similar manner; but we need to mention that, in the integration of the differential equation for t_6 in a neighbourhood of the SCM point P_0, the solution depends on two parameters and not on only one as in the other cases. The relation between them is obtained from the equation that relates t_6 to t_6^+ or, alternatively, the equation that relates t_6 to $t_6^{(6)}$.

Finally, it remains to determine the local parameters adapted to t_6 at P_2 and adapted to $t_6^{(6)}$ at P_7. For this, we shall use Proposition 6.5. Note that

$$w_2(P_0) = P_7 \quad \text{and} \quad w_6(P_4) = P_2;$$

thus, we may relate the local parameters adapted to t_6^+ at P_0 and P_7 to obtain the latter, and then lift it to a local parameter adapted to $t_6^{(6)}$ at P_7; and, similarly, relating the local parameters adapted to $t_6^{(6)}$ at P_4 and P_2, we obtain the latter, and then we lift it to a local parameter adapted to t_6 at P_2. ■

At this point, it would be natural to consider the adapted local parameter

$$q_A(z) = \left(k_A\,\frac{z - A}{z - \bar{A}}\right)^{e_A}$$

as a uniformizing variable in the neighbourhood of the point A. By doing this, we would obtain series developments

$$\begin{aligned}
t(z) &= \sum_{n=1}^{\infty} b_n q^n, && b_n := a_{ne},\ b_1 = 1, && \text{if } t(A) = 0,\\
t(z) &= t(A) + \sum_{n=1}^{\infty} b_n q^n, && b_n := a_{ne},\ b_1 = 1, && \text{if } t(A) \neq 0, \infty,\\
t(z) &= \sum_{n=-1}^{\infty} b_n q^n, && b_n := a_{ne},\ b_{-1} = 1, && \text{if } t(A) = \infty.
\end{aligned}$$

However, the arithmetic properties of the coefficients b_n can be improved. With this aim in mind, the functions q_A will be slightly modified in Section 7.

7. Local expansions. Series expansions for the uniformizing functions considered in the preceding sections will be given in a neighbourhood of the vertices of the fundamental half domain; all of them are elliptic points. Moreover, the uniformizing function t_6 will be developed in a neighbourhood of the SCM point P_0. Observe that P_0 is an elliptic point for $X_6^{(6)}$ and X_6^+, but it is not elliptic for X_6, $X_6^{(2)}$ and $X_6^{(3)}$.

Case $t(P) \neq \infty$. We begin by studying the uniformizing functions t at the points P where they take a finite value v. First, we consider developments of the shape

$$t(z) = \sum_{n=0}^{\infty} b'_n \, \frac{q(z)^n}{(en)!}, \qquad b'_1 = e!.$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Tables 9 and 10. Thus,

$$t(z) = \sum_{n=0}^{\infty} b''_n \, \frac{q(z)^n}{(en)!}, \qquad b''_1 = \nu e!.$$

Finally, we define the factor $\mathfrak{n}_v = \nu e!$, where $v = t(P)$, and normalize the generating function t by $t(P, q_P; z) := \mathfrak{n}_v^{-1} t(z)$ so that

$$t(P, q_P; z) = \sum_{n=0}^{\infty} c_n \, \frac{q_P(z)^n}{(en)!}, \qquad c_1 = 1, \qquad q_P(z) = \frac{1}{\nu_P}\left(k_P \, \frac{z-P}{z-\overline{P}}\right)^{e_P},$$

where the values of e_P, k_P and ν_P are listed in Tables 9 and 10.

