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Some identities for multiple Dedekind sums attached to Dirichlet characters

by

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1. Introduction. For any rational number x, we denote by $[x]$ the greatest integer not exceeding x and put

$$((x)) = \begin{cases} x - [x] - 1/2 & \text{if } x \text{ is not an integer,} \\ 0 & \text{otherwise.} \end{cases}$$

For integers h and k with $k > 0$, the classical Dedekind sum $s(h, k)$ is defined by

$$s(h,k) = \sum_{\mu \bmod k} \left(\left(\frac{\mu}{k}\right)\right)\left(\left(\frac{h\mu}{k}\right)\right).$$

In [5] Dedekind obtained the identity

$$s(ph,k) + \sum_{b=0}^{p-1} s(h+bk, pk) = (p+1)s(h,k)$$

for any prime number p ((28) of [5], (2.8) of [13]). In [7], Knopp extended this as

$$\sum_{\substack{ad=n \\ d>0}} \sum_{b=0}^{d-1} s(ah+bk, dk) = \sigma(n)s(h,k) \tag{1}$$

for any positive integer n, where $\sigma(n) = \sum_{d|n} d$. This identity was also extended to higher-order Dedekind sums in [12] and to Dedekind type sums in [1], [11] and [8].

It is known that (1) is equivalent to the following identity due to Subrahmanyam ([16]):

$$\sum_{b=0}^{n-1} s(h+bk, nk) = \sum_{d|n} \mu(d)\sigma\left(\frac{n}{d}\right) s(dh,k) \tag{2}$$

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for any positive integer n, where $\mu(n)$ is the Möbius function ([6], [8], [11], [14]). In [11], Nagasaka extended identities (1) and (2) and their equivalence to Dedekind type sums attached to Dirichlet characters. In [8], the author obtained somewhat more generalized results by elementary methods. In addition, making use of the generalized Euler numbers, he also deduced explicit extensions of (1) and (2) and their equivalence to higher-order Dedekind sums attached to Dirichlet characters.

In this paper, we generalize the above results to multiple Dedekind sums attached to Dirichlet characters. For that purpose, following mainly the methods in Section 3 of [8], we deduce an expression of multiple Dedekind sums by Euler numbers and obtain identities of certain rational functions, which will be transformed into our main results.

Throughout the paper, we denote by $\mathbb{Q}$, $\mathbb{Z}$ and $\mathbb{N}$, the rational number field, the ring of integers of $\mathbb{Q}$ and the set of positive integers respectively, as usual. We put $\bar{\mathbb{N}} = \mathbb{N} \cup \{0\}$. For any $m, n \in \mathbb{Z}$, we denote by $\delta(m,n)$ the greatest common divisor of m and n.

2. Definition of multiple Dedekind sums. Let B_m and $B_m(X)$ be the mth Bernoulli number and polynomial, respectively, defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \quad \text{and} \quad \frac{te^{tX}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(X) \frac{t^m}{m!}.$$

For $x \in \mathbb{Q}$, we put $\{x\} = x - [x]$ and define $\tilde{B}_m(x) = B_m(\{x\})$.

For any primitive Dirichlet character ψ, we denote by f_ψ the conductor of ψ. For any $x \in \mathbb{Q}$ with denominator relatively prime to f_ψ, we can define the value $\psi(x)$ by multiplicativity. We define the twisted Bernoulli function $\tilde{B}_{m,\psi}(x)$ attached to ψ by

$$\sum_{j=0}^{f_\psi - 1} \frac{\psi(\{x\} + j) t e^{(\{x\}+j)t}}{e^{f_\psi t} - 1} = \sum_{m=0}^{\infty} \tilde{B}_{m,\psi}(x) \frac{t^m}{m!},$$

or equivalently

$$\tilde{B}_{m,\psi}(x) = f_\psi^{m-1} \sum_{j \bmod f_\psi} \psi(x+j) \tilde{B}_m\left(\frac{x+j}{f_\psi}\right) \tag{3}$$

(cf. p. 301 of [15]).

