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Units in real Abelian fields

by

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1. Introduction. In this paper we consider the Galois module structure of the group of units of real absolutely Abelian number fields. Let N be such a field with Galois group $\Gamma = \mathcal{G}al(N/\mathbb{Q})$, let E_N be its unit group, and denote by $\overline{E}_N$ the torsion-free part of E_N, i.e. $\overline{E}_N = E_N/\{\pm 1\}$. Moreover h_N will be the class number of N.

We shall by $\mathcal{A}$ denote the natural order, i.e. the factor ring of the integral group ring $\mathbb{Z}\Gamma$ by the ideal I generated by $\sum_{\gamma\in\Gamma}\gamma$. This class of fields contains all Abelian fields of prime degree as well as certain fields of odd prime power degree with at most two ramified primes (A. Fröhlich [9], see also [4]). There are only a few results in the literature about the global Galois module structure for real Abelian fields (see [3], [6], [9] and [16]).

The aim of this paper is to describe the class of $\overline{E}_N$ in the locally free class group of the order $\mathcal{A}$ using the so called Hom description of the class group, introduced by A. Fröhlich [7].

To this end we introduce in Section 4 the logarithmic resolvent, which will play a central role in our paper. This tool enables us to represent the class of $\overline{E}_N$ in $\mathcal{A}$ as a homomorphism on the group of characters of Γ. This representation involves a unit of finite index in E_N and certain semilocal units generating $\mathbb{Z}_p \otimes \overline{E}_N$ (Theorem 4.5). In Theorem 4.6 we shall eliminate the global unit replacing it by Gaussian sums and values of Dirichlet and p-adic L-functions at 1.

In Section 5 we focus on real tame cyclic extensions $N/\mathbb{Q}$ of prime degree. In Propositions 5.2 and 5.3 we replace the semilocal generators of $\mathbb{Z}_p \otimes \overline{E}_N$ with generators of the full $\mathbb{Z}_p\Gamma$-modules of semilocal units of N, and subsequently in Proposition 5.5 with semilocal generators of the ring of integers of N. Finally in Theorem 5.6 we obtain a representation of the class of $\overline{E}_N$ in which Galois Gauss sums, $\mathbb{Q}_p$-irreducible characters and the orders of the

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Jordan–Hölder factor modules of the p-Sylow subgroups of the ideal class group of N will appear.

As an example of our approach we shall prove in Section 6 the following necessary and sufficient conditions for a real tame Abelian field N of prime degree to have a Minkowski unit (i.e. $\overline{E}_N \cong \mathcal{A}$):

THEOREM 6.1. *Let $N/\mathbb{Q}$ be a real, tame and cyclic extension of prime degree $l > 2$, and assume that l is regular, i.e. does not divide the class number h_l of the lth cyclotomic field. Then N has a Minkowski unit if and only if*

$$\nu_p(h_p^\chi)\Phi_{\mu,p}(1) = \nu_p(h_p^\mu)\Phi_{\chi,p}(1)$$

for any $\chi, \mu \in \widehat{\Gamma} \setminus \{1_\Gamma\}$, and for any prime $p \mid h_N$, $p \neq l$, where $\Phi_{\chi,p}$ is a $\mathbb{Q}_p$-irreducible character of Γ with χ as a summand and h_p^χ is the order of the $\Phi_{\chi,p}$-component of the class group of N.

Applying this theorem we get simple sufficient conditions for the existence of Minkowski units:

COROLLARY 6.2. *Let $N/\mathbb{Q}$ be a real, tame and cyclic extension of prime degree $l > 2$ and let l be regular. Then N has a Minkowski unit in the following two cases:*

(i) $h_N = 1$,
(ii) *any prime p dividing h_N is a primitive root of unity modulo l.*

This leads to new examples of fields having Minkowski units, like $\mathbb{Q}(\zeta_{47})^+$, $\mathbb{Q}(\zeta_{59})^+$, $\mathbb{Q}(\zeta_{83})^+$, $\mathbb{Q}(\zeta_{107})^+$ (Corollary 6.3(i), (ii)) and at least 611 fields derived from Schoof's tables in [18] (Corollary 6.3(iii), (iv)).

2. Notation and definitions. We shall adopt the standard notation from the book [20].

We shall deal with finite Abelian extensions $L/\mathbb{Q}$ of the rationals. For every such extension we shall fix its generator θ_L and denote by Γ_L its Galois group and by d_L its discriminant. The maximal real subfield of L will be denoted by L^+. If K is a subfield of L, then $\mathcal{N}_{L/K}$ will denote the norm map $L \to K$.

For a finite or infinite prime p we shall denote by $\Delta_{L,p}$ the corresponding decomposition group and by $T_{L,p}$ a set of representatives for the cosets of $\Delta_{L,p}$ in Γ_L. The ring of integers of L will be denoted by $\mathcal{O}_L$, E_L will be its unit group, and $\overline{E}_L$ will denote the factor group $E_L/\{\pm 1\}$ in the case when L is real.

The algebraic closure of the p-adic field $\mathbb{Q}_p$ will be denoted by $\overline{\mathbb{Q}}_p$, ν_p will be the exponential valuation of $\overline{\mathbb{Q}}_p$ satisfying $\nu_p(p) = 1$, and $R_p(L)$ and d_l will denote the p-adic regulator of L.

$\mathfrak{J}(L)$ and $\mathfrak{U}(L)$ will denote the group of ideles of L and the group of unit ideles of L, respectively.

By a *prime of* L we understand an equivalence class of valuations of L. A prime will be called *infinite* if it contains an Archimedean valuation, and *finite* otherwise. The infinite prime of $\mathbb{Q}$ will be denoted by ∞.

Let $\mathfrak{p}_L$ be a fixed prime of L lying above p. Then for any $t \in T_{L,p}$ we shall denote by $L_{t(\mathfrak{p}_L)}$ the completion $\mathbb{Q}_p(t(\theta_L))$ of L at $t(\mathfrak{p}_L)$. For $t \in T_{L,p}$ we shall also denote by $U_{t(\mathfrak{p}_L)}$ and $U^1_{t(\mathfrak{p}_L)}$ the units and the principal units of $L_{t(\mathfrak{p}_L)}$, respectively.

If $\mathfrak{p}$ lies above the infinite prime, then $L_{t(\mathfrak{p}_L)}$ is either $\mathbb{R}$ or $\mathbb{C}$, $U_{t(\mathfrak{p}_L)}$ is either $\mathbb{R}^*$ or $\mathbb{C}^*$, and $U^1_{t(\mathfrak{p}_L)}$ is either $\mathbb{R}^{*+}$ or $\mathbb{C}^*$, $\mathbb{R}^{*+}$ being the multiplicative group of positive real numbers.

By $|X|$ we shall denote the cardinality of the set X. For a positive integer m, ζ_m will be a primitive mth root of unity. For any ring R, we shall denote by R^* its group of units.

For any finite Abelian group G we put $\widehat{G} = \mathrm{Hom}(G, \mathbb{C}^*)$. For any prime number p, any $\chi \in \widehat{G}$ will be considered also as a character with values in $\overline{\mathbb{Q}}_p$, via a suitable embedding of the algebraic closure of $\mathbb{Q}$ into $\overline{\mathbb{Q}}_p$.

R_G will be the free additive group having $\widehat{G}$ for its set of free generators, and R'_G will denote the free subgroup of R_G generated by $\widehat{G} \setminus \{1_G\}$, 1_G being the trivial character.

For any character $\chi \in \widehat{G}$ we put

$$e_\chi = \frac{1}{|G|} \sum_{g \in G} \chi(g^{-1}) g.$$

If Λ is the automorphism group of a field containing $\{\chi(g) : g \in G, \chi \in \widehat{G}\}$, then one defines an action of Λ on the group $\widehat{G}$ by putting

$$\chi^\delta(g) = \delta(\chi(g)) \quad \text{for } \delta \in \Lambda.$$

Let Φ be a $\overline{\mathbb{Q}}_p$-valued character of Γ which is irreducible over $\mathbb{Q}_p$. This character is the sum of $\overline{\mathbb{Q}}_p$-irreducible and conjugate characters of Γ. If a character μ is a summand of Φ we shall write $\mu \,|\, \Phi$.

If we treat $\chi \neq 1_\Gamma$ as a $\overline{\mathbb{Q}}_p$-valued character, then $\Phi_{\chi,p}$ will be an irreducible character of Γ over $\mathbb{Q}_p$ having χ as a summand.

Let $\mathcal{H}_p$ be the p-Sylow subgroup of the class group of N. As $\mathcal{H}_p$ is a $\mathbb{Z}_p\Gamma$-module we can define h_p^χ to be the order of the $\Phi_{\chi,p}$-component of $\mathcal{H}_p$, i.e. the order of $e_{\Phi_{\chi,p}}\mathcal{H}_p$ where $e_{\Phi_{\chi,p}}$ is the idempotent corresponding to $\Phi_{\chi,p}$.

If R is a commutative ring, we put $\widetilde{G} = \sum_{g \in G} g \in RG$. Then for any RG-module M, the submodule $M^0 = \{m \in M : \widetilde{G} m = 0\}$ can be regarded as an $RG/(\widetilde{G})$-module with

$$\Big(\sum_{g\in G} x_g\tilde{g}\Big)m = \Big(\sum_{g\in G} x_g g\Big)m$$

for $x_g \in R$ and $m \in M$, where $\tilde{g} = g \bmod (\widetilde{G})$.

For any R-module M the submodule of all R-torsion elements of M will be denoted by $\mathrm{tor}_R(M)$.

For any prime p (finite or infinite) we define the following three submodules of $\mathfrak{J}(L)$:

$$L_p = \prod_{t\in T_{L,p}} L_{t(\mathfrak{p}_L)}, \quad U_{L,p} = \prod_{t\in T_{L,p}} U_{t(\mathfrak{p}_L)}, \quad U^1_{L,p} = \prod_{t\in T_{L,p}} U^1_{t(\mathfrak{p}_L)}.$$

Let V_r be the set of real embeddings of L into $\mathbb{C}$, and V_c the set of non-conjugate complex embeddings of L into $\mathbb{C}$. Then

$$L_\infty = \prod_{v\in V_r} \mathbb{R}_v \prod_{v\in V_c} \mathbb{C}_v \quad \text{and} \quad U^1_{L,\infty} = \prod_{v\in V_r} \mathbb{R}^{*+}_v \prod_{v\in V_c} \mathbb{C}^*_v$$

where $\mathbb{R}_v = \mathbb{R}$, $\mathbb{C}_v = \mathbb{C}$ and $\mathbb{R}^{*+}_v$ is the multiplicative group of positive real numbers. Now we define an action of Γ_L on L_p as follows:

For any $w, t \in T_{L,p}$ we define $\overline{wt} \in T_{L,p}$ and $\delta_{w,t} \in \Delta_{L,p}$ so that $wt = \delta_{w,t}\overline{wt}$. Now for $\gamma = \delta w$ and $u = (u_t)_{t\in T_L} \in L_p$, $\delta \in \Delta_{L,p}$, $w \in T_{L,p}$ we put

$$\gamma(u) = (\delta\delta_{w,t}(u_{\overline{wt}}))_t$$

where the automorphisms from $\Delta_{L,p}$ are extended to $L_{t(\mathfrak{p}_L)}$ by continuity.

We shall use the following observation which is a consequence of the action of Γ_L on L_p:

Remark 2.1. *Let L be an Abelian field containing the values of all characters of a finite Abelian group G and let p be a prime. Let $f\colon R'_G \to L_p$ be a homomorphism such that $f(\chi) = (f_0(\chi^t))_{t\in T_{L,p}}$ where $f_0 \in \mathrm{Hom}(R'_G, L_{\mathfrak{p}_L})$. Then*

$$f \in \mathrm{Hom}_{\Gamma_L}(R'_G, L_p) \quad \textit{if and only if} \quad f_0 \in \mathrm{Hom}_{\Delta_{L,p}}(R'_G, L_{\mathfrak{p}_L}).$$

Note that by the normality of $L/\mathbb{Q}$ the fields $L_{t(\mathfrak{p}_L)}$ do not depend on the ideal $\mathfrak{p}_L$ chosen, so f is defined correctly.

We shall need the following extension of the logarithm:

$$\mathrm{Log}_p\colon \prod_{t\in T_{L,p}} L^*_{t(\mathfrak{p}_L)} \to L_p, \quad \mathrm{Log}_p((a_t)_t) = (\log_p(a_t))_t \quad \text{for } (a_t)_t \in \prod_{t\in T_{L,p}} L^*_{t(\mathfrak{p}_L)}$$

where $\log_p$ is the Iwasawa p-adic logarithm (see e.g. [20]).

For the infinite prime we define

$$\mathrm{Log}_\infty : L^*_\infty \to L_\infty \quad \text{by} \quad \mathrm{Log}_\infty((u_v)_v) = (\log_\infty |u_v|)_v,$$

where $\log_\infty$ denotes the usual logarithm defined for positive reals.

Let Γ be a finite Abelian group. We define the orders $\mathcal{A} = \mathbb{Z}\Gamma/(\widetilde{\Gamma})$, $\mathcal{A}_p = \mathbb{Z}_p\Gamma/(\widetilde{\Gamma}), \mathcal{A}_\infty = \mathbb{R}\Gamma/(\widetilde{\Gamma})$ in the algebras $A = \mathbb{Q}\Gamma/(\widetilde{\Gamma})$, $A_p = \mathbb{Q}_p\Gamma/(\widetilde{\Gamma})$, $\mathbb{R}\Gamma/(\widetilde{\Gamma})$, respectively, and we shall write $\tilde{\gamma} = \gamma \bmod (\widetilde{\Gamma})$ in A or A_p. We also put $\mathfrak{U}(\mathcal{A}) = \prod_p \mathcal{A}_p^*$.

For any order $\mathcal{U}$ in a semisimple finite-dimensional algebra we shall denote by $\mathrm{Cl}(\mathcal{U})$ the locally free class group of $\mathcal{U}$ and by $(X)_{\mathcal{U}}$ the element of $\mathrm{Cl}(\mathcal{U})$ corresponding to the module X.

If F is an Abelian field containing $\{\chi(\gamma) : \gamma \in \Gamma,\ \chi \in \widehat{\Gamma}\}$ and $\Omega = \Gamma_F = \mathcal{G}al(F/\mathbb{Q})$, then we define a $\mathbb{Z}\Omega$-embedding $i_p : F \to F_p$ by

$$i_p(b) = (t(b))_t \quad \text{ with } t \in T_{F,p}.$$

3. Auxiliary results. In this section we state some results needed later. First we introduce the main ingredient of the Hom description of the class group, the notion of general determinant defined on the algebras A_p, $\mathbb{Z}_p\Gamma$ and $\mathbb{R}\Gamma$ (see [7]).

For any $a = \sum_{\gamma\in\Gamma} a_\gamma\tilde{\gamma} \in A$ and $\chi \in \widehat{\Gamma} \setminus \{1_\Gamma\}$ we put

$$\mathrm{Det}_\chi(a) = \sum_{\gamma\in\Gamma} a_\gamma\chi(\gamma).$$

Observe that if $a \in A^*$, then $\mathrm{Det}_\chi(a) \in F^*$ ($\mathrm{Det}_\chi(ab) = \mathrm{Det}_\chi(a)\,\mathrm{Det}_\chi(b)$ for $a, b \in A$). If $\mu \neq 1_\Gamma$ is another character we put

$$\mathrm{Det}_{\chi+\mu}(a) = \mathrm{Det}_\chi(a)\,\mathrm{Det}_\mu(a),$$

obtaining a homomorphism

$$\mathrm{Det}(a)\colon R'_\Gamma \to F^*, \quad \mathrm{Det}(a)(\chi) = \mathrm{Det}_\chi(a).$$

This also gives a homomorphism

$$\mathrm{Det}\colon A^* \to \mathrm{Hom}_\Omega(R'_\Gamma, F^*),$$

which can be extended to the homomorphism

$$\mathrm{Det}\colon A_p^* \to \mathrm{Hom}_\Omega(R'_\Gamma, F_p^*)$$

defined by $\mathrm{Det}_\chi(\sum_{\gamma\in\Gamma} a_\gamma\tilde{\gamma}) = \sum_{\gamma\in\Gamma} a_\gamma i_p(\chi(\gamma))$ for $a_\gamma \in \mathbb{Q}_p$.

After restricting from A_p^* to $\mathcal{A}_p^*$, we obtain

$$\mathrm{Det}\colon \mathcal{A}_p^* \to \mathrm{Hom}_\Omega(R'_\Gamma, U_{F,p}),$$

and finally we have

$$\mathrm{Det}\colon \mathfrak{U}(\mathcal{A}) \to \mathrm{Hom}_\Omega(R'_\Gamma, \mathfrak{J}(F)).$$

We shall also need the determinant map for the algebras $\mathbb{Z}_p\Gamma$ and $\mathbb{R}\Gamma$ defined by $\mathrm{Det}_\chi(\sum_{\gamma\in\Gamma} a_\gamma\gamma) = \sum_{\gamma\in\Gamma} a_\gamma i_p(\chi(\gamma))$ for $a_\gamma \in \mathbb{Q}_p$ or $\mathbb{R}$. We shall also use Det to denote the above map because of the evident identity

$$\mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} a_\gamma\gamma\Big) = \mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} a_\gamma\tilde{\gamma}\Big) \qquad \text{for } a_\gamma \in \mathbb{Q}_p \text{ or } \mathbb{R} \text{ and } \chi \neq 1_\Gamma. \tag{3.1}$$

Now we can state

THEOREM 3.1. *Let X be a locally free $\mathcal{A}$-module of rank one. Choose a free generator v of $\mathbb{Q}\otimes X$ over A and for each prime p choose a free generator x_p of $\mathbb{Z}_p \otimes X$ over $\mathcal{A}_p$. Then both v and x_p are free generators of $\mathbb{Q}_p \otimes X$ over A_p, and so*

$$x_p = \lambda_p v, \qquad \lambda_p \in A_p^*.$$

(i) *Let $h = (h_p) \in \mathrm{Hom}(R'_\Gamma, \prod_p F_p^*)$ be defined by $h_p(\chi) = h(\chi)_p = \mathrm{Det}_\chi(\lambda_p)$ for all p and $\chi \in R'_\Gamma$. Then $h \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(F))$ and its class $[h]$ modulo $\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$ depends only on the isomorphism class of X.*

(ii) *There is a unique isomorphism*

$$\mathrm{Cl}(\mathcal{A}) \cong \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))]$$

so that for every locally free rank one module X, the class $(X)_\mathcal{A}$ maps onto the corresponding class $[h]$ as constructed above.

Proof. Modify slightly the proof of the analogous theorem for orders in the group ring $\mathbb{Q}\Gamma$; see Theorem 1 in [7] and also [8]. ■

REMARK 3.2. *If $f \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(F))$, then its component f_p at the infinite prime $p = \infty$ is an element of $\mathrm{Det}(\mathcal{A}_\infty^*)$. Moreover the map f' defined by $f'_p(\chi) = f_p(\chi)$ for finite p and $f'_\infty(\chi) = 1$ for $\chi \neq 1_\Gamma$ has the same class $[f']$ modulo $\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$ as f.*

Proof. Define $\bar{f}_\infty \in \mathrm{Hom}_\Omega(R_\Gamma, F_\infty^*)$ by $\bar{f}_\infty(\chi) = f_\infty(\chi)$ for $\chi \neq 1_\Gamma$ and $\bar{f}_\infty(1_\Gamma) = 1$. According to Proposition 2.2 of [7] one has $\mathrm{Hom}_\Omega(R_\Gamma, F_\infty^*) \subseteq \mathrm{Det}((\mathbb{R}\Gamma)^*)$ and so there is $\alpha \in (\mathbb{R}\Gamma)^*$ such that $\bar{f}_\infty(\chi) = \mathrm{Det}_\chi(\alpha)$ for any $\chi \in \widehat{\Gamma}$. Now the first part of our assertion follows by noting that $f_\infty(\chi) = \mathrm{Det}_\chi(\bar{\alpha})$ where $\bar{\alpha}$ denotes the image of α in $\mathcal{A}_\infty^*$ ($\mathrm{Det}(\alpha)$ defined on $\mathbb{R}\Gamma$ has the same values as $\mathrm{Det}(\bar{\alpha})$ defined on $\mathcal{A}_\infty$).