Case $t(P) = \infty$. Finally, we study the uniformizing functions t at the point P_6, where all of them take the value ∞. First, we consider developments of the shape

$$t(z) = \sum_{n=-1}^{\infty} b'_n \, \frac{q(z)^n}{(2e(n+2))!}, \qquad b'_{-1} = (2e)!.$$

Next we renormalize the function q. We replace q by $\nu^{-1}q$, where the values of ν are listed in Tables 9 and 10. Thus,

$$t(z) = \sum_{n=-1}^{\infty} b''_n \, \frac{q(z)^n}{(2e(n+2))!}, \qquad b''_{-1} = \nu(2e)!.$$

Finally, we define the factor $\mathfrak{n}_\infty = \nu(2e)!$ and normalize the generating function t by $t(P, q_P; z) := \mathfrak{n}_\infty^{-1} t(z)$ so that

$$t(P, q_P; z) = \sum_{n=-1}^{\infty} c_n \, \frac{q_P(z)^n}{(2e(n+2))!}, \qquad c_{-1} = 1, \qquad q_P(z) = \frac{1}{\nu_P}\left(k_P \, \frac{z-P}{z-\overline{P}}\right)^{e_P},$$

where the values of e_P, k_P and ν_P are listed in Tables 9 and 10.

The following lemma, whose proof is left to the reader, justifies the choice of the local functions $q_P(z)^n/(2e(n+2))!$.

LEMMA 7.1. *Let*

$$f(q) := \sum_{n=1}^{\infty} \frac{a_n}{(en)!}\, q^n$$

be a power series such that $a_1 = e!$ *and* $a_n \in \mathbb{Z}$. *Define*

$$\frac{1}{f(q)} = \sum_{n=-1}^{\infty} \frac{b_n}{(2e(n+2))!}\, q^n.$$

Then $b_n \in (2e)!\mathbb{Z}$ *for any* $n \geq -1$. ∎

We note that each generating function $t(P, q_P; z)$ is a representative of the homothety class of the corresponding function t. The representatives depend on the point P and the relations between them are compiled in Table 11.

Table 11. Local uniformizing functions $t(P, q_P; z) = \mathfrak{n}_{t(P)}^{-1} \cdot t(z)$

X t	X_6^+ t_6^+	$X_6^{(2)}$ $t_6^{(2)}$	$X_6^{(3)}$ $t_6^{(3)}$	$X_6^{(6)}$ $t_6^{(6)}$	X_6 t_6
$\mathfrak{n}_\infty$	$3870720 = 2^{12} \cdot 3^3 \cdot 5 \cdot 7$ $t_6^+(P_6, q_{P_6})$	$30965760 = 2^{15} \cdot 3^3 \cdot 5 \cdot 7$ $t_6^{(2)}(P_6, q_{P_6})$	$48 = 2^4 \cdot 3$ $t_6^{(3)}(P_6, q_{P_6})$	$96 = 2^5 \cdot 3$ $t_6^{(6)}(P_6, q_{P_6})$	$384 = 2^7 \cdot 3$ $t_6(P_6, q_{P_6})$
$\mathfrak{n}_0$	$144 = 2^4 \cdot 3^2$ $t_6^+(P_0, q_{P_0})$	$\frac{27}{2} = 2^{-1} \cdot 3^3$ $t_6^{(2)}(P_3, q_{P_3})$	$10 = 2 \cdot 5$ $t_6^{(3)}(P_2, q_{P_2})$	$72 = 2^3 \cdot 3^2$ $t_6^{(6)}(P_0, q_{P_0})$	$\frac{3}{2} = 2^{-1} \cdot 3$ $t_6(P_3, q_{P_3})$
$\mathfrak{n}_1$	$40 = 2^3 \cdot 5$ $t_6^+(P_4, q_{P_4})$	$4 = 2^2$ $t_6^{(2)}(P_4, q_{P_4})$	$10 = 2 \cdot 5$ $t_6^{(3)}(P_4, q_{P_4})$	2 $t_6^{(6)}(P_4, q_{P_4})$	2 $t_6(P_4, q_{P_4})$
$\mathfrak{n}_2$	$*$ $*$	$*$ $*$	$*$ $*$	$72 = 2^3 \cdot 3^2$ $t_6^{(6)}(P_7, q_{P_7})$	$*$ $*$
$\mathfrak{n}_{-1}$	$*$ $*$	$*$ $*$	$*$ $*$	$*$ $*$	2 $t_6(P_2, q_{P_2})$
$\mathfrak{n}_i$	$*$ $*$	$*$ $*$	$*$ $*$	$*$ $*$	$12 = 2^2 \cdot 3$ $t_6(P_0, q_{P_0})$

Since they might be useful for further studies (as in the case of the modular j-function), we provide the starting coefficients for some developments of the complex uniformizing function t_6 of the curve X_6.

