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Extreme values of the Riemann zeta-function on short zero intervals

by

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1. Introduction. We are interested in the distribution of the extreme values taken by the function $|\zeta(1/2+it)|$ between adjacent zeros, conditional on the zero spacing. This study was initiated in [5], and continued by Steuding [7]. Suppose that $\{t_n : n \in \mathbb{N}\}$ denotes the sequence of zeros in $\mathbb{R}^+$ counted according to multiplicity and arranged in non-decreasing order, and $N_0(T) := \text{card}\{n : 0 < t_n \le T\}$. We write $l_n := t_{n+1} - t_n$, and we consider the intervals (t_n, t_{n+1}) satisfying the condition $l_n \le 2\pi\theta/\log t_n$; these are relatively short if $\theta \in \mathbb{R}^+$ is small, because we expect that $N_0(T) \sim (T/2\pi)\log T$, so that l_n equals approximately $2\pi/\log t_n$ on average, moreover we know unconditionally that $N_0(T) \asymp T\log T$. The question arises whether the zeta-function is also relatively small on such intervals, or if it has tall spikes, how often these occur. A complication in this problem is that we do not know the frequency of these short intervals. According to Montgomery's pair correlation conjecture, the number of the intervals specified above with $t_n \le T$ is $\ll \theta^3 N_0(T)$, but actually nothing has been proved in this direction.

Following [5], we define

$$M_n := \max\{|\zeta(1/2+it)| : t_n \le t \le t_{n+1}\}, \tag{1}$$

$$M^{(k)}(T,\theta) := \sum_{n\le N} \left\{ M_n^{2k} : l_n \le \frac{2\pi\theta}{\log T} \right\} \quad (k \ge 0). \tag{2}$$

In the sum (2), $N = N_0(T)$. Also $M^{(k)}(T,\infty)$ denotes the sum in which l_n is unrestricted. In [5], we showed that

$$M^{(k)}(T,\theta) \le H_k(\theta)\{1 + O(1/\log T)\}T\log^{k^2+1} T \quad (k = 1,2) \tag{3}$$

where $H_k(\theta)$ is an increasing, continuous, bounded function satisfying, in

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the two cases,

$$(4)\qquad \begin{aligned} H_1(\theta) &= \frac{\pi^3\theta^3}{480} \qquad \left(0 < \theta \le \theta_1 = \frac{5\sqrt{2}}{\pi\sqrt{3}}\right), \\ H_2(\theta) &= \frac{\pi\theta^3}{840} \qquad \left(0 < \theta \le \theta_2 = \frac{\sqrt{35}}{\pi\sqrt{3}}\right). \end{aligned}$$

In the range $\theta_k < \theta < \infty$, each $H_k(\theta)$ is a rather complicated transcendental function, which was evaluated for some typical values and which levels off towards the value obtained in the unrestricted case, respectively:

$$(5)\qquad \begin{aligned} H_1(\infty) &= \frac{5 + 2\sqrt{10}}{3\sqrt{75 + 60\sqrt{10}}} = .23200260\ldots, \\ H_2(\infty) &= \frac{28 + \sqrt{2086}}{6\pi^2\sqrt{2940 + 210\sqrt{2086}}} = .10968770\ldots/\pi^2. \end{aligned}$$

The first of the constants in (5) is not far from the best possible since Conrey and Ghosh [2] established that, on the Riemann Hypothesis, actually

$$(6)\quad M^{(1)}(T,\infty) = \left\{\frac{e^2-5}{4\pi} + o(1)\right\} T\log^2 T \qquad \left(\frac{e^2-5}{4\pi} = .19011504\ldots\right);$$

probably there is a similar formula when $h = 2$ with a constant not much smaller than that given in (5). Conrey [1] showed that

$$(7)\qquad \left\{\frac{\sqrt{21}}{90\pi^2} + o(1)\right\} T\log^5 T \le M^{(2)}(T) \le \left\{\frac{\sqrt{15}}{30\pi^2} + o(1)\right\} T\log^5 T,$$

these constants being $.0509175\ldots/\pi^2$ and $.1290994\ldots/\pi^2$ respectively. The upper bound in (7) is unconditional but the lower bound depends on the hypothesis that Hardy's function $Z(t)$ has only one stationary point in each interval (t_n, t_{n+1}); it is well known that this follows from the Riemann Hypothesis. For small θ we have $H_k(\theta) \ll \theta^3$ from (4), and recently Steuding [7] has given a simpler and more transparent proof of this result in the case $k = 2$, albeit with a weaker constant $\pi/140$. He also obtains $H_1(\theta) \le \pi\theta/6$. I cannot improve my bound for $H_1(\theta)$ for any value of θ, but I offer the following result about $H_2(\theta)$.

Theorem 1. *The inequality* (3) *is valid with*

$$(8)\qquad H_2(\theta) = \frac{\pi^3\theta^5}{100800} \qquad \left(\theta \le \theta_2 = \frac{6\sqrt{14}}{\pi\sqrt{17}} = 1.73316908\ldots\right).$$

In the range $\theta_2 \le \theta < \infty$, $H_2(\theta)$ *increases continuously towards the limit* $.1079199/\pi^2$, *with some values given in the following table.*

Table 1

τ	$\psi(\tau)$	u	θ	$\pi^2 H_2(\theta)$
0	5	0	1.7331690...	.04747811...
.1	4.88903944...	.03656627...	1.7468410...	.04935892...
.2	4.58070935...	.07223527...	1.7867108...	.05492413...
.3	4.12676086...	.10666825...	1.8472815...	.06350647...
.4	3.57041875...	.14059931...	1.9182403...	.07343739...
.5	2.93619379...	.17579172...	1.9892810...	.08269191...
.6	2.24438114...	.21472579...	2.0550451...	.08999126...
.7	1.52929734...	.26061358...	2.1186181...	.09531191...
.8	.84806601...	.31755016...	2.2024813...	.09968077...
.9	.28526769...	.38748063...	2.4188151...	.10440673...
1	0	.44061115...	∞	.10791999...