To prove the second part it suffices to observe that ff'^{-1} lies in $\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$. ■

Now let N be a real Abelian field of finite degree over $\mathbb{Q}$ with $\Gamma = \Gamma_N = \mathcal{G}al(N/\mathbb{Q})$. For this field we specify our notations in the following way: $\Delta_p = \Delta_{N,p}$, $T_p = T_{N,p}$, $\mathfrak{p} = \mathfrak{p}_N$, $U_p = U_{N,p}$ and $U_p^1 = U_{N,p}^1$. For a prime p we shall denote by e_p, f_p and $g_p = |T_p|$ the ramification index, the residue class degree of p in N and the number of prime ideals in $\mathcal{O}_N$ above p respectively.

In the case of a finite prime p we put

$$E^1_{N,p} = \{\varepsilon \in E_N : (t(\varepsilon))_{t\in T_p} \in U^1_p\},$$
$$\mathcal{E}_{N,p} = d_p(\mathbb{Z}_p \otimes E_N) = d_p(\mathbb{Z}_p \otimes E^1_{N,p}) \subset U^1_p$$

where $d_p : \mathbb{Z}_p \otimes E^1_{N,p} \to U^1_p$, $d_p(a \otimes \varepsilon) = (t(\varepsilon)^a)_{t\in T_p}$ for $a \in \mathbb{Z}_p$ and $\varepsilon \in E^1_{N,p}$.

Since N is real, the decomposition group of the infinite prime in N is trivial, hence $T_\infty = \Gamma$. We define the totally positive units by

$$E^+_N = \{\varepsilon \in E_N : (\gamma(\varepsilon))_{\gamma\in\Gamma} \in U^1_\infty\}$$

and we put

$$\mathcal{E}_{N,\infty} = d_\infty(\mathbb{R} \otimes E^+_N) = d_\infty(\mathbb{R} \otimes E_N) \subset U^1_\infty$$

where $d_\infty : \mathbb{R} \otimes E^+_N \to U^1_\infty$, $d_\infty(a \otimes \varepsilon) = (\gamma(\varepsilon)^a)_{\gamma\in\Gamma}$ for any $a \in \mathbb{R}$ and $\varepsilon \in E^+_N$.

THEOREM 3.3. *Let N be a real Abelian field. Then for a finite prime p,*

$$|\mathrm{tor}_{\mathbb{Z}_p}(U^1_p/\mathcal{E}_{N,p})| \stackrel{p}{=} \frac{pe_pR_p(N)}{b\sqrt{d_N}n_p},$$

where n_p is the absolute norm of $\prod_{t\in T_p} t(\mathfrak{p})$, $b = 1$ unless $p = 2$ and -1 is not a norm in $N_\mathfrak{p}$ for $\mathfrak{p}\,|\,2$, in which case $b = 2$. The relation $\stackrel{p}{=}$ means equality up to a p-adic unit factor. If $p = \infty$, then $|\mathrm{tor}(U^1_\infty/\mathcal{E}_{N,\infty})| = 1$.

Proof. For finite p it suffices to apply Corollary 2.6.1(ii)$_2$, Theorem 2.6.4 and Remark 2.6.5(i) in [11, Chapter III]. If $p = \infty$, then $T_\infty = \Gamma$ and the Γ-homomorphism

$$(u_\gamma)_\gamma \mapsto \sum_{\gamma\in\Gamma} \log(u_\gamma)\gamma^{-1}$$

gives $U^1_\infty/\mathcal{E}_{N,\infty} \cong \mathbb{R}\Gamma/(\mathbb{R}\Gamma)^0$, whence $\mathrm{tor}_\mathbb{R}(U^1_\infty/\mathcal{E}_{N,\infty}) = \{1\}$. ∎

THEOREM 3.4 (Ramachandra units). *Let $n \not\equiv 2 \bmod 4$, and let $n = \prod_{i=1}^s p_i^{e_i}$ be its prime factorization. Let I run through all proper subsets of $\{1,\dots,s\}$, and let $n_I = \prod_{i\in I} p_i^{e_i}$. For $1 < a < \frac{1}{2}n$ with $(a,n) = 1$ define*

$$\xi_a = \zeta_n^{l_a} \prod_I \frac{1-\zeta_n^{an_I}}{1-\zeta_n^{n_I}} \quad \text{where} \quad l_a = \frac{1}{2}(1-a)\sum_I n_I.$$

Then the elements ξ_a form a set of multiplicatively independent units generating a subgroup of finite index in the group of units of $\mathbb{Q}(\zeta_n)^+$.

Proof. This is Theorem 8.3 in [20]. ∎

THEOREM 3.5. *Let χ be an even nontrivial primitive Dirichlet character of conductor f and let m be an integer not divisible by f. Let $\tau(\chi) =$*

$\sum_{a=1}^{f} \chi(a)\zeta_f^a$ *be the Gauss sum. Then*

$$\sum_{\substack{a=1\\(a,f)=1}}^{f} \bar{\chi}(a)\log_p(1-\zeta_f^{am}) = \frac{pf}{\tau(\chi)}\,\frac{\chi(m)}{\chi(p)-p}L_p(1,\chi) \quad \textit{for a prime number } p.$$

For the infinite prime

$$\sum_{\substack{a=1\\(a,f)=1}}^{f} \bar{\chi}(a)\log_\infty|1-\zeta_f^{am}| = -\frac{\chi(m)fL(1,\chi)}{\tau(\chi)}.$$

Proof. In the case of finite p and $(m,f)=1$ this is Leopoldt's original formula (for a proof see Theorem 5.18 in [20]).

Thus we may confine ourselves to the case $(m,f)>1$ and $f\nmid m$. Observe that as $\chi(m)=0$ in this case, it suffices to show that the sum also vanishes. Write $m=m_1d$ where $d=(m,f)$. Then after putting $c=am_1$ we obtain

$$\sum_{\substack{a=1\\(a,f)=1}}^{f} \bar{\chi}(a)\log_p(1-\zeta_f^{am}) = \chi(m_1)\sum_{\substack{c=1\\(c,f)=1}}^{f} \bar{\chi}(c)\log_p(1-\zeta_f^{cd}).$$

Since χ is primitive and $d\,|\,f$ $(1<d<f)$, there is an integer b such that $b\equiv 1 \pmod{f/d}$, $(b,f)=1$, and $\chi(b)\neq 1$ (see [1, Theorem 2 on p. 469]).

As $b\equiv 1 \pmod{f/d}$ one has $\zeta_f^{cd}=\zeta_f^{bcd}$ and so after a suitable change of the summation index we have

$$\begin{aligned}\sum_{\substack{c=1\\(c,f)=1}}^{f} \bar{\chi}(c)\log_p(1-\zeta_f^{cd}) &= \sum_{\substack{c=1\\(c,f)=1}}^{f} \bar{\chi}(c)\log_p(1-\zeta_f^{bcd})\\ &= \chi(b)\sum_{\substack{c=1\\(c,f)=1}}^{f} \bar{\chi}(c)\log_p(1-\zeta_f^{cd}).\end{aligned}$$

Since $\chi(b)\neq 1$ it follows that

$$\sum_{\substack{c=1\\(c,f)=1}}^{f} \bar{\chi}(c)\log_p(1-\zeta_f^{cd})=0$$

and this completes the proof for finite primes.

If $p=\infty$ it suffices to apply the preceding argument to the well known classical formula (for a proof see Theorem 4.9 in [20]). ∎

We shall also need the following technical result:

LEMMA 3.6. *Let p be a prime number and let Φ be an irreducible character of Γ over $\mathbb{Q}_p$. Let $a \in \mathbb{Z}_p\Gamma$ and put $a = \sum_{\mu\in\widehat{\Gamma}} a_\mu e_\mu$ with $a_\mu \in \mathbb{Z}_p[\zeta_n]$, where n is the exponent of Γ. Then*

(i) *The elements of $\{a_\mu : \mu \mid \Phi\}$ are all conjugate over $\mathbb{Q}_p(\zeta_n)$, and the values $\nu_p(a_\mu)$ are equal for all $\mu \mid \Phi$.*
(ii) *If $a_\mu \neq 0$ for $\mu \mid \Phi$, then*

$$\nu_p(|e_\Phi(\mathbb{Z}_p\Gamma/\mathbb{Z}_p\Gamma a)|) = \nu_p(a_\chi)\Phi(1),$$

where χ is any summand of Φ.

Proof. (i) Let $a = \sum_{\gamma\in\Gamma} b_\gamma\gamma$ with $b_\gamma \in \mathbb{Z}_p$. Observe that, for any $\mu \in \widehat{\Gamma}$, $a_\mu = \sum_{\gamma\in\Gamma} b_\gamma\mu(\gamma)$, and if μ and χ are summands of Φ then $\mu = \chi^\sigma$ for some $\sigma \in \mathcal{G}al(\mathbb{Q}_p(\zeta_n)/\mathbb{Q}_p)$. Thus by the above equation we get $a_\mu = \sigma(a_\chi)$. The second part of (i) is an immediate consequence of the first.

(ii) Note that $|e_\Phi(\mathbb{Z}_p\Gamma/\mathbb{Z}_p\Gamma a)| = |e_\Phi\mathbb{Z}_p\Gamma/e_\Phi(\mathbb{Z}_p\Gamma a)|$. Since $e_\Phi(\mathbb{Z}_p\Gamma a)$ is the image of $e_\Phi\mathbb{Z}_p\Gamma$ under the $\mathbb{Q}_p$-linear transformation $\mathcal{L} : x \mapsto ax$, it follows that $|e_\Phi\mathbb{Z}_p\Gamma/e_\Phi(\mathbb{Z}_p\Gamma a)|$ modulo $\mathbb{Z}_p^*$ is equal to $\det(\mathcal{L})$. Thus calculating the determinant of the matrix of $\mathcal{L}$ relative to the basis $\{e_\mu : \mu \mid \Phi\}$ we obtain

$$|e_\Phi(\mathbb{Z}_p\Gamma/\mathbb{Z}_p\Gamma a)|\mathbb{Z}_p = \prod_{\mu\mid\Phi} a_\mu\mathbb{Z}_p = p^{\Phi(1)\nu_p(a_\chi)}\mathbb{Z}_p$$

where the last equality follows from (i) and the fact that Φ is the sum of $\Phi(1)$ irreducible characters over $\overline{\mathbb{Q}}_p$. ■

4. Logarithmic resolvent. From now on we assume that $F \supseteq N \cup \{\chi(\gamma) : \gamma \in \Gamma, \chi \in \widehat{\Gamma}\}$, and for any prime p, we choose a set of representatives for the decomposition group of p in the extension $F/\mathbb{Q}$ (recall that $\mathcal{G}al(F/\mathbb{Q}) = \Omega$) in the following way.

Let S_p be a set of coset representatives of the decomposition group for the prime $\mathfrak{p}_F$ over a prime $\mathfrak{p}$ of N in the extension F/N. Let $\check{T}_p \subseteq \Omega$ be a set of extensions of elements of the set T_p such that each $t \in T_p$ has exactly one extension $\check{t}$ in $\check{T}_p$. In this way $\check{T}_pS_p$ is a set of coset representatives of the decomposition group for $\mathfrak{p}_F$ over p in the extension $F/\mathbb{Q}$. Therefore we can write

$$F_p = \prod_{t\in T_p}\prod_{s\in S_p} F_{\check{t}s(\mathfrak{p}_F)} \quad \text{for finite } p,$$

$$F_\infty = \prod_{t\in T_\infty}\prod_{s\in S_\infty} \mathbb{C}_{\check{t},s} \quad \text{where } \mathbb{C}_{\check{t},s} = \mathbb{C}.$$

Using these sets of representatives we define the $\mathbb{Z}\Omega$-embeddings

$$j_p : N_p = \prod_{t\in T_p} N_{t(\mathfrak{p})} \to F_p \quad \text{by}$$

$$j_p((a_t)_t) = (a_{\check{t}s})_{\check{t}s} \quad \text{where } a_{\check{t}s} = a_t \in N_{t(\mathfrak{p})} \text{ for any } t \in T_p,\ s \in S_p.$$

We need to define Det on the rings $N_p\Gamma/(\widetilde{\Gamma})$ and $N_p\Gamma$. This can be done by putting

$$\mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} b_\gamma\tilde{\gamma}\Big) = \mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} b_\gamma\gamma\Big) = \sum_{\gamma\in\Gamma} j_p(b_\gamma)i_p(\chi(\gamma)) \quad \text{for } b_\gamma\in N_p, \chi\neq 1_\Gamma,$$

where i_p was defined at the end of Section 2.

Lemma 4.1. *Let p be a prime, $u \in (U_p^1)^0$ and $y \in \mathcal{A}_p$. Then*

$$\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(yu))\tilde{\gamma}^{-1} = y\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1}.$$

Proof. Put $y = \sum_{\delta\in\Gamma} y_\delta\tilde{\delta}$. Since $\mathrm{Log}_p(w^a) = a\,\mathrm{Log}_p(w)$ and $yw = \prod_{\delta\in\Gamma}\gamma(w)^{y_\delta}$, for any $a\in\mathbb{Z}_p$ or $a\in\mathbb{R}$ and $w\in U_p^1$, we get

$$\begin{aligned}\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(yu))\tilde{\gamma}^{-1} &= \sum_{\gamma\in\Gamma} \mathrm{Log}_p\Big(\gamma\prod_{\delta\in\Gamma}\delta(u)^{y_\delta}\Big)\tilde{\gamma}^{-1}\\ &= \sum_{\gamma\in\Gamma}\sum_{\delta\in\Gamma} y_\delta\,\mathrm{Log}_p(\delta\gamma(u))\tilde{\gamma}^{-1}.\end{aligned}$$

On the other hand

$$\begin{aligned}y\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1} &= \sum_{\gamma\in\Gamma}\sum_{\delta\in\Gamma} y_\delta\,\mathrm{Log}_p(\gamma(u))\tilde{\delta}\tilde{\gamma}^{-1}\\ &= \sum_{\delta\in\Gamma}\sum_{\alpha\in\Gamma} y_\delta\,\mathrm{Log}_p(\alpha\delta(u))\tilde{\alpha}^{-1} \quad \text{after substituting } \alpha=\gamma\delta^{-1}. \ \blacksquare\end{aligned}$$

Proposition 4.2. *Let p be a finite or infinite prime and let $u \in \mathcal{E}_{N,p}^0$ generate an $\mathcal{A}_p$-submodule of finite index in $\mathcal{E}_{N,p}$. Then*

$$\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1} \in (N_p\Gamma/(\widetilde{\Gamma}))^*.$$

Proof. First assume that $u = (t(\varepsilon))_t$, where $\varepsilon \in E_N^0$ generates an $\mathcal{A}_p$-submodule of finite index in E_N. It suffices to find an element $\sum_{\delta\in\Gamma} x_\delta\tilde{\delta} \in N_p\Gamma/(\widetilde{\Gamma})$ such that

$$\sum_{\delta\in\Gamma} x_\delta\tilde{\delta}\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma u)\tilde{\gamma}^{-1} = \tilde{1}, \quad \text{that is,} \quad \sum_{\delta\in\Gamma} x_\delta\delta\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma u)\gamma^{-1} = 1 + x\widetilde{\Gamma}$$

with $x\in N_p$. Hence after substituting $\alpha = \delta\gamma^{-1}$ we obtain

$$\sum_{\alpha\in\Gamma}\Big(\sum_{\delta\in\Gamma} x_\delta\,\mathrm{Log}_p(\delta\alpha^{-1}u)\Big)\alpha = 1 + x\widetilde{\Gamma}.$$

This is equivalent to the system of equations

$$\sum_{\delta\in\Gamma} x_\delta \operatorname{Log}_p(\delta\alpha^{-1}(u)) - x = 0 \quad \text{for } \alpha \neq 1,$$
$$\sum_{\delta\in\Gamma} x_\delta \operatorname{Log}_p \delta(u) - x = 1.$$

By putting $u = (t(\varepsilon))_t$, $x_\delta = (x_{\delta,t})_t \in N_p$ and $x = (x_t)_t \in N_p$ with $t \in T_p$, we get systems of equations (one for each $t \in T_p$)

$$\sum_{\delta\in\Gamma} x_{\delta,t} \log_p(t\delta\alpha^{-1}(\varepsilon)) - x_t = 0 \quad \text{for } \alpha \neq 1,$$
$$\sum_{\delta\in\Gamma} x_{\delta,t} \log_p(t\delta(\varepsilon)) - x_t = 1.$$

Thus after subtracting we get

$$\sum_{\delta\in\Gamma} x_{\delta,t} \log_p(t\delta\alpha^{-1}(\varepsilon)/t\delta(\varepsilon)) = -1 \quad \text{for } \alpha \neq 1,$$
$$\sum_{\delta\in\Gamma} x_{\delta,t} \log_p(t\delta(\varepsilon)) - x_t = 1.$$

According to Leopoldt's conjecture which is known to be true for Abelian fields and by the independence of $\{\gamma(\varepsilon) : \gamma \in \Gamma \setminus \{1\}\}$ in the Archimedean case we get

$$\det[\log_p(\delta\alpha^{-1}(\varepsilon))]_{\alpha,\delta\in\Gamma\setminus\{1\}} \neq 0.$$

Using this and Lemma 5.26 of [20] with $\sum_{\delta\in\Gamma} \log_p(\delta(\varepsilon)) = 0$ we arrive at

$$\det[\log_p(\delta\alpha^{-1}(\varepsilon)/\delta(\varepsilon))]_{\alpha,\delta\in\Gamma\setminus\{1\}} \neq 0,$$

showing that the systems of equations have solutions, which proves our proposition for $u = (t(\varepsilon))_t$.

Now we turn to the general case. Let $v \in \mathcal{E}_{N,p}^0$ generate the $\mathcal{A}_p$-module of finite index in $\mathcal{E}_{N,p}$ and choose $\varepsilon \in E_N^0$ generating an $\mathcal{A}_p$-submodule of finite index in E_N. Then $u = (t(\varepsilon))_t$ also generates a module of finite index in $\mathcal{E}_{N,p}$ and therefore for suitable $m \in \mathbb{Z}$ and $y \in \mathcal{A}_p$ we get $v^m = u^y$. By Lemma 4.1 we have

$$m \sum_{\gamma\in\Gamma} \operatorname{Log}_p(\gamma(v))\tilde{\gamma}^{-1} = y \sum_{\gamma\in\Gamma} \operatorname{Log}_p(\gamma(u))\tilde{\gamma}^{-1}.$$

Similarly $u^{m'} = v^z$ for some $m' \in \mathbb{Z}$ and $z \in \mathcal{A}_p$, so we obtain

$$m' \sum_{\gamma\in\Gamma} \operatorname{Log}_p(\gamma(u))\tilde{\gamma}^{-1} = z \sum_{\gamma\in\Gamma} \operatorname{Log}_p(\gamma(v))\tilde{\gamma}^{-1}.$$

Consequently, by combining the above equalities, we get

$$mm'\sum_{\delta\in\Gamma}\mathrm{Log}_p(\delta(u))\tilde{\delta}^{-1}=yz\sum_{\delta\in\Gamma}\mathrm{Log}_p(\delta(u))\tilde{\delta}^{-1}.$$

Since $\sum_{\delta\in\Gamma}\mathrm{Log}_p(\delta(u))\tilde{\delta}^{-1}$ is a unit of $(N_p\Gamma/(\widetilde{\Gamma}))$, as we have shown above, the last equality implies $mm'\tilde{1}=yz$ and so $(1/m)y\in(N_p\Gamma/(\widetilde{\Gamma}))^*$. Thus finally

$$\sum_{\delta\in\Gamma}\mathrm{Log}_p(v)\tilde{\delta}^{-1}=\frac{1}{m}y\sum_{\gamma\in\Gamma}\mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1}\in(N_p\Gamma/(\widetilde{\Gamma}))^*. \ \blacksquare$$

Now, for any finite or infinite prime p and $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$, we may define the p-logarithmic resolvent of $u\in(U_p)^0$ by

$$(u|\chi)_{N,p}=\mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma}\mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1}\Big)=\sum_{\gamma\in\Gamma}i_p(\overline{\chi}(\gamma))j_p(\mathrm{Log}_p(\gamma(u)))\in F_p.$$

We also write

$$(\varepsilon|\chi)_{N,p}=((t(\varepsilon))_t|\chi)_{N,p}\in F_p\quad\text{ for }\varepsilon\in E_N^0\text{ (or }(E_N^+)^0).$$

COROLLARY 4.3. *Let $u\in(\mathcal{E}_{N,p})^0$ generate an $\mathcal{A}_p$-submodule of finite index in $\mathcal{E}_{N,p}$ and $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$. Then*

$$(u|\chi)_{N,p}\in F_p^*,$$

and for every $x\in\mathcal{A}_p$,

$$(xu|\chi)_{N,p}=\mathrm{Det}_\chi(x)(u|\chi)_{N,p}.$$

Proof. By Proposition 4.2, there exists $\lambda\in N_p\Gamma/(\widetilde{\Gamma})$ such that

$$\lambda\sum_{\gamma\in\Gamma}\mathrm{Log}_p(\gamma(u))\tilde{\gamma}^{-1}=\tilde{1},$$

and since Det_χ is a ring homomorphism, the first assertion follows. The second assertion is an immediate consequence of Lemma 4.1. $\blacksquare$

Let $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$ and for any prime p let u_p be an element of U_p^0. We define the *logarithmic resolvent*, the main tool of our paper, by

$$((u_p)_p|\chi)_N=((u_p|\chi)_{N,p})_p\in\prod_p F_p^*.$$

If $\varepsilon\in E_N^0$, then we write for brevity

$$(\varepsilon|\chi)_N=((\varepsilon_p|\chi)_{N,p})_p,$$

with $\varepsilon_p=(t(\varepsilon))_t$, where $t\in T_p$. If $\eta_p\in(\mathbb{Z}_p\otimes E_N)^0$, then we also put $(\eta_p|\chi)_{N,p}=(d_p(\eta_p)|\chi)_{N,p}$.

