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Automorphisms with exotic orbit growth

by

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1. Introduction. Automorphisms of compact metric groups provide a simple family of dynamical systems with additional structure, rendering them particularly amenable to detailed analysis. On the other hand, they are rigid in the sense that they cannot be smoothly perturbed, and for a fixed compact metric group the group of automorphisms is itself countable and discrete. Thus it is not clear which, if any, of their dynamical properties can vary continuously. The most striking manifestation of this is that it is not known if the set of possible topological entropies is countable or is the set $[0,\infty]$ (this question is equivalent to Lehmer's problem in algebraic number theory; see Lind [13] or the monograph [7] for the details). The possible exponential growth rates for the number of closed orbits is easier to decide, and it is shown in [23] that for any $C \in [0,\infty]$ there is a compact group automorphism $T : X \to X$ with

$$\frac{1}{n}\log \mathsf{F}_T(n) \to C \tag{1.1}$$

as $n \to \infty$, where $\mathsf{F}_T(n) = |\{x \in X : T^n x = x\}|$. Unfortunately, the systems constructed to achieve this continuum of different growth rates are non-ergodic automorphisms of totally disconnected groups, and so cannot be viewed as natural examples from the point of view of dynamical systems. It is not clear if a result like (1.1) is possible within the more natural class of ergodic automorphisms on connected groups, unless C is a logarithmic Mahler measure (in which case there is a toral automorphism that achieves this).

Our purpose here is to indicate some of the diversity that is nonetheless possible for ergodic automorphisms of connected groups, for a measure of the growth in closed orbits that involves more averaging than does (1.1).

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To describe this, let $T : X \to X$ be a continuous map on a compact metric space with topological entropy $h = h(T)$. A *closed orbit* τ *of length* $|\tau| = n$ is a set $\{x, T(x), T^2(x), \dots, T^n(x) = x\}$ with cardinality n. Following the analogy between closed orbits and prime numbers advanced by work of Parry and Pollicott [18] and Sharp [21], asymptotics for the expression

$$\mathsf{M}_T(N) = \sum_{|\tau| \le N} \frac{1}{e^{h(T)|\tau|}}$$

may be viewed as dynamical analogues of Mertens' theorem. The expression $\mathsf{M}_T(N)$ measures in a smoothed way the extent to which the topological entropy reflects the exponential growth in closed orbits or periodic points. A simple illustration of how M_T reflects this is to note that if

$$\mathsf{F}_T(n) = C_1 e^{hn} + \mathrm{O}(e^{h'n})$$

for some $h' < h$, then

$$\mathsf{M}_T(N) = C_1 \sum_{n=1}^{N} \frac{1}{n} + C_2 + \mathrm{O}(e^{-h''N})$$

for some $h'' > 0$ (see [17]).

Writing $\mathsf{O}_T(n)$ for the number of closed orbits of length n, we have

$$\mathsf{F}_T(n) = \sum_{d|n} d\, \mathsf{O}_T(d)$$

and hence

$$\mathsf{O}_T(n) = \frac{1}{n} \sum_{d|n} \mu\left(\frac{n}{d}\right) \mathsf{F}_T(d) \tag{1.2}$$

by Möbius inversion.

For continuous maps on compact metric spaces, it is clear that all possible sequences arise for the count of orbits (see [19]; the same holds in the setting of C^∞ diffeomorphisms of the torus by a result of Windsor [25]). For algebraic dynamical systems the situation is far more constrained, and it is not clear how much freedom there is in possible orbit-growth rates. Our purpose here is to exhibit two different continua of growth rates, on two different speed scales:

- for any $\kappa \in (0, 1)$ there is an automorphism T of a one-dimensional compact metric group with $\mathsf{M}_T(N) \sim \kappa \log N$;
- for any $\delta \in (0, 1)$ and $k > 0$ there is an automorphism T of a one-dimensional compact metric group with $\mathsf{M}_T(N) \sim k(\log N)^\delta$.

While this plays no part in the argument, it is worth noting that there is a complete divorce between the topological entropy and the growth in closed orbits of these examples—they all have topological entropy $\log 2$.

2. The systems studied. We will study a family of endomorphisms (or automorphisms) of one-dimensional solenoids, all built as isometric extensions of the circle-doubling map. To describe these, let $\mathbb{P}$ denote the set of rational primes, and associate to any $S \subset \mathbb{P}$ the ring

$$R_S = \{r \in \mathbb{Q} : |r|_p \le 1 \text{ for all } p \in \mathbb{P} \setminus S\},$$

where $|\cdot|_p$ denotes the normalized p-adic absolute value on $\mathbb{Q}$, so that $|p|_p = p^{-1}$. Thus, for example, $R_\emptyset = \mathbb{Z}$, $R_{\{2,3\}} = \mathbb{Z}[1/6]$, and $R_{\mathbb{P}} = \mathbb{Q}$. Let $T = T_S$ denote the endomorphism of $\widehat{R_S}$ dual to the map $r \mapsto 2r$ on R_S. This map may be thought of as an isometric extension of the circle-doubling map, with topological entropy $h(T_S) = \sum_{p \in S \cup \{\infty\}} \max\{\log |2|_p, 0\} = \log 2$ (see [14] for an explanation of this formula, and for the simplest examples of how the set S influences the number of periodic orbits). Each element of S destroys some closed orbits, by lifting them to non-closed orbits in the isometric extension; see [5] for a detailed explanation in the case $S = \{2, 3\}$. This is reflected in the formula for the count of periodic points in the system,

$$F_{T_S}(n) = (2^n - 1) \prod_{p \in S} |2^n - 1|_p \tag{2.1}$$

(see [2] for the general formula being used here), showing that each inverted prime p in S in the dual group R_S removes the p-part of $2^n - 1$ from the total count of all points of period n. The effect of each inverted prime in R_S on the count of closed orbits via the relation (1.2) is more involved.

We write $|x|_S = \prod_{p \in S} |x|_p$ for convenience, and since we will be using the same underlying map throughout, we will replace $T = T_S$ by the parameter S defining the system in all of the expressions from Section 1. There are then three natural cases: the 'finite' case with $|S| < \infty$ and the 'co-finite' case with $|\mathbb{P} \setminus S| < \infty$, together producing countably many examples, and the more complex remaining 'infinite and co-infinite' case. A special case of the results in [3] is that for S finite we have

$$\mathsf{M}_{T_S}(N) = \mathsf{M}_S(N) = \sum_{|\tau| \le N} \frac{1}{e^{h|\tau|}} = k_S \log N + C_S + \mathrm{O}(N^{-1})$$

for some $k_S \in (0, 1] \cap \mathbb{Q}$ and constant C_S. For example, [3, Ex. 1.5] shows that

$$k_{\{3,7\}} = \frac{269}{576}.$$

Here we continue the analysis further, showing the following theorem.

THEOREM 1. *The set of possible values of the constant k_S with*

$$\mathsf{M}_S(N) = k_S \log N + C_S + \mathrm{O}(N^{-1}),$$

as S varies among the finite subsets of $\mathbb{P}$, is dense in $[0, 1]$.

If S is infinite, then more possibilities arise, but with less control of the error terms.

THEOREM 2. *For any $k \in (0,1)$, there is an infinite co-infinite subset S of $\mathbb{P}$ with*

$$\mathsf{M}_S(N) \sim k \log N.$$

We also give explicit examples of sets S for which the value of k arising in Theorem 2 is transcendental.

The co-finite case is very different in that $\mathsf{M}_S(N)$ converges as $N \to \infty$; other orbit-counting asymptotics better adapted to the polynomially bounded orbit-growth present in these systems are studied in [4] and [11]. The following result is more surprising, in that a positive proportion of primes may be omitted from S while still destroying so many orbits that $\mathsf{M}_S(N)$ is bounded.

PROPOSITION 3. *There is a subset S of $\mathbb{P}$ with natural density in $(0,1)$ such that*

$$\mathsf{M}_S(N) = C_S + \mathrm{O}(N^{-1}).$$

In fact there are such sets S with arbitrarily small non-zero natural density.

While it seems hopeless to describe fully the range of possible growth rates for $\mathsf{M}_S(N)$ as S varies, we are able to exhibit many examples whose growth lies strictly between that of the examples in Theorem 2 and that of the examples in Proposition 3.

THEOREM 4. *For any $\delta \in (0,1)$ and any $k > 0$, there is a subset S of $\mathbb{P}$ such that*

$$\mathsf{M}_S(N) \sim k(\log N)^{\delta}.$$

We also find a family of examples whose growth lies between that of the examples in Theorem 4 and of those in Proposition 3.