The first two columns are explained in the course of the proof below. The improvement obtained over the results in [5] decreases as θ increases: for example the new bound is better by a factor $7/72$ at the old $\theta_2 = \pi^{-1}\sqrt{35/3}$ but at infinity the results are barely distinguishable.

The correct interpretation of this result is not clear. We now have $H_1(\theta) \ll \theta^3$ and $H_2(\theta) \ll \theta^5$. If these bounds represented the true orders of magnitude then it would be awkward to match them with the pair correlation conjecture; indeed, $H_1(\theta)$ would be essentially the same as the frequency of the short intervals and for this to happen, the behaviour of the M_n would have to be more or less independent of θ. An alternative model would be that the M_n were usually smaller on these short intervals, with occasional large spikes dominating the sum $M^{(1)}(T,\theta)$. In either scenario $H_2(\theta)$ would appear to have to be of the order at least θ^3, whereas we know that $H_2(\theta) \ll \theta^5$.

As in [5] our method involves an inequality relating the maximum modulus of a function on an interval between zeros to certain integral means of the function and some of its derivatives. The new inequality occupies most of the paper. It involves the parameters λ, μ, ν which ideally would be chosen optimally, however I am not yet able to prove the inequality in the most general case; I choose some parameters that I can cope with, which may not be optimal. This affects the various constants given above and Table 1, but not the exponent 5 of θ.

2. An extremal problem. The results in [5] depended on the following inequality.

LEMMA 1. *Let $y(x)$ be real-valued on $[a,b]$, $y(a) = y(b) = 0$. Suppose that y is twice differentiable, $y'' \in L^2[a,b]$, and that*

$$(9) \qquad \int_a^b y(x)^2\,dx = A, \qquad \int_a^b y'(x)^2\,dx = B, \qquad \int_a^b y''(x)^2\,dx = C.$$

Put $M := \max\{|y(x)| : a < x < b\}$. *Then, for arbitrary* $\mu > \lambda > 0$, *we have*

$$(10) \qquad M^2 \le \frac{\lambda^2\mu^2 A + (\lambda^2+\mu^2)B + C}{2(\mu^2-\lambda^2)} \times \left\{ \frac{1}{\lambda}\tanh\frac{\lambda L}{2} - \frac{1}{\mu}\tanh\frac{\mu L}{2} \right\} \qquad (L := b-a).$$

There are two useful features here: first that upper bound is linear in A, B and C, which is essential for the application, and second that the factor involving L on the right is $\ll L^3$ for small L. The inequality is sharp in the sense that it becomes false in general if any factor < 1 be introduced on the right-hand side. The question as to whether a sharp bound for M in terms of A, B, C (in the case that (9) is internally consistent) may be derived from (10) by choosing λ and μ in an optimal fashion, is interesting in itself but not relevant to the application to the zeta-function (because we should, in optimizing, lose the linearity in A, B, C).

The idea in [5] was to apply Lemma 1 with $[a,b] = [t_n, t_{n+1}]$ and $y = Z, Z^2$ respectively to bound the sums $M^{(1)}(T,\theta)$ and $M^{(2)}(T,\theta)$. I have nothing to add when $k = 1$, but observe now that this strategy disregards some information when $k = 2$, namely that in this case $y = Z^2$ has *double* zeros at t_n and t_{n+1}. With this in mind, we look for a version of Lemma 1 containing the extra hypothesis that $y'(a) = y'(b) = 0$, and it emerges that (9) may be usefully supplemented by the equation

$$(11) \qquad \int_a^b y'''(x)^2\,dx = D;$$

clearly we need to add the hypothesis that y'' is differentiable and $y''' \in L^2[a,b]$. Since these moments of $Z(t)$ and its derivatives may all be evaluated, these are acceptable prices. We want an inequality corresponding to Lemma 1 of the following shape, in which the factor $F(\lambda,\mu,\nu;L)$ on the right is sharp and, for fixed λ,μ,ν, has the property that $F(\lambda,\mu,\nu;L) \ll L^5$ when $L \to 0$.

Conjectural inequality. *Let* $y(x)$ *be real-valued on* $[a,b]$ *and* $y(a) = y'(a) = y(b) = y'(b) = 0$. *Suppose that* y *is three times differentiable*, $y''' \in L^2[a,b]$, *and that*

$$(12) \qquad \begin{aligned} &\int_a^b y(x)^2\,dx = A, \qquad &&\int_a^b y''(x)^2\,dx = C,\\ &\int_a^b y'(x)^2\,dx = B, \qquad &&\int_a^b y'''(x)^2\,dx = D. \end{aligned}$$

Put $M := \max\{|y(x)| : a < x < b\}$. *Then, for arbitrary* $\nu > \mu > \lambda > 0$, *we have*

$$M^2 \le \{\lambda^2\mu^2\nu^2 A + (\lambda^2\mu^2 + \mu^2\nu^2 + \nu^2\lambda^2)B + (\lambda^2 + \mu^2 + \nu^2)C + D\} \times F(\lambda, \mu, \nu; L), \tag{13}$$

in which $L = b - a$.