Let $P = (p_1, \ldots, p_n, p) \in \mathbb{N}^{n+1}$, $H = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ and $k \in \mathbb{N}$. In addition, let $\Psi = (\psi_1, \ldots, \psi_n, \psi)$ be an $(n+1)$-tuple of primitive Dirichlet characters, put $f_\Psi = (\prod_{i=1}^n f_{\psi_i}) f_\psi$ and assume that $\delta(k, f_\Psi) = 1$. We define the multiple Dedekind sums attached to Ψ by

$$S(P,H,k,\Psi)=\sum_{a_1,\dots,a_n \bmod k}\left(\prod_{i=1}^{n}\tilde{B}_{p_i,\psi_i}\left(\frac{a_i}{k}\right)\right)\tilde{B}_{p,\psi}\left(\frac{a_1h_1+\cdots+a_nh_n}{k}\right).$$

Note that if $P=(1,1)$, $H=(h)$ and $\Psi=(1,1)$, then $S(P,H,k,\Psi)=s(h,k)+1/4$. If $P=(m+r-1,r)$ with $m\ge r\ge 1$, $H=(h)$ and $\Psi=(\chi,\psi)$, then $S(P,H,k,\Psi)$ equals the higher-order Dedekind sum $\tilde{S}^{(r)}_{m+1}(\chi,\psi,h,k)$ of [8]. We also note that in [3], Carlitz considered the case where $P=(1,\dots,1)\in\mathbb{N}^{n+1}$, $H=(h_1,\dots,h_n)$ and $\Psi=(1,\dots,1)$ to extend the well-known reciprocity formula for Dedekind sums, and that its further generalizations are studied in [4] and [9].

3. Expressions by Euler numbers. For a parameter u, we put

$$R(T,u)=\frac{u}{1+T-u}.$$

As in [10], we define the modified Euler numbers $E_m(u)$ by

$$R(e^t-1,u)=\frac{u}{e^t-u}=\frac{E_{-1}(u)}{t}+\sum_{m=0}^{\infty}E_m(u)\frac{t^m}{m!}.$$

Note that $E_{-1}(u)\neq 0$ only if $u=1$, and that we have $E_{-1}(1)=1$ and $mE_{m-1}(1)=B_m$ for $m\in\mathbb{N}$. It is known that

$$k^m\tilde{B}_m\left(\frac{a}{k}\right)=m\sum_{\zeta^k=1}E_{m-1}(\zeta)\zeta^a \tag{4}$$

for $a\in\mathbb{Z}$ and $k,m\in\mathbb{N}$ ((6.4) of [2]). In [8, (3.8)], we obtained a generalization of (4) for $\tilde{B}_{m,\psi}(x)$. Let us recall some basic formulas around it. For any primitive Dirichlet character ψ, we define the numbers $E_{m,\psi}(u)$ (modifications of the generalized Euler numbers of [17]) by

$$\sum_{a=0}^{f_\psi-1}\frac{\psi(a)u^{f_\psi-a}e^{at}}{e^{f_\psi t}-u^{f_\psi}}=\frac{E_{-1,\psi}(u)}{t}+\sum_{m=0}^{\infty}E_{m,\psi}(u)\frac{t^m}{m!}. \tag{5}$$

Note that $E_{-1,\psi}(u)\neq 0$ only if u is a primitive f_ψth root of unity. Let ζ_ψ be an arbitrarily chosen primitive f_ψth root of unity and put

$$\tau(\psi,\zeta_\psi)=\sum_{j \bmod f_\psi}\psi(j)\zeta_\psi^j,$$

the Gauss sum attached to ψ and ζ_ψ.

Lemma 3.1. *Let $k \in \mathbb{N}$ with $\delta(k, f_\psi) = 1$ and $m \in \bar{\mathbb{N}}$. Then*

$$\sum_{\rho^k=u} R(T,\rho) = kR((1+T)^k - 1, u), \tag{6}$$

$$E_{m,\psi}(u) = \frac{\tau(\psi,\zeta_\psi)}{f_\psi} \sum_{j \bmod f_\psi} \psi^{-1}(j) E_m(\zeta_\psi^j u), \tag{7}$$

$$\sum_{\rho^k=u} E_{m,\psi}(\rho) = k^{m+1}\psi(k) E_{m,\psi}(u), \tag{8}$$

$$\psi(k)k^m \tilde{B}_{m,\psi}\left(\frac{a}{k}\right) = m \sum_{\zeta^k=1} E_{m-1,\psi}(\zeta)\zeta^a \quad \textit{for } m \geq 1 \textit{ and } a \in \mathbb{Z}. \tag{9}$$

Proof. In general, for any polynomial $f(X)$ with degree less than k, we have

$$\frac{f(X)}{X^k - u} = \frac{1}{ku} \sum_{\rho^k=u} \frac{\rho f(\rho)}{X-\rho}. \tag{10}$$

Formula (6) follows from (10) by taking $X = 1 + T$ and $f(X) = ku$.