THEOREM 5. *For any $r \in \mathbb{N}$ and any $k > 0$, there is a subset S of $\mathbb{P}$ with*

$$\mathsf{M}_S(N) \sim k(\log\log N)^{r}.$$

Moreover, it is possible to achieve growth asymptotic to any suitable function growing slower than $\log\log N$. A byproduct of the constructions for Theorems 1 and 5 gives sets S such that both $\mathsf{M}_S(N)$ and $\mathsf{M}_{\mathbb{P}\setminus S}(N)$ are $\mathrm{o}(\log N)$.

The idea behind the proofs of all these results is rather similar. We choose our set of primes S so that it is easy to isolate a subseries of dominant terms in $\mathsf{M}_S(N)$ in such a way that the sum of the remaining terms converges, usually quickly (controlling this rate governs the error terms). We describe a general framework for dealing with such sets, and then the sets used to

carry out the constructions are defined by arithmetical criteria relying on properties of the set of primes p for which 2 has a given multiplicative order modulo p.

NOTATION. From (2.1), we have $\mathsf{M}_S(N) = \mathsf{M}_{S\cup\{2\}}(N)$ for any set S; thus, without loss of generality, we make the standing assumption that $2 \notin S$. We will use various global constants $C_1, C_2, \dots$, each independent of N and numbered consecutively. The symbols C and C_S denote local constants specific to the statement being made at the time. For an odd prime p, denote by m_p the multiplicative order of 2 modulo p; for a set T of primes, we write $m_T = \operatorname{lcm}\{m_p : p \in T\}$. We will also use Landau's big-O and little-o notation.

3. Asymptotic estimates. By (1.2) and (2.1) we have

$$\mathsf{M}_S(N) = \sum_{n\le N} \frac{1}{n2^n} \sum_{d\mid n} \mu\Big(\frac{n}{d}\Big) |2^d-1| \times |2^d-1|_S = \underbrace{\sum_{n\le N} \frac{|2^n-1|_S}{n}}_{=F_S(N)} + R_S(N),$$

where the last equality defines both $F_S(N)$ and $R_S(N)$.

LEMMA 6. $R_S(N) = C_S + \mathrm{O}(2^{-N/2})$.

Proof. By definition, $R_S(N)$ is the sum of two terms,

$$R_S(N) = -\sum_{n=1}^{N} \frac{|2^n-1|_S}{n2^n} + \sum_{n=1}^{N} \frac{1}{n2^n} \sum_{d\mid n,\, d<n} \mu\Big(\frac{n}{d}\Big)(2^d-1)|2^d-1|_S.$$

Since

$$\frac{|2^n-1|_S}{n2^n} \le \frac{1}{2^n} \quad\text{and}\quad \frac{1}{n2^n} \sum_{dn,\, d<n} (2^d-1)|2^d-1|_S \le \frac{1}{2^{n/2}},$$

both sums converge (absolutely) so $R_S(N)$ converges to some C_S. Moreover,

$$|R_S(N) - C_S| \le \sum_{n=N+1}^{\infty} \frac{1}{2^n} + \sum_{n=N+1}^{\infty} \frac{1}{2^{n/2}} = \mathrm{O}(2^{-N/2}). \ \blacksquare$$

Thus we think of $F_S(N)$ as a dominant term, and much of our effort will be aimed at understanding how $F_S(N)$ behaves as a function of S, which starts with understanding the arithmetic of $2^n - 1$. The main tool here is the elementary observation that, for p a prime and $n \in \mathbb{N}$,

$$\operatorname{ord}_p(2^n-1) = \begin{cases} \operatorname{ord}_p(2^{m_p}-1) + \operatorname{ord}_p(n) & \text{if } m_p \mid n, \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where $\operatorname{ord}_p(n)$ denotes the index of the highest power of p dividing n, so that $|n|_p = p^{-\operatorname{ord}_p(n)}$. In particular, if T is a finite set of primes and $n \in \mathbb{N}$,

then

$$|2^{nm_T}-1|_T = |2^{m_T}-1|_T|n|_T. \tag{3.2}$$

In the proof of [3, Proposition 5.3], a recipe is given for computing the coefficient of $\log N$ in the asymptotic expansion of $F_S(N)$, when S is finite. This is based on an inclusion-exclusion argument, splitting up the sum $F_S(N)$ according to the subsets of S. The disadvantage of this approach is that many subsets of S can lead to an empty sum: in principle, the splitting works for infinite S (since $F_S(N)$ is anyway a finite sum) but then the decomposition of the sum falls into an uncountable number of pieces. Here we take a different approach, splitting up the expressions arising according to the values m_T, for T a finite subset of S, rather than according to the subsets T. In this setting, several different subsets T may give the same value for m_T.

To this end, we set $\mathcal{M}_S = \{m_p : p \in S\}$ and denote by $\overline{\mathcal{M}}_S$ its closure under taking least common multiples:

$$\overline{\mathcal{M}}_S = \{\mathrm{lcm}(\mathcal{M}') : \mathcal{M}' \subseteq \mathcal{M}_S\} = \{m_T : T \subseteq S\}.$$

For $n \in \mathbb{N}$, we put

$$\bar{m}_n = \max\{\bar{m} \in \overline{\mathcal{M}}_S : \bar{m} \mid n\} = \mathrm{lcm}\{m_p \in \mathcal{M}_S : m_p \mid n\}.$$

Then, for $\bar{m} \in \overline{\mathcal{M}}$, set

$$\mathbb{N}_{\bar{m}} = \{n \in \mathbb{N} : \bar{m}_n = \bar{m}\} \quad \text{and} \quad S_{\bar{m}} = \{p \in S : m_p \mid \bar{m}\}.$$

Note that the sets $S_{\bar{m}}$ are finite, even if S is not. Then, using (3.2), we get

$$\begin{aligned} F_S(N) &= \sum_{\bar{m}\in\overline{\mathcal{M}}_S} \sum_{\substack{n\le N\\ n\in\mathbb{N}_{\bar{m}}}} \frac{|2^n-1|_S}{n} \\ &= \sum_{\bar{m}\in\overline{\mathcal{M}}_S} \frac{|2^{\bar{m}}-1|_{S_{\bar{m}}}}{\bar{m}} \sum_{\substack{n\le N/\bar{m}\\ m_p\nmid n\bar{m} \text{ for } p\in S\setminus S_{\bar{m}}}} \frac{|n|_{S_{\bar{m}}}}{n}. \end{aligned} \tag{3.3}$$

The asymptotic behaviour of these inner sums can be computed, at least in principle, using the results of [3, §5]. However, for a general set S of primes, this is cumbersome, so we will specialize to sets which are easier to deal with. In this, we are motivated by the next lemma, which follows one of the many paths used to prove Zsigmondy's theorem (see [6, §8.3.1] for the details).

LEMMA 7. *Fix $n \in \mathbb{N}$, and let $S \subset \mathbb{P}$ be a set of primes containing*

$$\{p \in \mathbb{P} : m_p = n\}.$$

Then

$$|2^n-1|_S \le \frac{n}{2^{\phi(n)-2}}.$$

Recall that a divisor of 2^n-1 is *primitive* (in the sequence $(2^n-1)_{n\geq 1}$) if it has no common factor with 2^m-1 for any m with $1 \leq m < n$. Thus $\{p \in \mathbb{P} : m_p = n\}$ is the set of primitive prime divisors of 2^n-1. This set is finite, since $m_p \geq \log_2 p$, but may be large—for example

$$m_{233} = m_{1103} = m_{2089} = 29.$$

Schinzel [20] proved that there are infinitely many n for which this set contains at least two elements, but it seems that not much more is known about it in general.