In order to define F we introduce the functions

$$C(\lambda, \mu, \nu; t) := \frac{\coth(\lambda t/2)}{(\mu^2 - \lambda^2)(\nu^2 - \lambda^2)\lambda} + \frac{\coth(\mu t/2)}{(\nu^2 - \mu^2)(\lambda^2 - \mu^2)\mu} + \frac{\coth(\nu t/2)}{(\lambda^2 - \nu^2)(\mu^2 - \nu^2)\nu}, \tag{14}$$

$$T(\lambda, \mu, \nu; t) := \frac{\tanh(\lambda t/2)}{(\mu^2 - \lambda^2)(\nu^2 - \lambda^2)\lambda} + \frac{\tanh(\mu t/2)}{(\nu^2 - \mu^2)(\lambda^2 - \mu^2)\mu} + \frac{\tanh(\nu t/2)}{(\lambda^2 - \nu^2)(\mu^2 - \nu^2)\nu}. \tag{15}$$

Notice that in each of the sums (14) and (15), there is one negative term, in the middle. As $t \in \mathbb{R}^+$ increases, $C(\lambda, \mu, \nu; t)$ decreases from ∞, and $T(\lambda, \mu, \nu; t)$ increases from 0, toward the common limit

$$\frac{\lambda + \mu + \nu}{\lambda\mu\nu(\lambda + \mu)(\mu + \nu)(\nu + \lambda)}. \tag{16}$$

To see the monotonicity property of T, observe that the function $\operatorname{sech}^2 \sqrt{x}$ is a convex function of x, which implies that

$$\operatorname{sech}^2 \frac{\mu t}{2} \le \frac{\nu^2 - \mu^2}{\nu^2 - \lambda^2} \operatorname{sech}^2 \frac{\lambda t}{2} + \frac{\mu^2 - \lambda^2}{\nu^2 - \lambda^2} \operatorname{sech}^2 \frac{\nu t}{2} \tag{17}$$

and the result follows on differentiating T. A similar argument shows that C is decreasing, actually convex on $\mathbb{R}^+$. For this we prove first that the function

$$y(x) := \sqrt{x} \operatorname{cosech}^2 \sqrt{x} \coth \sqrt{x} \tag{18}$$

is a convex function of x. We have, noticing that $\sqrt{x} \coth \sqrt{x} \ge 1$ on the second line,

$$\begin{aligned} y''(x) \sinh^4 \sqrt{x} &= \left\{ \frac{1}{2\sqrt{x}} - \frac{1}{8x\sqrt{x}} \right\} \sinh 2\sqrt{x} - \frac{1}{4x} \cosh 2\sqrt{x} - \frac{1}{2x} + \frac{3}{\sqrt{x}} \coth \sqrt{x} \\ &\ge \left\{ \frac{1}{2\sqrt{x}} - \frac{1}{8x\sqrt{x}} \right\} \sinh 2\sqrt{x} - \frac{1}{4x} \cosh 2\sqrt{x} + \frac{5}{2x} \\ &= \frac{2}{x} + d_0 + d_1 x + d_2 x^2 + \cdots \qquad \text{(say)}, \end{aligned} \tag{19}$$

and it emerges that all the coefficients d_j are positive. An inequality similar to (17) then establishes that $C''(\lambda,\mu,\nu;t) > 0$.

Since $T(\lambda,\mu,\nu;t)$ is increasing and bounded it cannot be convex: it is intuitive, but we shall not prove, that T'' has one sign change from positive to negative. We have the expansions

$$\begin{aligned} C(\lambda,\mu,\nu;t) &= \frac{2}{\lambda^2\mu^2\nu^2 t} + \frac{t^5}{15120} + \cdots, \\ T(\lambda,\mu,\nu;t) &= \frac{t^5}{240} - \frac{17(\lambda^2+\mu^2+\nu^2)t^7}{40320} + \cdots \end{aligned} \tag{20}$$

for small t, together with the relation

$$C(\lambda,\mu,\nu;t) + T(\lambda,\mu,\nu;t) = 2C(\lambda,\mu,\nu;2t). \tag{21}$$

We define

$$\begin{aligned} f(\lambda,\mu,\nu;t) &:= \left\{\frac{1}{C(\lambda,\mu,\nu;t)} + \frac{1}{T(\lambda,\mu,\nu;t)}\right\} \\ &= \frac{240}{t^5} + \frac{170(\lambda^2+\mu^2+\nu^2)}{7t^3} + \cdots, \end{aligned} \tag{22}$$

$$F(\lambda,\mu,\nu;L) := \frac{1}{f(\lambda,\mu,\nu;L/2)} = \frac{L^5}{7680} - \frac{17(\lambda^2+\mu^2+\nu^2)L^7}{5160960} + \cdots, \tag{23}$$

noticing that for $\delta \in \mathbb{R}^+$ we have the scaling formulae

$$\begin{aligned} C(\lambda,\mu,\nu;t) &= \delta^5 C(\delta\lambda,\delta\mu,\delta\nu;t/\delta), \quad T(\lambda,\mu,\nu;t) = \delta^5 T(\delta\lambda,\delta\mu,\delta\nu;t/\delta), \\ f(\lambda,\mu,\nu;t) &= \delta^{-5} f(\delta\lambda,\delta\mu,\delta\nu;t/\delta), \quad F(\lambda,\mu,\nu;t) = \delta^5 F(\delta\lambda,\delta\mu,\delta\nu;t/\delta). \end{aligned} \tag{24}$$

Finally we state, for each fixed triple λ,μ,ν:

HYPOTHESIS $\mathrm{A}(\lambda,\mu,\nu)$. *The function $f(\lambda,\mu,\nu;t)$ is convex for $t \in \mathbb{R}^+$.*

This seems particularly awkward to prove and it is the sticking point in our method. A consequence of the hypothesis is that $f(\lambda,\mu,\nu;t)$ is decreasing, as it converges to a finite limit as $t \to \infty$. Thus $F(\lambda,\mu,\nu;t)$ increases with t.

