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Mahler's classification of numbers compared with Koksma's

by

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1. Introduction. Mahler [7], in 1932, and Koksma [6], in 1939, introduced two related measures of the degree of approximation of a complex transcendental number ξ by algebraic numbers. For any integer $n \geq 1$, we denote by $w_n(\xi)$ the supremum of the exponents w for which

$$0 < |P(\xi)| < \mathrm{H}(P)^{-w}$$

has infinitely many solutions in integer polynomials $P(X)$ of degree at most n. Here, $\mathrm{H}(P)$ stands for the naïve height of the polynomial $P(X)$, that is, the maximum of the absolute values of its coefficients. Further, we set $w(\xi) = \limsup_{n\to\infty}(w_n(\xi)/n)$ and, according to Mahler [7], we say that ξ is an

- *S-number* if $w(\xi) < \infty$;
- *T-number* if $w(\xi) = \infty$ and $w_n(\xi) < \infty$ for any integer $n \geq 1$;
- *U-number* if $w(\xi) = \infty$ and $w_n(\xi) = \infty$ for some integer $n \geq 1$.

In the sense of the Lebesgue measure, almost all numbers are S-numbers. Liouville numbers are examples of U-numbers, but the existence of T-numbers remained an open problem during nearly forty years, until it was confirmed by Schmidt [10, 11].

Following Koksma [6], for any integer $n \geq 1$, we denote by $w_n^*(\xi)$ the supremum of the exponents w for which

$$0 < |\xi - \alpha| < \mathrm{H}(\alpha)^{-w-1}$$

has infinitely many solutions in complex algebraic numbers α of degree at most n. Here, $\mathrm{H}(\alpha)$ stands for the naïve height of α, that is, the naïve height of its minimal defining polynomial. Koksma [6] defined S^*-, T^*- and U^*-numbers as above, using w_n^* in place of w_n. He proved that this classification of numbers is equivalent to Mahler's (see e.g. the book of Schneider [14]).

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For more information on the functions w_n and w_n^*, the reader is directed to Wirsing [16] and Schmidt [13].

For any integer $n \geq 2$ and any complex transcendental number ξ we have

$$w_n^*(\xi) \leq w_n(\xi) \leq w_n^*(\xi) + n - 1. \tag{1}$$

The first inequality in (1) is easy (see e.g. [12, p. 44]), and the second one is due to Wirsing [16]. Thus, it is a natural question to ask whether there are complex numbers ξ such that $w_n^*(\xi) < w_n(\xi)$ for some integer $n \geq 2$. In 1976, R. C. Baker [1] gave a positive answer to this problem by proving that for any integer $n \geq 2$ the function $w_n - w_n^*$ can take any value in the interval $[0, (n-1)/n]$. He even succeeded in constructing real numbers ξ with prescribed values for $w_n(\xi)$ and $w_n^*(\xi)$ for all positive integers n.

In the present work, we improve upon Baker's result: we show that for any integer $n \geq 3$ the set of values taken by the function $w_n - w_n^*$ contains the interval $[0, n/4]$. As in [1], our method of proof originates in two papers by Schmidt [10, 11], where the existence of T-numbers is established. The main novelty introduced here is the use of integer polynomials having two zeros very close to each other.

Our results are stated in Section 2 and proved in Sections 5, 6 and 7. Further related comments are made in Section 8. Section 4 is devoted to auxiliary lemmas, and an independent remark on Koksma's classification is the purpose of Section 3.

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2. The main result. Theorem 1 asserts the existence of real numbers with special properties.

THEOREM 1. *Let $n \geq 3$ be an integer and set $F(n) = 2n^3 + 2n^2 + 3n - 1$. Let w_n and w_n^* be real numbers such that*

$$w_n^* \leq w_n \leq w_n^* + n/4, \qquad w_n > F(n). \tag{2}$$

Then there exists a real number ξ such that

$$w_n^*(\xi) = w_n^* \quad \text{and} \quad w_n(\xi) = w_n.$$

As in [1], ξ is obtained as the limit of a suitable sequence $\xi_j = (c_j + \gamma_j)/g_j$ of algebraic numbers, where the c_j's and g_j's are positive integers and the γ_j's are real algebraic numbers of degree n. Thanks to a rather tedious and complicated construction, the differences $|\xi - \xi_j|$ are precisely controlled and $w_n^*(\xi)$ satisfies

$$w_n^*(\xi) = \lim_{j \to \infty} \frac{-\log|\xi - \xi_j|}{\log \mathrm{H}(\xi_j)} - 1.$$

Our new idea is to take for the ξ_j's algebraic numbers having a complex conjugate ξ_j^σ very close to them. It then follows that $|\xi - \xi_j^\sigma|$ is very small and that $|P_j(\xi)|$ is much smaller than $|\mathrm{H}(\xi_j)|^{-w_n^*(\xi)}$, where $P_j(X)$ denotes the minimal defining polynomial of ξ_j. Consequently, $w_n(\xi)$ is larger than $w_n^*(\xi)$. Since a few important changes are needed in the argument of [1], we give the full details of the proof of Theorem 1. We point out that our method is effective, which is not the case of that of [1]; see Section 8 for explanations.

The fact that the function $n \mapsto F(n)$ occurring in the statement of Theorem 1 is of order of magnitude n^3 is due to technical constraints. Presumably, the same result holds true also when F is much smaller. Notice that Baker [1] proved Theorem 1 with (2) replaced by

$$w_n^* \le w_n \le w_n^* + (n-1)/n, \qquad w_n > n^3 + 2n^2 + 5n + 1,$$

and that Theorem 1 also holds when (2) is replaced by

$$w_n^* + (n-1)/n \le w_n \le w_n^* + n/4, \qquad w_n > 2n^3 + n - 1,$$

as will be clear from the proof.