REMARK 4.4. The logarithmic resolvent can be extended to elements of U_p by putting

$$(u|\chi)_{N,p} = \mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} \mathrm{Log}_p(\gamma(u))\gamma^{-1}\Big) = \sum_{\gamma\in\Gamma} i_p(\overline{\chi}(\gamma))j_p(\mathrm{Log}_p\,\gamma(u)).$$

Note also that the identity in Corollary 4.3 remains valid.

We shall also need the usual resolvent introduced by A. Fröhlich (see e.g. [7]) and used in the study of the additive Galois structure of rings of integers. For $a = (a_p)_p$ with $a_p \in \mathbb{Z}_p \otimes \mathcal{O}_N = \prod_{t\in T_p} \mathcal{O}_{\mathfrak{p}_t}$, we put

$$[a_p|\chi]_{N,p} = \mathrm{Det}_\chi\Big(\sum_{\gamma\in\Gamma} \gamma(a_p)\gamma^{-1}\Big) = \sum_{\gamma\in\Gamma} i_p(\overline{\chi}(\gamma))j_p(\gamma(a_p)) \in F_p$$

for $\chi \in \widehat{\Gamma}$. The resolvent of a is defined by

$$[a|\chi]_N = ([a_p|\chi]_{N,p})_p.$$

THEOREM 4.5. *Let N be a real Abelian field and assume that $\overline{E}_N$ is $\mathcal{A}$-locally free. Let $\varepsilon \in E_N^0$ generate an $\mathcal{A}$-submodule of E_N of finite index and for any prime p let ε_p be an element of $(\mathbb{Z}_p \otimes E_N)^0$ whose image is an $\mathcal{A}_p$-free generator of $\mathbb{Z}_p \otimes \overline{E}_N$. Then, for $\chi \in \widehat{\Gamma} \setminus \{1_\Gamma\}$, the map*

$$\chi \mapsto ((\varepsilon_p)_p|\chi)_N(\varepsilon|\chi)_N^{-1}$$

is a representative of $(\overline{E}_N)_\mathcal{A}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma, \mathfrak{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))].$$

Moreover we can choose ε_p in such a way that this map is equal to 1 at all primes not dividing the index $(E_N : \mathcal{A}\varepsilon)$.

Proof. First consider finite primes. Let $\vartheta_p : \mathbb{Z}_p \otimes E_N \to \mathbb{Z}_p \otimes \overline{E}_N$ denote the $\mathbb{Z}_p\Gamma$-homomorphism mapping $1 \otimes \eta$ to $1 \otimes \overline{\eta}$.

Applying Theorem 3.1 to $X = \overline{E}_N$, $v = 1 \otimes \overline{\varepsilon}$, $x_p = \vartheta_p(\varepsilon_p)$ we get

$$\vartheta_p(\varepsilon_p) = \lambda_p(1 \otimes \overline{\varepsilon}) = \lambda_p\vartheta_p(1 \otimes \varepsilon) \quad \text{ with } \lambda_p \in A_p^*.$$

As ϑ_p restricted to $(\mathbb{Z}_p \otimes E_N)^0$ is an $\mathcal{A}_p$-homomorphism and there is a positive integer m such that $p^m\lambda_p \in \mathcal{A}_p$, one has

$$\vartheta_p(p^m\varepsilon_p) = \vartheta_p(p^m\lambda_p(1 \otimes \varepsilon)).$$

Since the kernel of ϑ_p is $\mathbb{Z}_p$-torsion we can choose a positive integer $m' \geq m$ so that $p^{m'} \ker\vartheta_p = \{0\}$ and so $p^{m'}\varepsilon_p = p^{m'}\lambda_p(1\otimes\varepsilon)$. Now applying Corollary 4.3 to the above equation we obtain $\mathrm{Det}_\chi(p^{m'}\lambda_p)(1 \otimes \varepsilon|\chi)_{N,p} = (p^{m'}\varepsilon_p|\chi)_{N,p}$ and finally

$$\mathrm{Det}_\chi(\lambda_p) = (\varepsilon_p|\chi)_{N,p}(\varepsilon|\chi)_{N,p}^{-1}.$$

For $p = \infty$ the map $\vartheta_\infty : \mathbb{Z}\mathbb{R} \otimes E_N \to \mathbb{R} \otimes \overline{E}_N$ is an isomorphism so we obtain $\varepsilon_\infty = \lambda_\infty(1 \otimes \varepsilon)$, which as above gives the required formula.

To prove the last part of the theorem it suffices to put $\varepsilon_p = 1 \otimes \varepsilon$ for primes not dividing $(E_N : \mathcal{A}\varepsilon)$. Indeed, since for these primes

$$(\mathbb{Z}_p \otimes E_N)/\mathcal{A}_p(1 \otimes \varepsilon) \cong (\mathbb{Z}_p \otimes E_N)/(\mathbb{Z}_p \otimes \mathcal{A}\varepsilon) \cong \mathbb{Z}_p \otimes (E_N/\mathcal{A}\varepsilon) \cong \{1\},$$

we infer that $1 \otimes \varepsilon$ is a free generator of $\mathbb{Z}_p \otimes E_N$. Thus, by definition, $(\varepsilon_p|\chi)_{N,p} = (1 \otimes \varepsilon|\chi)_{N,p} = (d_p(1 \otimes \varepsilon)|\chi)_{N,p} = (t(\varepsilon)_t|\chi)_{N,p} = (\varepsilon|\chi)_{N,p}$ and so we can use this special choice of ε_p in the representing map. ∎

From now on we choose F to be a cyclotomic field $\mathbb{Q}(\zeta_n)$ containing N and the values of all characters from $\widehat{\Gamma}$. Thus $\Omega = \mathcal{G}al(F/\mathbb{Q}) = \mathcal{G}al(\mathbb{Q}(\zeta_n)/\mathbb{Q})$ and by putting $H = \mathcal{G}al(F/N)$, we obtain

$$\Omega/H \cong \mathcal{G}al(N/\mathbb{Q}) = \Gamma, \tag{4.1}$$

so we can identify Γ with Ω/H.

By (4.1) any character of Γ can be treated as a character of Ω which is trivial on H. Thus for any $\chi \in \widehat{\Gamma}$, there is $\chi_0 \in \widehat{\Omega}$ such that $\chi(gH) = \chi_0(g)$ for any $g \in \Omega$. We also identify the group Ω with $(\mathbb{Z}/n\mathbb{Z})^*$ and hence we can regard χ_0 as a Dirichlet character modulo n. Next we assign to this character the primitive character χ_*.

Thus $\chi \mapsto \chi_*$ gives a 1-1 correspondence between $\widehat{\Gamma}$ and the group of primitive Dirichlet characters associated with the field N.

In order to make the statement of the next theorem consistent for $p = \infty$ we put $\chi_*(p) = 0$ so that $\chi_*(p)/p - 1 = -1$. We also write $L_\infty(s, \chi_*) = L(s, \chi_*)$.

Theorem 4.6. *Let N and ε_p be as in Theorem* 4.5. *Then, for $\chi \in \widehat{\Gamma} \setminus \{1_\Gamma\}$, the map*

$$\chi \mapsto ((\varepsilon_p)_p|\chi)_N \left(\frac{[(\chi_*(p)/p - 1)\tau(\chi_*)]^\delta}{L_p(1, \chi_*^\delta)} \right)_{\delta,p} \quad \textit{with } \delta \in \check{T}_p S_p$$

is a representative of $(\overline{E}_N)_\mathcal{A}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma, \mathfrak{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))].$$

Proof. Let $\varepsilon \in E_N^0$ generate a module of finite index in E_N. By definition

$$(\varepsilon|\chi)_{N,p} = ((t(\varepsilon))_t|\chi)_{N,p} = \sum_{\gamma \in \Gamma} i_p(\bar{\chi}(\gamma)) j_p(\mathrm{Log}_p(\gamma(t(\varepsilon))_t)),$$

whence

$$(\varepsilon|\chi)_{N,p} = \Big(\sum_{\gamma \in \Gamma} \bar{\chi}^{\check{t}s}(\gamma) \log_p(\gamma t(\varepsilon)) \Big)_{\check{t}s} \quad \text{with } t \in T_p \text{ and } s \in S_p. \tag{4.2}$$

To simplify notation we write $(\varepsilon|\chi) = \sum_{\gamma \in \Gamma} \bar{\chi}(\gamma) \log_p(\gamma(\varepsilon))$ and then by (4.2) we have $(\varepsilon|\chi)_{N,p} = (t(\varepsilon)|\chi^{\check{t}s})_{\check{t}s}$.

Let W be a set of representatives for cosets of the subgroup H in Ω. Then by identification (4.1) of Γ with Ω/H and by the definition of χ_0 we can write

$$(t(\varepsilon)|\chi^{\check{t}s}) = \sum_{w\in W} \bar{\chi}^{\check{t}s}(wH)\log_p(wt(\varepsilon)) = \sum_{w\in W} \bar{\chi}_0^{\check{t}s}(w)\log_p(wt(\varepsilon)).$$

On the other hand $H \subseteq \ker\chi_0$ and automorphisms of H are identities on N, so we get

$$\begin{aligned}\sum_{g\in\Omega} \bar{\chi}_0^{\check{t}s}(g)\log_p(gt(\varepsilon)) &= \sum_{w\in W}\sum_{h\in H} \bar{\chi}_0^{\check{t}s}(wh)\log_p(wht(\varepsilon))\\ &= |H|\sum_{w\in W} \bar{\chi}_0^{\check{t}s}(w)\log_p(wt(\varepsilon))\end{aligned}$$

and consequently

$$(t(\varepsilon)|\chi^{\check{t}s}) = |H|^{-1}\sum_{g\in\Omega} \bar{\chi}_0^{\check{t}s}(g)\log_p(gt(\varepsilon)). \tag{4.3}$$

Let H^+ be a set of coset representatives of the subgroup $\mathcal{G}al(F/F^+)$ in H. Since $\mathcal{G}al(F^+/N) \cong H/\mathcal{G}al(F/F^+)$, the norm map $\mathcal{N}_{F^+/N}$ acts as multiplication by $\sum_{h\in H^+} h \in \mathbb{Z}\Omega$.

As we have

$$E_N^{(F^+:N)} \subseteq \mathcal{N}_{F^+/N}(E_{F^+}) \subseteq E_N,$$

it follows that $\mathcal{N}_{F^+/N}(E_{F^+})$ is of finite index in E_N. Hence there is an integer d and $\eta \in E_{F^+}$ such that $\varepsilon^d = \mathcal{N}_{F^+/N}(\eta) = \prod_{h\in H^+} h(\eta)$. Thus, by (4.3),

$$\begin{aligned}(t(\varepsilon)^d|\chi^{\check{t}s}) &= |H|^{-1}\sum_{g\in\Omega} \bar{\chi}_0^{\check{t}s}(g)\log_p\Big(\prod_{h\in H^+} g\check{t}h(\eta)\Big)\\ &= |H^{-1}|\sum_{h\in H^+}\Big(\sum_{g\in\Omega} \bar{\chi}_0^{\check{t}s}(g)\log_p(g\check{t}h(\eta))\Big),\end{aligned}$$

and after substituting $r = gh$ we get

$$\begin{aligned}&|H^{-1}|\sum_{h\in H^+}\sum_{r\in\Omega} \bar{\chi}_0^{\check{t}s}(h^{-1}r)\log_p(r\check{t}(\eta))\\ &\qquad = |H^{-1}|\,|H^+|\sum_{r\in\Omega} \bar{\chi}_0^{\check{t}s}(r)\log_p(r\check{t}(\eta)) \qquad (\ker\chi_0 \supseteq H),\end{aligned}$$

whence by $|H|/|H^+| = 2$ we obtain

$$(t(\varepsilon)|\chi^{\check{t}s}) = \frac{1}{2d}\sum_{g\in\Omega} \bar{\chi}_0^{\check{t}s}(g)\log_p(g\check{t}(\eta)). \tag{4.4}$$

Let χ_* be a primitive character for χ_0 with conductor n_* and let $\Omega_* = \{\delta \in \Omega : \delta \equiv 1 \pmod{n_*}\}$ (we identify Ω with $(\mathbb{Z}/n\mathbb{Z})^*$). As Ω_* is the kernel of the homomorphism $\Omega \to (\mathbb{Z}/n_*\mathbb{Z})^*$, we have $\Omega/\Omega_* \cong (\mathbb{Z}/n_*\mathbb{Z})^*$.

We now show that if F_* is the fixed field of Ω_*, then $F_* = \mathbb{Q}(\zeta_*)$, where $\zeta_* = \zeta_n^{n/n_*}$ is a primitive n_*th root of unity. Indeed, for any $\sigma \in \Omega_*$ we have $\sigma(\zeta_*) = \zeta_*$ and hence $\mathbb{Q}(\zeta_*) \subseteq F_*$. On the other hand

$$(F_* : \mathbb{Q}) = |\mathcal{G}al(F_*/\mathbb{Q})| = |\Omega/\Omega_*| = |(\mathbb{Z}/n_*\mathbb{Z})^*| = (\mathbb{Q}(\zeta_*) : \mathbb{Q}).$$

Note that $\chi_0^{ts},\ \chi_*^{ts},\ \Omega_*,\ n_*,\ \zeta_*$ mean the same for χ^{ts} as $\chi_0,\ \chi,\ \Omega_*,\ n_*,\ \zeta_*$ for χ.

Let R_* be a set of representatives for cosets of Ω_* in Ω. Since Ω_* is contained in the kernel of χ_0 and χ_0^{ts}, (4.5) gives

$$(t(\varepsilon)|\chi^{\check{t}s}) = \frac{1}{2d}\sum_{c\in R_*} \bar{\chi}_*^{\check{t}s}(c)\log_p(c\check{t}(\eta_*)) \tag{4.5}$$

where $\eta_* = \prod_{h\in\Omega_*} h(\eta) = \mathcal{N}_{F/F_*}(\eta)$ is a real unit of F_*.

For any $\delta \in \check{T}_pS_p$ we put $\delta(\zeta_*) = \zeta_*^{\delta_*}$, $\delta_* \in \mathbb{Z}$, and for $\check{t} \in T_p$ we also write $\check{t}_* = t_*$.

Now applying Theorem 3.4 for $n = n_*$ one has, for some integer d_*,

$$\check{t}(\eta_*)^{d_*} = \prod_{a\in V}\check{t}(\xi_a)^{x_a} = \prod_{a\in V}\Big(\prod_I \zeta_*^{l_a t_*}\frac{1-\zeta_*^{at_*n_I}}{1-\zeta_*^{t_*n_I}}\Big)^{x_a}$$

where $V = \{1 < a < \frac{1}{2}\varphi(n_*) - 1 : (a, n_*) = 1\}$ and $x_a \in \mathbb{Z}$. In order to apply Theorem 3.5 we consider characters $\chi_0^{\check{t}s}$ as functions defined on $\mathbb{Z}$ with $\chi_0^{\check{t}s}(c) = 0$ for $(c, n_*) > 1$. Using (4.5), $\log_p(\zeta_*) = 0$ and the fact that $c \in (\mathbb{Z}/n_*\mathbb{Z})^*$ acts on ζ_* by $\zeta_* \mapsto \zeta_*^c$ we obtain

$$\begin{aligned}(t(\varepsilon)|\chi^{\check{t}s}) &= \frac{1}{2dd_*}\sum_{c=1}^{n_*}\sum_{a\in V}\sum_I x_a\bar{\chi}_*^{\check{t}s}(c)\log_p\Big(\frac{1-\zeta_*^{act_*n_I}}{1-\zeta_*^{ct_*n_I}}\Big)\\ &= \frac{1}{2dd_*}\sum_{c=1}^{n_*}\sum_{a\in V}\sum_I x_a\bar{\chi}_*^{\check{t}s}(c)[\log_p(1-\zeta_*^{act_*n_I}) - \log_p(1-\zeta_*^{ct_*n_I})]\\ &= \frac{1}{2dd_*}\sum_I\sum_{a\in V} x_a\Big(\sum_{c=1}^{n_*}\bar{\chi}_*^{\check{t}s}(c)\log_p(1-\zeta_*^{act_*n_I}) - \sum_{c=1}^{n_*}\bar{\chi}_*^{\check{t}s}(c)\log_p(1-\zeta_*^{ct_*n_I})\Big).\end{aligned}$$

Now we apply Theorem 3.5 with $f = n_*$, $m = at_*n_I$ for $a \in V$ or $a = 1$ and $n_I \neq 1$. Hence

$$\sum_{c=1}^{n_*}\bar{\chi}_*^{\check{t}s}(c)\log_p(1-\zeta_*^{act_*n_I}) = 0 \qquad (\chi_*^{\check{t}s}(at_*n_I) = 0).$$

Thus

$$(t(\varepsilon)|\chi^{\check{t}s}) = \frac{1}{2dd_*}\sum_{a\in V} x_a\Big(\sum_{c=1}^{n_*}\bar{\chi}_*^{\check{t}s}(c)\log_p(1-\zeta_*^{act_*}) - \sum_{c=1}^{n_*}\bar{\chi}_*^{\check{t}s}(c)\log_p(1-\zeta_*^{ct_*})\Big)$$

and once again using Theorem 3.5,

$$(t(\varepsilon)|\chi^{\check{t}s}) = A(p,\check{t}s)\frac{\chi_*^{\check{t}s}(t_*)}{\tau(\chi_*^{\check{t}s})}\,\frac{n_*}{2dd_*}\sum_{a\in V} x_a(\chi_*^{\check{t}s}(a)-1)$$

where

$$A(p,\check{t}s) = \begin{cases} pL_p(1,\chi_*^{\check{t}s})/(\chi_*^{\check{t}s}(p)-p) & \text{if } p \text{ is finite,}\\ -L(1,\chi_*^{\check{t}s}) & \text{if } p \text{ is infinite.}\end{cases}$$

Since $(\varepsilon|\chi)_{N,p} = (t(\varepsilon)|\chi^{\check{t}s})_{\check{t}s}$ and the map

$$\chi \mapsto \Big(\check{t}s\Big(\frac{n_*}{2dd_*}\sum_{a\in V} x_a(\chi_*(a)-1)\Big)\Big)_{\check{t}s}$$

is an element of $\mathrm{Hom}_\Omega(R'_\Gamma, F^*)$ it follows from Theorem 4.5 that the map

$$\mathcal{R}:\chi\mapsto ((\varepsilon_p)_p|\chi)_N\Bigg(\Bigg(\frac{(\chi_*^{\check{t}s}(p)-p)\tau(\chi_*^{\check{t}s})}{pL_p(1,\chi_*^{\check{t}s})\chi_*^{\check{t}s}(t_*)}\Bigg)_{\check{t}s}\Bigg)_p \tag{4.6}$$

is a representative of $(\overline{E}_N)_{\mathcal{A}}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma,\mathcal{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma,F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))].$$

For $s\in S_p$ we put $s(\zeta_n)=\zeta_n^{s'}$ where $s'\in\mathbb{Z}$. Since $s(\zeta_*)=\zeta_n^{s_*n/n_*}$, it follows that $s'\equiv s_* \pmod{n_*}$. As $\chi_*(x \bmod n_*)=\chi_0(x)$ one has $H \bmod n_* \subseteq \ker\chi_*$, whence $S_p \bmod n_*\subseteq\ker\chi_*$ because $S_p\subseteq \mathcal{G}al(F/N)=H$. Therefore $\chi_*(s' \bmod n_*)=1$ for s' representing $s\in S_p$ and we get

$$\chi_*((\check{t}s)_*)=\chi_*(\check{t}_*)\chi_*(s_*)=\chi_*(t_*)\chi_*(s' \bmod n_*)=\chi_*(t_*).$$

On the other hand for any $\delta\in\check{T}_pS_p$ one has $\tau(\chi_*^\delta)=\chi_*^\delta(\delta_*)\delta[\tau(\chi_*)]$. Now it follows that $\tau(\chi_*^{ts})/\chi_*^{ts}(t_*)=\check{t}s(\tau(\chi_*))$, which together with (4.6) shows that $\mathcal{R}$ is the required homomorphism. ∎

5. Cyclic fields. From now on we assume that the field N is a tame cyclic extension of $\mathbb{Q}$ of prime degree $l\neq 2$ with $\Gamma=\mathcal{G}al(N/\mathbb{Q})=\langle\gamma_0\rangle$. Observe that the conductor of N is a square-free integer q such that $\phi(q)\equiv 0 \pmod{l}$, where ϕ denotes the Euler function, and q is also the conductor of χ_* for $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$. Thus we put $F=\mathbb{Q}(\zeta_{lq})$.

