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On the real roots of generalized Thue–Morse polynomials

by

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In this article we investigate real roots of real polynomials. By results of M. Kac [8–10] we know that a polynomial of degree n has on average $(2/\pi)\log n$ real zeros. See also results of Edelman and Kostlan [6] on the same subject. Some 10 years later Erdős and Offord [7] proved that the mean number of real roots of a random polynomial of degree n with coefficients ± 1 is again $(2/\pi)\log n$. This leads us to the following question: can we find sequences $(\alpha_i)_{i\in\mathbb{N}}$ with coefficients ± 1 such that the corresponding polynomials $\sum_{i=0}^{n}\alpha_i X^i$ have $O(\log n)$ real roots, and are these sequences random in some sense?

We introduce generalized Thue–Morse sequences whose corresponding polynomials of large degree n have at least $C\log n$ real roots, where C is an explicit positive constant. Finally, we discuss the spectral measure of these sequences.

1. Introduction. Erdős and Offord [7] established that the average number of real zeros of the degree n polynomial $\sum_{i=0}^{n}\pm X^i$ is equivalent to $(2/\pi)\log n$. A natural question could then be to find a sequence $(\alpha_i)_{i\in\mathbb{N}}$ of ± 1 such that

$$\frac{1}{N}\sum_{n=0}^{N-1}\varrho(g_n)\backsim\frac{2}{\pi}\log N \tag{1}$$

where $\varrho(g_n)$ is the number of real zeros of the polynomial

$$g_n(X)=\sum_{i=0}^{n}\alpha_i X^i.$$

In a previous article [5] we tested the Thue–Morse sequence $(\varepsilon_i)_{i\in\mathbb{N}}$, defined by $\varepsilon_i=(-1)^{\nu(i)}$ where $\nu(i)$ is the sum of the binary digits of i.

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Unfortunately, we proved that (1) does not hold in this case. More precisely we have

$$\frac{1}{N}\sum_{n=0}^{N-1}\varrho(f_n) \underset{N\to\infty}{\longrightarrow} \frac{11}{4},$$

with $f_n(X)=\sum_{i=0}^{n}\varepsilon_i X^i$.

In this paper we show the existence of families $(\varepsilon_{w,i})_{i\in\mathbb{N}}$ of (±1)-sequences for which

$$\liminf_{N\to\infty}\frac{1}{N\log N}\sum_{n=0}^{N-1}\varrho(f_{w,n})>0,$$

where $f_{w,n}(X)=\sum_{i=0}^{n}\varepsilon_{w,i}X^i$. These sequences have, to some extent, a similar structure to the Thue–Morse sequence. They can be obtained in a very similar way, by means of iterations of morphisms. We call them *generalized Thue–Morse sequences.*

Before explaining this we introduce two useful notations. Let w be a word of length ℓ on the alphabet $\{+,-\}$ and $i\le j$ be integers less than ℓ. Then ${}_iw_j$ represents the factor of w beginning at letter i and finishing at letter j of w. For instance ${}_iw_i$ is simply the letter at position i of w and ${}_0w_{\ell-1}=w$. We put also $w[i]=1$ if ${}_iw_i=+$ and $w[i]=-1$ if ${}_iw_i=-$.

Now let φ be the morphism on the alphabet $\{+,-\}$ defined by

$$\varphi:\begin{cases}+\to +-,\\ -\to -+\,.\end{cases}$$

The first iterations of φ are

$$\begin{aligned}\varphi(+)&=+-,\\ \varphi^2(+)&=+--+,\\ \varphi^3(+)&=+--+-++-\,.\end{aligned}$$

Let

$$\mathcal{E}=\lim_{n\to\infty}\varphi^n(+)$$

be the Thue–Morse word. There exists an obvious link between $\mathcal{E}$ and $(\varepsilon_i)_{i\in\mathbb{N}}$, i.e.

$$\varepsilon_i=\mathcal{E}[i].$$

In the next section we slightly modify the definition of φ to get a wider family of sequences.