However, the upper bound in (1) can be lowered when $w_n(\xi)$ is close to n. Namely, Wirsing [16] proved that for any integer $n \ge 2$ and any real transcendental number ξ we have

$$w_n(\xi) \le w_n^*(\xi)(w_n(\xi) - n + 1),$$

which is sharper than (1) for

$$w_n(\xi) \le \frac{n + \sqrt{n^2 + 4n - 4}}{2}.$$

It turns out that our method allows us to construct real numbers ξ with prescribed values for $w_n(\xi)$ and $w_n^*(\xi)$, for finitely many integers n. Suitable modifications of the proof of Theorem 1 yield the following result.

THEOREM 2. *Let $3 \le n_1 < \ldots < n_k$ be positive integers. Let $w_1^* \le \ldots \le w_k^*$ and $w_1 \le \ldots \le w_k$ be real numbers satisfying*

$$w_j^* \le w_j \le w_j^* + n_j/4, \qquad w_j > 2n_j^3 + 2n_j^2 + 3n_j - 1 \qquad (1 \le j \le k).$$

Then the set of real S-numbers ξ such that

$$w_{n_j}^*(\xi) = w_j^* \quad \textit{and} \quad w_{n_j}(\xi) = w_j \quad \textit{for any } 1 \le j \le k,$$

has positive Hausdorff dimension.

Since the proof of Theorem 1 is already very technical, we do not give a complete proof of Theorem 2. We merely describe and explain which changes are to be done. This is the content of Section 7. For an introduction to the theory of Hausdorff dimension, the reader is directed e.g. to the book of Falconer [3].

Finally, we would like to propose an open problem. For an S-number ξ, we define its *type* $t(\xi)$ and its $*$-*type* $t^*(\xi)$ by $t(\xi) = \limsup_{n\to\infty} w_n(\xi)/n$ and $t^*(\xi) = \limsup_{n\to\infty} w_n^*(\xi)/n$, respectively. We infer from (1) that $t^*(\xi) \le t(\xi) \le t^*(\xi) + 1$.

PROBLEM. *Do there exist real numbers ξ with $t^*(\xi) < t(\xi)$, that is, with*

$$\limsup_{n\to\infty} \frac{w_n^*(\xi)}{n} < \limsup_{n\to\infty} \frac{w_n(\xi)}{n}\ ?$$

3. A remark on Koksma's classification. The number $w_n^*(\xi)$ is defined by taking into account all the algebraic numbers which are close to ξ. However, when ξ is a real transcendental number, it would be more natural to consider only the *real* algebraic numbers which are close to ξ. The aim of Lemma 1 below is to prove that this makes however no difference. For integers $n \ge 1$ and $H \ge 1$, set

$$\begin{aligned} w_n^{*r}(H,\xi) &:= \min\{|\xi-\alpha| : \alpha \text{ real algebraic}, \deg(\alpha) \le n, \mathrm{H}(\alpha) \le H, \alpha \ne \xi\},\\ w_n(H,\xi) &:= \min\{|\xi-\alpha| : \alpha \text{ algebraic}, \deg(\alpha) \le n, \mathrm{H}(\alpha) \le H, \alpha \ne \xi\},\\ w_n^{*r}(\xi) &:= \limsup_{H\to\infty} \frac{-\log(Hw_n^{*r}(H,\xi))}{\log H}. \end{aligned}$$

It is easy to check that

$$w_n^*(\xi) = \limsup_{H\to\infty} \frac{-\log(Hw_n^*(H,\xi))}{\log H}.$$

Further, we have the inequality

$$w_n^{*r}(\xi) \le w_n^*(\xi),$$

which turns out to be an equality, as stated in the next lemma.

LEMMA 1. *For any integer $n \ge 1$ and any real transcendental number ξ, we have $w_n^{*r}(\xi) = w_n^*(\xi)$. Consequently, in order to determine $w_n^*(\xi)$, it is enough to consider the real algebraic numbers which are close to ξ.*

Proof. The idea of the proof is due to Maurice Mignotte. Let $n \ge 1$ be an integer, $H > 1$ be a real number and ξ be a real transcendental number. Let α_1 be an algebraic number of height at most H and of degree $n_1 \le n$ such that $w_n^*(H,\xi) = |\xi - \alpha_1|$. We may assume that α_1 is non-real, otherwise the lemma is clearly true. Then the minimal defining polynomial of α_1, denoted by $P_1(X)$, has two distinct roots α_1 and $\overline{\alpha}_1$ very near to ξ. Grace's complex version of Rolle's theorem (see e.g. [2, p. 25]) asserts that its derivative $P_1'(X)$ has a root α_2 in the closed disk centered at $(\alpha_1+\overline{\alpha}_1)/2$ and of radius $|\alpha_1 - \overline{\alpha}_1|\cot(\pi/n_1)/2$. Observe that this closed disk reduces to the point $(\alpha_1 + \overline{\alpha}_1)/2$ if $n_1 = 2$. Consequently, we have

$$|\xi - \alpha_2| \le |\xi - \operatorname{Re}\alpha_1| + \frac{|\alpha_1 - \overline{\alpha}_1|}{2} \cdot \frac{n}{2} \le \left(\frac{n}{2} + 1\right)|\xi - \alpha_1|,$$
$$\deg(\alpha_2) \le n_1 - 1 \quad \text{and} \quad \mathrm{H}(\alpha_2) \le 2^n\, \mathrm{H}(P_1') \le 2^n nH.$$

Indeed, the minimal defining polynomial of α_2 is a divisor of $P_1'(X)$, hence its height is less than or equal to $2^{\deg(P_1')}\mathrm{H}(P_1')$, by using the "Gelfond inequality", as stated e.g. in [15, Remark 2, p. 81]. If α_2 is non-real, we proceed further in the same way in order to construct an algebraic approximant α_3 of ξ whose degree is strictly less than the degree of α_2. We iterate this process as soon as we end up with a real approximant. This always happens since the degrees of the algebraic numbers we construct are strictly decreasing. Consequently, there exists a real number α with

$$\mathrm{H}(\alpha) \le (2^n n)(2^{n-1}(n-1)) \ldots (2H) \le 2^{n^2} n^n H$$

and

$$|\xi - \alpha| \le \left(\frac{n}{2} + 1\right) \ldots \left(\frac{2}{2} + 1\right)|\xi - \alpha_1| \le n^n |\xi - \alpha_1|.$$