Our strategy is to replace the logarithmic resolvent of ε_p in Theorem 4.6 by the logarithmic resolvent of a suitable element of U_p^1, and then by the resolvent of a $\mathbb{Z}_p\Gamma$-generator of $\mathbb{Z}_p\otimes\mathcal{O}_N$.

In order to calculate the p-logarithmic resolvent, for any prime p, we have to consider separately two cases: $g_p=1$ and $g_p=l$.

In case $g_p=1$ we have $\Delta_p=\Gamma$ and so $T_p=\{\mathrm{id}\}$, $N_p=N_{\mathfrak{p}}$, and $U_p^1=U_{\mathfrak{p}}^1$. Thus, for any $u\in U_{\mathfrak{p}}^1$ and $a_p\in\mathbb{Z}_p\otimes\mathcal{O}_N=\mathcal{O}_{\mathfrak{p}}$, one has

$$
\begin{aligned}
(u|\chi)_{N,p} &= (u|\chi^s)_s = \Big(\sum_{\gamma\in\Gamma} \bar{\chi}^s(\gamma)\log_p \gamma(u)\Big)_s, \\
[a_p|\chi]_{N,p} &= \Big(\sum_{\gamma\in\Gamma} \bar{\chi}^s(\gamma)\gamma(a_p)\Big)_s \quad \text{with } s\in S_p.
\end{aligned} \tag{5.1}
$$

In case $g_p = l$ we have $\Delta_p = \{\mathrm{id}\}$, so $T_p = \Gamma$ and hence $N_{\gamma(\mathfrak{p})} = \mathbb{Q}_p$, $U_p^1 = \prod_{\gamma\in\Gamma} U^1_{\gamma(\mathfrak{p})}$ with $U^1_{\gamma(\mathfrak{p})} = 1 + p\mathbb{Z}_p$. For any $u = (u_\gamma)_\gamma \in U_p^1$ and $a_p = (a_\gamma)_\gamma \in \mathbb{Z}_p \otimes \mathcal{O}_N = \prod_{\gamma\in\Gamma}\mathbb{Z}_p$ using the definition of the action of Γ on N_p, we obtain

$$
(u|\chi)_{N,p} = \Big(\sum_{\gamma\in\Gamma} \check{t}s(\bar{\chi}(\gamma))\log_p u_{\gamma t}\Big)_{\check{t}s} = \Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)\log_p u_{\gamma t}\Big)_{\check{t}s}
$$

and respectively

$$
(a_p|\chi)_{N,p} = \Big(\sum_{\gamma\in\Gamma} \check{t}s(\bar{\chi}(\gamma))a_{\gamma t}\Big)_{\check{t}s} = \Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)a_{\gamma t}\Big)_{\check{t}s}.
$$

By changing the summation variables we get

$$
\begin{aligned}
(u|\chi)_{N,p} &= (\chi^{\check{t}s}(t))_{\check{t}s}\Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)\log_p u_\gamma\Big)_{\check{t}s}, \\
[a_p|\chi]_{N,p} &= (\chi^{\check{t}s}(t))_{\check{t}s}\Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)a_\gamma\Big)_{\check{t}s} \quad \text{with } t\in T_p \text{ and } s\in S_p.
\end{aligned} \tag{5.2}
$$

For $p = \infty$ one has $g_\infty = l$ and $\Delta_\infty = \{\mathrm{id}\}$, so $T_\infty = \Gamma$ and hence $N_{\gamma(\mathfrak{p})} = \mathbb{R}$, $U^1_\infty = \prod_{\gamma\in\Gamma}\mathbb{R}^{*+}$. For any $u = (u_\gamma)_\gamma \in U^1_\infty$ and $a_\infty = (a_\gamma)_\gamma \in \mathbb{R}\otimes\mathcal{O}_N = \prod_{\gamma\in\Gamma}\mathbb{R}$ we get

$$
\begin{aligned}
(u|\chi)_{N,\infty} &= (\chi^{\check{t}s}(t))_{\check{t}s}\Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)\log_\infty u_\gamma\Big)_{\check{t}s}, \\
[a_\infty|\chi]_{N,p} &= (\chi^{\check{t}s}(t))_{\check{t}s}\Big(\sum_{\gamma\in\Gamma} \bar{\chi}^{\check{t}s}(\gamma)a_\gamma\Big)_{\check{t}s}
\end{aligned}
$$

with $t\in T_\infty$ and $s\in S_\infty$.

Now we examine the structure of U_p^1 as a $\mathbb{Z}_p\Gamma$-module.

Proposition 5.1. *Let $N/\mathbb{Q}$ be a tame real Abelian extension with an odd prime degree l. Then the module U_p^1 is $\mathbb{Z}_p\Gamma$-free for any prime $p\neq 2$ and $U_2^1 \cong \mathbb{Z}_2\Gamma\oplus(\mathbb{Z}/2\mathbb{Z})^{g_2}$ where Γ acts trivially on $\mathbb{Z}/2\mathbb{Z}$, and g_2 is the number of prime ideals above 2 in N.*

Proof. First consider the case $g_p = 1$ where $U_p^1 = U_{\mathfrak{p}}^1$.

Let $p\neq 2$. If $\zeta_{p^s}\in N_{\mathfrak{p}}$ and $s\geq 1$, then we would have

$$
p^{s-1}(p-1)\,|\,(N_{\mathfrak{p}}:\mathbb{Q}_p) = l,
$$

which is impossible, so $s=0$. Thus $N_{\mathfrak{p}}$ is regular (i.e. does not contain pth roots of unity). Consequently, by Théorème 17 in [14], U_p^1 is $\mathbb{Z}_p\Gamma$-free.

If $p=2$ and $g_2=1$, then the only 2^sth roots of unity in $N_{\mathfrak{p}}$ are ± 1. Since $\mathcal{N}_{N_{\mathfrak{p}}/\mathbb{Q}_p}(-1)=-1$, Theorem 1 of [2] implies that $U_{\mathfrak{p}}^1\cong \mathbb{Z}_2\Gamma\oplus(\mathbb{Z}/2\mathbb{Z})$.

In the case of finite p, $g_p=l$ and $p\neq 2$, we have $U_p^1=\prod_{\gamma\in\Gamma}U_\gamma^1$, where $U_\gamma^1=1+p\mathbb{Z}_p$. As $\log_p: 1+p\mathbb{Z}_p\cong p\mathbb{Z}_p$, the mapping

$$\Psi:(u_\gamma)_\gamma\mapsto \frac{1}{p}\sum_{\gamma\in\Gamma}\log_p(u_\gamma)\gamma^{-1},\qquad \text{where } u_\gamma\in 1+p\mathbb{Z}_p,$$

establishes an isomorphism $U_p^1\cong\mathbb{Z}_p\Gamma$.

For infinite p the mapping Ψ_∞ defined by $\Psi_\infty((u_\gamma)_\gamma)=\sum_{\gamma\in\Gamma}\log(u_\gamma)\gamma^{-1}$ for $u_\gamma\in\mathbb{R}^{*+}$ shows that $U_\infty^1\cong\mathbb{R}\Gamma$.

If $p=2$ and $g_2=l$, let $(\varphi_1,\varphi_2): 1+2\mathbb{Z}_2=(1+4\mathbb{Z}_2)\{\pm1\}\to\mathbb{Z}_2\oplus(\mathbb{Z}/2\mathbb{Z})$ be an isomorphism ($\varphi_1=\frac14\log_2$). Then the mapping

$$(u_\gamma)_\gamma\mapsto\Big(\sum_{\gamma\in\Gamma}\varphi_1(u_\gamma)\gamma^{-1},(\varphi_2(u_\gamma))_\gamma\Big)$$

gives $U_2^1\cong\mathbb{Z}_2\Gamma\oplus(\mathbb{Z}/2\mathbb{Z})^l$. ∎

Proposition 5.2. *Let $N/\mathbb{Q}$ be a tame real Abelian extension with a prime degree l and let γ_0 be a generator of Γ. Let $\mathcal{E}_{N,l}$ be an $\mathcal{A}_l$-free module with generator ε_l and let u_l be a $\mathbb{Z}_l\Gamma$-free generator of U_l^1. Then for any $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$,*

$$(\varepsilon_l|\chi)_{N,l}=\mathrm{Det}_\chi(1-\tilde\gamma_0)^{m_l}(u_l|\chi)_{N,l}\,\mathrm{Det}_\chi(\rho_l)$$

where $\rho_l\in\mathcal{A}_l^*$, $m_l=\nu_l(lR_l(N)/\sqrt{d_N}n_l)+1$ and n_l is defined in Theorem 3.3.

Proof. Let $\Psi_l: U_l^1\to\mathbb{Z}_l\Gamma$ be a $\mathbb{Z}_l\Gamma$-isomorphism. Since $l\neq 2$ one has $\mathcal{E}_{N,l}\subseteq(U_l^1)^0$, hence $\Psi_l(\mathcal{E}_{N,l})$ is a submodule of $(\mathbb{Z}_l\Gamma)^0=(1-\gamma_0)\mathbb{Z}_l\Gamma$.

Since every nontrivial submodule of $(1-\gamma_0)\mathbb{Z}_l\Gamma$ equals $(1-\gamma_0)^{k+1}\mathbb{Z}_l\Gamma$ with some $k\geq 0$, we get

$$\Psi(\mathcal{E}_{N,l})=(1-\gamma_0)^{k_l+1}\mathbb{Z}_l\Gamma\qquad\text{for some integer } k_l\geq 0,$$

and so

$$U_l^1/\mathcal{E}_{N,l}\cong\mathbb{Z}_l\Gamma/(1-\gamma_0)^{k_l+1}\mathbb{Z}_l\Gamma.$$

Because $\mathbb{Z}_l\Gamma=(1-\gamma_0)\mathbb{Z}_l\Gamma\oplus\mathbb{Z}_l$ as $\mathbb{Z}_l$-modules, we have

$$\begin{aligned}\mathrm{tor}_{\mathbb{Z}_l}(U_l^1/\mathcal{E}_{N,l})&\cong\mathrm{tor}_{\mathbb{Z}_l}\{[(1-\gamma_0)\mathbb{Z}_l\Gamma\oplus\mathbb{Z}_l]/(1-\gamma_0)^{k_l+1}\mathbb{Z}_l\Gamma\}\\&\cong\mathrm{tor}_{\mathbb{Z}_l}[(1-\gamma_0)\mathbb{Z}_l\Gamma/(1-\gamma_0)^{k_l+1}\mathbb{Z}_l\Gamma].\end{aligned}$$

Since $(1-\gamma_0)\mathbb{Z}_l\Gamma$ is isomorphic to $\mathbb{Z}_l[\zeta_l]$ as a $\mathbb{Z}_l\Gamma$-module, we get

$$|\mathrm{tor}_{\mathbb{Z}_l}(U_l^1/\mathcal{E}_{N,l})|=|\mathbb{Z}_l[\zeta_l]/(1-\zeta_l)^{k_l}\mathbb{Z}_l[\zeta_l]|=p^{k_l}.$$

On the other hand, Theorem 3.3 gives

$$k_l = \nu_l\biggl(\frac{lR_l(N)}{b\sqrt{d_N}n_l}\biggr). \tag{5.3}$$

Since ε_l and u_l are $\mathbb{Z}_l\Gamma$-generators for $\mathcal{E}_{N,l}$ and U_l^1 respectively, $\Psi_l(\varepsilon_l)$ and $(1-\gamma_0)^{k_l+1}\Psi_l(u_l)$ are $\mathbb{Z}_l\Gamma$-generators of $(1-\gamma_0)^{k_l+1}\mathbb{Z}_l\Gamma$. This implies that there exist $x, y \in \mathbb{Z}_l\Gamma$ such that $\Psi_l(\varepsilon_l) = (1-\gamma_0)^{k_l+1}x\Psi_l(u_l)$ as well as $(1-\gamma_0)^{k_l+1}\Psi_l(u_l) = y\Psi_l(\varepsilon_l)$, hence $\varepsilon_l = (1-\gamma_0)^{k_l+1}xu_l$ and

$$(1-\gamma_0)^{k_l+1}u_p = y\varepsilon_l. \tag{5.4}$$

Thus $\varepsilon_l = xy\varepsilon_l$, so $\varepsilon_l = \bar{x}\bar{y}\varepsilon_l$, where $\bar{x}$, $\bar{y}$ are the images of x, y in $\mathcal{A}_l$. As ε_l is an $\mathcal{A}_l$-free generator of $\mathcal{E}_{N,l}$, we get $\bar{x}\bar{y} = 1$, hence $\bar{x}, \bar{y} \in \mathcal{A}_l^*$.

Now by (5.4) we obtain $(\bar{y}\varepsilon_l|\chi)_{N,l} = ((1-\gamma_0)^{k_l+1}u_l|\chi)_{N,l}$ and applying Corollary 4.3 and Remark 4.4 one has

$$\mathrm{Det}_\chi(\bar{y})(\varepsilon_l|\chi)_{N,l} = \mathrm{Det}_\chi(1-\gamma_0)^{k_l+1}(u_l|\chi)_{N,l},$$

hence

$$(\varepsilon_l|\chi)_{N,l} = \mathrm{Det}_\chi(\bar{x})\,\mathrm{Det}_\chi(1-\gamma_0)^{k_l+1}(u_l|\chi)_{N,l}.$$

Finally using (5.4) with $\mathrm{Det}_\chi(1-\gamma_0) = \mathrm{Det}_\chi(1-\tilde{\gamma}_0)$, and putting $\rho_l = \bar{x}$, we arrive at the required formula. ∎

In the proof of the next proposition we shall use results in which two types of cyclotomic units of an Abelian field appear.

First we define the group of *formal cyclotomic units* of N (see [12])

$$C_N = \langle \pm\mathcal{N}_{\mathbb{Q}(\zeta_{n_K})/K}(1-\zeta_{n_K}^a) : K \subseteq N,\ (a, n_K) = 1\rangle \cap E_N$$

where K runs over all nontrivial cyclic subfields of N, n_K denotes the conductor of K and $\langle : \rangle$ denotes the subgroup of N^* generated by elements and their conjugates. The second kind of cyclotomic units are the *Sinnott units* ([19]), which could be defined (see [15]) by

$$C_N' = \langle \pm\mathcal{N}_{\mathbb{Q}(\zeta_r)/\mathbb{Q}(\zeta_r)\cap N}(1-\zeta_r^a) : 1 < r \mid q, (a, r) = 1\rangle \cap E_N$$

where q is the conductor of N.

Generally these groups of units are not always equal but in the case of prime degree of $N/\mathbb{Q}$ we have $C_N' = C_N$. Indeed, since $n_K \mid q$, the inclusion $\supseteq$ follows from

$$\mathcal{N}_{\mathbb{Q}(\zeta_{n_K})/K}(1-\zeta_{n_K}^a) = \mathcal{N}_{\mathbb{Q}(\zeta_{n_K})\cap N/K}(\mathcal{N}_{\mathbb{Q}(\zeta_{n_K})/\mathbb{Q}(\zeta_{n_K})\cap N}(1-\zeta_{n_K}^a)).$$

To get the inverse inclusion note that $\mathbb{Q}(\zeta_r) \cap N$ is either N or $\mathbb{Q}$ and so the generators of C_N' are of two types

$$\mathcal{N}_{\mathbb{Q}(\zeta_q)/N}(1-\zeta_q^a) \quad \text{or} \quad \mathcal{N}_{\mathbb{Q}(\zeta_q)/\mathbb{Q}}(1-\zeta_q^a).$$

The former are also generators of C_N and the latter can only generate ± 1 so $C_N' \subseteq C_N$.

PROPOSITION 5.3. *Let $N/\mathbb{Q}$ be a tame real Abelian extension with odd prime degree l. For any odd prime number $p \neq l$ let $\mathcal{E}_{N,p}$ be an $\mathcal{A}_p$-free module with generator ε_p and let u_p be a $\mathbb{Z}_p\Gamma$-free generator of U_p^1. Let $\mathcal{E}_{N,2}/\mathrm{tor}_{\mathbb{Z}_2}(\mathcal{E}_{N,2})$ be an $\mathcal{A}_2$-free module with free generator $\varepsilon_2 \,\mathrm{tor}_{\mathbb{Z}_2}(\mathcal{E}_{N,2})$ and let $u_2 \in U_2^1$ be such that its image is a free $\mathbb{Z}_2\Gamma$-generator of $U_2^1/\mathrm{tor}_{\mathbb{Z}_2}(U_2^1)$. Then for any prime number $p \neq l$ and any $\chi \in \widehat{\Gamma} \setminus \{1_\Gamma\}$,*

$$\begin{aligned}(\varepsilon_p|\chi)_{N,p} &= (p^{d_{p,\chi^\delta}})_\delta (u_p|\chi)_{N,p} \,\mathrm{Det}_\chi(\rho_p), \quad \textit{where}\\ d_{p,\chi} &= \nu_p(L_p(1,\chi_*)) - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1) \quad \textit{for odd primes } p,\\ d_{2,\chi} &= \nu_2(L_2(1,\chi_*)) - 1 - \nu_2(h_2^\chi)/\Phi_{\chi,2}(1) \quad \textit{and} \quad \rho_p \in \mathcal{A}_p^*,\end{aligned}$$

$\Phi_{\chi,p}$ is a $\mathbb{Q}_p$-irreducible character of Γ with summand χ and h_p^χ is the order of the $\Phi_{\chi,p}$ component of the class group for N.

Proof. Let p be an odd prime and let u_p generate U_p^1. Let Ψ_p be an isomorphism $U_p^1 \cong \mathbb{Z}_p\Gamma$ such that $\Psi_p(u_p) = 1$.