2. Generalized Thue–Morse sequences. Let w be a word on $\{+,-\}$ of length $\ell\ge 2$ beginning with $+$. We put

$$\varphi_w:\begin{cases}+\to w,\\ -\to \overline{w}\end{cases}$$

where $\overline{w}$, the "opposite" of w, is defined by

$$_i\overline{w}_i = \begin{cases} - & \text{if } _iw_i = +, \\ + & \text{if } _iw_i = -. \end{cases}$$

The Thue–Morse w-word, $\mathcal{E}_w$, is then

$$\mathcal{E}_w = \lim_{n\to\infty} \varphi_w^n(+).$$

Denote by $(\varepsilon_{w,i})_{i\in\mathbb{N}}$ the corresponding Thue–Morse w-sequence with coefficients ± 1, that is, the sequence satisfying

$$\varepsilon_{w,i} = \mathcal{E}_w[i].$$

For example if $w = ++$ then

$$\sum_{i=0}^{\infty} \varepsilon_{w,i} X^i = \frac{1}{1-X}.$$

For each word w of length ℓ we consider P_w its *associated polynomial* defined by

$$P_w(X) = \sum_{j=0}^{\ell-1} w[j] X^j.$$

LEMMA 1. *Let w be a word of length $\ell \geq 2$ on $\{+,-\}$ beginning with $+$ and ε_w its associated generalized Thue–Morse sequence. Then*

$$\sum_{i=0}^{\infty} \varepsilon_{w,i} X^i = \prod_{h=0}^{\infty} P_w(X^{\ell^h}). \tag{2}$$

Proof. Let v be some word of length t on $\{+,-\}$ and

$$P_v(X) = \sum_{j=0}^{t-1} v[j] X^j$$

its associated polynomial. It suffices to see that

$$P_{\varphi_w(v)}(X) = P_w(X) P_v(X^\ell).$$

Indeed,

$$_{j\ell}(\varphi_w(v))_{j\ell+\ell-1} = \begin{cases} w & \text{if } v[j] = 1, \\ \overline{w} & \text{if } v[j] = -1, \end{cases}$$

for all j in $[\![0, t-1]\!]$. So we get

$$P_{\varphi_w(v)}(X) = \sum_{j=0}^{t-1} P_w(X) v[j] X^{j\ell}$$

and $P_{\varphi_w(v)}(X) = P_w(X) P_v(X^\ell)$ as claimed. The lemma immediately follows. Note that the series converges in $]-1, 1[$. ∎

Using this we can give for $\varepsilon_{w,i}$ a more explicit meaning that generalizes the initial definition of the Thue–Morse sequence. Let $m_1, \ldots, m_q$ be the integers $j \in [\![0, \ell-1]\!]$ such that ${}_j w_j = -$. Then from (2) it is clear that

$$\varepsilon_{w,i} = (-1)^{\nu_{m_1}(i)+\ldots+\nu_{m_q}(i)}$$

where $\nu_{m_k}(i)$ represents the number of m_k's in the base ℓ expansion of i.

3. Real roots of generalized Thue–Morse polynomials. For the classical Thue–Morse polynomials the starting word is $w = +-$ so that its associated polynomial is $P_w(X) = 1 - X$. Therefore $\ell = 2$ and since the real roots of $P_w(X^{2^h})$ are -1 and 1, we cannot use the convergence of $\sum_{i=0}^{\infty} \varepsilon_i X^i$ on $]-1,1[$. Now the starting polynomial P_w may vanish on $]0,1[$ and the real roots of $P_w(X^{\ell^h})$ spread in this case along $[0,1[$. Using the convergence of (2) on $]-1,1[$ we show that $\varrho(f_{w,n})$ is in $C \log n$. Let us make this more precise.

Theorem 1. *Let w be a word of length $\ell \geq 2$ on $\{+,-\}$ beginning with $+$ such that P_w has only simple roots on $]-1,1[$, say t roots $\beta_1 < \ldots < \beta_t$ in $]0,1[$ and t' roots $\beta_{t+t'} < \ldots < \beta_{t+2} < \beta_{t+1}$ in $]-1,0[$. Let $f_{w,n}$ be the generalized Thue–Morse polynomials associated with ε_w. Assume that $\beta_t < \beta_1^{1/\ell}$. Suppose in addition that $\beta_{t+1}^{1/\ell} < \beta_{t+t'}$ if ℓ is odd. Then there exists $K > 0$ such that for all $\epsilon > 0$ there is an $N(\epsilon)$ such that for all $n \geq N(\epsilon)$ we have*

$$\varrho(f_{w,n}) \geq \frac{2(1-\epsilon)t \log n}{\log \ell} - K \qquad \textit{if } \ell \textit{ is even},$$

$$\varrho(f_{w,n}) \geq \frac{(1-\epsilon)(t+t') \log n}{\log \ell} - K \qquad \textit{if } \ell \textit{ is odd.}$$