Proof of Lemma 7. Writing $(2^n-1)^*$ for the maximal primitive divisor of 2^n-1, we certainly have

$$|2^n-1|_S^{-1} \geq (2^n-1)^*. \tag{3.4}$$

By factorizing x^n-1 we have

$$2^n - 1 = \prod_{d\,|\,n} \Phi_d(2), \tag{3.5}$$

where Φ_d is the dth cyclotomic polynomial. It follows that $(2^n-1)^*$ is a factor of $\Phi_n(2)$. If a prime p divides $\gcd(\Phi_n(2), \Phi_d(2))$ for some $d \,|\, n$ with $d < n$, then $p \,|\, 2^d - 1$. Then, from (3.1),

$$\operatorname{ord}_p(2^n-1) = \operatorname{ord}_p(2^d-1) + \operatorname{ord}_p(n/d),$$

and from (3.5),

$$\operatorname{ord}_p(2^n-1) \geq \operatorname{ord}_p(2^d-1) + \operatorname{ord}_p(\Phi_n(2)) \geq \operatorname{ord}_p(2^d-1) + 1,$$

so in particular p divides n/d; therefore p divides n and d divides n/p, so p divides $2^{n/p}-1$. Moreover

$$\operatorname{ord}_p(2^n-1) = \operatorname{ord}_p(2^{n/p}-1) + 1$$

and

$$\operatorname{ord}_p(2^n-1) \geq \operatorname{ord}_p(2^{n/p}-1) + \operatorname{ord}_p(\Phi_n(2)),$$

so in fact $\operatorname{ord}_p(\Phi_n(2)) = 1$. Thus $\gcd(\Phi_n(2), \prod_{d|n,\, d<n} \Phi_d(2))$ divides $\prod_{p|n} p$, which is at most n, and

$$(2^n-1)^* \geq \Phi_n(2)/n. \tag{3.6}$$

On the other hand, by Möbius inversion applied to (3.5),

$$\Phi_n(2) = \prod_{d|n} (2^d-1)^{\mu(n/d)}$$

so

$$\log(\Phi_n(2)) = \phi(n)\log 2 + \sum_{d|n} \mu(n/d)\log(1-2^{-d}),$$

where $\phi(n)$ is the Euler totient function. Now, using the Taylor expansion for the logarithm,

$$\Big|\sum_{d|n}\mu(n/d)\log(1-2^{-d})\Big| \le \sum_{d|n}\sum_{j=1}^{\infty}\frac{2^{-jd}}{j} = \sum_{j=1}^{\infty}\frac{2^{-j}}{j}\sum_{d|n}2^{-j(d-1)} \le 2\log 2,$$

so $\Phi_n(2) \ge 2^{\phi(n)-2}$ and the result follows by (3.4) and (3.6). ∎

This lemma will be used as follows. Instead of starting with a set S of primes, we begin with $\mathcal{M}$ a subset of $\mathbb{N}$ and put

$$S_{\mathcal{M}} = \{p \in \mathbb{P} : m_p \in \mathcal{M}\}.$$

Then

$$\sum_{n\le N}\frac{|2^n-1|_{S_{\mathcal{M}}}}{n} = \underbrace{\sum_{\substack{n\le N\\ n\notin\mathcal{M}}}\frac{|2^n-1|_{S_{\mathcal{M}}}}{n}}_{=D_{S_{\mathcal{M}}}(N)} + \underbrace{\sum_{\substack{n\le N\\ n\in\mathcal{M}}}\frac{|2^n-1|_{S_{\mathcal{M}}}}{n}}_{=Q_{S_{\mathcal{M}}}(N)},$$

and, by Lemma 7,

$$Q_{S_{\mathcal{M}}}(N) \le C_3\sum_{n\le N}\frac{1}{2^{\phi(n)}},$$

which converges since $\phi(n) \ge \sqrt{n}$ for $n \ge 6$. Moreover, the same observation shows that

$$Q_{S_{\mathcal{M}}}(N) = C_4 + \mathrm{O}(2^{-\sqrt{N}}).$$

Thus the asymptotic behaviour is governed by the dominant term $D_{S_{\mathcal{M}}}(N)$. From Lemma 6, we get

$$\mathsf{M}_{S_{\mathcal{M}}}(N) = \sum_{\substack{n\le N\\ n\notin\mathcal{M}}}\frac{|2^n-1|_{S_{\mathcal{M}}}}{n} + C_5 + \mathrm{O}(2^{-\sqrt{N}}). \tag{3.7}$$

All our examples will take this form.

Remarks 8. (i) We have set up maps $S \mapsto \mathcal{M}_S$ and $\mathcal{M} \mapsto S_{\mathcal{M}}$ between the power sets of $\mathbb{P}\setminus\{2\}$ and $\mathbb{N}$, which are order-preserving for inclusion. It is easy to check that $S_{\mathcal{M}_S} \supseteq S$, while $S_{\mathcal{M}_{S_{\mathcal{M}}}} = S_{\mathcal{M}}$. Similarly, we have $\mathcal{M}_{S_{\mathcal{M}}} = \mathcal{M}\setminus\{1,6\}$, since all but the first and sixth terms of the Mersenne sequence have primitive divisors, and $\mathcal{M}_{S_{\mathcal{M}_S}} = \mathcal{M}_S$. In particular, we can apply the decomposition (3.3) of $\mathsf{M}_{S_{\mathcal{M}}}(N)$ in tandem with (3.7). When we do so, we will replace $\overline{\mathcal{M}}_{S_{\mathcal{M}}}$ by the closure $\overline{\mathcal{M}}$ of $\mathcal{M}$ under least common multiples to get

$$\mathsf{M}_{S_{\mathcal{M}}}(N) = \sum_{\bar m\in\overline{\mathcal{M}}}\frac{|2^{\bar m}-1|_{S_{\bar m}}}{\bar m}\sum_{\substack{n\le N/\bar m,\ n\bar m\notin\mathcal{M}\\ m_p\nmid n\bar m \text{ for } p\in S_{\mathcal{M}}\setminus S_{\bar m}}}\frac{|n|_{S_{\bar m}}}{n} + C_6 + \mathrm{O}(2^{-\sqrt{N}}). \tag{3.8}$$

(ii) For a general set S, let $\mathcal{M}_S^o$ be the set of $m \in \mathcal{M}_S$ for which S contains all primes p with $m_p = m$, and put $S^o = S_{\mathcal{M}_S^o}$; this is the largest subset of S of the form $S_{\mathcal{M}}$. Similarly, put $\overline{S} = S_{\mathcal{M}_S}$, the smallest superset of S of the form $S_{\mathcal{M}}$. The techniques here can be applied to the sets S^o and $\overline{S}$ and, since

$$\mathsf{M}_{\overline{S}}(N) \le \mathsf{M}_S(N) \le \mathsf{M}_{S^o}(N),$$

we get some information on the asymptotic behaviour of $\mathsf{M}_S(N)$.

The formula (3.8) is particularly simple in the case that $\mathcal{M}$ is closed under multiplication by $\mathbb{N}$; that is, if $a \in \mathcal{M}$ and $b \in \mathbb{N}$, then $ab \in \mathcal{M}$. In this case $\mathcal{M}$ is closed under least common multiples and, for $n \notin \mathcal{M}$, we have $\bar{m}_n = 1$. Thus the inner sum in (3.8) is empty for $\bar{m} \ne 1$, and we get

$$\mathsf{M}_{S_{\mathcal{M}}}(N) = \sum_{\substack{n \le N \\ n \notin \mathcal{M}}} \frac{1}{n} + C_7 + \mathrm{O}(2^{-\sqrt{N}}). \tag{3.9}$$

Provided $\mathsf{M}_{S_{\mathcal{M}}}(N) \to \infty$ as $N \to \infty$, this implies that

$$\mathsf{M}_{S_{\mathcal{M}}}(N) \sim \sum_{\substack{n \le N \\ n \notin \mathcal{M}}} \frac{1}{n}.$$

Many, though not all, of our examples will be of this form. The task is then to choose sets $\mathcal{M}$ which are closed under multiplication by $\mathbb{N}$, and for which we can control the asymptotics of the sum in (3.9). One technique we will often use for this is *partial* (or *Abel*) *summation*: if we write

$$\pi_{\mathcal{M}}(x) = |\{n \le x : n \notin \mathcal{M}\}|$$

and f is a positive differentiable function on the positive reals, then

$$\sum_{\substack{n \le x \\ n \notin \mathcal{M}}} f(n) = \pi_{\mathcal{M}}(x) f(x) + \int_1^x \pi_{\mathcal{M}}(t) f'(t)\, dt,$$

with the dominant term generally coming from the integral. In several cases the asymptotics of $\pi_{\mathcal{M}}(x)$ are already well understood.

4. Finite sets of primes. In order to prove Theorem 1, we need to choose finite sets S of primes for which we can make good estimates for the coefficient of the leading term in Mertens' theorem. These calculations are simplified by considering only primes p for which m_p is prime.

Let $\mathcal{L}$ be a finite set of primes and take $\mathcal{M} = \mathcal{L}$, so that

$$S = S_{\mathcal{L}} = \{p \in \mathbb{P} : m_p \in \mathcal{L}\},$$

which is a finite set. By [3, Theorem 1.4], we have

$$\mathsf{M}_{S_{\mathcal{L}}}(N) = k_{\mathcal{L}} \log(N) + C_{\mathcal{L}} + \mathrm{O}(N^{-1})$$

for some $k_{\mathcal{L}} \in (0,1] \cap \mathbb{Q}$ and constant $C_{\mathcal{L}}$. The following lemma gives upper and lower bounds for $k_{\mathcal{L}}$.