The coefficients c_n $(1 \le n \le 10)$ of $t_6(P_3, q_{P_3}; z)$:

$$
\begin{aligned}
1 &= 1\\
0 &= 0\\
-48 &= -2^4 \cdot 3\\
0 &= 0\\
27504 &= 2^4 \cdot 3^2 \cdot 191\\
0 &= 0\\
-64498392 &= -2^3 \cdot 3^2 \cdot 7 \cdot 127973\\
0 &= 0\\
436272183216 &= 2^4 \cdot 3^4 \cdot 23 \cdot 229 \cdot 63913\\
0 &= 0
\end{aligned}
$$

The coefficients c_n $(0 \le n \le 10)$ of $t_6(P_4, q_{P_4}; z)$:

$$
\begin{aligned}
1/2 &= 2^{-1}\\
1 &= 1\\
20 &= 2^2 \cdot 5\\
1356 &= 2^2 \cdot 3 \cdot 113\\
227040 &= 2^5 \cdot 3 \cdot 5 \cdot 11 \cdot 43\\
74611380 &= 2^2 \cdot 3 \cdot 5 \cdot 1243523\\
42574294080 &= 2^6 \cdot 3^2 \cdot 5 \cdot 17 \cdot 19 \cdot 45767\\
38683567274400 &= 2^5 \cdot 3^2 \cdot 5^2 \cdot 5372717677\\
52554612744944640 &= 2^{10} \cdot 3^2 \cdot 5 \cdot 11 \cdot 23 \cdot 4507937111\\
101782604056899960000 &= 2^6 \cdot 3^4 \cdot 5^4 \cdot 139 \cdot 226002762361\\
270629344957362042528000 &= 2^8 \cdot 3^4 \cdot 5^3 \cdot 29 \cdot 16126171 \cdot 223259851
\end{aligned}
$$

The coefficients c_n $(0 \le n \le 10)$ of $t_6(P_0, q_{P_0}; z)$:

$$
\begin{aligned}
i/12 &= -i \cdot (1+i)^{-4} \cdot 3^{-1}\\
1 &= 1\\
-12i &= i \cdot (1+i)^4 \cdot 3\\
-226 &= -2 \cdot 113\\
5664i &= (1+i)^{10} \cdot 3 \cdot 59\\
160728 &= 2^3 \cdot 3 \cdot 37 \cdot 181\\
-5467296i &= -(1+i)^{10} \cdot 3 \cdot 56951\\
-211472208 &= -2^4 \cdot 3^5 \cdot 109 \cdot 499\\
9193300992i &= -i \cdot (1+i)^{20} \cdot 3^2 \cdot 571 \cdot 1747\\
445513958784 &= 2^7 \cdot 3^3 \cdot 128910289\\
-23734590202368i &= -(1+i)^{18} \cdot 3^4 \cdot 15919 \cdot 35951
\end{aligned}
$$

The coefficients c_n $(-1 \le n \le 10)$ of $t_6(P_6, q_{P_6}; z)$:

$$\begin{array}{r}
1\\
0\\
18480\\
0\\
12803590800\\
0\\
-817993722627081000\\
0\\
-1560789298453265580199500000\\
0\\
122859953407720110679241179380345000\\
0
\end{array}$$

and their factorizations:

$$\begin{array}{l}
1\\
0\\
2^4 \cdot 3 \cdot 5 \cdot 7 \cdot 11\\
0\\
2^4 \cdot 3^2 \cdot 5^2 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19\\
0\\
-2^3 \cdot 3^3 \cdot 5^3 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 47 \cdot 61\\
0\\
-2^4 \cdot 3^4 \cdot 5^5 \cdot 7^3 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19^2 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 13729\\
0\\
2^3 \cdot 3^5 \cdot 5^4 \cdot 7^4 \cdot 11^4 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 41 \cdot 43 \cdot 661 \cdot 59107\\
0
\end{array}$$

