Kummer theory of abelian varieties and reductions of Mordell–Weil groups

by

TOM WESTON (Berkeley, CA)

Let $A$ be an abelian variety over a number field $F$. We write $\text{red}_v : A(F) \to A(k_v)$ for the reduction map at a place $v$ of $F$ with residue field $k_v$. W. Gajda has posed the following question.

**Question.** Let $\Sigma$ be a subgroup of $A(F)$. Suppose that $x$ is a point of $A(F)$ such that $\text{red}_v x$ lies in $\text{red}_v \Sigma$ for almost all places $v$ of $F$. Does it then follow that $x$ lies in $\Sigma$?

In this paper we use methods of Kummer theory to provide the following partial answer to this question.

**Theorem.** Let $A$ be an abelian variety over a number field $F$ and assume that $\text{End}_F A$ is commutative. Let $\Sigma$ be a subgroup of $A(F)$ and suppose that $x \in A(F)$ is such that $\text{red}_v x \in \text{red}_v \Sigma$ for almost all places $v$ of $F$. Then $x \in \Sigma + A(F)_{\text{tors}}$.

It does not appear that the torsion ambiguity can be eliminated with our present approach, and it is not clear to the author how to modify the arguments for the non-commutative case. We note that our theorem applies in particular to products of non-isogenous elliptic curves.

Gajda’s question has its origins in the support problem of P. Erdős: if $x$ and $y$ are positive integers such that for any $n \geq 1$ the set of primes dividing $x^n - 1$ is the same as the set of primes dividing $y^n - 1$, then must $x$ equal $y$? Corrales-Rodrígánez and Schoof gave an affirmative answer to this question in [3] and also answered the corresponding question for elliptic curves; this was generalized by Banaszk, Gajda and Krasoń in [1] to certain abelian varieties with complex or real multiplication and $\text{End}_F A$ a commutative maximal order. Recently Larsen [7] has given a proof of the support problem for arbitrary abelian varieties; see also [6] for results of Kowalski on a closely
related question. In this context the support problem takes the following form.

**Question.** Let \( x, y \in A(F) \) be non-torsion points. Suppose that the order of \( \text{red}_v x \) divides the order of \( \text{red}_v y \) for almost all places \( v \) of \( F \). Does it follow that \( x \) and \( y \) satisfy an \( \text{End}_F A \)-linear relation in \( A(F) \)?

If we take \( \Sigma = \text{End}_F A \cdot y \), the support problem implies a weak form of our main theorem in the case when \( \Sigma \) is a cyclic \( \text{End}_F A \)-module. The more precise question of Gajda we consider is one possible modification of the support problem for abelian varieties to a non-cyclic setting. The approach we use here is quite different from that of [3] and [1], relying more on the study of the Mordell–Weil group of \( A \) as a module for \( \text{End}_F A \) and less on Galois cohomology.

We now give an overview of our argument in the simplest case. Assume that \( A \) is simple, that \( \mathcal{O} := \text{End}_F A \) is integrally closed (so that it is a Dedekind domain), and that \( A(F) \) is a free \( \mathcal{O} \)-module. With \( \Sigma \subseteq A(F) \) and \( x \in A(F) \) as in the theorem, it suffices to prove that \( x \in \Sigma \otimes \mathbb{Z}_{(p)} \) for every prime \( p \) (with \( \mathbb{Z}_{(p)} \) the localization of \( \mathbb{Z} \) away from \( p \)). Fix, then, a prime \( p \) and suppose that \( x \notin \Sigma \otimes \mathbb{Z}_{(p)} \). The first step, which is purely algebraic, is to show that under this assumption one can choose an \( \mathcal{O} \)-basis \( y_1, \ldots, y_r \) of \( A(F) \) such that \( \psi_1(x) \notin \psi_1(\Sigma) + p^a \mathcal{O} \) for some \( a > 0 \); here \( \psi_1 : A(F) \to \mathcal{O} \) is the projection onto the \( y_1 \)-coordinate.

The next step is to choose an appropriate place \( v \) of \( F \). We work instead over the extensions \( F(A[p^n]) \) of \( F \). Using Kummer theory and the Chebotarev density theorem, we show that there is a \( b > 0 \) such that for any sufficiently large \( n \) there is a place \( w \) of \( F(A[p^n]) \) with \( \text{red}_w y_2, \ldots, \text{red}_w y_r \in p^b A(k_w) \), while \( \text{red}_w y_1 \notin p^b A(k_w) \) for any \( i \); here \( p \mathcal{O} = p_1^{e_1} \cdots p_g^{e_g} \) is the ideal factorization of \( p \) in \( \mathcal{O} \).

Fix \( n \geq a + b \) and choose such a place \( w \). By hypothesis we have \( \text{red}_w x = \text{red}_w y \) for some \( y \in \Sigma \). If we expand in terms of our chosen basis of \( A(F) \), the choice of \( w \) implies that

\[
(\psi_1(x) - \psi_1(y)) \text{red}_w y_1 \in p^n A(k_w).
\]

On the other hand, using the properties of \( \psi_1 \) and of \( w \), one can show directly that

\[
(\psi_1(x) - \psi_1(y)) \text{red}_w y_1 \notin p^{a+b} A(k_w).
\]

As \( n \geq a + b \), we have a contradiction, so that we must have had \( x \in \Sigma \otimes \mathbb{Z}_{(p)} \). This completes our sketch of the argument in this case.