LEMMA 9. *Let $\mathcal{L}$ be a finite subset of $\mathbb{P}$.*

(i) *We have*

$$k_{\mathcal{L}} \le \prod_{\ell \in \mathcal{L}} \left(1 - \frac{1}{\ell} + \frac{1}{\ell(2^\ell - 1)}\right).$$

(ii) *For $\ell \in \mathbb{P} \setminus \mathcal{L}$, we have $(1 - 1/\ell)k_{\mathcal{L}} \le k_{\mathcal{L}\cup\{\ell\}}$.*

Proof. For $\mathcal{L}'$ a subset of $\mathcal{L}$, we write $m(\mathcal{L}') = \prod_{\ell\in\mathcal{L}'} \ell$. We break up the Mertens sum as in (3.8), noting that $\overline{\mathcal{M}} = \{m(\mathcal{L}') : \mathcal{L}' \subseteq \mathcal{L}\}$:

$$\mathsf{M}_{S_{\mathcal{L}}}(N) \sim \sum_{\mathcal{L}'\subseteq\mathcal{L}} \frac{|2^{m(\mathcal{L}')} - 1|_{S_{\mathcal{L}'}}}{m(\mathcal{L}')} \sum_{\substack{n \le N/m(\mathcal{L}') \\ \ell \nmid n \text{ for } \ell \in \mathcal{L}\setminus\mathcal{L}'}} \frac{|n|_{S_{\mathcal{L}'}}}{n}. \tag{4.1}$$

By [3, Proposition 5.2], we have

$$\sum_{n\in N} \frac{|n|_{S_{\mathcal{L}'}}}{n} = k'_{\mathcal{L}'} \log N + C'_{\mathcal{L}'} + \mathrm{O}(N^{-1}),$$

with $k'_{\mathcal{L}'} = \prod_{p\in S_{\mathcal{L}'}} p/(p+1)$. Moreover, by [3, Lemma 5.1],

$$\sum_{\substack{n\le N \\ \ell\nmid n \text{ for } \ell\in\mathcal{L}\setminus\mathcal{L}'}} \frac{|n|_{S_{\mathcal{L}'}}}{n} = k'_{\mathcal{L}'} \prod_{\ell\in\mathcal{L}\setminus\mathcal{L}'} \left(1 - \frac{|\ell|_{S_{\mathcal{L}'}}}{\ell}\right) \log N + C''_{\mathcal{L}'} + \mathrm{O}(N^{-1}).$$

In particular, the coefficient of the $\log N$ term is

$$\prod_{\ell\in\mathcal{L}\setminus\mathcal{L}'} \left(1 - \frac{1}{\ell}\right) \prod_{p\in S_{\mathcal{L}'}\setminus\mathcal{L}} \frac{p}{p+1},$$

which is at most $\prod_{\ell\in\mathcal{L}\setminus\mathcal{L}'}(1 - 1/\ell)$. Moreover, for $\ell \in \mathcal{L}'$ and p such that $m_p = \ell$, we have $\mathrm{ord}_p(2^{m(\mathcal{L}')} - 1) \ge \mathrm{ord}_p(2^\ell - 1)$ so

$$|2^{m(\mathcal{L}')} - 1|_{S_{\mathcal{L}'}} \le \prod_{\ell\in\mathcal{L}'} \frac{1}{2^\ell - 1}.$$

Putting everything back into (4.1), we see that the coefficient $k_{\mathcal{L}}$ of the $\log N$ term is bounded above by

$$\sum_{\mathcal{L}'\subseteq\mathcal{L}} \prod_{\ell\in\mathcal{L}'} \frac{1}{\ell(2^{\ell-1})} \prod_{\ell\in\mathcal{L}\setminus\mathcal{L}'} \left(1 - \frac{1}{\ell}\right) = \prod_{\ell\in\mathcal{L}} \left(1 - \frac{1}{\ell} + \frac{1}{\ell(2^{\ell-1})}\right).$$

This proves (i), and the proof of (ii) is similar but easier: we have

$$\mathsf{M}_{S_{\mathcal{L}\cup\{\ell\}}}(N) \sim \sum_{\mathcal{L}'\subseteq\mathcal{L}\cup\{\ell\}} \frac{|2^{m(\mathcal{L}')}-1|_{S_{\mathcal{L}'}}}{m(\mathcal{L}')} \sum_{\substack{n\le N/m(\mathcal{L}')\\ \gcd(n,\ell m(\mathcal{L}))\mid m(\mathcal{L}')}} \frac{|n|_{S_{\mathcal{L}'}}}{n},$$

and, for $\mathcal{L}'$ a subset contained in $\mathcal{L}$, the contribution of the sum corresponding to $\mathcal{L}'$ is $1-|\ell|_{S_{\mathcal{L}'}}/\ell$ times the contribution of the sum corresponding to $\mathcal{L}'$ in (4.1). In particular, the Mertens sum for $\mathcal{L}\cup\{\ell\}$ is at least $1-1/\ell$ times that for $\mathcal{L}$. ∎

Proof of Theorem 1. Let $k\in(0,1)$ and $\varepsilon>0$, and choose two primes $\ell_0 > 1+k/\varepsilon$ and $\ell_1>\ell_0$ such that

$$\prod_{\ell_0\le\ell<\ell_1}\left(1-\frac{1}{l}+\frac{1}{l(2^l-1)}\right)<k;$$

this is possible since the product over all primes greater than ℓ_0 converges to 0.

We choose recursively a subset $\mathcal{L}$ of $\{\ell\in\mathbb{P}:\ell<\ell_1\}$, using the greedy algorithm as follows. Let $\ell\in\mathbb{P}$ and suppose we have already defined $\mathcal{L}(\ell) := \mathcal{L}\cap\{1,\dots,\ell-1\}$. If $k\le k_{\mathcal{L}(\ell)}<k+\varepsilon$ then we are done and $\mathcal{L}=\mathcal{L}(\ell)$; otherwise $\ell\in\mathcal{L}$ if and only if $k_{\mathcal{L}(\ell)\cup\{\ell\}}\ge k$.

The claim is then that, for the subset $\mathcal{L}$ given by this algorithm, the leading coefficient $k_{\mathcal{L}}$ satisfies $k\le k_{\mathcal{L}}<k+\varepsilon$. The first inequality is clear from the definition, while the second follows from the following two observations:

(i) There is a prime ℓ with $\ell_0\le\ell<\ell_1$ such that $\ell\notin\mathcal{L}$: if not, by Lemma 9(i),

$$k_{\mathcal{L}}\le\prod_{\ell_0\le\ell<\ell_1}\left(1-\frac{1}{\ell}+\frac{1}{\ell(2^\ell-1)}\right)<k,$$

which is absurd.

(ii) With ℓ as in (i), we have $k_{\mathcal{L}(\ell)\cup\{\ell\}}<k$, since $\ell\notin\mathcal{L}$; thus, by Lemma 9(ii),

$$k_{\mathcal{L}(\ell)}\le\frac{\ell}{\ell-1}k_{\mathcal{L}(\ell)\cup\{\ell\}}<\frac{\ell_0}{\ell_0-1}k<k+\varepsilon.$$ ∎

Remark 10. Let $\mathcal{L}$ be an *infinite* set of primes such that $\sum_{\ell\in\mathcal{L}}1/\ell$ diverges and put $S=S_{\mathcal{L}}=\{p\in\mathbb{P}:m_p\in\mathcal{L}\}$. Then $\mathsf{M}_S(N)\le\mathsf{M}_{S_{\mathcal{L}'}}(N)$ for any finite subset $\mathcal{L}'$ of $\mathcal{L}$, so $\mathsf{M}_S(N)$ grows no more quickly than $k_{\mathcal{L}'}\log N$. Since $k_{\mathcal{L}'}$ is at most $\prod_{\ell\in\mathcal{L}'}\left(1-\frac{1}{\ell}+\frac{1}{\ell(2^\ell-1)}\right)$, by Lemma 9(i), there are finite subsets $\mathcal{L}'$ of $\mathcal{L}$ with $k_{\mathcal{L}'}$ arbitrarily close to 0, and we deduce that $\mathsf{M}_S(N)=\mathrm{o}(\log N)$.