Thus, for any real number $H \ge 1$ we have

$$w_n^{*r}(2^{n^2} n^n H, \xi) \le n^n w_n^*(H, \xi)$$

and

$$w_n^*(\xi) \le \limsup_{H\to\infty} \frac{-\log(Hn^{-n} w_n^{*r}(2^{n^2} n^n H, \xi))}{\log H} \le w_n^{*r}(\xi),$$

as asserted. ∎

Remark. The idea of the proof of Lemma 1 can also be applied to approximation in the p-adic field $\mathbb{Q}_p$. Recall that this field is not algebraically closed, and denote by $\overline{\mathbb{Q}}_p$ an algebraic closure of it. We can show that for any integer $n \ge 1$ and any transcendental number ξ in $\mathbb{Q}_p$ the supremum of the exponents w for which

$$0 < |\xi - \alpha| < \mathrm{H}(\alpha)^{-w-1}$$

has infinitely many solutions in algebraic numbers α in $\overline{\mathbb{Q}}_p$ of degree at most n is equal to the supremum of the exponents w for which the same inequality has infinitely many solutions in algebraic numbers α in $\mathbb{Q}_p$. Indeed, let $H > 1$ be a real number and let α_1 be an algebraic number in $\overline{\mathbb{Q}}_p$ of height at most H and degree $n_1 \le n$, such that

$$|\xi - \alpha_1| = \min\{|\xi - \alpha| : \alpha \text{ algebraic in } \overline{\mathbb{Q}}_p,\ \deg(\alpha) \le n,\ \mathrm{H}(\alpha) \le H,\ \alpha \ne \xi\}.$$

We may assume that α_1 is not in $\mathbb{Q}_p$, otherwise there is nothing to do. Denote by $\alpha_1^{(1)} := \alpha_1, \alpha_1^{(2)}, \ldots, \alpha_1^{(n_1)}$ the conjugates of α_1, numbered in such a way that

$$|\xi - \alpha_1| \le |\xi - \alpha_1^{(2)}| \le \ldots \le |\xi - \alpha_1^{(n_1)}|.$$

If $|\xi-\alpha_1| < |\xi-\alpha_1^{(2)}|$, then Krasner's Lemma (see e.g. [9, p. 130]) implies that α_1 lies in $\mathbb{Q}_p$, which we have excluded. Consequently, the minimal defining polynomial $P_1(X)$ of α_1 has two roots α_1 and $\alpha_1^{(2)}$ with $|\xi - \alpha_1| = |\xi - \alpha_1^{(2)}|$, and we deduce from the p-adic version of Rolle's theorem (see e.g. [9, p. 316]) that $P_1'(X)$ has a root α_2 with $\mathrm{H}(\alpha_2) \le 2^n \mathrm{H}(P_1') \le 2^n nH$ and $|\xi - \alpha_2| < p^2|\xi - \alpha_1|$. If α_2 does not lie in $\mathbb{Q}_p$, we iterate this process, exactly as in the proof of Lemma 1.

4. Auxiliary results. Lemma 2 below gives a version of the *Liouville inequality.*

LEMMA 2. *Let α and β be distinct algebraic numbers of degree at most m and n, respectively. Then there exists a positive constant $c(m,n) < 1$, depending only on m and n, such that*

$$|\alpha - \beta| \ge c(m,n)\mathrm{H}(\alpha)^{-n}\mathrm{H}(\beta)^{-m}.$$

An admissible value for $c(m,n)$ is $(m+1)^{-n-1}(n+1)^{-m-1}$.

Proof. This is a direct consequence of Theorems 6 and 7 of Güting [4]. ∎

In the next lemma we define a two-parameter infinite family of integer polynomials (found by Mignotte [8]) having two zeros very close to each other.

LEMMA 3. *Let $n \ge 3$ and $a \ge 10$ be integers. The polynomial*

$$P_{n,a}(X) := X^n - 2(aX-1)^2$$

is irreducible and has two real roots very close to each other, namely

$$\delta^+(n,a) := a^{-1} + a^{-(n+2)/2}/\sqrt{2} + \varepsilon^+(n,a),$$
$$\delta^-(n,a) := a^{-1} - a^{-(n+2)/2}/\sqrt{2} + \varepsilon^-(n,a),$$

where $|\varepsilon^+(n,a)|, |\varepsilon^-(n,a)| \le C(a/2)^{-n-1}$ for some absolute constant C. Further, it follows from Rouché's theorem that $P_{n,a}(X)$ has no other roots in the disk centered at the origin and of radius $1/2$.

Proof. The irreducibility of $P_{n,a}(X)$ follows from the Eisenstein criterion, and Rouché's theorem shows that $P_{n,a}(X)$ has exactly two roots in the disk centered at the origin and of radius $1/2$. Studying the function $x \mapsto P_{n,a}(a^{-1}+x)$ in a neighbourhood of the origin, we see that these two roots can be expressed as stated above. ∎

Lemma 4. *Let α be an algebraic number of degree $n \geq 1$ and let a, b and c be integers with $c \neq 0$. Then*

$$\mathrm{H}\left(\frac{a\alpha+b}{c}\right) \leq 2^{n+1}\mathrm{H}(\alpha)\max\{|a|,|b|,|c|\}^n.$$

Proof. Denoting by $P(X)$ the minimal defining polynomial of α, we see that $Q(X) := a^n P(cX/a - b/a)$ is the one of $(a\alpha+b)/c$. Since the height of $Q(X)$ is bounded from above by $2^{n+1}\mathrm{H}(\alpha)\max\{|a|,|b|,|c|\}^n$, the lemma is proved. ∎

Lemma 5 ([1, Lemma 4]). *Let n be a positive integer and let g be a prime number with $g > n$. Let $P(X)$ be a monic polynomial of degree n with integer coefficients. Then there is no integer a such that g divides each of $P(a), P(a+1), \ldots, P(a+n)$.* ∎