Proof. Put $\beta_{j,h} = \beta_j^{1/\ell^h}$ for all $h \in \mathbb{Z}$. The roots of

$$f_{w,\ell^k-1}(X) = \prod_{h=0}^{k-1} P_w(X^{\ell^h})$$

in $]0,1[$ are therefore

$$\begin{matrix} \beta_{1,0} & \cdots & \beta_{t,0}, \\ \vdots & & \vdots \\ \beta_{1,h} & \cdots & \beta_{t,h}, \\ \vdots & & \vdots \\ \beta_{1,k-1} & \cdots & \beta_{t,k-1}. \end{matrix}$$

Put also

$$\delta_{j,h} = \begin{cases} \sqrt{\beta_{j,h}\beta_{j+1,h}} & \text{for } j \in [\![1, t-1]\!],\ h \in \mathbb{Z}, \\ \sqrt{\beta_{t,h}\beta_{1,h+1}} & \text{for } j = t,\ h \in \mathbb{Z}. \end{cases}$$

The next lemma plays an important part in the following.

LEMMA 2. *Let u be the multiplicity of* 1 *as a root of P_w. Then there are two constants C_1 and C_2 such that for all large h, $k \geq h$ and $j \in [\![1, t]\!]$,*

$$|f_{w,\ell^k-1}(\delta_{j,h})| \geq C_1 C_2^h \ell^{-uh(h+1)/2}. \tag{3}$$

Proof of Lemma 2. First of all we determine $C_1(j)$ and $C_2(j)$ for any j in $[\![1, t]\!]$. Since there are only finitely many j's the lemma will follow immediately.

We remark that

$$f_{w,\ell^k-1}(\delta_{j,h}) = \prod_{s=0}^{k-1} P_w(\delta_{j,h}^{\ell^s}) = \prod_{i=h-k+1}^{h} P_w(\delta_{j,i}).$$

Now

$$P_w(\delta_{j,i}) = \frac{P_w^{(u)}(1)}{u!}(1-\delta_{j,i})^u + o((1-\delta_{j,i})^u)$$

when $1 - \delta_{j,i} \to 0$, so there are $C_5 \in \mathbb{R}_+$ and $i_0 > 0$ such that for all $i > i_0$,

$$|P_w(\delta_{j,i})| \geq C_5 |1-\delta_{j,i}|^u \geq C_5 \left|\frac{\log \delta_{j,0}}{2\ell^i}\right|^u$$

since $|1 - e^x| \geq |x|/2$ near 0. Thus

$$\prod_{i=i_0}^{h} |P_w(\delta_{j,i})| \geq C_4 C_3^h \ell^{-uh(h+1)/2}.$$

The factor

$$\prod_{i=h-k+1}^{i_0-1} |P_w(\delta_{j,i})|$$

leads us to study the behaviour of P_w near 0. Now P_w is locally either greater than 1 or less than 1. In the first case it is obvious that for a suitable constant C_1,

$$\prod_{i=h-k+1}^{h} |P_w(\delta_{j,i})| \geq C_1 C_2^h \ell^{-uh(h+1)/2}.$$

In the second case $\delta_{j,i} \geq \delta_{j,i}^2 \geq \ldots \geq \delta_{j,i}^{\ell-1}$ and these quantities are small in comparison with 1 for all large $|i|$. Thus

$$|P_w(\delta_{j,i}) - 1| \leq (\ell-1)\delta_{j,i}.$$

Since for $i \leq 0$ we have $\delta_{j,i} = \delta_{j,0}^{\ell^{|i|}}$, we obtain the convergence of

$$\prod_{i=-\infty}^{i_0-1} |P_w(\delta_{j,i})|.$$

The lemma is then proved. ∎

REMARK. When $P_w(1) \neq 0$ it is possible to replace $C_1 C_2^h \ell^{-uh(h+1)/2}$ in (3) by a positive constant independent of h.