5. Logarithmic growth for infinite sets of primes. In this section we will prove Theorem 2. Fix $\ell \in \mathbb{P}$, and let

$$\overline{\mathcal{M}}_\ell = \{n \in \mathbb{N} : \ell \,|\, n\},$$

so that

$$S_{\overline{\mathcal{M}}_\ell} = \overline{S}_\ell = \{p \in \mathbb{P} : \ell \,|\, m_p\}.$$

By Hasse [9, 10] these sets have a positive Dirichlet density within the set of primes: for $\ell > 2$ the density is $\ell/(\ell^2-1)$, and for $\ell = 2$ the density is $\frac{17}{24}$. They also have natural density by, for example, [24, Theorem 2]. Noting that $\overline{\mathcal{M}}_\ell$ is closed under multiplication by $\mathbb{N}$, by (3.9) we have

$$\mathsf{M}_{\overline{S}_\ell}(N) = \sum_{\substack{n \le N \\ \ell \nmid n}} \frac{1}{n} + C_9 + \mathrm{O}(2^{-\sqrt{N}}) = \left(1 - \frac{1}{\ell}\right) \log N + C_8 + \mathrm{O}(N^{-1}).$$

Indeed, if $\mathcal{L}$ is any *finite* set of primes, applying the same argument to

$$\overline{\mathcal{M}}_\mathcal{L} = \{n \in \mathbb{N} : \ell \,|\, n \text{ for some } \ell \in \mathcal{L}\} \quad \text{and} \quad \overline{S}_\mathcal{L} = \{p \in \mathbb{P} : m_p \in \overline{M}_\mathcal{L}\},$$

we get

$$\mathsf{M}_{\overline{S}_\mathcal{L}}(N) = \prod_{\ell \in \mathcal{L}} \left(1 - \frac{1}{\ell}\right) \log N + C_{10} + \mathrm{O}(N^{-1}).$$

Since the set $\{\prod_{\ell\in\mathcal{L}}(1-1/\ell) : \mathcal{L} \subset \mathbb{P} \text{ finite}\}$ is dense in $[0,1]$, this gives an easy way of getting a dense set of values for the leading coefficient in Mertens' theorem. Note however that Theorem 1 was more delicate, since the claim was that a dense set of values can be obtained using only *finite* sets S. Similarly, Theorem 2 claims more: *every* value in $(0,1)$ can be obtained as leading coefficient.

Proof of Theorem 2. Now let $\mathcal{L} \subset \mathbb{P}$ be any set of primes for which the product $k_\mathcal{L} := \prod_{\ell\in\mathcal{L}}(1-1/\ell)$ is non-zero, define $\overline{\mathcal{M}}_\mathcal{L}$ and $\overline{S}_\mathcal{L}$ as above, and apply the argument above to obtain

$$\mathsf{M}_{\overline{S}_\mathcal{L}}(N) \sim \sum_{\substack{n \le N \\ n \notin \overline{\mathcal{M}}_\mathcal{L}}} \frac{1}{n}.$$

Applying [22, Theorem I.3.11] we have $|\{n \le x : n \notin \overline{\mathcal{M}}_\mathcal{L}\}| \sim k_\mathcal{L} x$, thus, by partial summation, we get

$$\mathsf{M}_{\overline{S}_\mathcal{L}}(N) \sim k_\mathcal{L} \log N.$$

This gives Theorem 2 since $\{k_\mathcal{L} : \mathcal{L} \subseteq \mathbb{P}\} = [0,1]$. ∎

6. Sublogarithmic growth. Now we consider sets giving intermediate sublogarithmic growth, proving Theorem 4. We return to sets close to $\overline{\mathcal{M}}_\mathcal{L}$ and $\overline{S}_\mathcal{L}$ of §5 but now for infinite sets $\mathcal{L}$ of primes such that $\prod_{\ell\in\mathcal{L}}(1-1/\ell)$

$= 0$. We will need a result from analytic number theory that allows sets of primes to be selected with prescribed properties; we defer the proof to §10.

PROPOSITION 11. *For any $\delta \in (0,1]$, there is a set $\mathcal{L}$ of primes such that*

$$\sum_{\substack{\ell \le x \\ l \in \mathcal{L}}} \frac{\log \ell}{\ell} = \delta \log x + \mathrm{O}(1) \tag{6.1}$$

and, for any $c > 1$, there is a set $\mathcal{L}' \subseteq \mathcal{L}$ such that

$$\prod_{p \in \mathcal{L}'} \left(1 + \frac{1}{p}\right) = c \quad \textit{and} \quad \sum_{p \in \mathcal{L}'} \frac{\log p}{p} < \infty.$$

Proof of Theorem 4. Let $\delta \in (0,1]$ and $k > 0$, and let $\mathcal{L}$ be a set of primes satisfying (6.1). As before we put $\overline{\mathcal{M}}_{\mathcal{L}} = \{n \in \mathbb{N} : \ell \,|\, n \text{ for some } \ell \in \mathcal{L}\}$ and now we set

$$\overline{\mathcal{M}}'_{\mathcal{L}} = \{n \in \mathbb{N} : n \in \overline{\mathcal{M}}_{\mathcal{L}} \text{ or } n \text{ is not square-free}\}$$

and $\bar{S}'_{\mathcal{L}} = \{p \in \mathbb{P} : m_p \in \overline{\mathcal{M}}'_{\mathcal{L}}\}$. Note that $\overline{\mathcal{M}}'_{\mathcal{L}}$ is also closed under multiplication by $\mathbb{N}$ so that

$$\mathsf{M}_{\bar{S}'_{\mathcal{L}}}(N) = \sum_{\substack{n \le N \\ n \notin \overline{\mathcal{M}}'_{\mathcal{L}}}} \frac{1}{n} + C_{11} + \mathrm{O}(2^{-\sqrt{N}})$$

by (3.9). Now we apply [8, Theorem A.5] with, in the notation used there, the function

$$g(n) = \begin{cases} 1/n & \text{if } n \notin \mathcal{M}'_{\mathcal{L}}, \\ 0 & \text{otherwise.} \end{cases}$$

Note that, by (6.1), the hypotheses [8, (A.15–17)] of that theorem are indeed satisfied. We conclude that

$$\sum_{\substack{n \le N \\ n \notin \overline{\mathcal{M}}'_{\mathcal{L}}}} \frac{1}{n} = k_{\mathcal{L}} (\log N)^{\delta} + \mathrm{O}((\log N)^{\delta - 1}),$$

where $k_{\mathcal{L}} > 0$ is

$$k_{\mathcal{L}} = \frac{1}{\Gamma(\delta + 1)} \prod_{p \in \mathbb{P}} \left(1 - \frac{1}{p}\right)^{\delta} \prod_{p \notin \mathcal{L}} \left(1 + \frac{1}{p}\right)$$

by [8, (A.24)]. Notice that we can adjust $\mathcal{L}$ by any set of primes $\mathcal{L}'$ such that $\sum_{\ell \in \mathcal{L}'} (\log \ell)/\ell$ converges without affecting the hypothesis (6.1).

Assume now that $\mathcal{L}$ is the set of primes $\mathcal{L}$ constructed in Proposition 11. Let $\mathcal{L}'' \subseteq \mathbb{P} \setminus \mathcal{L}$ be a set of primes such that $k \prod_{p \in \mathcal{L}''} (1 + 1/p) \ge k_{\mathcal{L}}$. By

Proposition 11, there is a subset $\mathcal{L}'$ of $\mathcal{L}$ such that

$$\prod_{p\in\mathcal{L}'}\Big(1+\frac{1}{p}\Big)=\frac{k}{k_{\mathcal{L}}}\prod_{p\in\mathcal{L}''}\Big(1+\frac{1}{p}\Big).$$

In particular, putting $\mathcal{L}_0=(\mathcal{L}\setminus\mathcal{L}')\cup\mathcal{L}''$, we have $k_{\mathcal{L}_0}=k$ so the set $\overline{S}'_{\mathcal{L}_0}$ gives the required asymptotic. ∎

Remark 12. Since the sets $\overline{\mathcal{M}}_{\mathcal{L}}$ and $\overline{\mathcal{M}}'_{\mathcal{L}}$ coincide on the set of squarefree natural numbers, there is a constant $c_{\mathcal{L}}$ such that

$$\sum_{\substack{n\le N\\ n\notin\overline{\mathcal{M}}'_{\mathcal{L}}}}\frac{1}{n}\le\sum_{\substack{n\le N\\ n\notin\overline{\mathcal{M}}_{\mathcal{L}}}}\frac{1}{n}\le c_{\mathcal{L}}\sum_{\substack{n\le N\\ n\notin\overline{\mathcal{M}}_{\mathcal{L}}}}\frac{1}{n}.$$

In particular, the Mertens sum $\mathsf{M}_{\overline{S}_{\mathcal{L}}}(N)$ also grows like $(\log N)^{\delta}$. We have chosen to use $\overline{\mathcal{M}}'_{\mathcal{L}}$ here rather than $\overline{\mathcal{M}}_{\mathcal{L}}$ since it is for such a set that we were able to find an off-the-shelf reference [8, Theorem A.5] for the asymptotics.