In order to prove (7), we first note that replacing k and u by f_ψ and u^{f_ψ} respectively in (10) and taking $f(X) = \sum_{a=0}^{f_\psi-1} \psi(a) u^{f_\psi - a} X^a$, we obtain

$$\begin{aligned}
\sum_{a=0}^{f_\psi-1} \frac{\psi(a)u^{f_\psi-a}X^a}{X^{f_\psi} - u^{f_\psi}} &= \frac{1}{f_\psi u^{f_\psi}} \sum_{j \bmod f_\psi} \sum_{a=0}^{f_\psi-1} \frac{\zeta_\psi^j u \psi(a) u^{f_\psi-a}(\zeta_\psi^j u)^a}{X - \zeta_\psi^j u} \\
&= \frac{1}{f_\psi} \sum_{j \bmod f_\psi} \sum_{a=0}^{f_\psi-1} \psi(a)\zeta_\psi^{ja} \frac{\zeta_\psi^j u}{X - \zeta_\psi^j u} \\
&= \frac{\tau(\psi,\zeta_\psi)}{f_\psi} \sum_{j \bmod f_\psi} \frac{\psi^{-1}(j)\zeta_\psi^j u}{X - \zeta_\psi^j u}.
\end{aligned}$$

By taking $X = e^t$, formula (7) follows from the definitions of $E_m(u)$ and $E_{m,\psi}(u)$.

Next taking $T = e^t - 1$ in (6), we see that

$$\sum_{\rho^k=u} \frac{\rho}{e^t - \rho} = \frac{ku}{e^{kt} - 1},$$

which means

$$\sum_{\rho^k=u} E_m(\rho) = k^{m+1} E_m(u).$$

Then by (7), we deduce

$$\begin{aligned}\sum_{\rho^k=u} E_{m,\psi}(\rho) &= \frac{\tau(\psi,\zeta_\psi)}{f_\psi}\sum_{j \bmod f_\psi}\psi^{-1}(j)\sum_{\rho^k=u}E_m(\zeta_\psi^j\rho)\\ &= \frac{\tau(\psi,\zeta_\psi)}{f_\psi}\sum_{j \bmod f_\psi}\psi^{-1}(j)k^{m+1}E_m(\zeta_\psi^{kj}u)\\ &= \frac{\tau(\psi,\zeta_\psi)}{f_\psi}k^{m+1}\psi(k)\sum_{j \bmod f_\psi}\psi^{-1}(j)E_m(\zeta_\psi^j u)\\ &= k^{m+1}\psi(k)E_{m,\psi}(u).\end{aligned}$$

Thus we obtain (8).

Finally let us prove (9). By (3), (4) and (7), we see that

$$\begin{aligned}\psi(k)k^m\tilde{B}_{m,\psi}\left(\frac{a}{k}\right) &= \psi(k)k^m f_\psi^{m-1}\sum_{j \bmod f_\psi}\psi\left(\frac{a}{k}+j\right)\tilde{B}_m\left(\frac{a+kj}{kf_\psi}\right)\\ &= \frac{m}{f_\psi}\sum_{j \bmod f_\psi}\psi(a+kj)\sum_{\xi^{f_\psi k}=1}E_{m-1}(\xi)\xi^{a+kj}\\ &= \frac{m}{f_\psi}\sum_{j \bmod f_\psi}\psi(a+kj)\sum_{i=0}^{f_\psi-1}\sum_{\zeta^k=1}E_{m-1}(\zeta_\psi^i\zeta)\zeta_\psi^{i(a+kj)}\zeta^a.\end{aligned}$$

When j runs through all the residue classes modulo f_ψ, so does $a+kj$. Hence the above equals

$$\frac{\tau(\psi,\zeta_\psi)}{f_\psi}m\sum_{i=0}^{f_\psi-1}\psi^{-1}(i)\sum_{\zeta^k=1}E_{m-1}(\zeta_\psi^i\zeta)\zeta^a,$$

which also equals the right hand side of (9) on account of (7). This completes the proof.