Since f_{w,ℓ^k-1} has $t-1$ simple roots between $\delta_{1,h}$ and $\delta_{t,h}$, it follows that f_{w,ℓ^k-1} changes sign $t-1$ times, passing above and below the lines

$$y = C_1 C_2^h \ell^{-uh(h+1)/2} \quad \text{and} \quad y = -C_1 C_2^h \ell^{-uh(h+1)/2}.$$

For example, consider Figure 1 which displays $f_{w,728}$ built from the word $w = + - -$. As $P_w(1) = -1$ the remark ensures that $f_{w,728}$ and more generally $f_{w,3^k-1}$ winds itself round two absolute axes. Here they are $y = 0.067130$ and $y = -0.067130$ (bold lines on Figure 1).

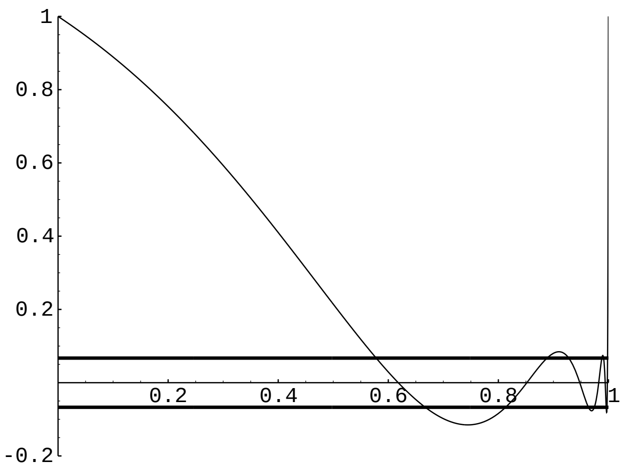

Fig. 1. $f_{w,728}(X)$ on $[0,1]$

Now if $n \geq \ell^k$ it is clear that on $]0,1[$,

$$|f_{w,n}(x) - f_{w,\ell^k-1}(x)| \leq \frac{x^{\ell^k}}{1-x}.$$

Let $\epsilon > 0$. Since $\delta_{t,\lfloor(1-\epsilon)k\rfloor}^{\ell^k} = \delta_{t,\lfloor(1-\epsilon)k\rfloor-k} \leq \delta_t^{\ell^{\epsilon k}}$, we have

$$\frac{\delta_{t,\lfloor(1-\epsilon)k\rfloor}^{\ell^k}}{1-\delta_{t,\lfloor(1-\epsilon)k\rfloor}} \cdot \frac{\ell^{uk(k+1)/2}}{C_1 C_2^k} \underset{k\to\infty}{\longrightarrow} 0.$$

So there is $k(\epsilon)$ such that for all $k \geq k(\epsilon)$,

$$\delta_{t,\lfloor(1-\epsilon)k\rfloor}^{\ell^k} - C_1 C_2^k \ell^{-uk(k+1)/2}(1-\delta_{t,\lfloor(1-\epsilon)k\rfloor}) \leq 0.$$

Moreover the function $x^{\ell^k} - C_1 C_2^k \ell^{-uk(k+1)/2}(1-x)$ is increasing on $]0,1[$.

So on $]0, \delta_{t,\lfloor (1-\epsilon)k \rfloor}]$

$$\frac{x^{\ell^k}}{1-x} < C_1 C_2^k \ell^{-uk(k+1)/2}.$$

This inequality ensures that $f_{w,n}$ is subject to the same oscillations as f_{w,ℓ^k-1} provided Lemma 2 holds. Indeed, for h and k large,

$$f_{w,n}(\delta_{j,h}) \geq f_{w,\ell^k-1}(\delta_{j,h}) - \frac{x^{\ell^k}}{1-x} > 0$$

when $f_{w,\ell^k-1}(\delta_{j,h}) > 0$ and

$$f_{w,n}(\delta_{j,h}) \leq f_{w,\ell^k-1}(\delta_{j,h}) + \frac{x^{\ell^k}}{1-x} < 0$$

when $f_{w,\ell^k-1}(\delta_{j,h}) < 0$.

Let $\varrho_1(f_{w,n})$ be the number of real roots of $f_{w,n}$ in $]0,1[$. So

$$\varrho_1(f_{w,n}) \geq \frac{(1-\epsilon)t \log n}{\log \ell} - K_1,$$

for a suitable absolute constant K_1, as soon as $n \geq N(\epsilon) = \ell^{k(\epsilon)}$.