We now review the contents of this paper in more detail. We begin in Section 1.1 with a review of Kummer theory and in Section 1.2 we adapt the methods of Bashmakov–Ribet as in [9] to prove that the cokernel of
the \( p \)-adic Kummer map is bounded. In Section 1.3 we discuss the relation
between Kummer theory and reduction maps.

In the sketch above we assumed that \( \mathcal{O} \) was an integrally closed domain
and that \( A(F) \) was free over \( \mathcal{O} \). The algebra required to eliminate these
assumptions is developed in Section 2. These results are combined with
Kummer theory to produce places \( w \) as above in Section 3.1, and the proof
of our main theorem is given in Section 3.2.

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1. Kummer theory

1.1. Review of Kummer theory. Let \( A \) be an abelian variety over a
number field \( F \); set \( \mathcal{O} = \text{End}_F A \). For \( \alpha \in \mathcal{O} \) we set \( F_\alpha = F(A[\alpha]) \) and
\( G_\alpha = \text{Gal}(F_\alpha/F) \). The Kummer map
\[
\kappa_\alpha : A(F)/\alpha \to \text{Hom}_{G_\alpha}(\text{Gal}(F/F_\alpha), A[\alpha])
\]
is defined as the composition
\[
A(F)/\alpha \hookrightarrow H^1(F, A[\alpha]) \overset{\text{res}}{\to} H^1(F_\alpha, A[\alpha])^{G_\alpha}
\]
with the first map a coboundary map for the \( \text{Gal}(F/F) \)-cohomology of the
Kummer sequence
\[
0 \to A[\alpha] \to A(F) \overset{\alpha}{\to} A(F) \to 0
\]
and the second map restriction to \( F_\alpha \). (Concretely, for \( x \in A(F) \), \( \kappa_\alpha(x) \) is
the homomorphism sending \( \gamma \in \text{Gal}(F/F_\alpha) \) to \( \gamma(\frac{x}{\alpha}) - \frac{x}{\alpha} \in A[\alpha] \) where \( \frac{x}{\alpha} \) is
some fixed \( \alpha \)-th root of \( x \) in \( A(F) \).)

If \( \Gamma \) is an \( \mathcal{O} \)-submodule of \( A(F) \) and \( \alpha \in \mathcal{O} \), we write \( F_\alpha(\frac{1}{\alpha}\Gamma) \) for the
extension of \( F_\alpha \) generated by all \( \alpha \)-th roots of elements of \( \Gamma \); alternately, \( F_\alpha(\frac{1}{\alpha}\Gamma) \) is the fixed field of the intersection of the kernels of the
homomorphisms \( \kappa_\alpha(\Gamma) \). The Galois group \( \mathfrak{g}_\alpha(\Gamma) := \text{Gal} \left( F_\alpha(\frac{1}{\alpha}\Gamma)/F_\alpha \right) \) is an
\( \mathcal{O}[G_\alpha] \)-module and \( \kappa_\alpha \) restricts to an \( \mathcal{O} \)-linear map
\[
\Gamma/\alpha \to \text{Hom}_{G_\alpha}(\mathfrak{g}_\alpha(\Gamma), A[\alpha]).
\]
We write the \( \mathcal{O}[G_\alpha] \)-dual of this map as
\[
\lambda_\alpha^\Gamma : \mathfrak{g}_\alpha(\Gamma) \hookrightarrow \text{Hom}_\mathcal{O}(\Gamma, A[\alpha]).
\]

1.2. \( p \)-adic Kummer theory. Fix a rational prime \( p \); set \( \mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p \) and
\( K_p = \mathcal{O} \otimes \mathbb{Q}_p \). The Tate module \( T_p A := \text{lim} A[p^n] \) (resp. Tate space \( V_p A := T_p A \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \) is naturally an \( \mathcal{O}_p[G_{p\infty}] \)-module (resp. \( K_p[G_{p\infty}] \)-module) where
\( G_{p\infty} = \text{Gal}(F(A[p^\infty])/F) \). It follows from \([8, \text{Section 19, Corollary 2}]\) that there is a decomposition
\[
K_p = \prod M_{n_i}K_i
\]
where $M_{n_i} K_i$ is the central simple algebra of $n_i \times n_i$-matrices over the division ring $K_i$. Corresponding to (1.1) is a decomposition $V_i A = \bigoplus V_i A^{n_i}$ of $V_i A$ into $K_i[G_{p\infty}]$-modules. By [5, Theorem 4], we have

$$(1.2) \quad \text{End}_{Q_p[G_{p\infty}]} V_i A = K_i$$

for each $i$; in particular, each $V_i A$ is an irreducible $K_i[G_{p\infty}]$-module. We record a second immediate consequence of (1.2) in the next lemma.

**Lemma 1.1.** Let $\Gamma$ be an $O$-module. Then the evaluation map

$$\Gamma \otimes O K_i \to \text{Hom}_{K_i[G_{p\infty}]}(\text{Hom}_O(\Gamma, V_i A), V_i A)$$

is an isomorphism.