As seen in the proof of (7), we have the following equation equivalent to (7):

$$\begin{aligned}(11)\qquad &\frac{\tau(\psi,\zeta_\psi)}{f_\psi}\sum_{j \bmod f_\psi}\psi^{-1}(j)R(\zeta_\psi^{-j}e^t-1,u)\\ &= \frac{\tau(\psi,\zeta_\psi)}{f_\psi}\sum_{j \bmod f_\psi}\frac{\psi^{-1}(j)\zeta_\psi^j u}{e^t-\zeta_\psi^j u} = \frac{E_{-1,\psi}(u)}{t}+\sum_{m=0}^{\infty}E_{m,\psi}(u)\frac{t^m}{m!}.\end{aligned}$$

Making use of (9), we can easily generalize formula (3.9) of [8] to multiple Dedekind sums as follows:

Proposition 3.2. *Let* $P=(p_1,\ldots,p_n,p)\in\mathbb{N}^{n+1}$, $H=(h_1,\ldots,h_n)\in\mathbb{Z}^n$ *and* $k\in\mathbb{N}$. *Let* $\Psi=(\psi_1,\ldots,\psi_n,\psi)$ *be an* $(n+1)$*-tuple of primitive Dirichlet characters and assume that* $\delta(k,f_\Psi)=1$. *Then*

$$k^{p_1+\cdots+p_n+p-n}(\psi_1\cdots\psi_n\psi)(k)S(P,H,k,\Psi) = p_1\cdots p_n p\sum_{\zeta^k=1}\Big(\prod_{i=1}^{n}E_{p_i-1,\psi_i}(\zeta^{h_i})\Big)E_{p-1,\psi}(\zeta^{-1}).$$

For any $m\in\mathbb{N}$, we put $mH=(mh_1,\ldots,mh_n)$. By Proposition 3.2 and formula (8) and by direct calculation, we also obtain

Corollary 3.3. *Let* P,H,k,Ψ *be as above. For any* $m\in\mathbb{N}$ *with* $\delta(m,f_\Psi)=1$, *we have*

$$m^{p_1+\cdots+p_n-n}(\psi_1\cdots\psi_n)(m)S(P,mH,mk,\Psi)=S(P,H,k,\Psi).$$

Note that Corollary 3.3 contains formula (2.3) of [5] (or equivalently (2.7) of [13]) and (4.1) of [2] as special cases (cf. [8, (2.3)] for the case of Dedekind type sums).

4. Main theorem. For any $m,N\in\mathbb{N}$ and any $(n+1)$-tuple of primitive Dirichlet characters $\Psi=(\psi_1,\ldots,\psi_n,\psi)$, we set

$$\sigma_{m,\Psi}(N)=\sum_{d\mid N}d^m(\psi_1\cdots\psi_n\psi)(d).$$

For any $d\in\mathbb{N}$, we put $I_d=\{(b_1,\ldots,b_n)\in\mathbb{Z}^n \mid 0\le b_1,\ldots,b_n\le d-1\}$. Now we state our main

Theorem 4.1. *Let* $P=(p_1,\ldots,p_n,p)\in\mathbb{N}^{n+1}$, $H\in\mathbb{Z}^n$ *and* $k,N\in\mathbb{N}$. *Let* $\Psi=(\psi_1,\ldots,\psi_n,\psi)$ *be an* $(n+1)$*-tuple of primitive Dirichlet characters and assume that* $\delta(kN,f_\Psi)=1$. *Put* $s(P)=p_1+\cdots+p_n+p-n$. *Then we have*

$$N^{s(P)-p}(\psi_1\cdots\psi_n)(N)\sum_{\substack{ad=N\\ d>0}}\sum_{B\in I_d}d^{p-n}\psi(d)S(P,aH+kB,dk,\Psi) = \sigma_{s(P),\Psi}(N)S(P,H,k,\Psi), \tag{12}$$

where we put $aH+kB=(ah_1+kb_1,\ldots,ah_n+kb_n)$ *for* $H=(h_1,\ldots,h_n)$ *and* $B=(b_1,\ldots,b_n)$. *We also have*

$$N^{s(P)-n}(\psi_1\cdots\psi_n\psi)(N)\sum_{B\in I_d}S(P,H+kB,Nk,\Psi) = \sum_{d\mid N}\mu(d)(\psi_1\cdots\psi_n)(d)d^{s(P)-p}\sigma_{s(P),\Psi}\Big(\frac{N}{d}\Big)S(P,dH,k,\Psi). \tag{13}$$

