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A remark on the Piatetski-Shapiro–Vinogradov theorem

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1. Introduction. In 1937 Vinogradov [7] proved the well known Goldbach–Vinogradov theorem: Every sufficiently large odd integer N can be represented as a sum of three primes. It can be stated in a more exact quantitative form: Let $T(N)$ denote the number of solutions of the equation

$$N = p_1 + p_2 + p_3.$$

Then

$$T(N) = \frac{C(N)N^2}{2\log^3 N} + O\biggl(\frac{N^2}{\log^A N}\biggr)$$

for any $A > 3$, where $C(N)$ denotes the singular series

$$C(N) = \prod_{p|N}\biggl(1 - \frac{1}{(p-1)^2}\biggr)\prod_{(p,N)=1}\biggl(1 + \frac{1}{(p-1)^3}\biggr).$$

Motivated by earlier work of Erdős and Nathanson [2] on sums of squares, some mathematicians considered the question of whether one could find thin subsets of primes which were still sufficient to obtain the Goldbach–Vinogradov theorem. In 1986, based on probability considerations, Wirsing [8] proved that there exists a subset S of primes with the property

$$\sum_{\substack{p\le x\\ p\in S}} 1 \ll (x\log x)^{1/3},$$

which serves this purpose. Although Wirsing's result is best possible apart from the logarithmic factor, it does not lead to a subset of primes which is constructive or recognizable.

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Primes of the form $[n^c]$, where $1 \le c < 2$, are called *Piatetski-Shapiro primes*. Let $\gamma = 1/c$ and

$$P_\gamma = \{p \mid p = [n^{1/\gamma}] = [n^c]\}.$$

In 1992 Balog and Friedlander [1] obtained a theorem which had two interesting corollaries:

Corollary 1. *For any fixed* $20/21 < \gamma \le 1$, *the Goldbach–Vinogradov theorem holds with primes in* P_γ.

Corollary 2. *For any fixed* $8/9 < \gamma \le 1$, *the Goldbach–Vinogradov theorem holds with one prime in* P_γ.

In 1995 Jia [4] improved Corollary 1 to

Theorem 1. *For any fixed* $15/16 < \gamma \le 1$, *the Goldbach–Vinogradov theorem holds with primes in* P_γ.

The purpose of this note is to present an approach different from that of Balog–Friedlander's which leads to an improvement of Corollary 2.

Theorem 2. *For any fixed* $205/243 < \gamma \le 1$, *the Goldbach–Vinogradov theorem holds with one prime in* P_γ.

2. Proof of Theorem 2. In order to prove Theorem 2 we need the following two lemmas.

Lemma 1 [6]. *For any fixed* $205/243 < \gamma \le 1$,

$$P_\gamma(x) = \sum_{\substack{x<p\le 2x\\ p=[n^{1/\gamma}]}} 1 \gg \frac{x^\gamma}{\gamma \log x}.$$

Lemma 2 [3]. *Let* $E(x)$ *denote the number of even integers in the interval* $[x/2, x]$ *which cannot be represented as a sum of two primes. Then for any* $A > 0$,

$$E(x) = O_A\left(\frac{x}{\log^A x}\right).$$

In 1975 Montgomery and Vaughan [5] improved Lemma 2 by showing that there exists an effective constant $\Delta > 0$ such that

$$E(x) = O(x^{1-\Delta}),$$

but for our purpose Lemma 2 is sufficient.

Proof of Theorem 2. Let

$$\mathcal{A} = \{N - p \mid N/3 < p \le 2N/3,\ p \in P_\gamma\}.$$

Then by Lemma 1 we have

$$|\mathcal{A}| \gg \frac{N^{\gamma}}{\gamma \log N}.$$

Let $E(\mathcal{A})$ denote the number of even integers in $\mathcal{A}$ which cannot be represented as a sum of two primes. Then by Lemma 2 we have

$$E(\mathcal{A}) \ll \frac{N^{\gamma}}{\log^3 N},$$

and

$$|\mathcal{A} \setminus E(\mathcal{A})| \gg \frac{N^{\gamma}}{\gamma \log N}.$$

For any $N - p \in \mathcal{A} \setminus E(\mathcal{A})$, there exist primes p_1, p_2 such that

$$N - p = p_1 + p_2,$$

hence

$$N = p + p_1 + p_2, \qquad N/3 < p \leq 2N/3,\ p \in P_{\gamma},$$

and Theorem 2 follows.

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