Fix an $O$-submodule $\Gamma$ of $A(F)$. The inverse limit $\mathfrak{g}_{p\infty}(\Gamma)$ of the $\mathfrak{g}_{p^n}(\Gamma)$ is naturally an $O_p[G_{p\infty}]$-module endowed with an injection

$$\lambda_{p\infty}^\Gamma : \mathfrak{g}_{p\infty}(\Gamma) \hookrightarrow \text{Hom}_O(\Gamma, T_{p A}).$$

More generally, since $O_p/p^n \cong O/p^n$ for all $n$, for any $O$-module $\Gamma \subseteq A(F) \otimes \mathbb{Z}_p$ we can still define $\mathfrak{g}_{p^n}(\Gamma)$ and $\lambda_{p^n}^\Gamma$ for $n \leq \infty$. In any case, there is a $K_p[G_{p\infty}]$-module decomposition

$$(1.3) \quad \mathfrak{g}_{p\infty}(\Gamma) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \bigoplus V_i A$$

(with $n_i$ as in (1.1)) into $K_i[G_{p\infty}]$-modules, and there are natural injections

$$\lambda_i^\Gamma : \mathfrak{g}_i(\Gamma) \hookrightarrow \text{Hom}_O(\Gamma, V_i A).$$

The decomposition (1.3) is functorial in the sense that there is a natural surjection $\mathfrak{g}_i(\Gamma) \twoheadrightarrow \mathfrak{g}_i(\Gamma')$ for any $O$-submodule $\Gamma'$ of $\Gamma$.

The main result of Kummer theory we need is the following. The proof is a straightforward adaptation of the methods of Bashmakov and Ribet.

**Proposition 1.2.** Fix a rational prime $p$ and let $\Gamma$ be an $O$-submodule of $A(F)$. Then the cokernel of $\lambda_{p^n}^\Gamma$ is bounded independent of $n$.

**Proof.** First consider the cyclic case $\Gamma = O \cdot x$ for $x \in A(F)$. If $\Gamma \cong O$, then $Z \cdot x$ is Zariski dense in $A$; the proposition thus follows from [2, Theorem 2] in this case. More generally, let $A'$ denote the largest abelian subvariety of $A$, defined over $F$, in which $Z \cdot x$ is Zariski dense; set $O' = \text{End}_F A'$. Using the Poincaré reducibility theorem (see [8, Section 19, Theorem 1]), one easily checks that

$$\text{Hom}_O(\Gamma, V_p A) \cong \text{Hom}_{O'}(\Gamma, V_p A'),$$

so that the general cyclic case follows from [2, Theorem 2] applied to $A'$. In fact, one has $\text{coker} \lambda_{p\infty}^{Ox} = \text{coker} \lambda_{p\infty}^{Ox'}$ whenever $x, x' \in A(F)$ are sufficiently $p$-adically congruent, so that the same arguments apply for arbitrary $x \in A(F) \otimes \mathbb{Z}_p$. 
For general $\Gamma$ it suffices to show that each of the injections $\lambda_{i}^{\Gamma}$ is an isomorphism. Suppose, then, that some $\lambda_{i}^{\Gamma}$ is not surjective. As $\text{Hom}_{\mathcal{O}}(\Gamma, V_{i}A)$ is a direct sum of copies of the irreducible $K_{i}[G_{p}]$-module $V_{i}A$ (and thus in particular is a semisimple $K_{i}[G_{p}]$-module), it follows that there exists a $K_{i}[G_{p}]$-surjection

$$\varphi : \text{Hom}_{\mathcal{O}}(\Gamma, V_{i}A) \to V_{i}A$$

annihilating $g_{i}(\Gamma)$. By Lemma 1.1 the map $\varphi$ is given by evaluation at some $x \in \Gamma \otimes_{\mathcal{O}} K_{i}$; using the injection $K_{i} \hookrightarrow K_{p}$ and scaling $\varphi$ if necessary, we may in fact assume that $x \in \Gamma \otimes \mathbb{Z}_{p}$. There is then a commutative diagram

$$\begin{array}{ccc}
g_{i}(\Gamma) & \xrightarrow{\lambda_{i}^{\Gamma}} & \text{Hom}_{\mathcal{O}}(\Gamma, V_{i}A) \\
\downarrow \hspace{1cm} & & \downarrow \varphi \\
g_{i}(\mathcal{O} \cdot x) & \xrightarrow{\lambda_{i}^{O \cdot x}} & V_{i}A
\end{array}$$

The clockwise composition is zero by construction, so that we must have $\lambda_{i}^{O \cdot x} = 0$ as well. By the cyclic case considered above this implies that $x$ maps to zero in $\Gamma \otimes_{\mathcal{O}} K_{i}$. But then $\varphi$, which is evaluation at $x$, is also zero. This contradicts the surjectivity of $\varphi$ and thus proves the proposition.

1.3. Reductions and Frobenius elements. We write $k_{w}$ for the residue field of a finite extension $F'$ of $F$ at a place $w$, and $\text{red}_{w} : A(F') \to A(k_{w})$ for the reduction map.

**Lemma 1.3.** Fix $\alpha \in \mathcal{O}$ and $x \in A(F')$. Let $w$ be a finite place of $F_{\alpha}$, relatively prime to $\alpha$, at which $A$ has good reduction. Then $\text{red}_{w} x$ lies in $\alpha A(k_{w})$ if and only if $\lambda_{\alpha}^{O \cdot x}(\text{Frob}_{w}) = 0$, where $\text{Frob}_{w} \in \text{Gal}(F_{\alpha}(\frac{x}{\alpha})/F_{\alpha})$ is the Frobenius element at $w$.