Furthermore (12) *and* (13) *can be deduced from each other.*

In order to prove the theorem, we show the following

PROPOSITION 4.2. *Let $H = (h_1, \ldots, h_n) \in \mathbb{Z}^n$ and $k, N \in \mathbb{N}$. We have the following equalities of rational functions in indeterminates $T_1, \ldots, T_n, T$:*

$$\sum_{\substack{ad=N\\ d>0}} \sum_{\zeta^{dk}=1} \sum_{(b_1,\ldots,b_n)\in I_d} \Big(\prod_{i=1}^{n} \frac{1}{d} R((1+T_i)^a - 1, \zeta^{ah_i+kb_i})\Big) R(T, \zeta^{-1}) = \sum_{d|N} \sum_{\zeta^{dk}=1} \Big(\prod_{i=1}^{n} R((1+T_i)^d - 1, \zeta^{dh_i})\Big) R(T, \zeta^{-1}) \tag{14}$$

and

$$\sum_{\zeta^{Nk}=1} \sum_{(b_1,\ldots,b_n)\in I_N} \Big(\prod_{i=1}^{n} \frac{1}{N} R(T_i, \zeta^{h_i+kb_i})\Big) R(T, \zeta^{-1}) = \sum_{d|N} \mu(d) \sum_{c|\frac{N}{d}} \sum_{\zeta^k=1} \Big(\prod_{i=1}^{n} R((1+T_i)^{cd} - 1, \zeta^{dh_i})\Big) cR((1+T)^c - 1, \zeta^{-1}). \tag{15}$$

Furthermore (14) *and* (15) *can be deduced from each other.*

Proof. For each $m \in \mathbb{N}$, we choose a primitive mth root of unity in such a manner that $\zeta_{mm'}^{m'} = \zeta_m$ for any $m, m' \in \mathbb{N}$. Note that for any $m, k \in \mathbb{N}$, we have

$$\{(d,\zeta) \mid d \,|\, m, \zeta^{dk} = 1\} = \Big\{\Big(\frac{m}{a}, \zeta_{mk}^{j}\Big) \,\Big|\, 0 \le j \le mk-1,\ a \,|\, \delta(m,j)\Big\}. \tag{16}$$

By applying (16) for $m = N$, the left-hand side of (14) equals

$$\sum_{j=0}^{Nk-1} \sum_{a|\delta(N,j)} \sum_{(b_1,\ldots,b_n)\in I_{N/a}} \Big(\prod_{i=1}^{n} \frac{a}{N} R((1+T_i)^a - 1, \zeta_{Nk}^{j(ah_i+b_ik)})\Big) R(T, \zeta_{Nk}^{-j}) = \sum_{j=0}^{Nk-1} \sum_{a|\delta(N,j)} \Big(\frac{a}{N}\Big)^n \Big(\prod_{i=1}^{n} \sum_{b=0}^{N/a-1} R((1+T_i)^a - 1, \zeta_{Nk}^{jah_i} \zeta_N^{jb})\Big) R(T, \zeta_{Nk}^{-j}).$$

Note that ζ_N^{jb} is an $N/\delta(N,j)$th root of unity and that when b runs through the integers with $0 \le b \le N/a - 1$, ζ_N^{jb} runs through all the $N/\delta(N,j)$th roots of unity $\frac{N/a}{N/\delta(N,j)}$ times. It follows that the left-hand side of (14) equals

$$\sum_{j=0}^{Nk-1} \sum_{a|\delta(N,j)} \Big(\frac{a}{N}\Big)^n \Big(\prod_{i=1}^{n} \frac{\delta(N,j)}{a} \sum_{\zeta^{N/\delta(N,j)}=1} R((1+T_i)^a - 1, \zeta_{Nk}^{jah_i} \zeta)\Big) R(T, \zeta_{Nk}^{-j}).$$

Taking $u=(\zeta_{Nk}^{jah_i}\zeta)^{\frac{N}{\delta(N,j)}}=\zeta_{Nk}^{jah_i\frac{N}{\delta(N,j)}}$ in (6), we see that this equals

$$\begin{aligned}(17)\qquad &\sum_{j=0}^{Nk-1}\sum_{a\mid\delta(N,j)}\Big(\frac{a}{N}\Big)^n\Big(\prod_{i=1}^n\frac{N}{a}R\Big((1+T_i)^{\frac{Na}{\delta(N,j)}}-1,\zeta_{Nk}^{jah_i\frac{N}{\delta(N,j)}}\Big)\Big)R(T,\zeta_{Nk}^{-j})\\ &=\sum_{j=0}^{Nk-1}\sum_{a\mid\delta(N,j)}\Big(\prod_{i=1}^nR\Big((1+T_i)^{\frac{N}{\delta(N,j)/a}}-1,\zeta_{Nk}^{jh_i\frac{N}{\delta(N,j)/a}}\Big)\Big)R(T,\zeta_{Nk}^{-j}).\end{aligned}$$

When a runs through the integers with $a\mid\delta(N,j)$, so does $\delta(N,j)/a$. Hence, by applying (16) again, (17) equals the right-hand side of (14), that is, (14) holds.