Since $p \neq 2$ one has $\widetilde{\Gamma}\varepsilon_p = 1$ ($\mathcal{E}_{N,p} \subseteq (U_p^1)^0$) so we can put

$$\varepsilon_p = \sum_{\mu\neq 1_\Gamma} p^{r_\mu} t_\mu e_\mu u_p \quad \text{with } t_\mu \in \mathbb{Z}_p[\zeta_l]^* \text{ and integers } r_\mu \geq 0. \tag{5.5}$$

Hence we obtain

$$\Psi_p(\varepsilon_p) = \sum_{\mu\neq 1_\Gamma} p^{r_\mu} t_\mu e_\mu = \sum_{\mu\neq 1_\Gamma} p^{r_\mu} e_\mu \sum_{\mu\neq 1_\Gamma} t_\mu e_\mu. \tag{5.6}$$

Since $\sum_{\mu\neq 1_\Gamma} p^{r_\mu} t_\mu e_\mu \in \mathbb{Z}_p\Gamma$, it follows from Lemma 3.6 that $r_\mu = r_\vartheta$ are equal, and t_μ and t_ϑ are conjugate over $\mathbb{Q}_p(\zeta_l)$ provided the characters μ and ϑ are summands of the same irreducible character of Γ over $\mathbb{Q}_p$.

Thus for any $\mathbb{Q}_p$-irreducible character Υ of Γ and $\sigma \in \mathcal{G}al(\mathbb{Q}_p(\zeta_l)/\mathbb{Q}_p)$ one has $\sigma(\sum_{\mu|\Upsilon} t_\mu \bar{\mu}(\gamma)) = \sum_{\mu|\Upsilon} t_{\mu^\sigma} \bar{\mu^\sigma}(\gamma)$, whence $\sum_{\mu|\Upsilon} t_\mu\bar{\mu}(\gamma) \in \mathbb{Z}_p$. Consequently, we have

$$\sum_{\mu\neq 1_\Gamma} t_\mu e_\mu = \frac{1}{l}\sum_{\gamma\in\Gamma}\Big(\sum_{\mu\neq 1_\Gamma} t_\mu\bar{\mu}(\gamma)\Big)\gamma = \frac{1}{l}\sum_{\gamma\in\Gamma}\Big(\sum_{\Upsilon}\sum_{\mu|\Upsilon} t_\mu\bar{\mu}(\gamma)\Big)\gamma \in \mathbb{Z}_p\Gamma$$

where Υ runs over all nontrivial irreducible characters of Γ over $\mathbb{Q}_p$.

Since $\sum_{\mu\neq 1_\Gamma} t_\mu e_\mu \sum_{\mu\neq 1_\Gamma} t_\mu^{-1} e_\mu = 1 - (1/l)\widetilde{\Gamma}$ and $t_\mu^{-1} \in \mathbb{Z}_p[\zeta_l]$, the image of $\sum_{\mu\neq 1_\Gamma} t_\mu e_\mu$ in $\mathcal{A}_p$ is a unit in $\mathcal{A}_p^*$. Then according to (5.5) and Corollary 4.3,

$$\begin{aligned}(\varepsilon_p|\chi)_{N,p} &= \Big(\sum_{\mu\neq 1_\Gamma} p^{r_\mu} t_\mu e_\mu u_p\Big|\chi\Big)_{N,p} = \mathrm{Det}_\chi\Big(\sum_{\mu\neq 1_\Gamma} p^{r_\mu} t_\mu e_\mu\Big)(u_p|\chi)_{N,p}\\ &= (p^{r_{\chi^\delta}})_\delta (u_p|\chi)_{N,p}\,\mathrm{Det}_\chi(\rho_p),\end{aligned}$$

where $\rho_p = \sum_{\mu\neq 1_\Gamma} t_\mu e_\mu \in \mathcal{A}_p^*$. In order to calculate r_χ we apply Lemma 3.6

and (5.6) to obtain

$$(5.7)\qquad \nu_p(|e_{\Phi_{\chi,p}}(U_p^1/\mathcal{E}_{N,p})|) = \nu_p(|e_{\Phi_{\chi,p}}(\mathbb{Z}_p\Gamma/[\mathbb{Z}_p\Gamma\Psi_p(\varepsilon_p)])|) = r_\chi \Phi_{\chi,p}(1).$$

Let $\mathcal{C}_{N,p} \subseteq \mathcal{E}_{N,p}$ be the closure of $C_N \cap E_{N,p}^1$ embedded by d_p into U_p^1, i.e. $\mathcal{C}_{N,p} = d_p(\mathbb{Z}_p \otimes (C_N \cap E_{N,p}^1)) = d_p(\mathbb{Z}_p \otimes C_N)$. Note that the last equality is a consequence of $\mathbb{Z}_p \otimes E_N \cong \mathbb{Z}_p \otimes E_{N,p}^1$ ($a \otimes \eta = a/(p^{f_p}-1) \otimes \eta^{p^{f_p}-1} \in \mathbb{Z}_p \otimes E_{N,p}^1$ for any $a \in \mathbb{Z}_p$ and any $\eta \in E_N$).

In [12, p. 157] R. Greenberg (see also [10, Théorème 1]) proved that for any odd prime number p,

$$(5.8)\qquad \nu_p(|e_{\Phi_{\chi,p}}(U_p^1/\mathcal{C}_{N,p})|) = \nu_p\Big(\prod_{\mu|\Phi_{\chi,p}} L_p(1,\mu_*)\Big).$$

We also need a conjecture of G. Gras which, as shown by R. Greenberg in [12], is a consequence of the Main Conjecture proved by B. Mazur and A. Wiles in [17]. It states that $\mathcal{H}_p$ and the p-Sylow subgroup of E_N/C_N have isomorphic Jordan–Hölder series as $\mathbb{Z}_p\Gamma$-modules for $p > 2$. This is equivalent to

$$(5.9)\qquad |e_\Phi(E_N/C_N)_p| = |e_\Phi \mathcal{H}_p|$$

where Φ is an irreducible nontrivial character of Γ over $\mathbb{Q}_p$ and the subscript p indicates that we are dealing with the p-component.

Since $\mathcal{E}_{N,p} = d_p(\mathbb{Z}_p \otimes E_{N,p}^1) = d_p(\mathbb{Z}_p \otimes E_N)$ and d_p is injective (see Theorem 3.6.2(vi) in [11, Chapter III]) one has

$$(E_N/C_N)_p \cong \mathbb{Z}_p \otimes (E_N/C_N) \cong (\mathbb{Z}_p \otimes E_N)/(\mathbb{Z}_p \otimes C_N) \cong \mathcal{E}_{N,p}/\mathcal{C}_{N,p},$$

so by (5.9),

$$|e_\Phi(\mathcal{E}_{N,p}/\mathcal{C}_{N,p})| = |e_\Phi \mathcal{H}_p|.$$

This and (5.8) yield now

$$\nu_p(|e_{\Phi_{\chi,p}}(U_p^1/\mathcal{E}_{N,p})|) = \nu_p\Big(\prod_{\mu|\Phi_{\chi,p}} L_p(1,\mu_*)\Big) - \nu_p(|e_{\Phi_{\chi,p}}\mathcal{H}_p|).$$

Since all $\mu \in \widehat{\Gamma} \setminus \{1_\Gamma\}$ with $\mu \,|\, \Phi_{\chi,p}$ are conjugate over $\overline{\mathbb{Q}}_p$, we have $\mu = \chi^\sigma$ for some $\sigma \in \Omega = \mathcal{G}al(\mathbb{Q}_p(\zeta_{lq})/\mathbb{Q}_p)$ (recall that $F = \mathbb{Q}_p(\zeta_{lq})$). By Theorem 3.5 and $\tau(\chi^\sigma) = \chi^\sigma(\sigma_*)\sigma(\tau(\chi))$, where σ_* is an integer defined by $\sigma(\zeta_q) = \zeta_q^{\sigma_*}$, we get $L_p(1,\mu_*) = \sigma(L_p(1,\chi_*))$, which gives $\nu_p(L_p(1,\mu_*)) = \nu_p(L_p(1,\chi_*))$. Thus we obtain

$$\nu_p(|e_{\Phi_{\chi,p}}(U_p^1/\mathcal{E}_{N,p})|) = \Phi_{\chi,p}(1)\nu_p(L_p(1,\mu_*)) - \nu_p(h_p^\chi),$$

which together with (5.7) gives

$$r_\chi = \nu_p(L_p(1,\mu_*)) - \nu_p(h_p^\chi)/\Phi_{\chi,p}(1).$$

Finally after putting $d_{p,\chi} = r_\chi$ we arrive at the required formula for odd primes.

Now let $p=2$. Denote by $\varXi$ the isomorphism $U_2^1 \cong \mathbb{Z}_2\varGamma \oplus (\mathbb{Z}/2\mathbb{Z})^{g_2}$. Since $u_2 \operatorname{tor}_{\mathbb{Z}_2}(U_2^1)$ is a free $\mathbb{Z}_2\varGamma$-generator of $U_2^1/\operatorname{tor}_{\mathbb{Z}_2}(U_2^1)$, the first component of $\varXi(u_2)$ is a unit of $\mathbb{Z}_2\varGamma$. Thus we may assume, without loss of generality, that this component is 1.

As the torsion submodules of U_2^1 and $\mathcal{E}_{N,2}$ are of exponent 2, u_2^2 and ε_2^2 are generators of the submodules $(U_2^1)^2$ and $(\mathcal{E}_{N,2})^2$, respectively. By the same argument we can treat $\varXi((U_2^1)^2)$ and $\varXi((\mathcal{E}_{N,2})^2)$ as submodules of $\mathbb{Z}_2\varGamma$. Observe that $\widetilde{\varGamma}\varepsilon_2^2 = 1$, hence there exist $x_\mu \in \mathbb{Z}_2[\zeta_l]^*$ and integers $b_\mu \geq 0$ such that

$$\varepsilon_2^2 = \sum_{\mu \neq 1_\varGamma} 2^{b_\mu} x_\mu e_\mu u_2^2.$$

Hence we obtain

$$\varXi(\varepsilon_2^2) = \sum_{\mu \neq 1_\varGamma} 2^{b_\mu} x_\mu e_\mu = \sum_{\mu \neq 1_\varGamma} 2^{b_\mu} e_\mu \sum_{\mu \neq 1_\varGamma} x_\mu e_\mu \tag{5.10}$$

and arguing as in the case of odd p we get

$$(\varepsilon_2|\chi)_{N,2} = (2^{b_{\chi^\delta}})_\delta (u_2|\chi)_{N,2} \operatorname{Det}_\chi(\rho_2) \quad \text{for some } \rho_2 \in \mathcal{A}_2^*. \tag{5.11}$$

We also have

$$\begin{aligned} e_{\varPhi_{\chi,2}}[U_2^1/(\mathcal{E}_{N,2})^2] &\cong e_{\varPhi_{\chi,2}}[(\mathbb{Z}_2\varGamma \oplus (\mathbb{Z}/2\mathbb{Z})^{g_2})/\mathbb{Z}_2\varGamma\varXi(\varepsilon_2^2)] \\ &\cong e_{\varPhi_{\chi,2}}[\mathbb{Z}_2\varGamma/\mathbb{Z}_2\varGamma\varXi(\varepsilon_2^2)] \oplus e_{\varPhi_{\chi,2}}(\mathbb{Z}/2\mathbb{Z})^{g_2} \cong e_{\varPhi_{\chi,2}}[\mathbb{Z}_2\varGamma/\mathbb{Z}_2\varGamma\varXi(\varepsilon_2^2)] \end{aligned}$$

where the last isomorphism is a consequence of the trivial $\varGamma$-action on $(\mathbb{Z}/2\mathbb{Z})^{g_2}$. Now, using Lemma 3.6 and (5.10), we get

$$\nu_2(|e_{\varPhi_{\chi,2}}(U_2^1/(\mathcal{E}_{N,2})^2)|) = b_\chi \varPhi_{\chi,2}(1). \tag{5.12}$$

As $\operatorname{tor}_{\mathbb{Z}}(E_{N,2}^1) = \{\pm 1\}$ and $\mathbb{Z}_2 \otimes \{\pm 1\} = \{1 \otimes (\pm 1)\}$ one has $\operatorname{tor}_{\mathbb{Z}_2}(\mathcal{E}_{N,2}) = \{(\pm 1)\}$ and so, by the assumption of the proposition, $\mathcal{E}_{N,2}/\{(\pm 1)\} \cong \mathbb{Z}_2\varGamma/(\widetilde{\varGamma})$. Hence $\mathcal{E}_{N,2} \cong \mathbb{Z}_2\varGamma/(\widetilde{\varGamma}) \oplus (\mathbb{Z}/2\mathbb{Z})$ and since $e_{\varPhi_{\chi,2}}(\mathbb{Z}/2\mathbb{Z}) = \{0\}$ $(1_\varGamma \nmid \varPhi_{\chi,2})$, we obtain

$$\begin{aligned} e_{\varPhi_{\chi,2}}(\mathcal{E}_{N,2}/(\mathcal{E}_{N,2})^2) &\cong e_{\varPhi_{\chi,2}}\{[\mathbb{Z}_2\varGamma/(\widetilde{\varGamma}) \oplus (\mathbb{Z}/2\mathbb{Z})]/2(\mathbb{Z}_2\varGamma/(\widetilde{\varGamma}))\} \\ &\cong e_{\varPhi_{\chi,2}}\{\mathbb{Z}_2\varGamma/(\widetilde{\varGamma})/2(\mathbb{Z}_2\varGamma/(\widetilde{\varGamma}))\}. \end{aligned}$$

From $[\mathbb{Z}_2\varGamma/(\widetilde{\varGamma})]/2[\mathbb{Z}_2\varGamma/(\widetilde{\varGamma})] \cong \mathbb{Z}_2\varGamma/[2\mathbb{Z}_2\varGamma + (\widetilde{\varGamma})]$ and $e_{\varPhi_{\chi,2}}\widetilde{\varGamma} = 0$ we get

$$e_{\varPhi_{\chi,2}}\{[\mathbb{Z}_2\varGamma/(\widetilde{\varGamma})]/2[\mathbb{Z}_2\varGamma/(\widetilde{\varGamma})]\} \cong e_{\varPhi_{\chi,2}}(\mathbb{Z}_2\varGamma/2\mathbb{Z}_2\varGamma),$$

hence by Lemma 3.6, $|e_{\varPhi_{\chi,2}}(\mathcal{E}_{N,2}/(\mathcal{E}_{N,2})^2)| = 2^{\varPhi_{\chi,2}(1)}$, and (5.12) gives

$$\nu_2(|e_{\varPhi_{\chi,2}}(U_2^1/\mathcal{E}_{N,2})|) = (b_\chi - 1)\varPhi_{\chi,2}(1). \tag{5.13}$$

Let $\mathcal{C}_{N,2} \subseteq \mathcal{E}_{N,2}$ be the closure of $C_N \cap E_{N,2}^1$ in U_2^1.

Now we shall need formulas analogous to (5.8) and (5.9) for $p=2$. The first one comes from Gillard [10, Théorème 1 for $l=2$, $\Gamma=\{1\}$] and gives

$$\nu_2(|e_{\Phi_{\chi,2}}(U^1_{N,2}/\mathcal{C}_{N,2})|)=\nu_2\Big(\prod_{\mu|\Phi_{\chi,2}}L_2(1,\mu_*)\Big)-\Phi_{\chi,2}(1).$$

The second

$$|e_{\Phi_{\chi,2}}(E_N/C'_N)_2|=|e_{\Phi_{\chi,2}}\mathcal{H}_2|2^{\Phi_{\chi,2}(1)}$$

is a special case of the formula in [13, Theorem 4.15]. This theorem is applied for $p=2$, $\Delta_2=1$ where $d(\chi')=\Phi_{\chi,2}(1)$ (in the notation of [13]) and $|(R:U)_{\chi'}|=1$ because $2\nmid(R:U)$ (Proposition 5.1 in [19]). As in our case $C'_N=C_N$ we can replace C'_N by C_N in this formula.

Applying these formulas together with (5.11), (5.13) and proceeding as for odd primes we obtain the asserted formula for $p=2$. ∎

LEMMA 5.4. *Let p be a prime number and let $\Gamma=\langle\gamma_0\rangle$ be a cyclic group of prime order $l\neq p$. Put $e_\chi=(1/l)\sum_{\gamma\in\Gamma}\chi(\gamma)\gamma^{-1}$, $e_1=(1/l)\widetilde{\Gamma}$ and $e_0=1-e_1$ for $\chi\in\widehat{\Gamma}\setminus\{1_\Gamma\}$.*

(i) *If I is an ideal of $\mathbb{Z}_p\Gamma$ such that $I\subseteq(\mathbb{Z}_p\Gamma)^0$ and* $\operatorname{rank}_{\mathbb{Z}_p}(I)=l-1$, *then*

$$I=\begin{cases}\bigoplus_{\chi\neq 1_\Gamma}p^{r_\chi}e_\chi\mathbb{Z}_p & \text{in case } l\,|\,p-1,\\ \mathbb{Z}_p\Gamma p^r e_0 & \text{in case } l\nmid p-1.\end{cases}$$

(ii) *If I is an ideal of $\mathbb{Z}_p\Gamma$ such that* $\operatorname{rank}_{\mathbb{Z}_p}(I)=l$, *then*

$$I=\begin{cases}\bigoplus_{\chi}p^{r_\chi}e_\chi\mathbb{Z}_p & \text{in case } l\,|\,p-1,\\ \mathbb{Z}_p\Gamma p^r e_0\oplus\mathbb{Z}_p\Gamma p^h e_1 & \text{in case } l\nmid p-1,\end{cases}$$

where r_χ, r and h are nonnegative integers.

Proof. (i) In case $l\,|\,p-1$ we have $\zeta_l\in\mathbb{Z}_p$ and hence $(\mathbb{Z}_p\Gamma)^0=\bigoplus_{\chi\neq 1_\Gamma}e_\chi\mathbb{Z}_p$.

Let I be a nontrivial ideal of $(\mathbb{Z}_p\Gamma)^0$ with $\operatorname{rank}_{\mathbb{Z}_p}(I)=l-1$. Then $\mathbb{Q}_pI=\bigoplus_{\chi\neq 1_\Gamma}e_\chi\mathbb{Q}_p$ and so, for any $\chi\neq 1_\Gamma$, there exists a nonnegative integer r_χ such that $p^{r_\chi}e_\chi\in I$. Choose r_χ to be minimal.

Suppose $x\in I$ and $x=\sum_{\chi\neq 1_\Gamma}a_\chi e_\chi$ with $a_\chi\in\mathbb{Z}_p$. Since, for each $\mu\neq 1_\Gamma$, $xe_\mu\in I$ and $xe_\mu=a_\mu e_\mu$, we have $p^{r_\mu}\,|\,a_\mu$. Thus $x\in\bigoplus_{\chi\neq 1_\Gamma}p^{r_\chi}e_\chi\mathbb{Z}_p$, so $I\subseteq\bigoplus_{\chi\neq 1_\Gamma}p^{r_\chi}e_\chi\mathbb{Z}_p$. As the opposite inclusion is obvious, the first case of (i) follows.

In the case $l\nmid p-1$, $\mathbb{Z}_p\Gamma e_0=\{\sum_{j=0}^{l-1}a_j\gamma_0^j\in\mathbb{Z}_p\Gamma:\sum_{j=0}^{l-1}a_j=0\}$, so the map defined by $\sum_{j=0}^{l-1}a_j\gamma_0^j\mapsto\sum_{j=0}^{l-1}a_j\zeta_l^j$ establishes a $\mathbb{Z}_p\Gamma$-isomorphism

$\mathbb{Z}_p\Gamma e_0 \cong \mathbb{Z}_p[\zeta_l]$. As any submodule of $\mathbb{Z}_p[\zeta_l]$ is of the form $\mathbb{Z}_p[\zeta_l]p^r$ it follows that each ideal of $\mathbb{Z}_p\Gamma$ contained in $(\mathbb{Z}_p\Gamma)^0$ has the desired form.