**Proof.** Fix an $\alpha$th root $\frac{x}{\alpha}$ of $x$ in $A(\overline{F})$ and a place $w'$ of $F_{\alpha}(\frac{x}{\alpha})$ over $w$. If $\lambda_{\alpha}^{O \cdot x}(\text{Frob}_{w}) = 0$, then $w'$ is completely split over $w$ so that $k_{w'} = k_{w}$. In particular, $\text{red}_{w'} \frac{x}{\alpha} \in A(k_{w'})$ lies in $A(k_{w})$; thus $\text{red}_{w} x \in \alpha A(k_{w})$ as claimed.

Conversely, if there is $y \in A(k_{w})$ with $\alpha y = \text{red}_{w} x$, then $y - \text{red}_{w'} \frac{x}{\alpha}$ lies in $A[\alpha]$. Since $y$ and $A[\alpha]$ are both in $A(k_{w})$ we conclude that $\text{red}_{w'} \frac{x}{\alpha}$ is in $A(k_{w})$ as well. In particular, we have

$$\text{Frob}_{w}(\text{red}_{w'} \frac{x}{\alpha}) - \text{red}_{w'} \frac{x}{\alpha} = 0. \quad (1.4)$$

On the other hand, $\text{Frob}_{w}(\frac{x}{\alpha}) - \frac{x}{\alpha}$ already lies in $A[\alpha]$, which injects into $A(k_{w'})$; (1.4) thus forces

$$\text{Frob}_{w}(\frac{x}{\alpha}) - \frac{x}{\alpha} = 0 \quad \text{in } A(\overline{F}).$$

This says exactly that $\lambda_{\alpha}^{O \cdot x}(\text{Frob}_{w}) = 0$, as claimed. ■
We assume now that $\mathcal{O}$ is commutative. Suppose that $a$ is an ideal of $\mathcal{O}$ such that $\beta a \subseteq \alpha \mathcal{O}$ for some $\alpha, \beta \in \mathcal{O}$. Multiplication by $\beta$ then yields a map $A[\alpha] \rightarrow A[a]$.

**Lemma 1.4.** Let $\alpha, \beta, a$ be as above and fix $x \in A(F)$. Let $w$ be a finite place of $F_\alpha$, relatively prime to $\alpha$, at which $A$ has good reduction. If $\beta \cdot \lambda^{\alpha \times x}_\alpha(\text{Frob}_w) \neq 0$, then $\text{red}_w x \not\in aA(k_w)$.

**Proof.** We prove the contrapositive. Suppose that $\text{red}_w x \in aA(k_w)$. Then
\[
\beta \text{red}_w x \in \beta aA(k_w) \subseteq \alpha A(k_w),
\]
so that there is $y \in A(k_w)$ with $\beta \text{red}_w x = \alpha y$. On the other hand, fixing an $\alpha$th root $\tilde{x}_\alpha$ of $x$ in $A(\tilde{F})$ and a place $w'$ of $F_\alpha(\tilde{x}_\alpha)$ lying above $w$, we also have $\beta \text{red}_w x = \alpha \beta \text{red}_{w'} \tilde{x}_\alpha$. Therefore
\[
y - \beta \text{red}_{w'} \tilde{x}_\alpha \in A[\alpha].
\]
From here the argument proceeds as in the second half of the proof of Lemma 1.3 above to show that $\beta \cdot \lambda^{\alpha \times x}_\alpha(\text{Frob}_w) = 0$. ■

We remark that the converse of Lemma 1.4 holds in the case when $\alpha \mathcal{O} = aa'$ with $a, a'$ relatively prime and $\beta \in a' \cap (1 - a)$.

### 2. Modules over commutative, reduced, finite, flat $\mathbb{Z}$-algebras

#### 2.1. Projections

Let $\mathcal{O}$ be a commutative, reduced, finite, flat $\mathbb{Z}$-algebra. The normalization $\tilde{\mathcal{O}}$ of $\mathcal{O}$ decomposes as a product $\prod_{j=1}^h \tilde{\mathcal{O}}_j$ of Dedekind domains. (See [4, Section 11.2], for example, for a discussion of the normalization of a reduced ring.) We say that a $\mathbb{Z}$-linear map $t : \mathcal{O} \rightarrow \mathbb{Z}$ is full if it is non-trivial on $\mathcal{O} \cap \tilde{\mathcal{O}}_j$ for each $j$. Note that such a map always exists; indeed, this is clear for $\tilde{\mathcal{O}}$ (simply take the sum of the trace maps $\tilde{\mathcal{O}}_j \rightarrow \mathbb{Z}$), and multiplying a full map for $\tilde{\mathcal{O}}$ by $[\tilde{\mathcal{O}} : \mathcal{O}]$ yields a full map $\mathcal{O} \rightarrow \mathbb{Z}$.

**Lemma 2.1.** Fix a full map $t : \mathcal{O} \rightarrow \mathbb{Z}$. Then the map
\[
\text{Hom}_\mathcal{O}(N, \mathcal{O}) \rightarrow \text{Hom}_\mathbb{Z}(N, \mathbb{Z}), \quad f \mapsto t \circ f
\]
has finite cokernel for any finitely generated $\mathcal{O}$-module $N$.

**Proof.** Since $\mathcal{O}$ has finite index in $\tilde{\mathcal{O}}$, it suffices to prove the result after replacing $\mathcal{O}$ by $\tilde{\mathcal{O}}$ and $N$ by $N \otimes_\mathcal{O} \tilde{\mathcal{O}}$. We may therefore assume that $\mathcal{O}$ decomposes as a product $\prod \mathcal{O}_i$ of Dedekind domains. There is then a corresponding decomposition $N = \bigoplus N_i$, and by the definition of a full map it suffices to prove the lemma for each factor $N_i$; that is, we may assume that $\mathcal{O}$ is a Dedekind domain.