Next, similarly to the way we have deduced (17), we see that the left-hand side of (15) equals

$$\begin{aligned}(18)\qquad &\sum_{j=0}^{Nk-1}\frac{1}{N^n}\Big(\prod_{i=1}^n\sum_{b=0}^{N-1}R(T_i,\zeta_{Nk}^{j(h_i+bk)})\Big)R(T,\zeta_{Nk}^{-j})\\ &=\sum_{j=0}^{Nk-1}\frac{1}{N^n}\Big(\prod_{i=1}^nNR((1+T_i)^{N/\delta(N,j)}-1,\zeta_{Nk}^{jh_iN/\delta(N,j)})\Big)R(T,\zeta_{Nk}^{-j})\\ &=\sum_{j=0}^{Nk-1}\Big(\prod_{i=1}^nR((1+T_i)^{N/\delta(N,j)}-1,\zeta_{k}^{jh_i/\delta(N,j)})\Big)R(T,\zeta_{Nk/\delta(N,j)}^{-j/\delta(N,j)}).\end{aligned}$$

We also see that the right-hand side of (15) equals

$$\begin{aligned}&\sum_{N_1\mid N}\sum_{\substack{cd=N_1\\ d>0}}\mu(d)\sum_{\zeta^k=1}\Big(\prod_{i=1}^nR((1+T_i)^{N_1}-1,\zeta^{dh_i})\Big)\sum_{\xi^c=\zeta}R(T,\xi^{-1})\\ &=\sum_{N_1\mid N}\sum_{c\mid N_1}\mu\Big(\frac{N_1}{c}\Big)\sum_{\xi^{ck}=1}\Big(\prod_{i=1}^nR((1+T_i)^{N_1}-1,\xi^{N_1h_i})\Big)R(T,\xi^{-1}).\end{aligned}$$

By applying (16) for $m=N_1$ and taking $\xi=\zeta_{N_1k}^j$, this equals

$$\begin{aligned}(19)\qquad &\sum_{N_1\mid N}\sum_{j=0}^{N_1k-1}\sum_{a\mid\delta(N_1,j)}\mu(a)\Big(\prod_{i=1}^nR((1+T_i)^{N_1}-1,\zeta_k^{jh_i})\Big)R(T,\zeta_{N_1k}^{-j})\\ &=\sum_{N_1\mid N}\sum_{\substack{0\le j\le N_1k-1\\ \delta(N_1,j)=1}}\Big(\prod_{i=1}^nR((1+T_i)^{N_1}-1,\zeta_k^{jh_i})\Big)R(T,\zeta_{N_1k}^{-j}).\end{aligned}$$

Note that

$$\begin{aligned}&\{(N/\delta(N,j),j/\delta(N,j))\mid 0\le j\le Nk-1\}\\ &\qquad=\{(N_1,j)\mid N_1\mid N,\ 0\le j\le N_1k-1,\ \delta(N_1,j)=1\}.\end{aligned}$$

Hence, (18) equals (19), that is, (15) holds.

Now, let us prove the equivalence between (14) and (15). We first note that substituting Nk for k transforms (14) into

$$\sum_{\substack{ad=N\\d>0}}\sum_{\zeta^{dNk}=1}\sum_{(b_1,\dots,b_n)\in I_d}\Big(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^a-1,\zeta^{a(h_i+b_idk)})\Big)R(T,\zeta^{-1}) \tag{20}$$
$$=\sum_{d|N}\sum_{\zeta^{dNk}=1}\Big(\prod_{i=1}^{n}R((1+T_i)^d-1,\zeta^{dh_i})\Big)R(T,\zeta^{-1}).$$