Lemma 6. *Let $P(X) := a_n(X-\alpha_1)\ldots(X-\alpha_n)$ be a polynomial with complex coefficients of degree n and whose leading coefficient a_n is a positive number. Let $\psi > 0$ and ξ be real numbers. Then there exist effective positive constants $c_1(n,\xi,\psi)$ and $c_2(n,\xi,\psi)$, depending only on n, ψ and ξ, such that*

$$c_1(n,\xi,\psi)\,\frac{\mathrm{H}(P)}{a_n} \leq \prod_{|\xi-\alpha_j|\geq\psi} |\xi-\alpha_j| \leq c_2(n,\xi,\psi)\,\frac{\mathrm{H}(P)}{a_n}.$$

Proof. This follows from Hilfssatz 2 of Wirsing [16]. ∎

5. The inductive construction. Theorem 3 below gives an explicit inductive construction of sequences $(\xi_j)_{j\geq 1}$ of real algebraic numbers of degree n. It will be proved in Section 6 that such sequences converge to real numbers having the property stated in Theorem 1. We use in Theorem 3 the same notation as in Lemma 3, namely we denote by $\delta^+(n,a)$ the root of the polynomial $P_{n,a}(X)$ defined in that lemma.

Theorem 3. *Let $n \geq 3$ be an integer and let μ, ν be real numbers with $0 \leq \mu \leq (n-2)/2$ and $\nu > 1$. Set $G(n) = 2n^3+1$ and let $\chi > G(n)$ be a real number. Then there exist a positive number $\lambda < 1/2$, prime numbers $g_1 \geq 11, g_2, \ldots$ and integers $c_1, c_2, \ldots$ such that the following conditions are satisfied, where we have set $\gamma_j := \delta^+(n,[g_j^\mu])$ for any integer $j \geq 1$:*

(I_j) *g_j does not divide the norm of $c_j+\gamma_j$ $(j \geq 1)$.*

(II_1) *$\xi_1 = (c_1+\gamma_1)/g_1 \in\,]1,2[$.*

(II_j) *$\xi_j = (c_j+\gamma_j)/g_j$ belongs to the interval I_{j-1} defined by*

$$\xi_{j-1} + \frac{1}{2}\,g_{j-1}^{-\nu} < x < \xi_{j-1} + \frac{3}{4}\,g_{j-1}^{-\nu} \qquad (j \geq 2).$$

(III_1) *$|\xi_1-\alpha| \geq 2\lambda\mathrm{H}(\alpha)^{-\chi}$ for any algebraic number $\alpha \neq \xi_1$ of degree $\leq n$.*

(III$_j$) $|\xi_j - \alpha| \geq \lambda \mathrm{H}(\alpha)^{-\chi}$ *for any algebraic number* $\alpha \notin \{\xi_1, \ldots, \xi_j\}$ *of degree* $\leq n$ $(j \geq 2)$.

As will be seen in Section 6, Theorem 3 covers the range of values $[(n-1)/n,\ n/4]$ for the function $w_n - w_n^*$. For the interval $[0, (n-1)/n]$, we need Theorem 3′ below.

THEOREM 3′. *Theorem* 3 *holds with the function* $G(n)$ *replaced by* $H(n) = 2n^3 + 2n^2 + 2n + 1$ *and* γ_j *by* $2^{1/n}[g_j^{\mu'}]$, *where* μ' *is any number in* $[0, 1]$.

We observe that the sequence $(\xi_j)_{j\geq 1}$ obtained in Theorem 3 is strictly increasing and bounded, hence it converges to a limit ξ. For any $j \geq 1$, we have $c_j \leq 2g_j$, thus, by Lemma 4 and the definition of γ_j, the height of ξ_j satisfies

$$\mathrm{H}(\xi_j) \leq c(n) g_j^{2n-2},$$

for some constant $c(n)$ depending only on n. Condition (II$_{j+1}$) then shows that the order of approximation of ξ by the algebraic number ξ_j depends only on ν and n. Further, conditions (III$_j$) imply that the other algebraic numbers of degree at most n are not too close to ξ. Hence, the precise order of approximation of ξ by algebraic numbers of degree at most n is controlled in terms of n, ν and χ.

The rôle of the parameter μ is to measure the gap between $w_n(\xi)$ and $w_n^*(\xi)$, as will be shown in Section 5.

To simplify the notation, in what follows we denote by α a real algebraic number of degree less than or equal to n. Let ε be a positive number such that

$$\chi > 2n^3 + 1 + 2n^2\varepsilon. \tag{3}$$

In order to prove Theorem 3, we add three extra conditions (IV$_j$), (V$_j$) and (VI$_j$), which should be satisfied by the numbers ξ_j. We denote by Leb the Lebesgue measure on the real line.

Let J_j denote the subset of I_j consisting of the real numbers $x \in I_j$ satisfying

$$|x - \alpha| \geq 2\lambda \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number α of degree $\leq n$, distinct from $\xi_1, \ldots, \xi_j, x$ and of height $\mathrm{H}(\alpha)$ sufficiently large, that is, satisfying

$$\mathrm{H}(\alpha) \geq (\lambda g_j^{\nu})^{1/\chi}.$$

The supplementary conditions are the following:

(IV$_j$) $\xi_j \in J_{j-1}$ $(j \geq 2)$.

(V$_j$) If $\mathrm{H}(\alpha) \leq g_j^{1/(n+1+\varepsilon)}$, then $|\xi_j - \alpha| \geq 1/g_j$ $(j \geq 1)$.

(VI$_j$) The measure of J_j satisfies $\mathrm{Leb}(J_j) \geq \mathrm{Leb}(I_j)/2$ $(j \geq 1)$.

We construct the numbers $\xi_1, \xi_2, \dots$ by induction. At the jth stage, there are two distinct steps. Step (A_j) consists in building an algebraic number $\xi_j = (c_j + \gamma_j)/g_j$ of degree n satisfying conditions (I_j) to (V_j). In step (B_j), we show that the number ξ_j constructed in (A_j) satisfies (VI_j) as well, provided that g_j is chosen large enough in terms of

$$n, \mu, \nu, \chi, \varepsilon, \lambda, \xi_1, \dots, \xi_{j-1}. \tag{4}$$

The symbols o, $\gg$ and $\ll$ used throughout steps (A_j) and (B_j) mean that the numerical implicit constants depend (at most) on the quantities (4). Furthermore, the symbol o implies "as g_j tends to infinity".