REMARK 1. The scaling formulae (24) show that $\mathrm{A}(\lambda,\mu,\nu)$ and $\mathrm{A}(\delta\lambda,\delta\mu,\delta\nu)$ are equivalent for every $\delta > 0$. So we can normalize, for example by assuming that $\lambda = 1$.

The key result required for our application is

THEOREM 2. *Suppose that $\nu > \mu > \lambda > 0$ are such that Hypothesis $\mathrm{A}(\lambda,\mu,\nu)$ is valid. Then the conjectural inequality* (13) *holds for every function y satisfying the conditions stated above together with* (12).

3. An easier extremal problem. In this section we tackle a supplementary problem, which we can solve completely, essentially by moving the maximum to one end of the interval.

THEOREM 3. *Suppose that $y(x)$ is real-valued and three times differentiable on $[0,t]$, that $y''' \in L^2[0,t]$, and that $y(0) = M$ and $y'(0) = y(t) = y'(t) = 0$. Then if λ, μ, ν are distinct positive numbers we have*

$$\int_0^t \{y'''(x)^2 + (\lambda^2 + \mu^2 + \nu^2) y''(x)^2 + (\lambda^2\mu^2 + \mu^2\nu^2 + \nu^2\lambda^2) y'(x)^2 + \lambda^2\mu^2\nu^2 y(x)^2\}\, dx \geq \frac{1}{2} f(\lambda, \mu, \nu; t) M^2. \tag{25}$$

Notice that we do not require M to be the maximum value of $|y|$ here, but it is intuitive that it actually is so in the extremal case, moreover that y is then positive and decreasing, with y'' changing from negative to positive at some point of the interval. We do not assume any of these propositions.

Proof of Theorem 3. Denote the integral on the left of (25) by $J(y)$. We expand $y'''(x)$ as a Fourier sine series on $[0,t]$ (with no claims about convergence). We may integrate this series term-by-term to obtain the Fourier cosine series of $y''(x)$, and we notice that there is no constant term, because $y'(0) = y'(t)$. Integrating term-by-term again we obtain the Fourier sine series of $y'(x)$, and, after a final integration we have (say)

$$y(x) = \frac{1}{2} a_0 + a_1 \cos \frac{\pi x}{t} + a_2 \cos \frac{2\pi x}{t} + a_3 \cos \frac{3\pi x}{t} + \cdots \tag{26}$$

with equality in (26) because y has bounded variation and is continuous. In particular, we have

$$M = \frac{1}{2} a_0 + a_1 + a_2 + a_3 + \cdots, \qquad 0 = \frac{1}{2} a_0 - a_1 + a_2 - a_3 + \cdots, \tag{27}$$

whence

$$\frac{1}{2} M = \frac{1}{2} a_0 + a_2 + a_4 + \cdots = a_1 + a_3 + a_5 + \cdots. \tag{28}$$

Put

$$\begin{aligned} b(n) &:= \frac{\pi^6 n^6}{t^6} + (\lambda^2 + \mu^2 + \nu^2) \frac{\pi^4 n^4}{t^4} \\ &\quad + (\lambda^2\mu^2 + \mu^2\nu^2 + \nu^2\lambda^2) \frac{\pi^2 n^2}{t^2} + \lambda^2\mu^2\nu^2 \\ &= \left(\frac{\pi^2 n^2}{t^2} + \lambda^2\right)\left(\frac{\pi^2 n^2}{t^2} + \mu^2\right)\left(\frac{\pi^2 n^2}{t^2} + \nu^2\right), \end{aligned} \tag{29}$$

and observe that

$$J(y) = \frac{t}{2}\left\{\frac{1}{2}\,b(0)a_0^2 + \sum_{n=1}^{\infty} b(n)a_n^2\right\}. \tag{30}$$

We apply Cauchy's inequality to each part of (28), to obtain

$$\frac{1}{4}\,M^2 \le \left\{\frac{1}{2}\,b(0)a_0^2 + b(2)a_2^2 + b(4)a_4^2 + \cdots\right\}\left\{\frac{1}{2b(0)} + \frac{1}{b(2)} + \frac{1}{b(4)} + \cdots\right\} \tag{31}$$

and

$$\frac{1}{4}\,M^2 \le \{b(1)a_1^2 + b(3)a_3^2 + b(5)a_5^2 + \cdots\}\left\{\frac{1}{b(1)} + \frac{1}{b(3)} + \frac{1}{b(5)} + \cdots\right\}. \tag{32}$$

We deduce from (30)–(32) that

$$\begin{aligned} J(y) \ge \frac{t}{8}\Bigg\{&\left(\frac{1}{2b(0)} + \frac{1}{b(2)} + \frac{1}{b(4)} + \cdots\right)^{-1} \\ &+ \left(\frac{1}{b(1)} + \frac{1}{b(3)} + \frac{1}{b(5)} + \cdots\right)^{-1}\Bigg\}M^2. \end{aligned} \tag{33}$$