7. Doubly logarithmic growth. Here we consider sets giving doubly logarithmic growth or slower, in particular proving Theorem 5. In the case $r=1$, the proof is based on taking the set $\mathcal{M}$ of §3 to be the set $\mathbb{N}\setminus\mathbb{P}$ of composite natural numbers, so that $S=S_{\mathcal{M}}$ is the set of primes p such that m_p is composite. Since $\mathcal{M}$ is closed under multiplication by $\mathbb{N}$, applying (3.9) we have

$$\mathsf{M}_S(N)=\sum_{\substack{p\le N\\ p\in\mathbb{P}}}\frac{1}{p}+C_{12}+\mathrm{O}(2^{-\sqrt{N}}).$$

By Mertens' original theorem [15], we have

$$\sum_{\substack{p\le N\\ p\in\mathbb{P}}}\frac{1}{p}=\log\log N+C_{13}+\mathrm{O}((\log N)^{-1}),\tag{7.1}$$

and hence

$$\mathsf{M}_S(N)=\log\log N+C_{14}+\mathrm{O}((\log N)^{-1}),$$

which is an improved form (i.e. with error term) of Theorem 5 with $k=1$ and $r=1$.

Remark 13. The complement of this set S is the set of primes p for which m_p is prime so $\mathsf{M}_{\mathbb{P}\setminus S}(N)=\mathrm{o}(\log N)$, by Remark 10. Thus both $\mathsf{M}_S(N)$ and $\mathsf{M}_{\mathbb{P}\setminus S}(N)$ are $\mathrm{o}(\log N)$.

For the general case of Theorem 5 we will need the following lemma, which gives asymptotics for the number of integers with exactly r prime factors (counted with multiplicity), all from a fixed set of primes. For $\mathcal{L}$

a set of primes and $n \in \mathbb{N}$ we denote by $\Omega_{\mathcal{L}}(n)$ the number of prime factors of n in $\mathcal{L}$ (counted with multiplicity), and abbreviate $\Omega(n) = \Omega_{\mathbb{P}}(n)$.

LEMMA 14. *Let $\mathcal{L}$ be a set of primes of natural density δ and $r \in \mathbb{N}$. Then*

$$|\{n \le x : \Omega_{\mathcal{L}}(n) = \Omega(n) = r\}| \sim \delta^r \frac{x}{\log x} \frac{(\log\log x)^{r-1}}{(r-1)!}. \tag{7.2}$$

Proof. When $r = 1$, the case $\mathcal{L} = \mathbb{P}$ is the prime number theorem and the case of general $\mathcal{L}$ follows immediately, since $\mathcal{L}$ has density δ. For $r > 1$, the case $\mathcal{L} = \mathbb{P}$ is a result of Landau [12, XIII, §56, (5)], proved by induction on r. The proof (using the prime number theorem and partial summation – see [16, §7.4] for a sketch) works equally well for any set $\mathcal{L}$ as in the lemma, and the result follows. ∎

Proof of Theorem 5. Let $r \in \mathbb{N}$ and $k > 0$. We pick a natural number m such that

$$k_m := \sum_{d|m} \frac{1}{d} > k(r!)$$

and a set $\mathcal{L}$ of primes with natural density $\delta = (k(r!)/k_m)^{1/r}$. Denote by $\mathcal{M}_{r,\mathcal{L},m}$ the set of natural numbers n such that either $\Omega(n/\gcd(m,n)) > r$ or $n/\gcd(m,n)$ has a prime factor outside $\mathcal{L}$.

We put $S = S_{\mathcal{M}_{r,\mathcal{L},m}}$ and apply (3.9) to get

$$\mathsf{M}_S(N) \sim \sum_{\substack{n \le N \\ n \notin \mathcal{M}_{r,\mathcal{L},m}}} \frac{1}{n} = \sum_{d|m} \frac{1}{d} \sum_{\substack{n \le N/d \\ n \notin \mathcal{M}_{r,\mathcal{L},1}}} \frac{1}{n}. \tag{7.3}$$

On the other hand, by (7.2), we have

$$\pi_{\mathcal{M}_{r,\mathcal{L},1}}(x) = |\{n \le x : \Omega(n) = \Omega_{\mathcal{L}}(n) \le r\}| \sim \delta^r \frac{x}{\log x} \frac{(\log\log x)^{r-1}}{(r-1)!}.$$

Applying partial summation gives

$$\sum_{\substack{n \le x \\ n \notin \mathcal{M}_{r,\mathcal{L},1}}} \frac{1}{n} \sim \delta^r \frac{(\log\log x)^r}{r!},$$

and substituting this into (7.3) gives the result, because of the choice of δ. ∎

REMARK 15. Let θ be any positive, increasing, differentiable function on the positive reals such that, for large enough x, both $\theta(x) \le \log\log x$ and $\theta'(x) \le 1/(x\log x)$. Then there is a set $\mathcal{L}_\theta$ of primes such that

$$\sum_{\substack{p \le x \\ p \in \mathcal{L}_\theta}} \frac{1}{p} \sim \theta(x)$$

and hence, putting $S_\theta = \{p \in \mathbb{P} : m_p \notin \mathcal{L}_\theta\}$ and applying (3.9), we have

$$\mathsf{M}_{S_\theta}(N) \sim \theta(x).$$

The existence of such a set $\mathcal{L}_\theta$ comes from (7.1) and the following lemma, whose proof using the greedy algorithm is straightforward but technical so is omitted. It seems almost certain that a lemma of this sort exists in the literature but we have not been able to find it.

LEMMA 16. *Suppose f is a positive, increasing, differentiable function on the positive reals and a_n are non-negative reals converging to 0 such that $\sum_{n\le x} a_n \sim f(x)$. We write $\delta(x) = \sum_{n\le x} a_n - f(x)$. Suppose we have a positive, increasing, differentiable function θ on the positive reals such that:*

(i) *there is an $x_0 > 0$ such that $\theta(x) \le f(x)$ and $\theta'(x) \le f'(x)$ for all $x > x_0$;*
(ii) *$\delta = \mathrm{o}(\theta)$.*

Then there is a subset $\mathcal{N}_\theta \subset \mathbb{N}$ such that

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}_\theta}} a_n \sim \theta(x).$$

8. Convergence for co-infinite sets of primes

Proof of Proposition 3. Fix a prime ℓ and set $\overline{\mathcal{M}}_\ell^c = \{n \in \mathbb{N} : \ell \nmid n\}$, the complement of the set $\overline{\mathcal{M}}_\ell$ considered in §5; thus

$$S = S_{\overline{\mathcal{M}}_\ell^c} = \{p \in \mathbb{P} : \ell \nmid m_p\}$$

is a set of primes with positive natural density. Although $\overline{\mathcal{M}}_\ell^c$ is not closed under multiplication by $\mathbb{N}$, it is closed under least common multiples; moreover, for $m \in \overline{\mathcal{M}}_\ell^c$, we have $\mathbb{N}_m = \{m\ell^e : e \ge 0\}$, in the notation of §3. Thus, from (3.8),

$$\mathsf{M}_S(N) = \sum_{\substack{m\in\mathbb{N}\\ \ell\nmid m}} \frac{|2^m-1|_{S_m}}{m} \sum_{1<\ell^e\le N/m} \frac{|\ell|_{S_m}^e}{\ell^e} + C_{15} + \mathrm{O}(2^{-\sqrt{N}}). \tag{8.1}$$

Now

$$\sum_{1<\ell^e\le N/m} \frac{|\ell|_{S_m}^e}{\ell^e} \le \sum_{e\ge1} \frac{1}{\ell^e} = \frac{1}{(\ell-1)}.$$

Thus the terms of the (outer) sum in (8.1) converge and, since $|2^m-1|_{S_m} = (2^m-1)^{-1}$, the difference between each term and its limit is

$$\frac{|2^m-1|_{S_m}}{m} \sum_{\ell^e>N/m} \frac{|\ell|_{S_m}^e}{\ell^e} \le \frac{m}{2^m-1} \sum_{e>\log(N/m)/\log\ell} \frac{1}{\ell^e} \le C_{16}\frac{1}{2^m N}.$$

Plugging this back into the sum in (8.1), we see that it converges and the difference between it and its limit is bounded by

$$\frac{C_{16}}{N}\sum_{\substack{m\in\mathbb{N}\\ \ell\nmid m}}\frac{1}{2^m} = \mathrm{O}(N^{-1}). \ \blacksquare$$

REMARK 17. It is straightforward to generalize this proof to the case where $\mathcal{M}$ is the complement of the set $\overline{\mathcal{M}}_{\mathcal{L}}$ considered in §5, for any finite set of primes $\mathcal{L}$, so that $S_{\mathcal{M}}$ is the set of primes p such that m_p is not divisible by any $\ell \in \mathcal{L}$. By a special case of a very general result of Wiertelak [24, Theorem 2], when $\mathcal{L}$ consists only of odd primes the set $S_{\mathcal{M}}$ has natural density $\prod_{\ell\in\mathcal{L}}(1-\ell/(\ell^2-1))$. In particular, this density can be arbitrarily close to 0.