In [3], the preceding method has been applied in the modular case, providing natural series expansions for all the triangle modular functions (including the elliptic modular function j) around any elliptic point or cusp. Thus, in this case, each of these functions also supplies a family of natural arithmetic developments.

8. Canonical models. In this section we derive explicit canonical models for the curves X_6, $X_6^{(2)}$, $X_6^{(3)}$, $X_6^{(6)}$, X_6^+. Actually, the models we have computed up to now are complex models or, more precisely, models defined over the field $\mathbb{Q}(\zeta_{12})$, where ζ_{12} is a 12th primitive root of unity (cf. Corollary 8.2 and Remark 8.3).

Note that X_6 has no real points; in particular, although it is of genus zero and defined over $\mathbb{Q}$, the field $\mathbb{Q}(X_6)$ cannot be generated by a single function valued in $\mathbf{P}^1(\mathbb{C})$.

We know by Shimura [10] that there exist canonical rational models $j_6 : \overline{\Gamma}_6\backslash\mathcal{H} \to X_6(\mathbb{C})$, $j_6^{(d)} : \overline{\Gamma}_6^{(d)}\backslash\mathcal{H} \to X_6^{(d)}(\mathbb{C})$, $d = 2, 3, 6$, and $j_6^+ : \overline{\Gamma}_6^+\backslash\mathcal{H} \to X_6^+(\mathbb{C})$, which are compatible with the projections on the left hand side and the covering mappings on the right hand side.

Lemma 8.1. *The points $j_6(P_2)$ and $j_6(P_4)$ are rational over $\mathbb{Q}(\sqrt{-3})$. The points $j_6(P_3)$ and $j_6(P_6)$ are rational over $\mathbb{Q}(\sqrt{-1})$. The points $j_6(P_0)$ and $j_6(P_7)$ are rational over $\mathbb{Q}(\sqrt{-6}, \sqrt{-3})$.*

Proof. The points $j_6(P_2)$ and $j_6(P_4)$ are elliptic of order 3; therefore, they are rational over the field $\mathbb{Q}(\sqrt{-3})$ (cf. [4]). Analogously, the points $j_6(P_3)$ and $j_6(P_6)$ are elliptic of order 2; therefore, they are rational over the field $\mathbb{Q}(\sqrt{-1})$. On the other hand, the points $j_6(P_0)$ and $j_6(P_7)$ are complex multiplication points for the quadratic field $\mathbb{Q}(\sqrt{-6})$. They are the fixed points for the embeddings $\mathbb{Q}(\sqrt{-6}) \hookrightarrow \mathbb{H}_6$ given, respectively, by $\sqrt{-6} \mapsto -3\mathrm{J} + \mathrm{K}$ and $\sqrt{-6} \mapsto \mathrm{I} - 3\mathrm{J}$. According to the model of Shimura, we have

$$\begin{aligned}\mathbb{Q}(j_6(P_0)) \cdot \mathbb{Q}(\sqrt{-6}) &= \mathbb{Q}(j_6(P_7)) \cdot \mathbb{Q}(\sqrt{-6}) \\ &= \mathrm{HCF}(\mathbb{Q}(\sqrt{-6})) = \mathbb{Q}(\sqrt{-6}, \sqrt{-3}). \ \blacksquare\end{aligned}$$

Corollary 8.2. *The models given by the functions t_6^+, $t_6^{(2)}$, $t_6^{(3)}$, $t_6^{(6)}$, and t_6 are defined over $k := \mathbb{Q}(\zeta_{24}) = \mathbb{Q}(\sqrt{-3}, \sqrt{-1}, \sqrt{2})$.*