Conversely, by substituting $N\cdot H=(Nh_1,\dots,Nh_n)$ for $H=(h_1,\dots,h_n)$ in (20) and noting that

$$\sum_{(\zeta^{-1})^N=\rho^{-1}}R(T,\zeta^{-1})=NR((1+T)^N-1,\rho^{-1}),$$

(20) gets transformed into

$$\sum_{\substack{ad=N\\d>0}}\sum_{\rho^{dk}=1}\sum_{(b_1,\dots,b_n)\in I_d}\Big(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^a-1,\rho^{ah_i+b_ik})\Big)NR((1+T)^N-1,\rho^{-1})$$
$$=\sum_{d|N}\sum_{\rho^{dk}=1}\Big(\prod_{i=1}^{n}R((1+T_i)^d-1,\rho^{dh_i})\Big)NR((1+T)^N-1,\rho^{-1}).$$

Replacing T by $(1+T)^{1/N}-1$, we see that this formula is equivalent to (14). Consequently, (14) is equivalent to (20). Similarly, (15) is equivalent to

$$\sum_{\zeta^{N^2k}=1}\sum_{(b_1,\dots,b_n)\in I_N}\Big(\prod_{i=1}^{n}\frac{1}{N}R(T_i,\zeta^{h_i+b_iNk})\Big)R(T,\zeta^{-1}) \tag{21}$$
$$=\sum_{d|N}\mu(d)\sum_{c|\frac{N}{d}}\sum_{\zeta^{Nk}=1}\Big(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\zeta^{dh_i})\Big)cR((1+T)^c-1,\zeta^{-1}).$$

Hence, it is sufficient to prove the equivalence between (20) and (21).

The left-hand side of (20) equals

$$\sum_{\substack{ad=N\\d>0}}\sum_{\xi^{d^2k}=1}\sum_{(b_1,\dots,b_n)\in I_d}\Big(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^a-1,\xi^{h_i+b_idk})\Big)\sum_{\zeta^a=\xi}R(T,\zeta^{-1})$$
$$=\sum_{d|N}\sum_{\xi^{d^2k}=1}\sum_{(b_1,\dots,b_n)\in I_d}\Big(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^{N/d}-1,\xi^{h_i+b_idk})\Big)$$
$$\times\frac{N}{d}R((1+T)^{N/d}-1,\xi^{-1}).$$

Replacing T_i with $1\le i\le n$ by $(1+T_i)^{1/N}-1$ and T by $(1+T)^{1/N}-1$, we see that (20) is equivalent to

$$\sum_{d|N}\sum_{\xi^{d^2k}=1}\sum_{(b_1,\dots,b_n)\in I_d}\left(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^{1/d}-1,\xi^{h_i+b_idk})\right)\times\frac{1}{d}R((1+T)^{1/d}-1,\xi^{-1})$$
$$=\frac{1}{N}\sum_{d|N}\sum_{\zeta^{dNk}=1}\left(\prod_{i=1}^{n}\frac{1}{d}R((1+T_i)^{d/N}-1,\zeta^{dh_i})\right)R((1+T)^{1/N}-1,\zeta^{-1}).$$

By the Möbius inversion formula, this is also equivalent to

$$\sum_{\zeta^{N^2k}=1}\sum_{(b_1,\dots,b_n)\in I_N}\left(\prod_{i=1}^{n}\frac{1}{N}R((1+T_i)^{1/N}-1,\zeta^{h_i+b_iNk})\right)\times\frac{1}{N}R((1+T)^{1/N}-1,\zeta^{-1})$$
$$=\sum_{d|N}\mu(d)\frac{d}{N}\sum_{c|\frac{N}{d}}\sum_{\zeta^{ckN/d}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd/N}-1,\zeta^{ch_i})\right)\times R((1+T)^{d/N}-1,\zeta^{-1}),$$

that is, the left-hand side of (21) equals

$$\sum_{d|N}\mu(d)d\sum_{c|\frac{N}{d}}\sum_{\zeta^{ckN/d}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\zeta^{ch_i})\right)R((1+T)^{d}-1,\zeta^{-1}).$$

By (6) this also equals

$$\sum_{d|N}\mu(d)d\sum_{c|\frac{N}{d}}\sum_{\xi^{kN/d}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\xi^{h_i})\right)\sum_{\zeta^{c}=\xi}R((1+T)^{d}-1,\zeta^{-1})$$
$$=\sum_{d|N}\mu(d)d\sum_{c|\frac{N}{d}}\sum_{\xi^{kN/d}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\xi^{h_i})\right)cR((1+T)^{cd}-1,\xi^{-1})$$
$$=\sum_{d|N}\mu(d)d\sum_{c|\frac{N}{d}}\sum_{\xi^{kN/d}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\xi^{h_i})\right)\frac{c}{d}\sum_{\rho^{d}=\xi}R((1+T)^{c}-1,\rho^{-1})$$
$$=\sum_{d|N}\mu(d)\sum_{c|\frac{N}{d}}\sum_{\rho^{kN}=1}\left(\prod_{i=1}^{n}R((1+T_i)^{cd}-1,\rho^{dh_i})\right)cR((1+T)^{c}-1,\rho^{-1})$$
$$=\text{the right-hand side of (21).}$$

Thus, we see the equivalence between (20) and (21). This completes the proof.