9. Transcendental constants. Our first example of a transcendental constant comes from an elementary result in analytic number theory. Let $\mathcal{M}$ be the set of non-square-free natural numbers; then a theorem of Landau gives

$$\pi_{\mathcal{M}}(x) = |\{n \le x : n \notin \mathcal{M}\}| = \frac{6}{\pi^2}x + \mathrm{o}(\sqrt{x})$$

(see for example [12, XLIV, §162] or [22, Theorem I.3.10]). Thus, by partial summation and (3.9), we get

$$\mathsf{M}_{S_{\mathcal{M}}}(N) = \frac{6}{\pi^2}\log N + C_{17} + \mathrm{o}(N^{-1/2}).$$

For our second example, fix a prime ℓ and set $\mathcal{M}_{(\ell)} = \{\ell^e : e \ge 0\}$, so that $S = S_{\mathcal{M}_{(\ell)}}$ is the infinite set of primes p for which m_p is a power of ℓ. This is a thin set of primes, that is, it has density zero. As in the previous section, the set $\mathcal{M}_{(\ell)}$ is closed under least common multiples, but not under multiplication by $\mathbb{N}$. Applying (3.8), we get

$$\mathsf{M}_S(N) = \sum_{e=0}^{\infty}\frac{1}{\ell^e}\sum_{\substack{2\le n\le N\\ \mathrm{ord}_\ell(n)=e}}\frac{|2^n-1|_{S_e}}{n} + C_{18} + \mathrm{O}(2^{-\sqrt{N}}), \tag{9.1}$$

where S_e is the finite set of primes dividing $2^{\ell^e}-1$. (This set was denoted by S_{ℓ^e} in (3.8).) Noting that $\ell \notin S$, we observe that, for any $e \ge 0$, $n \in \mathbb{N}$ such that $\mathrm{ord}_\ell(n) = e$, and prime p dividing $2^{\ell^e}-1$, by (3.1) we have

$$\mathrm{ord}_p(2^n-1) = \mathrm{ord}_p(2^{\ell^e}-1) + \mathrm{ord}_p(n).$$

Since every prime divisor of $2^{\ell^e}-1$ lies in S_e, we deduce that

$$|2^n-1|_{S_e} = \frac{|n|_{S_e}}{2^{\ell^e}-1}.$$

Thus the sum in (9.1) becomes

$$\sum_{e=0}^{\infty} \frac{1}{\ell^e(2^{\ell^e}-1)} \sum_{\substack{2\le n\le N/\ell^e \\ \ell\nmid n}} \frac{|n|_{S_e}}{n}. \tag{9.2}$$

Now, by [3, Proposition 5.2, Lemma 5.1], we have

$$\sum_{\substack{2\le n\le N \\ \ell\nmid n}} \frac{|n|_{S_e}}{n} = \left(1-\frac{1}{\ell}\right) k_e \log N + \mathrm{O}_e(1), \tag{9.3}$$

where $k_e = \prod_{p\in S_e} p/(p+1)$. Here we need to control the error terms uniformly in e. For this, we use the following lemma, which we will prove at the end of the section.

Lemma 18. *For S' any finite set of primes put*

$$k'_{S'} = \prod_{p\in S'} \frac{p}{p+1} \quad \text{and} \quad f_{S'}(N) = \sum_{n\le N} \frac{|n|_{S'}}{n} - k'_{S'} \log N.$$

By [3, Proposition 5.2], *there exists $A_{S'} > 4$ such that $|f_{S'}(N)| \le A_{S'}$ for all $N > 1$.*

Now fix S', let $p \in \mathbb{P} \setminus S'$ and put $S'' = S' \cup \{p\}$. Then $|f_{S''}(N)| \le 2A_{S'}$ for all $N > 1$.

In particular the $\mathrm{O}_e(1)$ error in (9.3) is $\mathrm{O}(2^{|S_e|})$, with an implied constant independent of e, and $2^{|S_e|} \le \prod_{p\in S_e} p \le 2^{\ell^e} - 1$. Thus the error in each term of the outside sum in (9.2) is $\mathrm{O}(1/\ell^e)$ and the sum of these errors converges. Thus (9.1) and (9.2) give

$$\mathsf{M}_S(N) \sim k_S \log N,$$

with

$$k_S = \sum_{e=0}^{\infty} \frac{\ell-1}{\ell^{e+1}(2^{\ell^e}-1)} \prod_{p\in S_e} \frac{p}{p+1}.$$

Now the partial sums give infinitely many rational approximations a/b of k_S with error $\mathrm{O}(b^{-\ell})$; thus, provided $\ell \ge 3$, we deduce that k_S is transcendental by Roth's theorem.

It only remains to prove Lemma 18.

Proof of Lemma 18. We have

$$\sum_{n\le N} \frac{|n|_{S''}}{n} = \sum_{r=0}^{\lfloor \log N/\log p\rfloor} \frac{1}{p^{2r}} \sum_{\substack{n\le N/p^r \\ p\nmid n}} \frac{|n|_{S'}}{n}$$

and

$$\sum_{\substack{n\le N\\ p\nmid n}} \frac{|n|_{S'}}{n} = \left(1-\frac{1}{p}\right)k'_{S'}\log N - \frac{k'_{S'}\log p}{p} + f_{S'}(N) - \frac{1}{p}f_{S'}(N/p).$$

Putting these together, we get

$$f_{S''}(N) = \frac{k'_{S'}(p-1)}{p}\sum_{r>\lfloor \log N/\log p\rfloor}\frac{1}{p^{2r}}\log N - \frac{k'_{S'}(p-1)\log p}{p}\sum_{r=0}^{\lfloor \log N/\log p\rfloor}\frac{r}{p^{2r}}$$
$$- \frac{k'_{S'}\log p}{p}\sum_{r=0}^{\lfloor \log N/\log p\rfloor}\frac{1}{p^{2r}} + \sum_{r=0}^{\lfloor \log N/\log p\rfloor}\frac{1}{p^{2r}}\left(f_{S'}(N/p^r) - \frac{1}{p}f_{S'}(N/p^{r+1})\right).$$

Using $0 < k'_{S'} \le 1$, $p \ge 3$ and $N \ge 2$, we show that the first three terms are absolutely bounded by

$$\frac{pk'_{S'}}{(p+1)\log N} < \frac{1}{\log 2}, \quad \frac{pk'_{S'}}{(p+1)(p^2-1)} < \frac{3}{32} \quad \text{and} \quad \frac{pk'_{S'}\log p}{(p^2-1)} < \frac{3\log 3}{8}$$

respectively, whose sum is bounded by 2. The final term is bounded in absolute value by $\frac{p}{p-1}A_{S'} < \frac{3}{2}A_{S'}$ and the result follows from the assumption that $A_{S'} > 4$. ∎

Remark 19. In fact [3, Proposition 5.2] says that $f_{S'}(N) = C'_{S'} + \mathrm{O}(N^{-1})$ so a finer analysis of the errors in (9.3) along the lines of Lemma 18 should allow one to get an asymptotic expression for $M_S(N)$ with an error term.

10. Existence of suitable sets of primes. It remains to prove Proposition 11.

Proof of Proposition 11. Let $\delta \in (0,1]$. We seek first a set of primes $\mathcal{L}$ such that

$$\sum_{\substack{\ell\le x\\ l\in\mathcal{L}}}\frac{\log \ell}{\ell} = \delta\log x + \mathrm{O}(1).$$

For rational δ, such a set exists from Dirichlet's theorem on primes in arithmetic progression (see [1, Theorem 7.3]); thus $\mathcal{L}$ would be a set of primes defined by congruence conditions and δ would in fact be the natural density of $\mathcal{L}$. For arbitrary δ a more delicate construction is needed. Let S be the set of primes in the union of intervals

$$\bigcup_{n\in\mathbb{N}}(2^n, 2^{n+\delta}].$$

Now the prime number theorem implies that

$$\pi_{\mathbb{P}}(x) := |\{p \le x : p \in \mathbb{P}\}| = \frac{x}{\log x} + \frac{x}{(\log x)^2} + 2\frac{x}{(\log x)^3} + \mathrm{O}\bigg(\frac{x}{(\log x)^4}\bigg)$$

and applying partial summation gives

$$\sum_{2^n < p \le 2^{n+\delta}} \frac{\log p}{p} = \delta \log 2 + \mathrm{O}\bigg(\frac{1}{n^2}\bigg).$$

Summing over n gives the required asymptotic.