Proof. Since all of our curves are of genus zero and all of them have k-rational points, their fields of rational functions over k are generated by a single function. Now, since the automorphism group of the field is $\mathbf{PSL}(2, k)$, the generator is uniquely determined by its value at three k-rational points. But our functions t_6^+, $t_6^{(2)}$, $t_6^{(3)}$, $t_6^{(6)}$ and t_6 take k-rational values 0, 1, ∞ at points that are k-rational in the canonical model. Therefore, they generate the function fields over k. $\blacksquare$

Remark 8.3. Actually, the minimal field of definition of the points $j_6(P_0)$ and $j_6(P_7)$ is $\mathbb{Q}(\sqrt{-3})$ (cf. Theorem 8.7). For the minimal field of definition of each of the preceding models see the discussion preceding Theorem 8.7.

Now, we prove that the model computed for X_6^+ is the canonical model.

Proposition 8.4. *For the curve X_6^+ we have:*

(a) $j_6^+(P_0) = j_6^+(P_7)$, $j_6^+(P_2) = j_6^+(P_4)$, $j_6^+(P_3) = j_6^+(P_6) \in X_6^+(\mathbb{Q})$.
(b) $X_6^+(\mathbb{Q}) \simeq \mathbf{P}^1(\mathbb{Q})$.
(c) $\mathbb{Q}(X_6^+) = \mathbb{Q}(t_6^+)$.

Proof. The points $j_6^+(P_0) = j_6^+(P_7)$, $j_6^+(P_2) = j_6^+(P_4)$ and $j_6^+(P_3) = j_6^+(P_6)$ are the only points of X_6^+ that are elliptic, and they are of different orders; therefore, they must be rational. Since X_6^+ is defined over $\mathbb{Q}$ and of genus 0, we obtain (b). Finally, t_6^+ takes rational values at the three rational points $j_6^+(P_0)$, $j_6^+(P_4)$, $j_6^+(P_6)$ and we obtain (c). ■

Next, we show that there are $\mathbb{Q}$-rational points in any of the curves $X_6^{(2)}$, $X_6^{(3)}$ and $X_6^{(6)}$.

PROPOSITION 8.5. *For the curves $X_6^{(2)}$, $X_6^{(3)}$ and $X_6^{(6)}$, we have:*

(a) $j_6^{(2)}(P_2) = j_6^{(2)}(P_4)$, $j_6^{(2)}(P_0) = j_6^{(2)}(P_7) \in X_6^{(2)}(\mathbb{Q})$.
(b) $j_6^{(3)}(P_3) = j_6^{(3)}(P_6)$, $j_6^{(3)}(P_0) = j_6^{(3)}(P_7) \in X_6^{(3)}(\mathbb{Q})$.
(c) $j_6^{(6)}(P_2) = j_6^{(6)}(P_4)$, $j_6^{(6)}(P_3) = j_6^{(6)}(P_6) \in X_6^{(6)}(\mathbb{Q})$.

Therefore, there exist isomorphisms $X_6^{(2)}(\mathbb{Q}) \simeq \mathbf{P}^1(\mathbb{Q})$, $X_6^{(3)}(\mathbb{Q}) \simeq \mathbf{P}^1(\mathbb{Q})$ and $X_6^{(6)}(\mathbb{Q}) \simeq \mathbf{P}^1(\mathbb{Q})$.

Proof. (a) The point $j_6^{(2)}(P_2) = j_6^{(2)}(P_4)$ is the unique elliptic point of order 3 of $X_6^{(2)}$; therefore, it must be rational. Moreover, the cover $X_6^{(2)} \to X_6^+$ ramifies only at the points $j_6^+(P_0)$ and $j_6^+(P_4)$, which are rational. Since one of the ramification points, namely $j_6^{(2)}(P_4)$, is rational, the other must also be rational; that is, $j_6^{(2)}(P_0) = j_6^{(2)}(P_7) \in X_6^{(2)}(\mathbb{Q})$.