(ii) If $l \mid p-1$, then it suffices to apply the same argument as in the proof of (i). If $l \nmid p-1$, then $\mathbb{Q}_p I = \mathbb{Q}_p\Gamma e_0 \oplus \mathbb{Q}_p e_1$, where e_0 and e_1 are primitive idempotents of $\mathbb{Q}_p\Gamma$. Observe that there exists a nonnegative integer h such that $p^h e_1 \in I$ and choose h to be minimal. Let $x \in I$. As $I \subseteq \mathbb{Z}\Gamma_p = \mathbb{Z}_p\Gamma e_0 \oplus \mathbb{Z}_p e_1$, there exist $a_0 \in \mathbb{Z}_p\Gamma$ and $a_1 \in \mathbb{Z}_p$ such that $x = a_0e_0 + a_1e_1$. Since $xe_1 = a_1e_1 \in I$, we then have $p^h \mid a_1$ and therefore $x \in I^0 \oplus \mathbb{Z}_p p^h e_1$. As $\mathrm{rank}_{\mathbb{Z}_p}(I^0) = l-1$ it follows from (i) that $I \subseteq \mathbb{Z}_p\Gamma p^r e_0 \oplus \mathbb{Z}_p p^h e_1$. This completes the proof of (ii) because the opposite inclusion is obvious. ∎

PROPOSITION 5.5. *Let $N/\mathbb{Q}$ be a tame real Abelian extension of odd prime degree l. Let a_p be a free $\mathbb{Z}_p\Gamma$-generator of $\mathbb{Z}_p \otimes \mathcal{O}_N$ and for any prime $p \neq 2$ let u_p be a free $\mathbb{Z}_p\Gamma$-generator of U_p^1. Let $u_2 \in U_2^1$ be such that its image is a free $\mathbb{Z}_2\Gamma$-generator of $U_2^1/\mathrm{tor}_{\mathbb{Z}_p}(U_2^1)$. Then for any prime p and any $\chi \in \widehat{\Gamma} \setminus \{1_\Gamma\}$ there exists $w_p \in \mathcal{A}_p^*$ such that*

$$(u_p|\chi)_{N,p} = p^{s_p}[a_p|\chi]_{N,p}\,\mathrm{Det}_\chi(w_p)$$

where $s_p = 1$ with two exceptions:

- *$s_p = 0$ when $g_p = f_p = 1$ and $l \mid p-1$,*
- *$s_2 = 2$ when $g_2 = l$ or $f_2 = 1$.*

Proof. Assume first $g_p = l$. Then $U_p^1 = \prod_{\gamma\in\Gamma}(1+p\mathbb{Z}_p)$ and $\mathbb{Z}_p \otimes \mathcal{O}_N = \prod_{\gamma\in\Gamma}\mathbb{Z}_p$. Consider the case $p \neq 2$ and let Ψ be the isomorphism $U_p^1 \cong \mathbb{Z}_p\Gamma$ defined in Proposition 5.1.

Put $\Psi(u_p) = \sum_{\gamma\in\Gamma} w_\gamma\gamma$ and $u_p = (u_\gamma)_\gamma$. Observe that $\Psi(u_p)$ is the image of a free generator, so it lies in $(\mathbb{Z}_p\Gamma)^*$. Thus

$$\Psi(u_p)e_\chi = \frac{1}{p}\sum_{\gamma\in\Gamma}\log_p(u_\gamma)\gamma^{-1}e_\chi = \frac{1}{p}\sum_{\gamma\in\Gamma}\bar{\chi}(\gamma)\log_p(u_\gamma)e_\chi$$

but $\Psi(u_p)e_\chi = \sum_{\gamma\in\Gamma} w_\gamma\chi(\gamma)e_\chi$. Therefore

$$\frac{1}{p}\sum_{\gamma\in\Gamma}\bar{\chi}(\gamma)\log_p(u_\gamma) = \sum_{\gamma\in\Gamma} w_\gamma\chi(\gamma)$$

and by (5.2) we get

$$(u_p|\chi)_{N,p} = (\chi^{\check{t}s}(t))_{t,s}\Big(p\sum_{\gamma\in\Gamma} w_\gamma\chi^{\check{t}s}(\gamma)\Big)_{\check{t}s} = p\,\mathrm{Det}_\chi(w)(\chi^{\check{t}s}(t))_{\check{t}s},$$

where $w = \sum_{\gamma\in\Gamma} w_\gamma\tilde{\gamma} \in \mathcal{A}_p^*$.

On the other hand for $a_p = (a_\gamma)_\gamma \in \prod_{\gamma\in\Gamma}\mathbb{Z}_p$, using arguments as above and the isomorphism $\prod_{\gamma\in\Gamma}\mathbb{Z}_p \cong \mathbb{Z}_p\Gamma$ given by $(a_\gamma)_\gamma \mapsto \sum_{\gamma\in\Gamma} a_\gamma\gamma^{-1}$ we obtain

$$[a_p|\chi]_{N,p} = (\chi^{\check{t}s}(t))_{\check{t}s}\Big(\sum_{\gamma\in\Gamma} a_\gamma \bar{\chi}^{\check{t}s}(\gamma)\Big)_{\check{t}s} = \mathrm{Det}_\chi(\alpha)(\chi^{\check{t}s}(t))_{\check{t}s}$$

with $\alpha = \sum_{\gamma\in\Gamma} a_\gamma\tilde{\gamma} \in \mathcal{A}_p^*$.

Thus $(u_p|\chi)_{N,p} = p[a_p|\chi]_{N,p}\,\mathrm{Det}_\chi(w\alpha^{-1})$, i.e. $s_p = 1$, which shows the required formula in the case under consideration.

In the case $p = 2$ and $g_2 = l$ let $\Psi_2\colon \prod_{\gamma\in\Gamma}\mathbb{Z}_2 \to \mathbb{Z}_2\Gamma$ be the $\mathbb{Z}_2\Gamma$-homomorphism defined by

$$\Psi_2 : (u_\gamma)_\gamma \mapsto \frac{1}{4}\sum_{\gamma\in\Gamma}\log_2(u_\gamma)\gamma^{-1} \quad \text{where } u_\gamma \in 1+2\mathbb{Z}_2.$$

As $\log_2(1+2\mathbb{Z}_2) = \log_2(\{\pm1\}(1+4\mathbb{Z}_2)) = 4\mathbb{Z}_2$, Ψ_2 is surjective and its kernel is $\mathrm{tor}_{\mathbb{Z}_2}(U_2^1)$.

Since, by assumption, $u_2\,\mathrm{tor}_{\mathbb{Z}_2}(U_2^1)$ is a free $\mathbb{Z}_2\Gamma$-generator of $U_2^1/\mathrm{tor}_{\mathbb{Z}_2}(U_2^1)$, $\Psi_2(u_2)$ is a unit of $\mathbb{Z}_2\Gamma$. Indeed, let $u \in U_2^1$ be such that $\Psi_2(u) = 1$. Then there is $\beta \in \mathbb{Z}_2\Gamma$ such that $u\ker\Psi_2 = (\beta u_2)\ker\Psi_2$, whence $\beta\Psi_2(u_2) = \Psi_2(u) = 1$. Now proceeding as in the case $g_p = l$ and $p \neq 2$ with $\Psi_2(u_2)$ we get

$$(u_2|\chi)_{N,2} = 4[a_2|\chi]_{N,2}\mathrm{Det}_\chi(w_2) \quad \text{for some } w_2 \in \mathcal{A}_2^*, \text{ i.e. } s_2 = 2.$$

Now suppose that $g_p = 1$, $p \neq 2$ and note that $\mathbb{Z}_p \otimes \mathcal{O}_N = \mathcal{O}_\mathfrak{p}$ and $U_p^1 = U_\mathfrak{p}^1$ with $\mathfrak{p}$ being the only prime ideal of $\mathcal{O}_N$ above p.

First assume that $f_p = l$ and $p \neq 2$. Then $e_p = 1$, so $1 > e_p/(p-1)$ and the map $(1/p)\log_p : U_p^1 \to \mathcal{O}_\mathfrak{p}$ is a $\mathbb{Z}_p\Gamma$-isomorphism. Consequently, $(1/p)\log_p(u_p)$ is a free $\mathbb{Z}_p\Gamma$-generator of the ring of integers $\mathcal{O}_\mathfrak{p}$ so $\log_p(u_p) = px_pa_p$ for some $x_p \in (\mathbb{Z}_p\Gamma)^*$, and hence by (5.1),

$$(u_p|\chi)_{N,p} = (u_p|\chi^s)_s = p[a_p|\chi^s]_s\,\mathrm{Det}_\chi(x_p), \tag{5.14}$$

so $s_p = 1$.

In order to examine the case $f_p = 1$ and $p \neq 2$ assume first that $l \nmid p-1$. In this case let Ψ_0 denote a $\mathbb{Z}_p\Gamma$-isomorphism $U_p^1 \to \mathbb{Z}_p\Gamma$. Then, by Lemma 5.4(ii), we obtain

$$\Psi_0(U_p^m) = \mathbb{Z}_p\Gamma p^{r_m}e_0 \oplus \mathbb{Z}_p p^{h_m}e_1 = \mathbb{Z}_p\Gamma(p^{r_m}e_0 + p^{h_m}e_1)$$

for integers $m \geq 1$. Note that $r_1 = h_1 = 0$ and $r_{m+1} \geq r_m$, $h_{m+1} \geq h_m$.

As $|U_p^j/U_p^{j+1}| = p^{f_p}$ for integers $j \geq 1$, we have

$$|U_p^1/U_p^m| = p^{f_p(m-1)}. \tag{5.15}$$

On the other hand

$$|U_p^1/U_p^m| = |\mathbb{Z}_p\Gamma/\Psi_0(U_p^m)| = |\mathbb{Z}_p\Gamma/(\mathbb{Z}_p\Gamma p^{r_m}e_0 \oplus \mathbb{Z}_p p^{h_m}e_1)| = p^{(l-1)r_m+h_m},$$

whence by (5.15),

$$(l-1)r_m + h_m = f_p(m-1) = m-1. \tag{5.16}$$

Since $\Psi_0(u_p)(p^{r_m}e_0+p^{h_m}e_1)$ is a generator of $\Phi_0(U_p^m)$, it follows that $u_{p,m}=(p^{r_m}e_0+p^{h_m}e_1)u_p$ is a generator of U_p^m. Note that $u_{p,1}=u_p$. Consequently, for any $\chi\neq 1_\Gamma$ we have $(u_{p,m}|\chi)_{N,p}=\mathrm{Det}_\chi(p^{r_m}e_0+p^{h_m}e_1)(u_p|\chi)_{N,p}=p^{r_m}(u_p|\chi)_{N,p}$ because $\mathrm{Det}_\chi(e_1)=0$. Thus

$$(u_{p,m}|\chi)_{N,p}=p^{r_m}(u_p|\chi)_{N,p}. \tag{5.17}$$

Now we put $m=2$ in (5.16), whence $(l-1)r_2+h_2=1$ and so $r_2=0$, $h_2=1$. By the formula $|U_p^j/U_p^{j+1}|=p$ for integers $j\geq 1$ one has $(l-1)(r_{j+1}-r_j)+h_{j+1}-h_j=1$, whence $r_{j+1}=r_j$ and $h_{j+1}=h_j+1$. This implies $r_m=0$ and consequently for $m=l$, by (5.17), we obtain $(u_p|\chi)_{N,p}=(u_{p,l}|\chi)_{N,p}$. Since $e_p/(p-1)=l/(p-1)<l$, the map $(1/p)\log_p:U_p^l\to\mathcal{O}_\mathfrak{p}$ is a $\mathbb{Z}_p\Gamma$-isomorphism and thus $(1/p)\log_p(u_{p,l})$ is a free generator of $\mathcal{O}_\mathfrak{p}$, giving the formula (5.14), i.e. $s_p=1$.

Now let $l\,|\,p-1$. Let Θ be a $\mathbb{Z}_p\Gamma$-isomorphism $\mathcal{O}_\mathfrak{p}\cong\mathbb{Z}_p\Gamma$ such that $\Theta(a_p)=1$ and let $\widehat{\mathfrak{p}}$ denote the closure of $\mathfrak{p}$ in $\mathcal{O}_\mathfrak{p}$. Since $\zeta_l\in\mathbb{Z}_p$, Lemma 5.4 gives

$$\Theta(\widehat{\mathfrak{p}})=\bigoplus_{\chi\in\hat\Gamma}\mathbb{Z}_p p^{t_\chi}e_\chi=\mathbb{Z}_p\Gamma\sum_{\chi\in\hat\Gamma}p^{t_\chi}e_\chi\quad\text{ with some integers } t_\chi\geq 0.$$

This and $|\mathcal{O}_\mathfrak{p}/\widehat{\mathfrak{p}}|=p$ imply

$$\sum_{\chi\in\hat\Gamma}t_\chi=1. \tag{5.18}$$

On the other hand as $e_p/(p-1)<1$ the mapping $\log_p$ establishes a $\mathbb{Z}_p\Gamma$-isomorphism $U_p^1\cong\widehat{\mathfrak{p}}$ and thus $\Theta(\log_p(u_p))$ is a $\mathbb{Z}_p\Gamma$-free generator of $\Theta(\widehat{\mathfrak{p}})$. This means that $\Theta(\log_p(u_p))=x_p\sum_{\chi\in\hat\Gamma}p^{t_\chi}e_\chi$ for some $x_p\in(\mathbb{Z}_p\Gamma)^*$ and consequently $\log_p(u_p)=x_p\sum_{\chi\in\hat\Gamma}p^{t_\chi}e_\chi a_p$. Thus

$$(u_p|\chi)=[\log_p(u_p)|\chi]=p^{t_\chi}\,\mathrm{Det}_\chi(x_p)[a_p|\chi]. \tag{5.19}$$

By (5.18) in order to show that all t_χ are equal to 0 for $\chi\in\hat\Gamma\setminus\{1_\Gamma\}$ it suffices to prove that $t_{1_\Gamma}\geq 1$. To this end note that $\widetilde\Gamma\Theta(\widehat{\mathfrak{p}})=\mathbb{Z}_p p^{t_{1_\Gamma}}e_{1_\Gamma}$ and $\widetilde\Gamma\,\widehat{\mathfrak{p}}\subseteq\widehat{\mathfrak{p}}\cap\mathbb{Z}_p=p\mathbb{Z}_p$, whence $\Theta(\widetilde\Gamma\,\widehat{\mathfrak{p}})=\mathbb{Z}_p p^{t_{1_\Gamma}}e_{1_\Gamma}\subseteq p\mathbb{Z}_p\Gamma$, and so $t_{1_\Gamma}\geq 1$, showing that $s_p=0$.

Now we consider the case $p=2$ and $g_2=1$. For any integer $m\geq 1$ put $\overline{U}_2^m=(U_2^m\{\pm1\})/\{\pm1\}$ and observe that $\overline{U}_2^1=U_2^1/\{\pm1\}$ with $\overline{u}_2=u_2\{\pm1\}$ as free $\mathbb{Z}_2\Gamma$-generator. Let $\overline{\Psi}_2$ denote a $\mathbb{Z}_2\Gamma$-isomorphism $\overline{U}_2^1\cong\mathbb{Z}_2\Gamma$. Then by Lemma 5.4(ii) (the case $l\nmid p-1$),

$$\overline{\Psi}_2(\overline{U}_2^m)=\mathbb{Z}_2\Gamma 2^{r_m}e_0\oplus\mathbb{Z}_2 2^{h_m}e_1=\mathbb{Z}_2\Gamma(2^{r_m}e_0+2^{h_m}e_1),\qquad m\geq 1,$$

and again as in the case $p\neq 2$, $r_1=h_1=0$ and $r_{m+1}\geq r_m$, $h_{m+1}\geq h_m$.

Note that

$$|\overline{U}_2^1/\overline{U}_2^m| = |U_2^1/U_2^m\{\pm 1\}|,$$

and $-1 \in U_2^m$ if and only if $1 \le m \le e_2$. Since the kernel of the homomorphism $U_2^1/U_2^m \to U_2^1/U_2^m\{\pm 1\}$ is nontrivial and equal to $\{\pm 1 \cdot U_2^m\}$ precisely when $-1 \notin U_2^m$, by (5.15) (also true for $p = 2$) we have

$$|\overline{U}_2^1/\overline{U}_2^m| = \begin{cases} 2^{f_2(m-1)}, & 1 \le m \le e_2, \\ 2^{f_2(m-1)-1}, & m > e_2. \end{cases}$$

On the other hand

$$|\overline{U}_2^1/\overline{U}_2^m| = |\mathbb{Z}_2\Gamma/\overline{\Psi}_2(\overline{U}_2^m)| = |\mathbb{Z}_2\Gamma/(\mathbb{Z}_2\Gamma 2^{r_m}e_0 \oplus \mathbb{Z}_2 2^{h_m}e_1)| = 2^{(l-1)r_m+h_m},$$

whence

$$(5.20) \qquad (l-1)r_m + h_m = \begin{cases} f_2(m-1), & 1 \le m \le e_2, \\ f_2(m-1)-1, & m > e_2. \end{cases}$$

Since $\overline{\Psi}_2(\overline{u}_2)(2^{r_m}e_0 + 2^{h_m}e_1)$ is a generator of $\overline{\Psi}_2(\overline{U}_2^m)$, it follows that $u_{2,m}\{\pm 1\}$ is a generator of $\overline{U}_2^m$, where $u_{2,m} = (2^{r_m}e_0 + 2^{h_m}e_1)u_2$. Note that $u_{2,1}\{\pm 1\} = \overline{u}_2$. Consequently, for any $\chi \ne 1_\Gamma$ we have $(u_{2,m}|\chi)_{N,2} = \mathrm{Det}_\chi(2^{r_m}e_0 + 2^{h_m}e_1)(u_2|\chi)_{N,2} = 2^{r_m}(u_2|\chi)_{N,2}$ because $\mathrm{Det}_\chi(e_1) = 0$. Thus

$$(5.21) \qquad (u_{2,m}|\chi)_{N,2} = 2^{r_m}(u_2|\chi)_{N,2}.$$

To consider the case $p = 2$ and $f_2 = l$ we put $m = 2$ in (5.20). As $e_2 = 1$ we obtain $(l-1)r_2 + h_2 = l - 1$, whence $r_2 \in \{0, 1\}$.

We shall prove that $r_2 = 1$. Since the extension $N_{\mathfrak{p}}/\mathbb{Q}_2$ is unramified, the norm map is surjective on U_2^2, i.e. $\widetilde{\Gamma} U_2^2 = 1 + 2^2\mathbb{Z}_2$. The above and $1 + 2\mathbb{Z}_2 = (1 + 4\mathbb{Z}_2)\{\pm 1\}$ imply

$$\widetilde{\Gamma}\overline{U}_2^1 = (\widetilde{\Gamma}U_2^1)/\{\pm 1\} = (1+2\mathbb{Z}_2)/\{\pm 1\} = ((1+4\mathbb{Z}_2)\{\pm 1\})/\{\pm 1\} = \widetilde{\Gamma}\overline{U}_2^2$$

and so $\overline{\Psi}_2(\widetilde{\Gamma}\overline{U}_2^1) = \overline{\Psi}_2(\widetilde{\Gamma}\overline{U}_2^2)$. Thus from $\widetilde{\Gamma}\overline{\Psi}_2(\overline{U}_2^1) = \mathbb{Z}_2 e_1$ and $\widetilde{\Gamma}\overline{\Psi}_2(\overline{U}_2^2) = \mathbb{Z}_2 2^{h_2} e_1$ we infer that $h_2 = 0$, whence $r_2 = 1$ and so by (5.21), $(u_{2,2}|\chi)_{N,2} = 2(u_2|\chi)_{N,2}$.

Because $-1 \notin U_2^2$ ($2 > e_2$), we have $\overline{U}_2^2 \cong U_2^2$ and $u_{2,2}$ is a free $\mathbb{Z}_2\Gamma$-generator of U_2^2. Since $e_2 = 1$, it follows that $\frac{1}{4}\log_2$ is a $\mathbb{Z}_2\Gamma$-isomorphism $U_2^2 \cong \mathcal{O}_2$. Therefore

$$[\log_2(u_{2,2})|\chi]_{N,2} = 4[a_2|\chi]_{N,2}\,\mathrm{Det}_\chi(y) \quad \text{for some } y \in (\mathbb{Z}_2\Gamma)^*$$

and so by $[\log_2(u_{2,2})|\chi]_{N,2} = (u_{2,2}|\chi)_{N,2} = 2(u_2|\chi)_{N,2}$ we have

$$(u_2|\chi)_{N,2} = 2[a_2|\chi]_{N,2}\,\mathrm{Det}_\chi(y) \quad \text{for some } y \in (\mathbb{Z}_2\Gamma)^*,$$

proving that $s_2 = 1$ for $f_2 = l$.