If ℓ is even then $f_{w,n}$ has at least $\lfloor (1-\epsilon)k \rfloor t$ roots in $]-1,0[$ for large k and $n \geq \ell^k$. Then

$$\varrho(f_{w,n}) \geq \frac{2(1-\epsilon)t \log n}{\log \ell} - K,$$

for all large n.

If ℓ is odd we apply what we have just seen to $P_w(-X)$. This polynomial has by hypothesis t' roots in $]0,1[$ so that $f_{w,n}(-X)$ has at least $\lfloor (1-\epsilon)k \rfloor t'$ roots in $]0,1[$. Thus

$$\varrho(f_{w,n}) \geq \frac{(1-\epsilon)(t+t') \log n}{\log \ell} - K,$$

for all large n.

This ends the proof of Theorem 1. ∎

Under the assumptions of Theorem 1 we have the following inequalities.

Corollary. *If ℓ is even then*

$$\liminf_{N\to\infty} \frac{1}{N \log N} \sum_{n=0}^{N-1} \varrho(f_{w,n}) \geq \frac{2t}{\log \ell}.$$

If ℓ is odd then

$$\liminf_{N\to\infty} \frac{1}{N \log N} \sum_{n=0}^{N-1} \varrho(f_{w,n}) \geq \frac{t+t'}{\log \ell}.$$

Proof. Assuming ℓ to be even, we apply Theorem 1 to obtain

$$\frac{1}{N\log N}\sum_{n=0}^{N-1}\varrho(f_{w,n}) \ge \frac{1}{N\log N}\cdot\frac{2(1-\epsilon)t}{\log\ell}\Big(\sum_{n=N(\epsilon)}^{N-1}\log n - NK\Big)$$
$$\ge \frac{2(1-\epsilon)t}{\log\ell} - \frac{K'}{\log N}.$$

When N tends to infinity we get

$$\liminf_{N\to\infty}\frac{1}{N\log N}\sum_{n=0}^{N-1}\varrho(f_{w,n}) \ge \frac{2(1-\epsilon)t}{\log\ell}.$$

Since this is true for all $\epsilon > 0$ we deduce that

$$\liminf_{N\to\infty}\frac{1}{N\log N}\sum_{n=0}^{N-1}\varrho(f_{w,n}) \ge \frac{2t}{\log\ell}.$$

If ℓ is odd the same argument works. ■

So generalized Thue–Morse sequences yield polynomials with many real roots; but what can we say on their random behaviour? In the next section we say a few words about this.

4. Spectral measure of the generalized Thue–Morse sequences. A good tool to evaluate the random nature of a sequence is to study its spectral measure defined for instance in [11] and in [1]. We recall basic definitions, and then we give just the important results without all intermediate steps.

Let $\gamma_w(h)$ be the correlation function of $(\varepsilon_{w,n})_{n\in\mathbb{N}}$ defined by

$$\gamma_w(h) = \lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\varepsilon_{w,n}\varepsilon_{w,n+h}.$$

This limit always exists for generalized Thue–Morse sequences. The spectral measure $d\sigma_w$ is linked to $\gamma_w(h)$ by the formula

$$\gamma_w(h) = \int_0^1 e^{2i\pi hx}\,d\sigma_w(x).$$

For all $i \in [\![1, \ell-1]\!]$, γ_w satisfies the recurrence relations

$$\gamma_w(\ell k) = \gamma_w(k),$$
$$\gamma_w(\ell k + i) = \frac{a_{w,i}}{\ell}\gamma_w(k) + \frac{a_{w,\ell-i}}{\ell}\gamma_w(k+1),$$

where $a_{w,i} = \sum_{j=0}^{\ell-i-1} w[j]w[i+j]$.

To establish that $d\sigma_w$ is continuous we know [1] that it is sufficient to prove that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=0}^{N-1}\gamma_w(n)^2=0.$$