Proof of Theorem 4.1. Let $F(T_1,\dots,T_n,T)$ be the rational function expressed by (14). Put

$$I_\Psi=\{(j_1,\dots,j_n,j)\in\mathbb{Z}^{n+1}\mid 0\le j_i\le f_{\psi_i}-1 \text{ for } 1\le i\le n,\ 0\le j\le f_\psi-1\}.$$

By taking $T_1=\zeta_{\psi_1}^{-j_1}e^{t_1}-1,\dots,T_n=\zeta_{\psi_n}^{-j_n}e^{t_n}-1$ and $T=\zeta_\psi^{-j}e^t-1$ for each $(j_1,\dots,j_n,j)\in I_\Psi$, let us consider the function

$$\text{(22)}\qquad \Big(\prod_{i=1}^n\frac{\tau(\psi_i,\zeta_{\psi_i})}{f_{\psi_i}}\Big)\frac{\tau(\psi,\zeta_\psi)}{f_\psi}$$
$$\times\sum_{(j_1,\dots,j_n,j)\in I_\psi}\Big(\prod_{i=1}^n\psi_i^{-1}(j_i)\Big)\psi^{-1}(j)F(\zeta_{\psi_1}^{-j_1}e^{t_1}-1,\dots,\zeta_{\psi_n}^{-j_n}e^{t_n}-1,\zeta_\psi^{-j}e^t-1).$$

Since $(1+T_i)^a-1=\zeta_{\psi_i}^{-aj_i}e^{at_i}-1$ for $1\le i\le n$, writing $\psi_i^{-1}(aj)\psi_i(a)=\psi_i^{-1}(j)$ and replacing $t_i^{p_i-1}$ by $a^{p_i-1}t_i^{p_i-1}$, we see from (11) that the coefficient of $t_1^{p_1-1}\cdots t_n^{p_n-1}t^{p-1}$ in the expansion of the function (22) equals

$$\sum_{\substack{ad=N\\ d>0}}\sum_{\zeta^{dk}=1}\sum_{(b_1,\dots,b_n)\in I_d}\frac{1}{d^n}\Big(\prod_{i=1}^n\psi_i(a)E_{p_i-1,\psi_i}(\zeta^{ah_i+b_ik})\Big)$$
$$\times E_{p-1,\psi}(\zeta^{-1})a^{p_1+\dots+p_n-n}$$
$$=\sum_{d\mid N}\sum_{\zeta^{dk}=1}\Big(\prod_{i=1}^n\psi_i(d)E_{p_i-1,\psi_i}(\zeta^{dh_i})\Big)E_{p-1,\psi}(\zeta^{-1})d^{p_1+\dots+p_n-n}.$$

Then by Proposition 3.2 and Corollary 3.3, we obtain (12). Similarly (13) is deduced from (15).

Finally let us prove the equivalence between (12) and (13). We first note that (12) is equivalent to

$$\text{(23)}\qquad \sum_{d\mid N}\sum_{B\in I_d}d^{s(P)-n}(\psi_1\cdots\psi_n\psi)(d)S(P,H+dkB,d^2k,\Psi)$$
$$=\sigma_{s(P),\Psi}(N)S(P,H,Nk,\Psi).$$

In fact, replacing k by Nk in (12) and making use of Corollary 3.3, we obtain (23) from (12). Conversely, (12) is deduced from (23) by replacing $H=(h_1,\dots,h_n)$ by $N\cdot H=(Nh_1,\dots,Nh_n)$ and making use of Corollary 3.3. Similarly (13) is equivalent to

$$\text{(24)}\qquad N^{s(P)-n}(\psi_1\cdots\psi_n\psi)(N)\sum_{B\in I_d}S(P,H+NkB,N^2k,\Psi)$$
$$=\sum_{d\mid N}\mu(d)\sigma_{s(P),\Psi}\Big(\frac{N}{d}\Big)S(P,H,Nk/d,\Psi).$$

Now the equivalence between (23) and (24) is obvious by the Möbius inversion formula. This completes the proof.

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