(b) We argue similarly, with P_4 replaced by P_6.

(c) Now, $j_6^{(6)}(P_2) = j_6^{(6)}(P_4)$ is the unique elliptic point of order 3, and the ramification points are $j_6^+(P_4)$ and $j_6^+(P_6)$. ■

COROLLARY 8.6. *There exist automorphic functions u_2, u_3, u_6 such that $\mathbb{Q}(X_6^{(2)}) = \mathbb{Q}(u_2)$, $\mathbb{Q}(X_6^{(3)}) = \mathbb{Q}(u_3)$ and $\mathbb{Q}(X_6^{(6)}) = \mathbb{Q}(u_6)$.* ■

To compute these functions u_2, u_3, u_6, we shall take in account the ramification of the coverings. Since $X_6^{(2)} \to X_6^+$ is ramified only at the points such that t_6^+ takes the values 0, 1, the field extension $\mathbb{Q}(X_6^{(2)})|\mathbb{Q}(X_6^+) = \mathbb{Q}(t_6^+)$ is ramified exactly at t_6^+ and $t_6^+ - 1$. Therefore, we can choose u_2 such that $u_2^2 = r_2 t_6^+(t_6^+ - 1)^{-1}$, where r_2 is a square free integer. On the other hand, since $X_6^{(3)} \to X_6^+$ is ramified only at the points such that t_6^+ takes the values $0, \infty$, the field extension $\mathbb{Q}(X_6^{(3)})|\mathbb{Q}(X_6^+)$ is ramified exactly at t_6^+ and ∞. Therefore, we can choose u_3 such that $u_3^2 = r_3 t_6^+$, where r_3 is a square free integer. And finally, since $X_6^{(6)} \to X_6^+$ is ramified only at the points such that t_6^+ takes the values $1, \infty$, the field extension $\mathbb{Q}(X_6^{(6)})|\mathbb{Q}(X_6^+)$ is ramified exactly at $t_6^+ - 1$ and ∞. Therefore, we can choose u_6 such that $u_6^2 = r_6(t_6^+ - 1)$, where r_6 is a square free integer. Moreover, the three quadratic subfields of the field $\mathbb{Q}(X_6^{(3)}) \cdot \mathbb{Q}(X_6^{(6)}) = \mathbb{Q}(X_6)$ being $\mathbb{Q}(X_6^{(2)})$, $\mathbb{Q}(X_6^{(3)})$ and $\mathbb{Q}(X_6^{(6)})$, we may take the constants r_2, r_3 and r_6 such that one of them equals the product of the other two.

Now, observe that the equations in Theorem 4.3 imply the following ones:

$$\left(\frac{t_6^{(2)}+1}{t_6^{(2)}-1}\right)^2 = \frac{t_6^+}{t_6^+-1}, \qquad (2t_6^{(3)}-1)^2 = t_6^+, \qquad (t_6^{(6)}-1)^2 = 1-t_6^+.$$

Therefore, we have

$$u_2 = \sqrt{r_2}\,\frac{t_6^{(2)}+1}{t_6^{(2)}-1}, \qquad u_3 = \sqrt{r_3}\,(2t_6^{(3)}-1), \qquad u_6 = \sqrt{-r_6}\,(t_6^{(6)}-1).$$

Next we compute the constants r_2, r_3 and r_6. Since $t_6^{(2)}$ takes values in $\mathbb{Q}(\sqrt{-1})$ for three $\mathbb{Q}(\sqrt{-1})$-rational points of $X_6^{(2)}$, namely, the points $j_6^{(2)}(P_3)$, $j_6^{(2)}(P_4)$, $j_6^{(2)}(P_6)$, we have $\mathbb{Q}(\sqrt{-1})(X_6^{(2)}) = \mathbb{Q}(\sqrt{-1})(t_6^{(2)})$. Therefore,

$$\mathbb{Q}(\sqrt{-1})(u_2) = \mathbb{Q}(\sqrt{-1})(X_6^{(2)}) = \mathbb{Q}(\sqrt{-1})(t_6^{(2)}) = \mathbb{Q}(\sqrt{-1})(\sqrt{r_2}\,u_2).$$