Recall that (except at the poles)

$$\pi\coth\pi x = \frac{1}{x} + 2x\sum_{n=1}^{\infty}\frac{1}{n^2+x^2}. \tag{34}$$

In order to employ (34) it is convenient to write $\lambda_1 := t\lambda/\pi$ etc., so that we have

$$\begin{aligned} &\frac{1}{2b(0)} + \frac{1}{b(2)} + \cdots \\ &\quad= \frac{t^6}{\pi^6}\left\{\frac{1}{2\lambda_1^2\mu_1^2\nu_1^2} + \sum_{\text{even } n\in\mathbb{N}}\frac{1}{(n^2+\lambda_1^2)(n^2+\mu_1^2)(n^2+\nu_1^2)}\right\} \\ &\quad= \frac{t^6}{\pi^6}\Bigg\{\frac{1}{(\mu_1^2-\lambda_1^2)(\nu_1^2-\lambda_1^2)}\left\{\frac{1}{2\lambda_1^2} + \frac{1}{2^2+\lambda_1^2} + \frac{1}{4^2+\lambda_1^2} + \cdots\right\} \\ &\qquad+ \frac{1}{(\nu_1^2-\mu_1^2)(\lambda_1^2-\mu_1^2)}\left\{\frac{1}{2\mu_1^2} + \frac{1}{2^2+\mu_1^2} + \frac{1}{4^2+\mu_1^2} + \cdots\right\} \\ &\qquad+ \frac{1}{(\lambda_1^2-\nu_1^2)(\mu_1^2-\nu_1^2)}\left\{\frac{1}{2\nu_1^2} + \frac{1}{2^2+\nu_1^2} + \frac{1}{4^2+\nu_1^2} + \cdots\right\}\Bigg\} \end{aligned} \tag{35}$$

by partial fractions. We write $\sum^{(3)}$ followed by the first of three terms to indicate a sum like (35) where the second and third terms are obtained by permuting the variables λ_1, μ_1, ν_1 cyclically. We apply (34), and recall (14), to find that the sum in (35) equals

$$(36)\quad \frac{t^6}{4\pi^5}\sum\nolimits^{(3)}\frac{1}{(\mu_1^2-\lambda_1^2)(\nu_1^2-\lambda_1^2)\lambda_1}\coth\frac{\lambda_1\pi}{2} = \frac{t}{4}\sum\nolimits^{(3)}\frac{1}{(\mu^2-\lambda^2)(\nu^2-\lambda^2)\lambda}\coth\frac{\lambda t}{2} = \frac{t}{4}C(\lambda,\mu,\nu;t).$$

A similar argument involving (15) establishes that

$$(37)\quad \frac{1}{b(1)}+\frac{1}{b(3)}+\frac{1}{b(5)}+\cdots = \frac{t}{4}\sum\nolimits^{(3)}\frac{1}{(\mu^2-\lambda^2)(\nu^2-\lambda^2)\lambda}\tanh\frac{\lambda t}{2} = \frac{t}{4}T(\lambda,\mu,\nu;t),$$

and we insert (36) and (37) into (33) to obtain (25).

The inequality in (25) is sharp, moreover we may identify the extremal function as

$$(38)\quad y(x)=\frac{M}{2}\sum\nolimits^{(3)}\frac{1}{(\mu^2-\lambda^2)(\nu^2-\lambda^2)\lambda} \times\left\{\frac{\cosh\lambda(t/2-x)}{C(\lambda,\mu,\nu;t)\sinh(\lambda t/2)}+\frac{\sinh\lambda(t/2-x)}{T(\lambda,\mu,\nu;t)\cosh(\lambda t/2)}\right\}.$$

To track this function down we observe that in the extreme case, each application of Cauchy's inequality in (31) and (32) must be sharp; that is to say, a_n must be proportional to $1/b(n)$ when n is even or odd (or zero), but the constants of proportionality may be (and are) different in the various cases, and are determined by the boundary conditions $y(0)=M$, $y(t)=0$. It is easy to see in (38) that $y'(0)=y'(t)=0$, as the hyperbolic functions cancel at the end-points when this expression is differentiated. This completes the proof.

Remark 2. We expect the extremal function to be a linear combination of the six functions $e^{\pm\lambda x}$ etc. because these are the independent solutions of the Euler–Lagrange equation associated with (25).

4. Proof of Theorem 2. We may assume that $a=0$, $b=L$. The boundary conditions are $y(0)=y(L)=y'(0)=y'(L)=0$ and we put

$$(39)\quad I(y):=\int_0^L\{y'''(x)^2+(\lambda^2+\mu^2+\nu^2)y''(x)^2 + (\lambda^2\mu^2+\mu^2\nu^2+\nu^2\lambda^2)y'(x)^2+\lambda^2\mu^2\nu^2y(x)^2\}\,dx.$$

Moreover we suppose that $|y(x)|$ attains its maximum value M at the point $x=t$, indeed that $y(t)=M$, replacing y by $-y$ if necessary. Then $y'(t)=0$

and we see that we can split the integral in (39) into two ranges, $[0,t]$ and $[t,L]$, writing

$$I(y) =: I_1(y) + I_2(y). \tag{40}$$

On each range we have a problem of the type considered in Section 3. In the first range we apply Theorem 3 to the function $y(t-x)$ $(0 \le x \le t)$ to deduce from (22) that

$$I_1(y) \ge \tfrac{1}{2} f(\lambda,\mu,\nu;t)M^2, \tag{41}$$

similarly we apply Theorem 3 to the function $y(x-t)$ $(t \le x \le L)$ to obtain

$$I_2(y) \ge \tfrac{1}{2} f(\lambda,\mu,\nu;L-t)M^2, \tag{42}$$

whence

$$I(y) \ge \tfrac{1}{2}\{f(\lambda,\mu,\nu;t) + f(\lambda,\mu,\nu;L-t)\}M^2. \tag{43}$$