Step (A_1) is rather easy. Let $P(X)$ denote the minimal defining polynomial of γ_1 and observe that (I_1) is satisfied if, and only if, g_1 does not divide $P(-c_1)$. Thus, by Lemma 5, if the prime number g_1 is larger than n, then there are $\gg g_1$ numbers $\xi_1 = (c_1 + \gamma_1)/g_1$ in the interval $]1, 2[$ satisfying condition (I_1). These $\gg g_1$ numbers have mutual distances at least g_1^{-1}, and since there are only $o(g_1)$ algebraic numbers α of degree at most n satisfying $\mathrm{H}(\alpha) \le g_1^{1/(n+1+\varepsilon)}$, one can choose ξ_1 such that (V_1) is satisfied. We point out that there are $\gg g_1$ choices for c_1, where the constant implied in $\gg$ depends only on n. Further, by Lemma 2, we have

$$|\xi_1 - \alpha| \ge 2\lambda \mathrm{H}(\alpha)^{-n},$$

with $\lambda = c(n, n)\mathrm{H}(\xi_1)^{-n}/2$, for any real algebraic numbers $\alpha \ne \xi_1$ of degree at most n. Thus (I_1), (II_1), (III_1) and (V_1) are satisfied.

Let $j \ge 2$ be an integer and assume that $\xi_1, \dots, \xi_{j-1}$ have been constructed. Step (A_j) is much harder to verify, since we have no control on the set J_{j-1}. Thus, it seems difficult to check that condition (IV_j) holds. To overcome this problem, we follow Schmidt's argument [11], also used by Baker [1]. We set $\xi_j = (c_j + \gamma_j)/g_j$ for some positive integers c_j and $g_j > 8g_{j-1}$ and we introduce the set J'_{j-1} formed by the real numbers $x \in I_{j-1}$ satisfying

$$|x - \alpha| \ge 2\lambda \mathrm{H}(\alpha)^{-\chi}$$

for any algebraic number α of degree $\le n$, distinct from $\xi_1, \dots, \xi_j, x$, and whose height $\mathrm{H}(\alpha)$ satisfies the inequalities

$$(\lambda g_{j-1}^{\nu})^{1/\chi} \le \mathrm{H}(\alpha) \le (c_2(n)^{-1} g_j^{2n(n-1)})^{1/(\chi-n)}. \tag{5}$$

Since, by (3), we have

$$\chi - n > 2n(n-1)(n+1),$$

the exponent of g_j in the right member of (5) is strictly less than $1/(n+1)$. Thus, there are $o(g_j)$ algebraic numbers α satisfying (5), and we observe that, unlike J_{j-1}, the set J'_{j-1} is a finite union of intervals, and more precisely, a union of $o(g_j)$ intervals.

We will prove that for g_j large enough we have $\gg g_j$ choices for c_j in order that conditions (I_j) to (V_j) are satisfied.

Let α be an algebraic number of degree $\leq n$. Since $\mathrm{H}(\gamma_j) \leq 2g_j^{n-2}$ and $c_j \leq 2g_j$, we infer from Lemmas 2 and 4 that there exist positive constants $c_1(n)$ and $c_2(n)$ such that

$$|\xi_j - \alpha| \geq c_1(n)\mathrm{H}(\xi_j)^{-n}\mathrm{H}(\alpha)^{-n} \geq c_2(n)g_j^{-2n(n-1)}\mathrm{H}(\alpha)^{-n}. \tag{6}$$

In particular, using $2\lambda < 1$, we have

$$|\xi_j - \alpha| \geq 2\lambda\mathrm{H}(\alpha)^{-\chi} \tag{7}$$

as soon as

$$\mathrm{H}(\alpha)^{\chi-n} \geq c_2(n)^{-1}g_j^{2n(n-1)}. \tag{8}$$

By (VI_{j-1}) and since $J'_{j-1} \supset J_{j-1}$, we have $\mathrm{Leb}(J'_{j-1}) \gg 1$. Since the set J'_{j-1} is the union of $o(g_j)$ intervals, if g_j is a sufficiently large prime number, then, using Lemma 5 as in step (A_1), we find that there exist $\gg g_j$ numbers $\xi_j = (c_j + \gamma_j)/g_j$ in J'_{j-1} such that (I_j) is satisfied. Such ξ_j's also belong to J_{j-1}, since (8) implies (7), and condition (IV_j) is satisfied.

Thus, we are left with $\gg g_j$ suitable algebraic numbers ξ_j, mutually distant by at least g_j^{-1}. Only $o(g_j)$ algebraic numbers α of degree at most n satisfy

$$\mathrm{H}(\alpha) \leq g_j^{1/(n+1+\varepsilon)}, \tag{9}$$

thus one can choose ξ_j in such a way that $|\xi_j - \alpha| \geq 1/g_j$ for the numbers α satisfying (9). Consequently, there are $\gg g_j$ algebraic numbers ξ_j satisfying (I_j), (II_j), (IV_j) and (V_j).