Let

$$\Gamma_w(h)=\lim_{N\to\infty}\frac{1}{N}\sum_{m<N}\gamma_w(m)\gamma_w(m+h),$$

and we are left to show $\Gamma_w(0)=0$. Following the ideas of [1, Appendix I] we obtain the system

$$\mathcal{S}:\begin{cases} c_{11}(w)\Gamma_w(0)+c_{12}(w)\Gamma_w(1)=0,\\ c_{21}(w)\Gamma_w(0)+c_{22}(w)\Gamma_w(1)=0,\end{cases}$$

where

$$c_{11}(w)=1-\frac{1}{\ell}-2\sum_{j=1}^{\ell-1}\frac{a_{w,j}^2}{\ell^3},$$

$$c_{12}(w)=-2\sum_{j=1}^{\ell-1}\frac{a_{w,j}a_{w,\ell-j}}{\ell^3},$$

$$c_{21}(w)=\frac{2a_{w,1}}{\ell^2}+2\sum_{j=1}^{\ell-2}\frac{a_{w,j}a_{w,j+1}}{\ell^3},$$

$$c_{22}(w)=-1+\frac{2a_{w,\ell-1}}{\ell^2}+\sum_{j=1}^{\ell-2}\frac{a_{w,j+1}a_{w,\ell-j}+a_{w,j}a_{w,\ell-j-1}}{\ell^3}.$$

LEMMA 3. *Let w be word of length $\ell\ge 2$. We say w is of type $++$ if $w=+\ldots+$ (ℓ times), and of type $+-+$ if ℓ is odd and $w=+(-+)\ldots(-+)$ ($(\ell-1)/2$ brackets). Then the determinant $\Delta(w)$ of $\mathcal{S}$ vanishes if and only if w is of type $++$ or $+-+$.*

Proof. It is almost immediate that

$$|c_{11}(w)|\ge\frac{\ell^3-\ell}{3\ell^3},\qquad |c_{12}(w)|\le\frac{\ell^3-\ell}{3\ell^3},$$
$$|c_{21}(w)|\le\frac{2\ell^3-2\ell}{3\ell^3},\qquad |c_{22}(w)|\ge\frac{2\ell^3-2\ell}{3\ell^3}.$$

For example, let us show the last inequality. We know that

$$c_{22}(w)=-1+\frac{2a_{w,\ell-1}}{\ell^2}+\sum_{j=1}^{\ell-2}\frac{a_{w,j+1}a_{w,\ell-j}+a_{w,j}a_{w,\ell-j-1}}{\ell^3}.$$

Since

$$\left|\sum_{j=1}^{\ell-2} \frac{a_{w,j+1}a_{w,\ell-j} + a_{w,j}a_{w,\ell-j-1}}{\ell^3}\right| \le \sum_{j=1}^{\ell-2} \frac{(\ell-j-1)j + (\ell-j)(j+1)}{\ell^3} \le \frac{\ell^3-4\ell}{3\ell^3},$$

we have

$$|c_{22}(w)| \ge 1 - \frac{2|a_{w,\ell-1}|}{\ell^2} - \frac{\ell^3-4\ell}{3\ell^3} \ge \frac{2\ell^3-2\ell}{3\ell^3}.$$

We also notice that the last inequality is an equality if and only if

$$\begin{cases} |a_{w,j}| = \ell - j & \text{for } j \in [\![0, \ell-1]\!], \\ a_{w,\ell-1} = 1. \end{cases} \tag{4}$$

If (4) is not satisfied, then $|c_{22}(w)| > (2\ell^3 - 2\ell)/(3\ell^3)$ and

$$|\Delta(w)| \ge |c_{11}(w)c_{22}(w)| - |c_{12}(w)c_{21}(w)| > 0,$$

which implies that $\Gamma_w(0) = 0$. Now if (4) holds then $w[j]w[j+1]$ does not depend on $j \in [\![0, \ell-2]\!]$. Therefore $w[j] = 1$ or $w[j] = (-1)^j$ for all $j \in [\![0, \ell-1]\!]$. When $w[j] = (-1)^j$ the relation $a_{w,\ell-1} = w[0]w[\ell-1] = 1$ shows that ℓ is necessarily odd. Obviously if w is of type $++$ or $+-+$ then $\Delta(w) = 0$, so that the result is proved. ■

This lemma ensures that the spectral measure of a generalized Thue–Morse sequence is continuous, except for trivial $++$ or $+-+$ cases. However, although continuous, $d\sigma_w$ is singular (see Theorem 6 of [4]). Therefore $d\sigma_w$ is not absolutely continuous, which would be a true random behaviour. Nonetheless this ensures that $(\varepsilon_{w,n})_{n\in\mathbb{N}}$ is pseudo-random in the sense of Bass [2] and Bertrandias [3].

Finally, when w is of type $++$ its spectral measure is the Dirac mass $\delta_0(x)$. If w is of type $+-+$, then $d\sigma_w$ is $\delta_{1/2}(x)$.

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