We do not know the value of t and so we require the minimum of the right-hand side as a function of t. On the assumption of Hypothesis A(λ,μ,ν) we see that this occurs in the middle of the range, that is, $I(y) \ge f(\lambda,\mu,\nu;L/2)$. We multiply this inequality by $F(\lambda,\mu,\nu;L)$ as defined in (23) to deduce (13) as required. This completes the proof.

5. A special case. If the ratios $\lambda:\mu:\nu$ are rational then we can find κ (and suppose it to be as large as possible) so that λ, μ and ν are integer multiples of κ, whence $\tanh(\lambda t/2)$, $\tanh(\mu t/2)$, $\tanh(\nu t/2)$ are rational functions of $\tanh(\kappa t/2)$. So therefore are $C(\lambda,\mu,\nu;t)$ and $T(\lambda,\mu,\nu;t)$; moreover if we put $\tanh(\kappa t/2) = \tau$ then clearly $dt/d\tau$ is also a rational function of τ. This means that in this case, Hypothesis A(λ,μ,ν) reduces to an elementary, if perhaps lengthy, calculus problem.

Consider the case $\mu = 2\lambda$, $\nu = 3\lambda$, in which $\kappa = \lambda$ and so

$$\begin{aligned} C(\lambda,2\lambda,3\lambda;t) &= \frac{1}{\lambda^5}\left\{\frac{1}{24\tau} - \frac{1}{30}\cdot\frac{1+\tau^2}{2\tau} + \frac{1}{120}\cdot\frac{1+3\tau^2}{3\tau+\tau^3}\right\} \\ &= \frac{5-\tau^2}{60\lambda^5(3\tau+\tau^3)}, \\ T(\lambda,2\lambda,3\lambda;t) &= \frac{1}{\lambda^5}\left\{\frac{\tau}{24} - \frac{1}{30}\cdot\frac{2\tau}{1+\tau^2} + \frac{1}{120}\cdot\frac{3\tau+\tau^3}{1+3\tau^2}\right\} \\ &= \frac{2\tau^5}{15\lambda^5(1+\tau^2)(1+3\tau^2)}. \end{aligned} \tag{44}$$

We have

$$\begin{aligned} &C(\lambda,2\lambda,3\lambda;t)^{-1} + T(\lambda,2\lambda,3\lambda;t)^{-1} \\ &\qquad = \frac{15\lambda^5\{5+20\tau^2+14\tau^4+20\tau^6+5\tau^8\}}{2\tau^5(5-\tau^4)}, \end{aligned} \tag{45}$$

and we denote the right-hand side of (45) by $(15\lambda^5/2)g(\tau)$. For this to be a convex function of t it is necessary and sufficient that for $0 < \tau < 1$ we should have

$$(1-\tau^2)g''(\tau) - 2\tau g'(\tau) > 0. \tag{46}$$

A calculation shows that

$$-g'(\tau) = \frac{5(1+\tau^2)(1-\tau^2)(25+60\tau^2+30\tau^4+12\tau^6+\tau^8)}{\tau^6(5-\tau^4)^2} > 0. \tag{47}$$

Now we differentiate (47) logarithmically to obtain

$$\frac{g''(\tau)}{g'(\tau)} = \frac{-4\tau^3}{1-\tau^4} - \frac{6}{\tau} + \frac{8\tau^3}{5-\tau^4} + \frac{120\tau+120\tau^3+72\tau^5+8\tau^7}{25+60\tau^2+30\tau^4+12\tau^6+\tau^8}. \tag{48}$$

The third and fourth terms increase on $[0,1]$ and so contribute at most 4.5 to the sum, whereas the first and second terms contribute less than -6. Hence the right-hand side of (48) is negative and since $g'(\tau)$ is also negative, we find that $g''(\tau)$ is positive, that is, both terms in (46) are positive. This is all we need.

6. Proof of Theorem 1. We suppose that λ, μ, ν are such that Hypothesis $\mathrm{A}(\lambda,\mu,\nu)$ is valid and apply Theorem 2 with $y = Z^2$, $a = t_n$, $b = t_{n+1} =: a + l_n$. Since $F^{-1} = f$ by (23), we see that (13) yields

$$M_n^4 f(\lambda,\mu,\nu;l_n) \le \int\limits_{t_n}^{t_{n+1}} \bigg\{\lambda^2\mu^2\nu^2 Z(t)^4 + (\lambda^2\mu^2+\mu^2\nu^2+\nu^2\lambda^2)\bigg\{\frac{d}{dt}Z(t)^2\bigg\}^2 + (\lambda^2+\mu^2+\nu^2)\bigg\{\frac{d^2}{dt^2}Z(t)^2\bigg\}^2 + \bigg\{\frac{d^3}{dt^3}Z(t)^2\bigg\}^2\bigg\}\,dt. \tag{49}$$

We add all these inequalities, to obtain

$$\sum_{n=1}^{N} M_n^4 f(\lambda,\mu,\nu;l_n) \le \int\limits_0^U \bigg\{\lambda^2\mu^2\nu^2 Z(t)^4 + \cdots + \bigg\{\frac{d^3}{dt^3}Z(t)^2\bigg\}^2\bigg\}\,dt, \tag{50}$$

in which $U := t_{N+1}$. Hardy and Littlewood [6] proved that $t_{N+1} - t_N \ll_\varepsilon t_N^{1/4+\varepsilon}$ and so we have $T < U \le T + T^{1/3}$ for large T. (In fact all we require in what follows is that $U \le T + O(T/\log T)$.)