It remains to show that such a ξ_j also satisfies (III_j). To this end, it suffices to prove that

$$|\xi_j - \alpha| \geq \lambda\mathrm{H}(\alpha)^{-\chi}$$

for the algebraic numbers α of degree $\leq n$ which are different from $\xi_1, \ldots, \xi_j$ and whose height $\mathrm{H}(\alpha)$ satisfies

$$\mathrm{H}(\alpha) < (\lambda g_{j-1}^{\nu})^{1/\chi}.$$

Since the sequence $(g_t)_{t\geq 1}$ is increasing, either we have

$$g_1^{-\nu} < \lambda\mathrm{H}(\alpha)^{-\chi}, \tag{10}$$

or there exists an integer t with $2 \leq t < j$ such that

$$g_t^{-\nu} < \lambda\mathrm{H}(\alpha)^{-\chi} \leq g_{t-1}^{-\nu}. \tag{11}$$

In the former case, we infer from (III_1) and (10) that

$$|\xi_j - \alpha| \geq |\xi_1 - \alpha| - |\xi_j - \xi_1| \geq 2\lambda\mathrm{H}(\alpha)^{-\chi} - g_1^{-\nu} > \lambda\mathrm{H}(\alpha)^{-\chi}.$$

In the latter case, (IV_t) and (11) yield

$$|\xi_j - \alpha| \geq |\xi_t - \alpha| - |\xi_j - \xi_t| \geq 2\lambda \mathrm{H}(\alpha)^{-\chi} - g_t^{-\nu} > \lambda \mathrm{H}(\alpha)^{-\chi}.$$

The upper estimates $|\xi_j - \xi_1| \leq g_1^{-\nu}$ and $|\xi_j - \xi_t| \leq g_t^{-\nu}$ used above follow from (II_j) and the assumption $g_l > 8g_{l-1}$, valid for any integer l with $2 \leq l \leq j$. Consequently, condition (III_j) holds and the proof of step (A_j) is complete.

Let $j \geq 1$ be an integer. For the proof of step (B_j), we first establish that if g_j is large enough and if x lies in I_j, then

$$|x - \alpha| \geq 2\lambda \mathrm{H}(\alpha)^{-\chi} \tag{12}$$

for any algebraic number $\alpha \neq \xi_j$ such that

$$(\lambda g_j^{\nu})^{1/\chi} \leq \mathrm{H}(\alpha) \leq g_j^{\nu/(\chi-n-1-\varepsilon)}. \tag{13}$$

Let then $\alpha \neq \xi_j$ be an algebraic number of degree $\leq n$ satisfying (13) and let x be in I_j, that is,

$$\frac{1}{2}\, g_j^{-\nu} < x - \xi_j < \frac{3}{4}\, g_j^{-\nu}. \tag{14}$$

If $g_j^{\nu/(\chi-n-1-\varepsilon)} \leq g_j^{1/(n+1+\varepsilon)}$, then $\mathrm{H}(\alpha) \leq g_j^{1/(n+1+\varepsilon)}$ and it follows from (V_j), (13) and (14) that

$$|x - \alpha| \geq |\xi_j - \alpha| - |\xi_j - x| \geq g_j^{-1} - g_j^{-\nu} \geq 2g_j^{-\nu} \geq 2\lambda \mathrm{H}(\alpha)^{-\chi}. \tag{15}$$

Otherwise, we have

$$g_j^{\nu/(\chi-n-1-\varepsilon)} > g_j^{1/(n+1+\varepsilon)}, \tag{16}$$

and, by (6), we get

$$\begin{aligned} |x - \alpha| &\geq |\xi_j - \alpha| - |\xi_j - x| \geq c_2(n) g_j^{-2n(n-1)} \mathrm{H}(\alpha)^{-n} - g_j^{-\nu} \\ &\geq c_2(n) g_j^{-2n(n-1)} \mathrm{H}(\alpha)^{-n}/2. \end{aligned} \tag{17}$$

To check the last inequality, we have to verify that

$$2g_j^{-\nu} \leq c_2(n) g_j^{-2n(n-1)} \mathrm{H}(\alpha)^{-n}. \tag{18}$$

In view of (13), (18) is true as soon as

$$2g_j^{n\nu/(\chi-n-1-\varepsilon)} \leq c_2(n) g_j^{\nu} g_j^{-2n(n-1)},$$

which, by (14), holds for g_j large enough when

$$\frac{n}{\chi - n - 1 - \varepsilon} < 1 - 2n(n-1)\, \frac{n+1+\varepsilon}{\chi - n - 1 - \varepsilon},$$

in particular when χ satisfies (3).

Moreover, we have

$$c_2(n) g_j^{-2n(n-1)} \mathrm{H}(\alpha)^{-n} \geq 4\lambda \mathrm{H}(\alpha)^{-\chi}. \tag{19}$$

Indeed, by (13), $\lambda < 1$ and (16), we get

$$\begin{aligned}\mathrm{H}(\alpha)^{\chi-n} &\geq (\lambda g_j^{\nu})^{(\chi-n)/\chi} \geq \lambda g_j^{(\chi-n)(\chi-n-1-\varepsilon)/(\chi n+\chi+\chi\varepsilon)} \\ &\geq 4\lambda c_2(n)^{-1} g_j^{2n(n-1)},\end{aligned}$$

since we infer from (3) that

$$(\chi - n)(\chi - n - 1 - \varepsilon) > 2\chi n(n + 1 + \varepsilon)(n - 1).$$

Combining (17) and (19), we have checked that

$$|x - \alpha| \geq 2\lambda \mathrm{H}(\alpha)^{-\chi}$$

when (16) holds; hence, by (15), (12) is true if $\alpha \neq \xi_j$ satisfies (13). Consequently, if g_j is large enough, then the complement J_j^{c} of J_j in I_j is contained in the union of the intervals

$$E(\alpha) :=]\alpha - 2\lambda\mathrm{H}(\alpha)^{-\chi}, \alpha + 2\lambda\mathrm{H}(\alpha)^{-\chi}[,$$

where α runs over the set of algebraic numbers of degree $\leq n$ and with height greater than $g_j^{\nu/(\chi-n-1-\varepsilon)}$. The Lebesgue measure of J_j^{c} is then

$$\ll \sum_{H > g_j^{\nu/(\chi-n-1-\varepsilon)}} H^{n-\chi} = o(g_j^{-\nu}) = o(\mathrm{Leb}(I_j)).$$

Thus, we conclude that we can find g_j large enough such that $\mathrm{Leb}(J_j) \geq \mathrm{Leb}(I_j)/2$. This completes step (B_j) as well as the proof of Theorem 3. ■