In this case every finitely generated $\mathcal{O}$-module has a free submodule of finite index; this allows one to reduce to the case when $N$ is free, and then
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We now fix a finitely generated \( \mathcal{O} \)-module \( N \) and a \( \mathbb{Z} \)-submodule \( M \) of \( N \) containing the \( \mathbb{Z} \)-torsion submodule \( N_{\text{tors}} \) of \( N \).

**Lemma 2.2.** Fix \( x \in N \) and suppose that \( p \) is a rational prime such that \( x \not\in M \otimes \mathbb{Z}_{(p)} \). Then there is an \( \mathcal{O} \)-linear map \( \psi : N \to \mathcal{O} \) such that \( \psi(x) \not\in \psi(M) + p^n\mathcal{O} \) for sufficiently large \( n \).

**Proof.** Choose a \( \mathbb{Z} \)-basis \( y_1, \ldots, y_r \in N \) of \( N/N_{\text{tors}} \) such that there are integers \( d_1, \ldots, d_r \) with

\[
M = \langle d_1 y_1, \ldots, d_r y_r \rangle + N_{\text{tors}}.
\]

(Of course, some of the \( d_i \) may be zero.) Writing \( x = a_1 y_1 + \ldots + a_r y_r + t \) with \( a_i \in \mathbb{Z} \) and \( t \in N_{\text{tors}} \), the fact that \( x \not\in M \otimes \mathbb{Z}_{(p)} \) implies that there is some index \( i \) such that

\[
\text{ord}_p a_i < \text{ord}_p d_i.
\]

Let \( \psi_0 : N \to \mathbb{Z} \) be \( \#N_{\text{tors}} \) times projection onto \( y_i \); this is a well defined map, and it follows from (2.3) that \( \psi_0(x) \not\in \psi_0(M) + p^n\mathbb{Z} \) for sufficiently large \( n \). (In fact, \( n > \text{ord}_p(a_i \cdot \#N_{\text{tors}}) \) suffices.)

Fix a full map \( t : \mathcal{O} \to \mathbb{Z} \). By Lemma 2.1, we can find a non-zero integer \( b \) such that \( bt \) is in the image of (2.1). Thus there is an \( \mathcal{O} \)-linear map \( \psi : N \to \mathcal{O} \) with \( bt = t \circ \psi \). Since \( t(p^n\mathcal{O}) \subseteq p^n\mathbb{Z} \), we conclude that \( \psi(x) \not\in \psi(M) + p^n\mathcal{O} \) for sufficiently large \( n \), as desired. ■

**2.2. Pre-bases.** We continue with \( M \subseteq N \) as before. Fix \( y \in N \) not in \( N_{\text{tors}} \) and let \( \varphi : \mathcal{O} \to \mathcal{O} \cdot y \) be the \( \mathcal{O} \)-linear surjection sending 1 to \( y \). We define \( \eta_0(y) \) to be the least positive integer \( m \) such that there exists an \( \mathcal{O} \)-linear map \( \psi : \mathcal{O} \cdot y \to \mathcal{O} \) with the composition

\[
\mathcal{O} \cdot y \xrightarrow{\psi} \mathcal{O} \xrightarrow{\varphi} \mathcal{O} \cdot y
\]

multiplication by \( m \). (Let \( K_j \) denote the fraction field of \( \tilde{\mathcal{O}}_j \); since \( \mathcal{O} \otimes \mathbb{Q} = \prod K_j \), to see that any maps \( \psi \) as above exist it suffices to prove the corresponding fact after replacing \( \mathcal{O} \) by \( \prod K_j \). In this context the map \( \varphi \) identifies with the quotient map

\[
\prod K_j 
\]
for some non-empty subset $J$ of $\{1, \ldots, h\}$, so that the existence of $\psi$ is obvious.)

We say that $y_1, \ldots, y_r \in N$ are an $O$-pre-basis of $N$ if:

- $y_i \notin N_{\text{tors}}$ for all $i$;
- $(O \cdot y_1) \oplus \ldots \oplus (O \cdot y_r)$ injects into $N$ with finite cokernel.

(Note that we do not require that the corresponding map $O' \to N$ is injective.) Let $\eta'(y_1, \ldots, y_r)$ be the order of this cokernel and define

$$\eta(y_1, \ldots, y_r) = \eta'(y_1, \ldots, y_r) \cdot \eta_0(y_1) \ldots \eta_0(y_r).$$

It then follows from the definition of $\eta_0(y_i)$ that there are $O$-linear maps

$$\psi_{i_1, \ldots, i_r}^y : N \to O$$

for $i = 1, \ldots, r$ such that

$$(2.4) \quad \eta(y_1, \ldots, y_r)y = \psi_{i_1, \ldots, i_r}^y(y_1 + \ldots + \psi_{i_1, \ldots, i_r}^y(y_1 \ldots y_r)$$

for all $y \in N$. We usually just write $\eta$ and $\psi_i$ if the pre-basis $y_1, \ldots, y_r$ is clear from context. A standard inductive procedure shows that pre-bases always exist.

**Proposition 2.3.** Fix $x \in N$ and suppose that $p$ is a rational prime such that $x \notin M \otimes \mathbb{Z}(p)$. Then there is an $O$-pre-basis $y_1, \ldots, y_r$ of $N$ such that $\psi_1(x) \notin \psi_1(M) + p^nO$ for sufficiently large $n$.