In the case $f_2 = 1$ and $e_2 = l$ we put $m = 2$ in (5.20) to get $(l-1)r_2 + h_2 = 1$, whence $r_2 = 0$, $h_2 = 1$. It turns out that $r_j = 0$ for each j. In order to prove this we apply (5.20) for $l \ge j \ge 2$ and obtain $(l-1)r_j + h_j = j - 1$, whence $(l-1)(r_j-r_{j-1})+h_j-h_{j-1}=1$, which implies $r_j=0$ and $h_j=j-1$ for $l\ge j\ge 2$. For $j\ge l+1$ we have $(l-1)r_j+h_j=j-2$, which gives $(l-1)r_{l+1}+h_{l+1}=l-1$ and so $r_{l+1}=0$ because $h_{l+1}\ge h_l=l-1$. After subtracting suitable equations we obtain $(l-1)(r_{j+1}-r_j)+h_{j+1}-h_j=1$ for $j\ge l+1$, proving that $r_j=0$ for $j\ge l+1$.

Now by putting $m=2l$ in (5.21) we obtain $(u_{2,2l}|\chi)_{N,2}=(u_2|\chi)_{N,2}$. Since $-1\notin U_2^{2l}$ $(2l>e_2)$, we get $\overline{U}_2^{2l}\cong U_2^{2l}$ and $u_{2,2l}$ is a free $\mathbb{Z}_2\Gamma$-generator of U_2^{2l}.

On the other hand in our case $2l>e_2$ and so $\log_2$ establishes an isomorphism $U_2^{2l}\cong\widehat{\mathfrak{p}}^{2l}=4\mathcal{O}_{\mathfrak{p}}$. Therefore

$$[\log_2(u_{2,2l})|\chi]_{N,2}=4[a_2|\chi]_{N,2}\,\mathrm{Det}_\chi(x)\quad\text{ for some } x\in(\mathbb{Z}_2\Gamma)^*,$$

whence

$$(u_2|\chi)_{N,2}=4[a_2|\chi]_{N,2}\,\mathrm{Det}_\chi(x)\quad\text{ for some } x\in(\mathbb{Z}_2\Gamma)^*,$$

showing that $s_2=2$ for $f_2=1$.

Finally if $p=l$, then $e_p=1$ as $N/\mathbb{Q}$ is tame, $e_p/(p-1)<1$ and so the map $(1/p)\log_p:U_p^1\to\mathcal{O}_{\mathfrak{p}}$ is a $\mathbb{Z}_p\Gamma$-isomorphism. Consequently, $(1/p)\log_p(u_p)$ is a free generator of $\mathcal{O}_{\mathfrak{p}}$ and proceeding as in the preceding cases we obtain

$$(u_p|\chi)_{N,p}=p[a_p|\chi^s]_s\,\mathrm{Det}_\chi(x_p)\quad\text{ for some } x_p\in(\mathbb{Z}_p\Gamma)^*,$$

completing the proof of our proposition. ∎

Now we can formulate the main theorem of this section.

THEOREM 5.6. *Let N be a tame real cyclic field of prime degree $l>2$ and let γ_0 generate Γ. For any prime $p\ne l$ put*

$$\omega_{p,\chi}=p^{d_{p,\chi}+s_p},\qquad \omega_{l,\chi}=\mathrm{Det}_\chi(1-\tilde{\gamma}_0)^{m_l}l,\qquad \omega_{\infty,\chi}=1,$$

where $d_{p,\chi}$, s_p, and m_l are from Propositions 5.2, 5.3 *and* 5.5. *Then the map*

$$\chi\mapsto\tau(N/\mathbb{Q},\chi)\bigg(\frac{[(\chi_*(p)/p-1)\tau(\chi_*)]^\delta}{L_p(1,\chi_*^\delta)}\omega_{p,\chi^\delta}\bigg)_{\delta,p}\qquad\text{with } \delta\in\check{T}S$$

is a representative of $(\overline{E}_N)_{\mathcal{A}}$ in

$$\mathrm{Hom}_\Omega(R'_\Gamma,\mathfrak{J}(F))/[\mathrm{Hom}_\Omega(R'_\Gamma,F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))],$$

where $\tau(N/\mathbb{Q},\chi)$ is the Galois Gauss sum and χ_* is the primitive character for χ.

The definition and the properties of Galois Gauss sum can be found in [7]. We recall that for $p=\infty$ we put $\chi_*(p)/p-1=-1$ and $L_\infty(s,\chi_*)=L(s,\chi_*)$.

Proof. Note that as $\mathcal{A}\cong\mathbb{Z}[\zeta_l]$ is a maximal order it follows that $\overline{E}_N$ is $\mathcal{A}$-locally free, i.e. for any prime p, $\mathbb{Z}_p\otimes\overline{E}_N$ and $\mathbb{R}\otimes\overline{E}_N$ are $\mathcal{A}_p$-free and $\mathcal{A}_\infty$-free respectively. Since N is tame, Noether's theorem implies that $\mathcal{O}_N$ is $\mathbb{Z}$-locally free, i.e. $\mathbb{Z}_p\otimes\mathcal{O}_N$ and $\mathbb{R}\otimes\mathcal{O}_N$ are $\mathbb{Z}_p\Gamma$-free and $\mathbb{R}\Gamma$-free respectively.

To apply Theorem 4.6 we shall prove, for each prime p, the existence of $\varepsilon_p \in (\mathbb{Z}_p \otimes E_N)^0$ whose image is a free $\mathcal{A}_p$-generator of $\mathbb{Z}_p \otimes \overline{E}_N$.

Assume first that p is odd. Then

$$\mathbb{Z}_p \otimes \overline{E}_N \cong (\mathbb{Z}_p \otimes E_N)/(\mathbb{Z}_p \otimes \{\pm 1\}) \cong \mathbb{Z}_p \otimes E_N$$

and $(\mathbb{Z}_p \otimes E_N)^0 = \mathbb{Z}_p \otimes E_N$ because for any $a \otimes \eta \in \mathbb{Z}_p \otimes E_N$ one has $\widetilde{\Gamma}(a \otimes \eta) = a \otimes (\pm 1) = (a/2) \otimes 1 = 1 \otimes 1$. Therefore there is $\varepsilon_p \in (\mathbb{Z}_p \otimes E_N)^0$ whose image is a free $\mathcal{A}_p$-generator of $\mathbb{Z}_p \otimes \overline{E}_N$ for odd p.

For $p = 2$ we have $\mathbb{Z}_2 \otimes \{\pm 1\} = \{1 \otimes (\pm 1)\}$ and so $\mathbb{Z}_2 \otimes \overline{E}_N \cong \mathbb{Z}_2 \otimes E_N/\{1 \otimes (\pm 1)\}$. Let $\varepsilon_2 \in \mathbb{Z}_2 \otimes E_N$ be such that $\varepsilon_2 \bmod \{1 \otimes (\pm 1)\}$ is a free $\mathcal{A}_2$-generator of $\mathbb{Z}_2 \otimes E_N/\{1 \otimes (\pm 1)\}$. Then $\widetilde{\Gamma}\varepsilon_2 \in \{1 \otimes (\pm 1)\}$ and if we had $\widetilde{\Gamma}\varepsilon_2 = 1 \otimes (-1) \neq 1 \otimes 1$, then we can take $\varepsilon_2(1 \otimes (-1))$ which belongs to $(\mathbb{Z}_2 \otimes E_N)^0$ (because $|\Gamma|$ is odd) and its image generates $\mathbb{Z}_2 \otimes \overline{E}_N$.

In the infinite case one has $\mathbb{R} \otimes E_N \cong \mathbb{R} \otimes \overline{E}_N$. Then we put $\varepsilon_\infty = 1 \otimes \varepsilon_0 \in \mathbb{R} \otimes E_N = (\mathbb{R} \otimes E_N)^0$, where ε_0 and its conjugates generate a subgroup of finite index in E_N.

Thus according to Theorem 4.6 for the above defined ε_p we infer that the map

$$\chi \mapsto ((\varepsilon_p)_p|\chi)_N \left(\frac{[(\chi_*(p)/p - 1)\tau(\chi_*)]^\delta}{L_p(1, \chi_*^\delta)} \right)_{\delta,p} \quad \text{with } \delta \in \check{T}_p S_p$$

represents the class of $\overline{E}_N$.

Observe that $d_p(\varepsilon_p)$ is a free $\mathcal{A}_p$-generator of $\mathcal{E}_{N,p}$ for odd or infinite prime p, and $d_2(\varepsilon_2)\{(\pm 1)\}$ is a free $\mathcal{A}_2$-generator of $\mathcal{E}_{N,2}/\{(\pm 1)\}$. By definition $(\varepsilon_p|\chi)_{N,p} = (d_p(\varepsilon_p)|\chi)_{N,p}$ and after applying Propositions 5.2 and 5.3 for $d_p(\varepsilon_p)$ and $d_2(\varepsilon_2)\{(\pm 1)\}$ we obtain

$$(\varepsilon_l|\chi)_{N,l} = \mathrm{Det}_\chi(1 - \tilde{\gamma}_0)^{m_l}(u_l|\chi)_{N,l}\,\mathrm{Det}_\chi(\rho_l),$$
$$(\varepsilon_p|\chi)_{N,p} = (p^{d_{p,\chi^\delta}})_\delta(u_p|\chi)_{N,p}\,\mathrm{Det}_\chi(\rho_p),$$

where $\rho_p \in \mathcal{A}_p^*$ and the existence of free generators u_p and $u_2 \operatorname{tor}_{\mathbb{Z}_2}(U_2^1)$ is a consequence of Proposition 5.1.

Next, by Proposition 5.5 we obtain

$$(\varepsilon_p|\chi)_{N,p} = \mathrm{Det}_\chi(x_p)\omega_{p,\chi}[a_p|\chi]_{N,p}$$

for some $x_p \in \mathbb{Z}_p\Gamma^*$ and some a_p that is a free $\mathbb{Z}_p\Gamma$-generator of $\mathbb{Z}_p \otimes \mathcal{O}_N$ (the existence of a_p follows from Noether's theorem on tame extensions).

Thus for $p \neq l$ the map

$$\chi \mapsto (\omega_{p,\chi^{\check{t}s}}[(\chi_*(p)/p - 1)\tau(\chi_*)]^{\check{t}s}/L_p(1, \chi_*^{\check{t}s}))_{\check{t}s}[a_p|\chi]_{N,p}$$

is the pth component of the map which represents the class of $\overline{E}_N$ in $\mathrm{Cl}(\mathcal{A})$.

If $p = l > 2$, then a reasoning as for $p \neq l$, together with Propositions 5.1 and 5.3, gives the lth component of the above map.

In the infinite case let $u_\infty = (u_\gamma)_\gamma$ be a free $\mathbb{R}\Gamma$-generator of U_∞^1, and Ψ_∞ be the $\mathbb{R}\Gamma$-isomorphism $U_\infty^1 \cong \mathbb{R}\Gamma$ defined in the proof of Proposition 5.1. Then since

$$\Psi_\infty(\mathcal{E}_{N,\infty}) \subseteq \Psi_\infty((U_{N,\infty}^1)^0) = (\mathbb{R}\Gamma)^0 = \mathbb{R}\Gamma(\gamma_0 - 1)$$

and $\mathcal{E}_{N,\infty} = d_\infty(\mathbb{R}\otimes E_N)$ has $\mathbb{R}$-dimension $|\Gamma|-1$, we infer that $\Psi_\infty(\mathcal{E}_{N,\infty}) = \mathbb{R}\Gamma(\gamma_0 - 1)$. Hence $\Psi_\infty(d_\infty(\varepsilon_\infty))$ and $\Psi_\infty(u_\infty(\gamma_0 - 1))$ are $\mathcal{A}_\infty$-generators of $\Psi_\infty(\mathcal{E}_{N,\infty})$. Then, by arguing as in the proof of Proposition 5.3, there exists $x \in \mathcal{A}_\infty^*$ such that $d_\infty(\varepsilon_\infty) = x(\gamma_0 - 1)u_\infty$, whence $(d_\infty(\varepsilon_\infty)|\chi)_{N,\infty} = \mathrm{Det}_\chi(x(\gamma_0-1))(u_\infty|\chi)_{N,\infty}$. As $\mathrm{Log}_\infty : U_{N,\infty}^1 \to \mathbb{R}\otimes\mathcal{O}_N = \prod_{\gamma\in\Gamma}\mathbb{R}$ is an $\mathbb{R}\Gamma$-isomorphism it follows that $a_\infty = (\log_\infty(u_\gamma))_\gamma$ is a free generator of $\mathbb{R}\otimes\mathcal{O}_N$. Note that, by definition, $(\varepsilon_\infty|\chi)_{N,\infty} = (d_\infty(\varepsilon_\infty)|\chi)_{N,\infty}$ and $(u_\infty|\chi)_{N,\infty} = [a_\infty|\chi]_{N,\infty}$, so $(\varepsilon_\infty|\chi)_{N,\infty} = \mathrm{Det}_\chi(x(\gamma_0 - 1))[a_\infty|\chi]_{N,\infty}$. Since $\tilde{\gamma}_0 - \tilde{1} \in \mathcal{A}_\infty^*$, the map

$$\chi \mapsto ([\tau(\chi_*)]^{ts}/L(1,\chi_*^{ts}))_{t,s}[a_\infty|\chi]_{N,\infty}$$

is the coordinate at ∞ of the map considered above.

By the formula (5.24) of Chapter I and Theorem 6 of the book [7] the map $\chi \mapsto [a|\chi]_N\tau(N/\mathbb{Q},\chi)^{-1}$ is a representative of the class of $\mathcal{O}_N$ in the locally free class group

$$\mathrm{Cl}(\mathbb{Z}\Gamma) \cong \mathrm{Hom}_\Omega(R_\Gamma, \mathcal{J}(F))/[\mathrm{Hom}_\Omega(R_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathbb{Z}\Gamma))].$$

On the other hand since $N/\mathbb{Q}$ is an Abelian tame extension, N has a normal integral basis, i.e. $\mathcal{O}_N \cong \mathbb{Z}\Gamma$ and therefore the class of $(\mathcal{O}_N)$ is trivial. Consequently, the maps $\chi \mapsto [a|\chi]_N$ and $\chi \mapsto \tau(N/\mathbb{Q},\chi)$ are equal modulo $\mathrm{Hom}_\Omega(R_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathbb{Z}\Gamma))$ and so equal modulo $\mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$. Thus after replacing $[a|\chi]_N$ by $\tau(N/\mathbb{Q},\chi)$ we obtain the required formula. ∎

6. Minkowski units in fields of a prime degree. Now we are able to give sufficient and necessary conditions for real cyclic fields of prime degree to have the simplest multiplicative Galois module structure, i.e. to have their groups of units $\mathcal{A}$-free modulo torsion. This is equivalent to the existence of units which together with their conjugates are fundamental units; such units are called *Minkowski units.*

Using these conditions we also give new examples of fields having Minkowski units. Let h_l denote the class number of the lth cyclotomic field.

THEOREM 6.1. *Let $N/\mathbb{Q}$ be a real, tame and cyclic extension of prime degree $l > 2$ and let l be regular, i.e. h_l is prime to l. Then N has a Minkowski unit if and only if*

$$\nu_p(h_p^\chi)\Phi_{\mu,p}(1) = \nu_p(h_p^\mu)\Phi_{\chi,p}(1)$$

for any $\chi,\mu \in \widehat{\Gamma}\setminus\{1_\Gamma\}$, and for any odd prime $p \neq l$ dividing h_N.

Proof. Because $\mathcal{A}$ is commutative it satisfies the Eichler condition (see Proposition 51.2 in [5]) which is sufficient for $\mathcal{A}$ to have locally free cancellation. This means that $(X)_{\mathcal{A}} = (Y)_{\mathcal{A}}$ implies $X \cong Y$ for $\mathcal{A}$-locally free modules X and Y ([5, pp. 303–304]). Thus in order to show that $\overline{E}_N \cong \mathcal{A}$, it suffices to prove that the class $(\overline{E}_N)_{\mathcal{A}}$ in $\mathrm{Cl}(\mathcal{A})$ is trivial. Since $\mathcal{A}$ and $\mathbb{Z}[\zeta_l]$ are isomorphic rings, we have $|\mathrm{Cl}(\mathcal{A})| = h_l$, and from the assumption that h_l is prime to l, one concludes that $(\overline{E}_N)_{\mathcal{A}} = 1$ if and only if $(\overline{E}_N)^l_{\mathcal{A}} = 1$.

Let $f \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, \mathfrak{J}(F))$ be a homomorphism representing the class $(\overline{E}_N)_{\mathcal{A}}$ in $\mathrm{Cl}(\mathcal{A})$ as defined in Theorem 5.6. Thus in order to examine whether $\overline{E}_N \cong \mathcal{A}$ it suffices to consider the function f^l.

In view of Remark 3.2 we can always assume that each homomorphism representing a class in $\mathrm{Cl}(\mathcal{A})$ has its component at infinity equal to 1.

For any $\chi \neq 1_{\Gamma}$ we put

$$f(\chi) = \tau(N/\mathbb{Q}, \chi)(\omega_{p,\chi^{\delta}}/W_{\chi^{\delta},p})_{p,\delta} \tag{6.1}$$

where

$$W_{\chi^{\delta},p} = \frac{L_p(1,\chi_*^{\delta})}{[(\chi_*(p)/p-1)\tau(\chi_*)]^{\delta}} = \frac{L_p(1,\chi_*^{\delta})\chi_*^{\delta}(\delta_*)}{(\chi_*^{\delta}(p)/p-1)\tau(\chi_*^{\delta})}$$

with an integer δ_* such that $\delta(\zeta_q) = \zeta_q^{\delta_*}$.

Since we need only an evaluation of $\nu_p(W^l_{\chi^{\delta},p})$ it suffices to consider

$$W_{\chi,p} = \frac{L_p(1,\chi_*)p}{(\chi_*(p)-p)\tau(\chi_*)} = \frac{1}{q}\sum_{a=1}^{q}\overline{\chi}_*(a)\log_p(1-\zeta_q^a)$$

where the last equality is a consequence of Theorem 3.5.

Hence for any $\sigma \in \mathcal{G}al(\mathbb{Q}_p(\zeta_{lq})/\mathbb{Q}_p)$ and $\chi \neq 1_{\Gamma}$, one has

$$\sigma(W_{\chi,p}) = \chi_*(\sigma_*)W_{\chi^{\sigma},p} \tag{6.2}$$

where σ_* is an integer defined by $\sigma(\zeta_q) = \zeta_q^{\sigma_*}$.

Now using Leopoldt's p-adic class number formula (see Theorem 5.24 in [20])

$$\frac{2^{l-1}h_N R_l(N)}{\sqrt{d_N}} = \prod_{\chi_*\neq 1}\left(1-\frac{\chi_*(l)}{l}\right)^{-1} L_l(1,\chi_*)$$

and $\prod_{\chi\in\widehat{\Gamma}}\tau(\chi_*) = \sqrt{d_N}$, we get $2^{l-1}h_N R_l(N)/d_N = \prod_{\chi\neq 1_{\Gamma}} W_{\chi,l}$.