The proof of Theorem 3′ follows the same lines as that of Theorem 3, the only difference being that the estimate $\mathrm{H}(\gamma_j) \leq 2g_j^n$ should replace $\mathrm{H}(\gamma_j) \leq 2g_j^{n-2}$. Thus, the assumption (3) should be modified and we have to replace $G(n)$ by $H(n)$. ■

6. Completion of the proof of Theorem 1. We first deal with the range of values $[(n-1)/n, n/4]$. Let Δ be in $[(n-1)/n, n/4]$ and set

$$\mu = \frac{2(n\Delta - n + 1)}{n - 2}.$$

We observe that μ is in $[0, (n-2)/2]$. Let $w_n > 2n^3 + n - 1$ and set $w_n^* = w_n - \Delta$. The sequence $(\xi_j)_{j\geq 1}$ obtained in Theorem 3 is strictly increasing and bounded, thus it converges towards a real number denoted by ξ. Set $\nu = n(w_n^* + 1)$ and set $\chi = w_n - n + 2$ so that $\chi > 2n^3 + 1$. Let $\xi_1, \xi_2, \ldots$ be as in Theorem 3 and denote by ξ the limit of the strictly increasing sequence $(\xi_j)_{j\geq 1}$.

We write $A \ll B$ if there exists a constant $c(n)$, depending only on n, such that $|A| < c(n)B$, and we write $A \asymp B$ if both $A \ll B$ and $B \ll A$.

Our choice of γ_j implies that the minimal defining polynomial of ξ_j is

$$Q_j(X) := (g_j X - c_j)^n - 2([g_j^{\mu}](g_j X - c_j) - 1)^2.$$

This polynomial is indeed irreducible and primitive by (I_j) and the first statement of Lemma 3. Since $\mu \le (n-2)/2$, we have $\mathrm{H}(\xi_j) \asymp g_j^n$.

Moreover, for any $j \ge 1$,

$$\xi_j + g_j^{-\nu}/2 < \xi < \xi_j + g_j^{-\nu},$$

and we deduce that

$$|\xi - \xi_j| \asymp \mathrm{H}(\xi_j)^{-\nu/n} \asymp \mathrm{H}(\xi_j)^{-w_n^*-1}. \tag{20}$$

Further, if α is of degree $\le n$ and is not one of the ξ_j's, then $|\xi - \alpha| \ge \lambda \mathrm{H}(\alpha)^{-\chi}$, whence

$$|\xi - \alpha| \ge \mathrm{H}(\alpha)^{-w_n^*-1}, \tag{21}$$

since $\chi \le w_n^* + 1$. It follows from (20), (21) and Lemma 1 that $w_n^*(\xi) = w_n^*$.

It now remains to prove that $w_n(\xi) = w_n$. Denote by $\xi_j = \beta_{j1}, \ldots, \beta_{jn}$ the roots of $Q_j(X)$, numbered in such a way that $|\xi_j - \beta_{j2}| \asymp g_j^{-\mu(n+2)/2-1}$. Denote by $\delta_3, \ldots, \delta_k$ the roots of $P_{n,[g_j^\mu]}(X)$ other than $\delta^+(n, [g_j^\mu])$ and $\delta^-(n, [g_j^\mu])$. For $k \ge 3$, we have

$$|\xi_j - \beta_{jk}| \asymp \frac{|1/[g_j^\mu] - \delta_k|}{g_j}.$$

Consequently, we get

$$\begin{aligned} |Q_j(\xi)| &= g_j^n |\xi - \xi_j|\, |\xi - \beta_{j2}| \prod_{3 \le k \le n} |\xi - \beta_{jk}| \\ &\asymp g_j^2 \mathrm{H}(\xi_j)^{-w_n^*-1} g_j^{-\mu(n+2)/2-1} \prod_{3 \le k \le n} |1/a - \delta_k| \\ &\asymp \mathrm{H}(\xi_j)^{-w_n^*-1} g_j^{-\mu(n+2)/2+1} [g_j^\mu]^2, \end{aligned}$$

by Lemma 6 and the last statement of Lemma 3. Thus,

$$|Q_j(\xi)| \asymp \mathrm{H}(Q_j)^{-w_n^*-1-\mu(n-2)/(2n)+1/n},$$

and we see that

$$w_n(\xi) \ge w_n^* + 1 + \mu(n-2)/(2n) - 1/n; \tag{22}$$

hence, by definition of μ, we obtain

$$w_n(\xi) \ge w_n^* + \Delta. \tag{23}$$

In order to show that the inequalities in (22) and (23) are indeed equalities, we argue exactly as in Baker [1]. Let $P(X)$ be an integer polynomial of degree $\le n$ which is not a multiple of some $Q_j(X)$. Write

$$P(X) = aR_1(X) \ldots R_p(X),$$

where a is an integer and the polynomials $R_i(X)$ are primitive and irreducible. Since $R_i(\xi) \ne 0$, if k denotes the degree of the polynomial $R_i(X)$, then, by [4, Theorem 4], this polynomial has a root θ satisfying

$$|R_i(\xi)| \gg \mathrm{H}(R_i)^{2-k}|\xi-\theta| \gg \lambda\mathrm{H}(R_i)^{-\chi-k+2} \gg \lambda\mathrm{H}(R_i)^{-w_n}. \tag{24}$$

Consequently, it follows from (24) and the "Gelfond inequality" (see e.g. [15, Remark 2, p. 81]) that

$$|P(\xi)| \gg (\mathrm{H}(R_1)\dots\mathrm{H}(R_p))^{-w_n} \gg \mathrm{H}(P)^{-w_n},$$

and we get $w_n(\xi)=w_n$, as claimed.