LEMMA 2. *We have*

$$\int\limits_0^T \bigg\{\frac{d^k}{dt^k}Z(t)^2\bigg\}^2 dt = \frac{12}{(2k+1)(2k+2)(2k+3)(2k+4)\pi^2}\bigg\{1 + O\bigg(\frac{1}{\log T}\bigg)\bigg\} T\log^{2k+4} T \tag{51}$$

for each $k = 0, 1, \ldots$ *as* $T \to \infty$.

We shall not prove this result here, but remark that we have derived the form of the main term from Conrey's formula [1], which is purely asymptotic, and relied on the method set out in [3] to provide an error term.

Put $\lambda = u\log T,\ \mu = v\log T,\ \nu = w\log T$, and recall that $\mathrm{A}(\lambda,\mu,\nu)$ and $\mathrm{A}(u,v,w)$ are equivalent. Then (50) and (51) give

$$\sum_{n=1}^{N} M_n^4 f(\lambda,\mu,\nu;l_n) \le \frac{12}{\pi^2}\left\{\frac{u^2v^2w^2}{24}+\frac{u^2v^2+v^2w^2+w^2u^2}{360}+\frac{u^2+v^2+w^2}{1680}+\frac{1}{5040}\right\}\left\{1+O\left(\frac{1}{\log T}\right)\right\}T\log^{10}T. \tag{52}$$

By the scaling formulae (24), we have $f(\lambda,\mu,\nu;l_n) = f(u,v,w;l_n\log T)\log^5 T$. Since f decreases, we have $f(u,v,w;2\pi\theta) \le f(u,v,w;l_n\log T)$ whenever $l_n \le 2\pi\theta/\log T$ and so we may deduce from (52) that

$$\sum_{n\le N}\left\{M_n^4 : l_n \le \frac{2\pi\theta}{\log T}\right\} \le \frac{1}{2\pi^2}F(u,v,w;2\pi\theta)\left\{u^2v^2w^2+\frac{u^2v^2+v^2w^2+w^2u^2}{15}+\frac{u^2+v^2+w^2}{70}+\frac{1}{210}\right\} \times\left\{1+O\left(\frac{1}{\log T}\right)\right\}T\log^5 T. \tag{53}$$

We also have

$$\sum_{n\le N} M_n^4 \le \frac{1}{2\pi^2}\,F(u,v,w;\infty)\left\{u^2v^2w^2+\frac{u^2v^2+v^2w^2+w^2u^2}{15}+\frac{u^2+v^2+w^2}{70}+\frac{1}{210}\right\}\left\{1+O\left(\frac{1}{\log T}\right)\right\}T\log^5 T, \tag{54}$$

where, from (16), (22) and (23),

$$F(u,v,w;\infty) := \frac{u+v+w}{2uvw(u+v)(v+w)(w+u)}. \tag{55}$$

At this point our method is restricted by the fact that we have verified Hypothesis $\mathrm{A}(\lambda,\mu,\nu)$ in the case $1:2:3$ only. We put $v=2u$, $w=3u$ and obtain

$$\sum_{n\le N}\left\{M_n^4 : l_n \le \frac{2\pi\theta}{\log T}\right\} \le h(\theta,u)\left\{1+O\left(\frac{1}{\log T}\right)\right\}T\log^5 T, \tag{56}$$

where

$$h(\theta,u) = \frac{1}{2\pi^2}\,F(u,2u,3u;2\pi\theta)\left\{36u^6+\frac{49}{15}\,u^4+\frac{1}{5}\,u^2+\frac{1}{210}\right\}. \tag{57}$$

In the unrestricted case (55) implies

$$h(\infty, u) = \frac{1}{240\pi^2}\left\{36u + \frac{49}{15u} + \frac{1}{5u^3} + \frac{1}{210u^5}\right\}. \tag{58}$$

The next step is to choose $u = u(\theta)$ to minimize $h(\theta, u)$ for each fixed $\theta \in \mathbb{R}^+$ and for $\theta = \infty$: we find that $u(\infty) = .44061115\ldots$, which yields $H_2(\theta) = .010934581\ldots$, just marginally better than the $.011113587\ldots$ obtained in [5].

Let us turn to (56). From the definitions (14) and (15), we have

$$\begin{aligned} &C(u, 2u, 3u; \pi\theta) \\ &\quad = \left\{\frac{1}{24}\coth\frac{\pi u\theta}{2} - \frac{1}{30}\coth \pi u\theta + \frac{1}{120}\coth\frac{3\pi u\theta}{2}\right\}u^{-5} \sim \frac{1}{18u^6\pi\theta}, \end{aligned} \tag{59}$$

$$\begin{aligned} &T(u, 2u, 3u; \pi\theta) \\ &\quad = \left\{\frac{1}{24}\tanh\frac{\pi u\theta}{2} - \frac{1}{30}\tanh \pi u\theta + \frac{1}{120}\tanh\frac{3\pi u\theta}{2}\right\}u^{-5} \sim \frac{\pi^5\theta^5}{240}, \end{aligned} \tag{60}$$

for fixed $\theta \in \mathbb{R}^+$ and $u \to 0$. From (22) and (23),

$$F(u, 2u, 3u; 2\pi\theta) \sim \frac{\pi^5\theta^5}{240}\left\{1 - \frac{17}{24}\pi^2\theta^2u^2 + \cdots\right\} \qquad (\theta \in \mathbb{R}^+,\ u \to 0), \tag{61}$$

whence from (57),

$$h(\theta, 0) = \frac{\pi^3\theta^5}{100800} \qquad (\theta \in \mathbb{R}^+). \tag{62}$$

We insert this into (56) to obtain the first part of Theorem 1.