Thus, $\sqrt{r_2} \in \mathbb{Q}(\sqrt{-1})$. Working similarly for $t_6^{(3)}$ and $X_6^{(3)}$, we find that $\sqrt{r_3} \in \mathbb{Q}(\sqrt{-3})$. Since r_2 and r_3 are square free, we must have $r_2 = -1$ or $r_2 = 1$, and $r_3 = -3$ or $r_3 = 1$. To discard the value $r_3 = 1$, we observe that $\mathbb{Q}(X_6) = \mathbb{Q}(u_3, u_6)$, and if $r_3 = 1$, then the two functions u_3, u_6 would take rational values $(1, 0)$ at P_4, while $j_6(P_4)$ is not a rational point of X_6. This fixes $r_3 = -3$. Similarly, if $r_2 = 1$, then the two functions u_2 and $1/u_6$ would take rational values $(1, 0)$ at P_6, while $j_6(P_6)$ is not a rational point of X_6 and $\mathbb{Q}(X_6) = \mathbb{Q}(u_2, 1/u_6)$. This fixes $r_2 = -1$ and $r_6 = 3$.

We are now in a position to make precise the minimal field of definition of the models in Corollary 8.2: it is $\mathbb{Q}$ for t_6^+; $\mathbb{Q}(\sqrt{-1})$ for $t_6^{(2)}$; $\mathbb{Q}(\sqrt{-3})$ for $t_6^{(3)}$ and $t_6^{(6)}$; and $\mathbb{Q}(\sqrt{-3}, \sqrt{-1})$ for t_6.

We have proven the following result. In particular, our functions satisfy the algebraic equation for X_6 given by Ihara (cf. [6]).

THEOREM 8.7. *The functions* $j_6^+, j_6^{(2)}, j_6^{(3)}, j_6^{(6)} : \mathcal{H} \to \mathbf{P}^1(\mathbb{C})$ *given by*

$$j_6^+ = t_6^+, \qquad j_6^{(2)} = u_2 = \sqrt{-1}\,\frac{t_6^{(2)}+1}{t_6^{(2)}-1},$$

$$j_6^{(3)} = u_3 = \sqrt{-3}\,(2t_6^{(3)}-1), \qquad j_6^{(6)} = u_6 = \sqrt{-3}\,(t_6^{(6)}-1),$$

and the function $j_6 : \mathcal{H} \to \mathbf{P}^2(\mathbb{C})$ *given by*

$$j_6 = (u_3 : u_6 : 1) = (u_2 : 1 : 1/u_6) = (1 : 1/u_2 : 1/u_3)$$

define canonical models of the curves X_6^+, $X_6^{(2)}$, $X_6^{(3)}$, $X_6^{(6)}$, *and* X_6, *respectively. For the canonical model of the curve* X_6, *we have* $\mathbb{Q}(X_6) = \mathbb{Q}(j_6) =$

$\mathbb{Q}(u_3, u_6) = \mathbb{Q}(u_2, u_6) = \mathbb{Q}(u_2, u_3)$ *and the equations*

$$u_3^2 + u_6^2 + 3 = 0, \qquad u_2^2 + 1 + \frac{3}{u_6^2} = 0, \qquad 1 + \frac{1}{u_2^2} + \frac{3}{u_3^2} = 0$$

provide a set of affine rational charts for the projective curve $X_6/\mathbb{Q}$. *Moreover*, $j_6(P_0) = (0 : -\sqrt{-3} : 1)$ *and* $j_6(P_7) = (0 : \sqrt{-3} : 1)$. ∎

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