We now consider Δ' in the range $[0,(n-1)/n]$ and set

$$\mu' = \frac{n-1-n\Delta'}{n-1}.$$

We argue as above except that we use Theorem 3′ instead of Theorem 3. The polynomials $Q_j(X)$ are replaced by

$$R_j(X) := (g_jX-c_j)^n - 2[g_j^{\mu'}]^n,$$

which have been used by Baker [1], and, proceeding as above, we show that

$$|R_j(\xi)| \asymp g_j^n|\xi-\xi_j|g_j^{(\mu-1)(n-1)} \asymp \mathrm{H}(R_j)^{-w_n^*-(1-\mu)(n-1)/n}$$

and

$$w_n(\xi) = w_n^*(\xi) + (1-\mu)\,\frac{n-1}{n} = w_n^*(\xi)+\Delta'.$$

The proof of Theorem 1 is now finished. ∎

7. Outline of the proof of Theorem 2. As pointed out at the end of steps (A_1) and (A_j), the integers c_j occurring in the inductive construction of Theorem 3 are far from being uniquely determined. Indeed, as stated by Baker [1, p. 29], it turns out that, if g_j is sufficiently large, we have at each step (A_j) with $j\ge 2$ at least

$$\frac{g_jg_{j-1}^{-\nu}}{32n} \tag{25}$$

suitable choices for ξ_j. Observe that $g_{j-1}^{-\nu}/4$ is the length of the interval I_{j-1} and that the n occurring in the denominator of (25) is a consequence of Lemma 5. No particular importance has to be attached to the constant 32. This shows that we obtain an uncountable set of real numbers ξ with the property stated in Theorem 1. Moreover, using the method described in Section 5 of [1], it can be shown that they form a set with positive h-measure for some function $h: t\mapsto t^\delta$ with $\delta>0$ (in [1], the ν_j's are unbounded, thus h has to grow faster than any function $t\mapsto t^\delta$ in a neighborhood of the origin), thus with positive Hausdorff dimension. Since the sets of T- and U-numbers have Hausdorff dimension zero (see e.g. [5]), it follows that, for any fixed integer $n\ge 2$, there exist S-numbers ξ with the property stated in Theorem 1.

In order to control simultaneously finitely many $w_n(\xi)$ and $w_n^*(\xi)$, only slight modifications of Theorem 3 are needed. Essentially, it is sufficient to use at the jth inductive step the algebraic number $\gamma_j := \delta^+(n_{l+1}, [g_j^\mu])$, where l is the remainder in the Euclidean division of j by k.

8. Further remarks

• *Comparison with the work of Baker.* Let $n \geq 1$ be an integer and let w_n^* be a given real number. Clearly, it is an easy matter to construct a real number ξ with $w_n^*(\xi) \geq w_n^*$, e.g. by using a nested interval construction. However, it is much more difficult to get an upper bound for $w_n^*(\xi)$, and in particular to ensure that $w_n^*(\xi) = w_n^*$.

Theorem 1 of Baker [1] depends on the following deep result of Schmidt [11, 12].

THEOREM S. *Let β be a real algebraic number and let $\eta > 0$ be real. Let $n \geq 1$ be an integer. Then there exists an ineffective positive constant $C_1(\beta, n, \eta)$, depending only on β, n and η, such that*

$$|\beta - \alpha| \geq C_1(\beta, n, \eta)\mathrm{H}(\alpha)^{-n-1-\eta} \tag{26}$$

for all real algebraic numbers $\alpha \neq \beta$ of degree at most n.

Wirsing [17] obtained a slightly weaker result (with $-2n - \eta$ in the exponent of $\mathrm{H}(\alpha)$), which turns out to be sufficient to confirm the existence of T-numbers (see [10, 11]). Indeed, the crucial point is that the exponent of $\mathrm{H}(\alpha)$ in (26) does not depend on β.

If we use Theorem S with $\eta = 1$ instead of Lemma 2 in (6), we deduce that there exists a constant $c(\gamma_j)$ such that

$$|\xi_j - \alpha| = \frac{1}{g_j}\,|\gamma_j - (g_j\alpha - c_j)| \geq c(\gamma_j) g_j^{-n^2-1}\mathrm{H}(\alpha)^{-n-2}.$$

However, γ_j depends on g_j, and we do not have any estimate for $c(\gamma_j)$. Thus we cannot argue as in (7), (8), etc. Consequently, the method used in the present paper does not allow us to construct real numbers ξ satisfying the conclusion of Theorem 1 for *every* integer $n \geq 3$.

In his paper, Baker used for γ_j the numbers $2^{1/n}[g_j^\mu]$, where μ runs over $[0, 1]$ and allows him to control the difference between w_n and w_n^*. In this case, Theorem S can be applied since

$$\begin{aligned} |\xi_j - \alpha| &= \frac{1}{g_j}\,|2^{1/n}[g_j^\mu] - (g_j\alpha - c_j)| = \frac{[g_j^\mu]}{g_j}\left|2^{1/n} - \frac{g_j\alpha - c_j}{[g_j^\mu]}\right| \\ &\geq c(2^{1/n}) g_j^{-n^2-2n-1}\mathrm{H}(\alpha)^{-n-2}. \end{aligned} \tag{27}$$

Here, $c(2^{1/n})$ does not depend on j, thus one can argue as in (7), (8), etc. The exponent of g_j in (27) is slightly larger than in (6), thus we get a better lower bound for χ, whose order of magnitude is n^3, however.

• *Approximation in the complex field.* It is possible to adapt the proof of Theorem 1 to construct complex non-real numbers ξ with $w_n^*(\xi) < w_n(\xi)$, as Baker [1] did in his Theorem 2. Our method allows us to show that for any integer $n \geq 3$ the set of values taken by the function $w_{2n} - w_{2n}^*$ evaluated at complex non-real numbers contains the interval $[0, n/8]$. Presumably, it should be possible to show that non-real numbers ξ with $w_n^*(\xi) < w_n(\xi)$ for some *odd* integer $n \geq 5$ exist; the problem is to find suitable polynomials to replace $P_{n,a}(X)$.

• *Approximation in p-adic fields.* Presumably, the method can be carried over to the p-adic field $\mathbb{Q}_p$ without too much difficulty in order to prove that there exist p-adic numbers ξ with $w_n(\xi) \neq w_n^*(\xi)$. The polynomials $X^n - 2(X - p^a)^2$ would then play the rôle of the polynomials $P_{n,a}(X)$ defined in Lemma 3.

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