**Proof.** By Lemma 2.2, we may choose an $O$-linear map $\psi : N \to O$ such that $\psi(x) \notin \psi(M) + p^nO$ for sufficiently large $n$. Let $K'$ denote the image of $\psi \otimes \mathbb{Q}$; we have $K' = \prod_{j \in J} K_j$ for some non-empty subset $J$ of $\{1, \ldots, h\}$. In particular, $K'$ is a projective $\prod K_j$-module, so that there exists a map $\varphi_0 : K' \to N \otimes \mathbb{Q}$ such that $(\psi \otimes \mathbb{Q}) \circ \varphi_0$ is the identity on $K'$. Scaling $\varphi_0$ by an integer we obtain an $O$-linear map $\varphi : \tilde{O}' \to N$ such that $\psi \circ \varphi$ is multiplication by some non-zero integer; here $\tilde{O}' = \prod_{j \in J} \tilde{O}_j$.

Set $y_1 = \varphi(1)$ and choose an $O$-pre-basis $y_2, \ldots, y_r$ for $\ker \psi$. Then $y_1, \ldots, y_r$ is an $O$-pre-basis of $N$ and $\psi_1 = m\psi$ for some non-zero integer $m$. It thus follows from the definition of $\psi$ that $\psi_1(x) \notin \psi_1(M) + p^nO$ for sufficiently large $n$, as desired. $
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**2.3. Ideals.** We continue with $O$ as above. Fix a rational prime $p$ and write the $\mathbb{Z}$-exponent of $\tilde{O}/O$ as $cp^d$ with $d \geq 0$ and $c$ relatively prime to $p$. Let

$$p^{\tilde{O}} = \tilde{p}_1^{e_1} \ldots \tilde{p}_g^{e_g}$$

be the factorization of $p\tilde{O}$ into prime ideals of $\tilde{O}$; for each $i \in \{1, \ldots, g\}$ we let $\mu_p(i)$ denote the unique $j \in \{1, \ldots, h\}$ such that $\tilde{p}_i$ is the pullback of a prime ideal on $\tilde{O}_j$. For $y \in N$ we define $I_p(y) \subseteq \{1, \ldots, g\}$ to be the set of indices $i$ such that the image of $y$ in $N \otimes O \tilde{O}_{\tilde{p}_i}$ is non-torsion. In fact, since
every proper ideal of each $\tilde{\mathcal{O}}_j$ has finite index, we have
\begin{equation}
I_p(y) = \{ i; \text{rank}_\mathbb{Z}(\langle \mathcal{O} \cap \tilde{\mathcal{O}}_{\mu_p(i)} \rangle \cdot y) > 0 \}.
\end{equation}
For $i = 1, \ldots, g$ and any $n$, we define ideals of $\mathcal{O}$ by
$$p_{i,n} = \tilde{p}_i^{e_{i,n}} \cap \mathcal{O}.$$  

The reader is invited to focus on the case $d = 0$, when $p_{i,n} = p_{i,1}^n$ and the analysis below is quite a bit simpler. In the general case, we have $cp^d\tilde{p}_i^{e_{i,n}} \subseteq p_{i,n}$; since the $\tilde{p}_i$ are relatively prime, it follows that
\begin{equation}
c^{-1}p^{d(g-1)} \mathcal{O} \subseteq p_{i,n} + \prod_{j \neq i} p_{j,n}
\end{equation}
for all $n$. Furthermore, $p^n\tilde{\mathcal{O}} \cap \mathcal{O} \subseteq p^{n-d}\mathcal{O}$ for $n \geq d$, so that
\begin{alignat}{2}
p^n \mathcal{O} &\subseteq p_{1,n} \cap \cdots \cap p_{g,n} \subseteq p^{n-d}\mathcal{O}, \\
c^n p^{n+d} \mathcal{O} &\subseteq p_{1,n} \cdots p_{g,n} \subseteq p^{n-d}\mathcal{O}
\end{alignat}
for any $n \geq d$.

**Lemma 2.4.** Let $N$ be a finitely generated $\mathcal{O}$-module. Fix $\alpha \in \mathcal{O}$ and $x \in N$. Suppose that there is an index $i$ and non-negative integers $a, b$ such that:

1. $\alpha \not\in p_{i,a}$;
2. $x \not\in p_{i,b} N$;
3. $N[p^{a+b}] \subseteq p^b N$.

Then $\alpha x \not\in p^{a+b+d} N$.

**Proof.** We first replace $\mathcal{O}$ by $\lim O / \mathcal{O}/p_{i,n}$, $N$ by $\lim N / p_{i,n}$, and $\tilde{\mathcal{O}}$ by $\lim \tilde{\mathcal{O}} / \tilde{p}_i^n$. Let $\bar{\mathcal{O}}$ denote the maximal ideal of $\tilde{\mathcal{O}}$, so that $\tilde{p}_i^{e_i} = p_\mathcal{O}$, and set $\bar{p}_n = \tilde{p}_i^{e_{i,n}} \cap \mathcal{O}$. With this notation we have $\alpha \not\in p_a$ and $x \not\in p_b N$, and it suffices to prove that $\alpha x \not\in p^{a+b+d} N$. Note that $\alpha \not\in \bar{p}_i^{e_{i,a}}$, so that there is some $\beta \in \bar{\mathcal{O}}$ with $\alpha \beta = p^a$.