Observe that for $p = l$ all characters $\chi \in \widehat{\Gamma}\setminus\{1_{\Gamma}\}$ are conjugate over $\mathbb{Q}_l$ since the decomposition group of l in $\mathbb{Q}(\zeta_l)$ is the whole group $\mathcal{G}al(\mathbb{Q}(\zeta_l)/\mathbb{Q})$. This in turn by (6.2) shows that $\nu_l(W_{\chi,l}) = \nu_l(W_{\mu,l})$ for all $\chi,\mu \in \widehat{\Gamma}\setminus\{1_{\Gamma}\}$. Thus the above and $l \nmid d_N$ ($N/\mathbb{Q}$ is tame) imply

$$(l-1)\nu_l(W_{\chi,l}) = \nu_l\left(\frac{2^{l-1}h_N R_l(N)}{d_N}\right) = \nu_l(h_N R_l(N)) \quad \text{for } \chi \neq 1_{\Gamma}. \tag{6.3}$$

Since $\nu_p(W_{\chi,p}) = \nu_p(L_p(1,\chi_*)) + 1 - \nu_p(\tau(\chi_*))$, Theorem 5.6 and (6.3) imply

$$\begin{aligned}\nu_l\left(\frac{\omega_{l,\chi}}{W_{\chi,l}}\right) &= \frac{1}{l-1}(1-\nu_l(h_N)),\\ \nu_2\left(\frac{\omega_{2,\chi}}{W_{\chi,2}}\right) &= s_2 - 2 + \nu_2(\tau(\chi_*)) - \nu_2(h_2^\chi)/\varPhi_{\chi,2}(1),\\ \nu_p\left(\frac{\omega_{p,\chi}}{W_{\chi,p}}\right) &= s_p - 1 + \nu_p(\tau(\chi_*)) - \nu_p(h_p^\chi)/\varPhi_{\chi,p}(1) \quad \text{for other primes } p.\end{aligned}$$

This and the fact that $s_p = 1$ for all unramified primes p in N (see Proposition 5.5) show that $\nu_p(\omega_{p,\chi}/W_{\chi,p}) = 0$ for any $\chi \neq 1_\varGamma$ and almost all primes p. Thus we obtain

$$\left(\frac{\omega_{p,\chi^\delta}}{W_{\chi^\delta,p}}\right)_{p,\delta} \in \mathcal{J}(F), \qquad \delta \in \check{T}_p S_p.$$

If we take any $\sigma \in \varOmega$ which is the identity on $\mathbb{Q}_p(\zeta_l)$, then (6.2) gives $\sigma(W_{\chi,p}^l) = W_{\chi,p}^l$, showing that $W_{\chi,p}^l \in \mathbb{Q}_p(\zeta_l)$ and consequently $(\omega_{p,\chi}/W_{\chi,p})^l \in \mathbb{Q}_p(\zeta_l)$.

Note that for any prime $\mathfrak{p}_F$ in F over p one has $\mathbb{Q}_p(\zeta_l) \subseteq F_{\mathfrak{p}_F}$, so each δ from the decomposition group $\varDelta_{F,p}$ is an extension of some automorphism of $\mathbb{Q}_p(\zeta_l)$. This and the fact that $\nu_p(L_p(1,\chi_*^\sigma)) = \nu_p(\sigma L_p(1,\chi_*))$ for $\sigma \in \mathcal{G}al(\mathbb{Q}_p(\zeta_l)/\mathbb{Q}_p)$ imply that $\omega_{p,\chi^\delta} = \delta(\omega_{p,\chi})$ for each $\delta \in \varDelta_{F,p}$. Similarly using (6.2) we obtain $(W_{\chi^\delta,p})^l = \delta(W_{\chi,p})^l$ for $\delta \in \varDelta_{F,p}$. Now applying Remark 2.1 to the mapping defined by

$$\mathcal{D}(\chi) = (\mathcal{D}_p(\chi^\delta))_{\delta,p} = \left(\frac{\omega_{p,\chi^\delta}}{W_{\chi^\delta,p}}\right)^l_{p,\delta} \qquad \text{with } \delta \in \check{T}S$$

we deduce that $\mathcal{D} \in \mathrm{Hom}_\varOmega(R'_\varGamma, \mathcal{J}(\mathbb{Q}(\zeta_l)))$.

As $(\omega_{p,\chi}/W_{\chi,p})^l$ and $\tau(\chi_*)^l \in \mathbb{Q}_p(\zeta_l)$, for odd $p \neq l$ the numbers

$$\begin{aligned}a_{\chi,p} &= \nu_p((\omega_{p,\chi}/W_{\chi,p})^l) - \nu_p(\tau(\chi_*)^l) - l(s_p - 1) = -l\nu_p(h_p^\chi)/\varPhi_{\chi,p}(1),\\ a_{\chi,2} &= \nu_2((\omega_{2,\chi}/W_{\chi,2})^l) - \nu_2(\tau(\chi_*)^l) - l(s_2 - 2) = -l\nu_2(h_2^\chi)/\varPhi_{\chi,2}(1)\end{aligned}$$

are rational integers. Thus we can define a mapping $\mathcal{F}$ on $R'_\varGamma$ by putting $\mathcal{F}(\chi) = (\mathcal{F}_p(\chi^\delta))_{p,\delta}$ where $\mathcal{F}_p(\chi) = p^{a_{\chi,p}}$ for $p \neq l$ and $\mathcal{F}_l(\chi) = 1$.

We also define mappings $\mathcal{B}(\chi) = (\mathcal{B}_p(\chi^\delta))_{p,\delta}$ and $\mathcal{G}(\chi) = (\mathcal{G}_p(\chi^\delta))_{p,\delta}$ by

$$\begin{aligned}&\mathcal{B}_p(\chi) = p^{b_p} \quad \text{with } b_2 = l(s_2 - 2),\ b_p = l(s_p - 1) \text{ for odd prime } p \neq l,\\ &\mathcal{B}_l(\chi) = (1 - \chi(\gamma_0))^{l(1-\nu_l(h_N))}.\end{aligned}$$

The mapping $\mathcal{G}$ is defined by

$$\mathcal{G}_p(\chi) = \mathcal{D}_p(\chi)\mathcal{F}_p^{-1}(\chi)\mathcal{B}_p^{-1}(\chi)\tau(\chi_*)^{-l}. \tag{6.4}$$

Since $\mathcal{B}_p(\chi) = 1$ for almost all p and $\mathcal{B}_p(\chi^\sigma) = \sigma(\mathcal{B}_p(\chi))$ for $\sigma \in \Omega$, it follows that $\mathcal{B} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(\mathbb{Q}(\zeta_l)))$. Similarly $\tau(\chi_*^\sigma)^l = \sigma(\tau(\chi_*)^l) \in \mathbb{Q}(\zeta_l)$ for $\sigma \in \Omega$, so the map $\chi \mapsto (\tau(\chi_*^\delta)^l)_{p,\delta}$ is in $\mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(\mathbb{Q}(\zeta_l)))$.

Using similar arguments applied to $\omega_{p,\chi}$ we get $a_{\chi^\delta,p} = a_{\chi,p}$ and so $\mathcal{F}_p(\chi^\delta) = \delta(\mathcal{F}_p(\chi))$ for any $\delta \in \Delta_{F,p}$. Hence by Remark 2.1 we infer that $\mathcal{F} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(\mathbb{Q}(\zeta_l)))$, which together with (6.4) gives $\mathcal{G} \in \mathrm{Hom}_\Omega(R'_\Gamma, \mathcal{J}(\mathbb{Q}(\zeta_l)))$. The definition of $\mathcal{G}$ in (6.4) and $\nu_l(\tau_*(\chi)) = 0$ show that $\mathcal{G}_p(\chi) \in \mathbb{Z}_p[\zeta_l]^*$ for any p and $\chi \neq 1_\Gamma$. Since, for any representative $t \in T_{\mathbb{Q}(\zeta_l),p}$, the completion of $\mathbb{Q}(\zeta_l)$ at $t(\mathfrak{p}_{\mathbb{Q}(\zeta_l)})$ is $\mathbb{Q}_p(t(\zeta_l)) = \mathbb{Q}_p(\zeta_l)$, we have

$$\begin{aligned}\mathcal{G} &\in \mathrm{Hom}_\Omega\Big(R'_\Gamma, \prod_p \prod_t \mathcal{O}^*_{\mathbb{Q}(\zeta_l)_{\mathfrak{p}_t}}\Big)\\ &= \mathrm{Hom}_\Omega\Big(R'_\Gamma, \prod_p \prod_t \mathbb{Z}_p[\zeta_l]^*\Big) \subseteq \mathrm{Hom}_\Omega\Big(R'_\Gamma, \prod_p F_p^*\Big)\end{aligned}$$

where t runs over $T_{\mathbb{Q}(\zeta_l),p}$ and $\mathfrak{p}_t$ denotes $t(\mathfrak{p}_{\mathbb{Q}(\zeta_l)})$ for short.

We shall prove that $\mathcal{G} \in \mathrm{Det}(\mathfrak{U}(\mathcal{A}))$.

For any prime $p \neq l$ define $\mathcal{G}'_p \in \mathrm{Hom}_\Omega(R_\Gamma, \prod_t \mathbb{Z}_p[\zeta_l]^*)$ to be $\mathcal{G}'_p(\chi) = \mathcal{G}_p(\chi)$ for $\chi \neq 1_\Gamma$ and $\mathcal{G}'_p(1_\Gamma) = 1$. Since $p \nmid l = |\Gamma|$, $\mathbb{Z}_p\Gamma$ is a maximal order in $\mathbb{Q}_p\Gamma$ and by Proposition 2.2 in [7] we get $\mathrm{Hom}_\Omega(R_\Gamma, \prod_t \mathbb{Z}_p[\zeta_l]^*) = \mathrm{Det}((\mathbb{Z}_p\Gamma)^*)$. Consequently, $\mathcal{G}'_p \in \mathrm{Det}((\mathbb{Z}_p\Gamma)^*)$ and so $\mathcal{G}_p \in \mathrm{Det}(\mathcal{A}_p^*)$ because the image (under $\gamma \mapsto \tilde{\gamma}$) of any unit of $\mathbb{Z}_p\Gamma$ is a unit of $\mathcal{A}_p$ and they have the same Det_χ.

For $p = l$ let $\chi \neq 1_\Gamma$ and let γ_0 be a generator of Γ with $\chi(\gamma_0) = \zeta_l$. Put $\mathcal{G}_l(\chi) = \sum_{j=0}^{l-2} a_j \zeta_l^j \in \mathbb{Z}_p[\zeta_l]^*$. Since $\sum_{j=0}^{l-1} x_j \tilde{\gamma}_0^j \mapsto \sum_{j=0}^{l-1} x_j \zeta_l^j$ is a ring isomorphism $\mathcal{A}_l \cong \mathbb{Z}_l[\zeta_l]$, we have $\sum_{j=0}^{l-1} a_j \tilde{\gamma}_0^j \in \mathcal{A}_l^*$. Also note that $\mathrm{Det}_\chi(\sum_{j=0}^{l-1} a_j \tilde{\gamma}_0^j) = (\mathcal{G}_l(\chi^\sigma))_\sigma$. Let μ be any nontrivial character of Γ. Since, in this case, all nontrivial characters of Γ are conjugate over $\mathbb{Q}_l$, we have $\mu = \chi^\rho$ for some $\rho \in \mathcal{G}al(\mathbb{Q}_l(\zeta_l)/\mathbb{Q}_l)$ and so

$$\begin{aligned}\mathrm{Det}_\mu\Big(\sum_{j=0}^{l-1} a_j \tilde{\gamma}_0^j\Big) &= \Big(\sum_{j=0}^{l-1} a_j \chi^{\rho\sigma}(\gamma_0^j)\Big)_\sigma\\ &= (\rho(\mathcal{G}_l(\chi^\sigma)))_\sigma = (\mathcal{G}_l(\chi^{\rho\sigma}))_\sigma = (\mathcal{G}_l(\mu^\sigma))_\sigma,\end{aligned}$$

which shows that $\mathcal{G}_l \in \mathrm{Det}(\mathcal{A}_l^*)$. Thus $\mathcal{G} = (\mathcal{G}_p)_p \in \mathrm{Det}(\prod_p \mathcal{A}_p^*) = \mathrm{Det}(\mathfrak{U}(\mathcal{A}))$.

We will show that $\mathcal{B} \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$. To see this observe that $\mathcal{B}_p(\chi) = 1$ for almost primes so we can put $\widetilde{\mathcal{B}}(\chi) = \prod_p \mathcal{B}_p(\chi) \in F^*$ for any $\chi \neq 1_\Gamma$ and define the map $\mathcal{B}' : \chi \mapsto (\widetilde{B}(\chi^\sigma))_{p,\sigma}$. It is clear that $\mathcal{B}' \in \mathrm{Hom}_\Omega(R'_\Gamma, F^*)$. As $\mathcal{B}_p(\chi)$ is a power of p for $p \neq l$ and $\mathcal{B}_l(\chi)$ is a power of $1 - \zeta_l$ times a unit of $\mathbb{Z}[\zeta_l]$ we deduce that $\mathcal{B}/\mathcal{B}' \in \mathrm{Hom}_\Omega(R'_\Gamma, \prod_p \prod_t \mathbb{Z}_p[\zeta_l]^*)$ where t runs over $T_{\mathbb{Q}(\zeta_l),p}$. Then after proceeding as for $\mathcal{G}$ we obtain $\mathcal{B}/\mathcal{B}' \in$

$\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$, whence

(6.5) $$\mathcal{B} = \mathcal{B}'(\mathcal{B}/\mathcal{B}') \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A})).$$

Since the Galois Gauss sum $\tau(\chi, N/\mathbb{Q})$ is an algebraic number in F and $\tau(\chi^{\sigma}, N/\mathbb{Q})/\sigma(\tau(\chi, N/\mathbb{Q}))$ is an lth root of unity for any $\sigma \in \Omega$ (see Theorem 20B(ii) in [8]),

$$\chi \mapsto \tau(\chi, N/\mathbb{Q})^l \text{ is an element of } \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*).$$

We can say the same about the function

$$\chi \mapsto (\tau(\chi_*^{\sigma})^l)_{p,\sigma}.$$

Thus by (6.1), the definition of $\mathcal{D}$, (6.4) and (6.5) we may write

(6.6) $$f^l = \mathcal{F}\mathcal{F}' \quad \text{with} \quad \mathcal{F}' \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A})).$$

Now assume that the conditions of our theorem are satisfied. Then for any prime p and any $\sigma \in \Omega$ we have $\mathcal{F}_p(\chi^{\sigma}) = \sigma(\mathcal{F}_p(\chi))$. Since $\mathcal{F}_p(\chi)$ is always a power of p and equals 1 for almost all p, we can apply the same arguments as used for $\mathcal{B}$ to get $\mathcal{F} \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$. Thus by (6.6) we have $f^l \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$, which proves that $\overline{E}_N$ is $\mathcal{A}$-free.

Conversely, suppose that $\overline{E}_N$ is $\mathcal{A}$-free. Then the representing function f of the class $(\overline{E}_N)_{\mathcal{A}}$ belongs to $\mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$ and so does f^l. Consequently, according to (6.6) we have $\mathcal{F} \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)\,\mathrm{Det}(\mathfrak{U}(\mathcal{A}))$. Thus, by the definition of $\mathcal{F}$, we may write

$$(\mathcal{F}_p(\chi^{\sigma}))_{\sigma} = (p^{a_{p,\chi^{\sigma}}})_{\sigma} = (\mathcal{Z}(\chi^{\sigma}))_{\sigma}(y_{p,\chi^{\sigma}})_{\sigma} \in \prod_{\sigma \in T_{F,p}} F^*_{\sigma(\mathfrak{p}_F)}$$

where $\mathcal{Z} \in \mathrm{Hom}_{\Omega}(R'_{\Gamma}, F^*)$ and the mapping $\chi \mapsto (y_{p,\chi^{\sigma}})_{\sigma}$ is an element of $\mathrm{Det}((\mathcal{A}_p^*))$. Hence $y_{p,\chi^{\sigma}} \in \mathcal{O}^*_{F_{\sigma(\mathfrak{p}_F)}}$ and since $p^{a_{p,\chi^{\sigma}}} = \mathcal{Z}(\chi^{\sigma})y_{p,\chi^{\sigma}}$ one has $y_{p,\chi^{\sigma}} \in F \cap \mathcal{O}^*_{F_{\sigma(\mathfrak{p}_F)}}$. It follows that

$$\nu_{\sigma(\mathfrak{p}_F)}(p^{a_{p,\chi^{\sigma}}}) = \nu_{\sigma(\mathfrak{p}_F)}(\sigma(\mathcal{Z}(\chi))) \quad \text{and so} \quad e a_{p,\chi^{\sigma}} = \nu_{\mathfrak{p}_F}(\mathcal{Z}(\chi))$$

where $\nu_{\mathfrak{p}_F}$ denotes the normalized $\mathfrak{p}_F$-adic exponential valuation (i.e. with image $\mathbb{Z}$) in $F_{\mathfrak{p}_F}$ and e is the ramification index of p in F. The above equalities show that for any σ from $T_{f,p}$, the set of representatives of the decomposition group of p in F, and for any nontrivial character χ one has $a_{p,\chi^{\sigma}} = a_{p,\chi}$. This is also true for any σ from the decomposition group for p, as a consequence of the Ω-invariance of $(\mathcal{F}_p(\chi^{\sigma}))_{\sigma}$ and Remark 2.1. Thus for any p, $\chi \neq 1_{\Gamma}$ and $\sigma \in \Omega$ one has $a_{p,\chi^{\sigma}} = a_{p,\chi}$, and our theorem follows. ∎

Now we apply the above theorem to give examples of fields having Minkowski units. This will be done in two corollaries.

Corollary 6.2. *Let $N/\mathbb{Q}$ be a real, tame and cyclic extension of prime degree $l > 2$ and let l be regular. Then N has a Minkowski unit in the following two cases:*

(i) $h_N = 1$,

(ii) *any prime p dividing h_N is a primitive root modulo l.*

Proof. (i) In the case $h_N = 1$, the homomorphism $\mathcal{F}$ defined in the proof of Theorem 6.1 has all values equal to 1, so $\mathcal{F}$ represents the trivial class.

(ii) If p is a primitive root of unity modulo l, then the decomposition group of p is the whole group $\mathcal{G}al(\mathbb{Q}(\zeta_l)/\mathbb{Q}) \cong \mathcal{G}al(\mathbb{Q}_l(\zeta_l)/\mathbb{Q}_l)$ and so all nontrivial characters of Γ are conjugate over $\mathbb{Q}_p$. Consequently, $\Phi_{\chi,p} = \Phi_{\mu,p}$ for all $\chi, \mu \in \widehat{\Gamma} \setminus \{1_\Gamma\}$ and for $p \mid h_N$. Thus the condition of Theorem 6.1 is satisfied and the existence of a Minkowski unit follows. ■

The next corollary provides concrete examples of fields with Minkowski units. As is well known, all real Abelian fields of prime degree over $\mathbb{Q}$ less than 23 have Minkowski units (see [3]); below we shall give fields of degree $l \geq 23$. Let $\mathbb{Q}(\zeta_q)^+$ denote the maximal real subfield of $\mathbb{Q}(\zeta_q)$ and h_q^+ its class number.

Corollary 6.3. *Let l and q be odd prime numbers such that l is regular and $q \equiv 1 \pmod{l}$. Let N be the unique real subfield of $\mathbb{Q}(\zeta_q)$ such that $(N : \mathbb{Q}) = l$. Then N has a Minkowski unit for the following pairs (l, q):*

(i) $(23, 47)$, $(29, 59)$ (*i.e.* $N = \mathbb{Q}(\zeta_{47})^+$, $\mathbb{Q}(\zeta_{59})^+$). *In this case* $h_N = 1$ (*see tables in* [20]).

(ii) $(41, 83)$, $(53, 107)$ (*i.e.* $N = \mathbb{Q}(\zeta_{83})^+$, $\mathbb{Q}(\zeta_{107})^+$), $(23, 139)$ (*tables in* [20]) *on the assumption that the generalized Riemann hypothesis holds.*

(iii) $l \mid q - 1$ *and q is a prime less than* 10 000 *from Schoof's table* ([18]) *such that $h_q^+ = 1$. There are* 564 *such pairs.*

(iv) $l \mid q - 1$ *and q is a prime less than* 10 000 *from Schoof's table* ([18]) *such that $h_q^+ > 1$ and all prime factors of h_q^+ (possible factors of h_N) are primitive roots of unity modulo l. There are* 47 *such pairs.*

The correctness of the examples in (iii), which is highly probable, depends on whether the entries in Schoof's table are equal to h_q^+ for suitable q's (see the discussion in [20, pp. 420–421]).

Proof. All the above fields satisfy one of the conditions of Corollary 6.2. In case N is a proper subfield of $\mathbb{Q}(\zeta_q)^+$ we use the relation $h_N \mid h_q^+$ (see Theorem 22 in [20]). Then if $h_q^+ > 1$ we consider prime factors of h_q^+ as possible factors of h_N and check whether they are primitive roots of unity modulo l. ■

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