Now we consider the minimization problem in (56). Put

$$\tau := \tanh\frac{\pi u\theta}{2}, \tag{63}$$

so that from (45), we have

$$F(u, 2u, 3u; 2\pi\theta) = \frac{2\tau^5(5 - \tau^4)}{15u^5(5 + 20\tau^2 + 14\tau^4 + 20\tau^6 + 5\tau^8)} \qquad (0 \le \tau \le 1) \tag{64}$$

and

$$\begin{aligned} &h(\theta, u) \\ &= \frac{\tau^5(5 - \tau^4)}{15\pi^2(5 + 20\tau^2 + 14\tau^4 + 20\tau^6 + 5\tau^8)}\left\{36u + \frac{49}{15u} + \frac{1}{5u^3} + \frac{1}{210u^5}\right\}. \end{aligned} \tag{65}$$

As u increases, τ increases and so does the rational function of τ in (65) (see the proof of $\mathrm{A}(1, 2, 3)$ above). So we must choose u in the range where the second factor in (65) decreases, say $0 \le u \le u_2$. Notice that $u_2 = u(\infty) = .44061115\ldots$.

We differentiate $h(\theta,u)$ logarithmically with respect to u and find that

$$(66)\qquad u\,\frac{h'(\theta,u)}{h(\theta,u)} = \left\{5+\frac{4\tau^4}{5-\tau^4}-\frac{40\tau^2+56\tau^4+120\tau^6+40\tau^8}{5+20\tau^2+14\tau^4+20\tau^6+5\tau^8}\right\}(1-\tau^2)\,\frac{\operatorname{arctanh}\tau}{\tau} - \frac{5+126u^2+686u^4-7560u^6}{1+42u^2+686u^4+7560u^6} =: \psi(\tau)-\phi(u) \quad \text{(say)},$$

and $h(\theta,u)$ is decreasing if $\phi(u)>\psi(\tau)$. It is easy to see by differentiation that $\phi(u)$ decreases from 5 to 0 on the range $[0,u_2]$, also we have $\psi(0)=5$, $\psi(1-0)=0$; we claim that $\psi(\tau)$ decreases on $[0,1]$, but this is a little awkward. First we observe that the right-hand factor $(1-\tau^2)\tau^{-1}\operatorname{arctanh}\tau$ decreases: if $\tau=\tanh\xi$ it equals $2\xi/\sinh 2\xi$, which clearly decreases. We write the left-hand factor in the form $5-4m(x)$ where $x=\tau^2$ and

$$(67)\qquad m(x)=\frac{50x+65x^2+120x^3+22x^4-50x^5-15x^6}{(5-x^2)(5+20x+14x^2+20x^3+5x^4)},$$

so that $m(0)=0$, $m(1)=3/4$. After a calculation we find that

$$(68)\qquad \Delta x\,\frac{m'(x)}{m(x)} = 10(1-x)(125+450x+1675x^2+3495x^3 + 3625x^4+1943x^5+777x^6+197x^7+6x^8-5x^9)$$

where Δ denotes the denominator of $m'(x)/m(x)$, that is, the product of the three factors in (67). We see from (68) that $m(x)$ increases on $[0,1]$ and so $5-4m(x)$ decreases. Thus $\psi(\tau)$ decreases as required. We deduce that $u=\phi^{-1}\{\psi(\tau)\}$ is a one-to-one function mapping $[0,1]$ onto $[0,u_2]$.

We now simplify our calculations: rather than solve the equation $g'(\theta,u) = 0$ for fixed θ we compute $\psi(\tau)$ for a range of values of τ, as in Table 1, and then solve the equation $\phi(u)=\psi(\tau)$ for u. (This is a cubic equation for u^2, suitable for a calculator: it is an easy matter to enhance its output by two or three significant figures.) Then we have, for this τ and u,

$$(69)\qquad \theta=\frac{1}{\pi u}\log\left(\frac{1+\tau}{1-\tau}\right),$$

which yields the results listed in Table 1. Notice that τ and u tend to 0 together and θ converges to a limit θ_2 which is computed by comparing the Maclaurin expansions of $\psi(\tau)$ and $\phi(u)$. We find that $\theta_2=6\sqrt{14}/\pi\sqrt{17}$; for $\theta\le\theta_2$ we cannot improve on (62). The graph of $H_2(\theta)$ must change from convex to concave beyond (if not at) θ_2 and indeed flatten off pretty quickly: this is demonstrated by the table.

A point of caution is that we have not demonstrated that for the values of θ obtained in this calculation we have actually found the optimal u: certainly we have $\psi(\tau) - \phi(u) = 0$ by construction, but it is conceivable that this is not the only local minimum of $h(u, \theta)$. However the method leads to an upper bound $H_2(\theta)$ as required and I preferred it to a computer search for a minimum.

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