Set $C = \bar{\mathcal{O}} / \mathcal{O}$ and $\bar{N} = N \otimes_{\mathcal{O}} \bar{\mathcal{O}}$; $C$ is killed by $p^d$ and there is an exact sequence
\begin{equation}
\operatorname{Tor}^\mathcal{O}(N, C) \to N \xrightarrow{\iota} \bar{N} \xrightarrow{\iota} N \otimes_{\mathcal{O}} C \to 0.
\end{equation}
Suppose now that $\alpha x \in p^{a+b+d} N$. Applying $\iota$ and multiplying by $\beta$, we find that $p^a \iota(x) \in p^{a+b+d} \bar{N}$. By (2.9) we have $p^d \bar{N} \subseteq \iota(N)$, so that this implies that $p^a x - p^{a+b} n \in \ker \iota$ for some $n \in N$. Again by (2.9) this kernel is killed by $p^d$; we conclude that $p^{a+b} x \in p^{a+b+d} N$.

Thus
$$x \in p^b N + N[p^{a+b}] \subseteq p^b N \subseteq p_b N.$$  
Since $x \not\in p_b N$ by hypothesis, this yields the desired contradiction.
3. Reductions of Mordell–Weil groups

3.1. Galois elements. Let $A$ be an abelian variety over a number field $F$. By [8, Section 19, Corollary 2] the ring $\mathcal{O} := \text{End}_F$ is a reduced, finite, flat $\mathbb{Z}$-algebra. We further assume that it is commutative; we fix a rational prime $p$, and we continue with the notations of Section 2 for this ring $\mathcal{O}$ and prime $p$. By (2.6) we may fix $a_{i,n} \in \mathfrak{p}_{i,n}$ and $b_{i,n} \in \prod_{j \neq i} \mathfrak{p}_{j,n}$ such that $a_{i,n} + b_{i,n} = c^{g-1}p^d(g-1)$. By (2.8) the following map is well defined:

$$\varphi_n : A[p^{n-d}] \to A[\mathfrak{p}_{1,n}] \oplus \ldots \oplus A[\mathfrak{p}_{g,n}], \quad t \mapsto (b_{1,n}t, \ldots, b_{g,n}t).$$

Lemma 3.1. The cokernel of $\varphi_n$ is bounded independent of $n$.

Proof. Since $p^n \in \mathfrak{p}_{i,n}$ we can define a map

$$\psi_n : A[\mathfrak{p}_{1,n}] \oplus \ldots \oplus A[\mathfrak{p}_{g,n}] \to A[p^{n-d}], \quad (t_1, \ldots, t_g) \mapsto p^d(t_1 + \ldots + t_g).$$

As $c^{g-1}p^d(g-1) - b_{i,n} \in \mathfrak{p}_{i,n}$, the map $\varphi_n \circ \psi_n$ is just multiplication by $c^{g-1}p^d$. The lemma follows from this. 

For an $\mathcal{O}$-submodule $\Gamma$ of $A(F)$, we now write

$$\lambda^\Gamma_{\mathfrak{p}_{i,n+d}} : \mathfrak{g}_{p^n}(\Gamma) \to \text{Hom}_\mathcal{O}(\Gamma, A[\mathfrak{p}_{i,n+d}])$$

for the composition of $\lambda^\Gamma_{\mathfrak{p}_n}$ with $\varphi_{n+d}$ and projection to $A[\mathfrak{p}_{i,n+d}]$. In the next lemma we use the natural map $\mathfrak{g}_{p^n}(\Gamma) \to \mathfrak{g}_{p^m}(\Gamma)$ (corresponding to multiplication by $p^{n-m}$ from $\text{Hom}_\mathcal{O}(\Gamma, A[\mathfrak{p}_{i,n+d}])$) to $\text{Hom}_\mathcal{O}(\Gamma, A[\mathfrak{p}_{i,m+d}])$) to regard $\lambda^\Gamma_{\mathfrak{p}_{i,m+d}}$ as a map from $\mathfrak{g}_{p^n}(\Gamma)$ for $n \geq m$.

Lemma 3.2. Let $y_1, \ldots, y_r$ be an $\mathcal{O}$-pre-basis of $A(F)$. Then there is an integer $b$ such that for all sufficiently large $n$ there is a $\sigma_n \in \mathfrak{g}_{p^n}(A(F))$ with

$$\lambda^\mathcal{O}_{p^n} y_j(\sigma_n) = 0 \quad \text{for } j = 2, \ldots, r; \quad \lambda^\mathcal{O}_{p^n} y_1(\sigma_n) \neq 0 \quad \text{for all } i \in I_p(y_1).$$

Proof. The cokernel of the natural map

$$\pi : \text{Hom}_\mathcal{O}(A(F), A[p^n]) \to \bigoplus_{j=1}^r \text{Hom}_\mathcal{O}(\mathcal{O} \cdot y_j, A[p^n])$$

is bounded independent of $n$ by the definition of a pre-basis. Combined with Proposition 1.2, this implies that the cokernel of

$$\pi \circ \lambda^{A(F)}_{p^n} : \mathfrak{g}_{p^n}(A(F)) \to \bigoplus_{j=1}^r \text{Hom}_\mathcal{O}(\mathcal{O} \cdot y_j, A[p^n])$$

is bounded independent of $n$. Finally, by Lemma 3.1 we conclude that the cokernel of the map

$$\mathfrak{g}_{p^n}(A(F)) \to \left( \bigoplus_{i \in I_p(y_1)} \text{Hom}_\mathcal{O}(\mathcal{O} \cdot y_1, A[\mathfrak{p}_{i,n+d}]) \right) \oplus \left( \bigoplus_{j=2}^r \text{Hom}_\mathcal{O}(\mathcal{O} \cdot y_j, A[p^n]) \right)$$

is bounded independent of $n$. 

By the definition of the set $I_p(y_i)$, for each $i \in I_p(y_1)$ there is some $m > 0$ such that $p^{n+d-m} \text{Hom}_O(O \cdot y_1, A[p_{i,n+d}]) \neq 0$ for sufficiently large $n$. (That is, these groups grow with $n$.) Since the cokernel of (3.1) is bounded, it follows that there is an integer $b$ such that for sufficiently large $n$ there is

$$\sigma_n \in \mathfrak{g}_{p^n}(A(F))$$

with

$$\sigma_n | \text{Hom}_O(O \cdot y_j, A[p^n]) = 0 \quad \text{for} \quad j = 2, \ldots, r;$$

$$p^{n+d-b} \sigma_n | \text{Hom}_O(O \cdot y_1, A[p_{i,n+d}]) \neq 0 \quad \text{for all} \quad i \in I_p(y_1).$$

By the remarks preceding the lemma, this $\sigma_n$ is the required element of $\mathfrak{g}_{p^n}(A(F))$.

**Lemma 3.3.** Let $y_1, \ldots, y_r$ be an $O$-pre-basis of $A(F)$. Then there is an integer $b$ such that for all sufficiently large $n$ there are infinitely many places $w$ of $F[p^n]$ with

$$\text{red}_w y_j \in p^n A(k_w) \quad \text{for} \quad j = 2, \ldots, r; \quad \text{red}_w y_1 \not\in p_{i,b} A(k_w) \quad \text{for} \quad i \in I_p(y_1).$$

**Proof.** Let $n$ be sufficiently large and fix $\sigma_n$ as in Lemma 3.2. If $w$ is a place of $F[p^n]$ with $\text{Frob}_w = \sigma_n$ in $\mathfrak{g}_{p^n}(A(F))$, then $w$ satisfies the conditions of the lemma by Lemmas 1.3 and 1.4. Since the Chebotarev density theorem guarantees the existence of infinitely many such $w$, the lemma follows.

**3.2. Reduction of subgroups.** We are now in a position to prove our main result.

**Proposition 3.4.** Let $A$ be an abelian variety over a number field $F$; assume that $O = \text{End}_F A$ is commutative. Fix a rational prime $p$ and let $\Sigma$ be a subgroup of $A(F)$ containing $A(F)_{\text{tors}}$. Suppose that $x \in A(F)$ is such that

$$\text{red}_v x \in \text{red}_v \Sigma \quad \text{for almost all places} \quad v \quad \text{of} \quad F.$$ 

Then $x$ lies in $\Sigma \otimes \mathbb{Z}(p)$.

**Proof.** Suppose that $x \not\in \Sigma \otimes \mathbb{Z}(p)$. By Proposition 2.3 we can then choose an $O$-pre-basis $y_1, \ldots, y_r$ of $A(F)$ such that there is an integer $a$ with

$$\psi_1(x) \not\in \psi_1(\Sigma) + p^a O.$$ 

Let $b$ be the integer determined by $y_1, \ldots, y_r$ in Lemma 3.3 and fix $n > a + b + 2d$. Let $w$ be a place of $F[p^n]$ as in Lemma 3.3; by (3.2) we may further assume that there is a $y \in \Sigma$ with $\text{red}_w x = \text{red}_w y$. Multiplying by $\eta$, by (2.4) we have

$$\psi_1(x) \text{red}_w y_1 + \ldots + \psi_r(x) \text{red}_w y_r = \psi_1(y) \text{red}_w y_1 + \ldots + \psi_r(y) \text{red}_w y_r.$$ 

Thus

$$\psi_1(x) - \psi_1(y) \text{red}_w y_1 \in p^n A(k_w)$$

by the definition of $w$. 

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Set $\alpha = \psi_1(x) - \psi_1(y)$; by (3.3) and (2.7), $\alpha \notin p_{i,a+d}$ for some $i$. Fix such an $i$. Since $\alpha \in \text{im} \psi_1$, by (2.5) we have $i \in I_P(y_1)$; thus we also have $\text{red}_w y_1 \notin p_{i,b} A(k_w)$ by the definition of $w$. Since $A(k_w)[p^{a+2d}] \subseteq p^b A(k_w)$ (as $A[p^n] \subseteq A(k_w)$ and $a + b + 2d < n$), we may therefore apply Lemma 2.4 to conclude that $\alpha \text{red}_w y_1 \notin p^{a+b+2d} A(k_w)$. Since $a + b + 2d < n$, this contradicts (3.4), and thus proves the proposition.

**Corollary 3.5.** Let $A$ be an abelian variety over a number field $F$ and assume that $\text{End}_F A$ is commutative. Let $\Sigma$ be a subgroup of $A(F)$ containing $A(F)_{\text{tors}}$ and suppose that $x \in A(F)$ is such that $\text{red}_v x \in \text{red}_v \Sigma$ for almost all places $v$ of $\Sigma$. Then $x \in \Sigma$.

**Proof.** This is immediate from Proposition 3.4 applied for all primes $p$. ■

**References**


Department of Mathematics
University of California, Berkeley
Berkeley, CA, 94720-3840, U.S.A.
E-mail: weston@math.berkeley.edu

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