For $\mathcal{L}' \subseteq \mathcal{L}$, write $\Sigma(\mathcal{L}') = \sum_{p \in \mathcal{L}'} (\log p)/p$. Taking logarithms, the statement now sought is that any $a > 0$ can be written as

$$\left.\begin{aligned} a &= \sum_{p \in \mathcal{L}'} \log\bigg(1 + \frac{1}{p}\bigg), \\ \Sigma(\mathcal{L}') &< \infty. \end{aligned}\right\}$$

The basic idea is to use the greedy algorithm but on a subset of $\mathcal{L}$ which is forced to be sparse enough to ensure the convergence of $\Sigma(\mathcal{L}')$.

Let

$$\mathcal{L}' := \mathbb{P} \cap \Big(\bigcup_{X<m<Y} (2^m, 2^{m+\delta}] \cup (2^{Y+\delta/4}, 2^{Y+\delta}] \cup \bigcup_{n=1}^{\infty} (2^{R_n}, 2^{R_n r_n}] \Big), \tag{10.1}$$

where

$$R_n := [2^{n/2} Y].$$

Here we assume that $X, Y \in \mathbb{N}$ and $r_n \in \mathbb{R}$ are parameters satisfying

$$3 \le X < Y$$

and

$$1 < r_n < 1 + \frac{\delta}{R_n}$$

for all $n \ge 1$, so that the union on the right-hand side of (10.1) is disjoint and $\mathcal{L}' \subseteq \mathcal{L}$. The construction involves choosing the parameters X, Y and r_n appropriately. More precisely, we show that, provided X is large enough, there are choices of Y and r_n such that the corresponding set $\mathcal{L}'$ has the required properties.

We first derive asymptotic estimates for the sums of $\log(1 + 1/p)$ over $2^m < p \le 2^{m+\delta}$, $2^Y < p \le 2^{Y+\delta/4}$ and $2^{R_n} < p \le 2^{R_n r_n}$. We have

$$\begin{aligned} \sum_{2^m < p \le 2^{m+\delta}} \log\bigg(1 + \frac{1}{p}\bigg) &= \sum_{2^m < p \le 2^{m+\delta}} \bigg(\frac{1}{p} + \mathrm{O}\bigg(\frac{1}{p^2}\bigg)\bigg) \\ &= \sum_{2^m < p \le 2^{m+\delta}} \frac{1}{p} + \mathrm{O}(2^{-m}). \end{aligned} \tag{10.2}$$

Using the prime number theorem with error term in the well-known form (see for example [22, Part II, §4.1, Theorem 1])

$$\pi(x) = \int_2^x \frac{dt}{\log t} + \mathrm{O}(x \exp(-C_{19}\sqrt{\log x})),$$

C_{19} being a suitable positive constant, and partial summation and integration, we deduce that

$$\begin{aligned}
(10.3)\qquad \sum_{2^m<p\le 2^{m+\delta}} \frac{1}{p} &= \frac{1}{2^{m+\delta}}\Big(\sum_{2^m<p\le 2^{m+\delta}} 1\Big) + \int_{2^m}^{2^{m+\delta}} \frac{1}{t^2}\Big(\sum_{2^m<p\le t} 1\Big)\,dt \\
&= \frac{1}{2^{m+\delta}} \int_{2^m}^{2^{m+\delta}} \frac{dt}{\log t} + \int_{2^m}^{2^{m+\delta}} \frac{1}{t^2} \int_{2^m}^{t} \frac{dy}{\log y}\,dt + \mathrm{O}(\exp(-C_{20}m^{1/2})) \\
&= \int_{2^m}^{2^{m+\delta}} \frac{dt}{t\log t} + \mathrm{O}(\exp(-C_{20}m^{1/2})) \\
&= \log\log 2^{m+\delta} - \log\log 2^m + \mathrm{O}(\exp(-C_{20}m^{1/2})) \\
&= \log\Big(1+\frac{\delta}{m}\Big) + \mathrm{O}(\exp(-C_{20}m^{1/2})),
\end{aligned}$$

for a suitable positive constant C_{20}. Combining (10.2) and (10.3), we obtain

$$(10.4)\qquad \sum_{2^m<p\le 2^{m+\delta}} \log\Big(1+\frac{1}{p}\Big) = \log\Big(1+\frac{\delta}{m}\Big) + \mathrm{O}(\exp(-C_{20}m^{1/2})).$$

Similarly, we derive

$$(10.5)\qquad \sum_{2^Y<p\le 2^{Y+\delta/4}} \log\Big(1+\frac{1}{p}\Big) = \log\Big(1+\frac{1}{4}\,\frac{\delta}{Y}\Big) + \mathrm{O}(\exp(-C_{20}Y^{1/2}))$$

and

$$\begin{aligned}
(10.6)\qquad b_n &:= \sum_{2^{R_n}<p\le 2^{R_n}r_n} \log\Big(1+\frac{1}{p}\Big) = \log r_n + \mathrm{O}(\exp(-C_{20}R_n^{1/2})) \\
&= \log r_n + \mathrm{O}(\exp(-C_{20}2^{n/4}Y^{1/2})).
\end{aligned}$$

Now assume that X is large enough so that

$$\sum_{2^m<p\le 2^{m+\delta}} \log\Big(1+\frac{1}{p}\Big) < a$$

for all $m > X$. Let Y be the unique natural number satisfying

$$\sum_{X<m\le Y} \sum_{2^m<p\le 2^{m+\delta}} \log\Big(1+\frac{1}{p}\Big) < a \le \sum_{X<m\le Y+1} \sum_{2^m<p\le 2^{m+\delta}} \log\Big(1+\frac{1}{p}\Big).$$

Set

$$a' := a - \Bigg(\sum_{X<m<Y} \sum_{2^m<p\le 2^{m+\delta}} \log\Big(1+\frac{1}{p}\Big) + \sum_{2^{Y+\delta/4}<p\le 2^{Y+\delta}} \log\Big(1+\frac{1}{p}\Big)\Bigg).$$

Using (10.4) and (10.5), we have

$$\begin{aligned}(10.7)\qquad \frac{1}{5}\,\frac{\delta}{Y} &< \log\Big(1+\frac{1}{4}\,\frac{\delta}{Y}\Big) + \mathrm{O}(\exp(-C_{20}Y^{1/2})) \\ &= \sum_{2^Y<p\le 2^{Y+\delta/4}} \log\Big(1+\frac{1}{p}\Big) \\ &< a' \le \sum_{2^Y<p\le 2^{Y+\delta/4}} \log\Big(1+\frac{1}{p}\Big) + \sum_{2^{Y+1}<p\le 2^{Y+1+\delta}} \log\Big(1+\frac{1}{p}\Big) \\ &= \log\Big(1+\frac{1}{4}\,\frac{\delta}{Y}\Big) + \log\Big(1+\frac{\delta}{Y+1}\Big) + \mathrm{O}(\exp(-C_{20}Y^{1/2})) \\ &< \frac{5}{4}\,\frac{\delta}{Y} + \mathrm{O}(\exp(-C_{20}Y^{1/2})) < \frac{4}{3}\log\Big(1+\frac{\delta}{Y}\Big),\end{aligned}$$

provided Y is sufficiently large (which is the case if X is sufficiently large).

Now write

$$r_1 = \exp(a'/2)$$

and then define

$$r_n = \exp\Big(\frac{a' - (b_1+\cdots+b_{n-1})}{2}\Big)$$

for all $n \ge 2$, where b_j is defined as in (10.6).

We wish to show by induction that, if X (and hence Y) is chosen large enough, then the following three properties hold for every $n \ge 1$:

$$(10.8)\qquad 1 < r_n < 1 + \frac{\delta}{R_n},$$

$$(10.9)\qquad a'\Big(1 - \frac{1}{2^n} - f(n)\Big) < b_1 + \cdots + b_n < a'\Big(1 - \frac{1}{2^n} + f(n)\Big)$$

and

$$(10.10)\qquad \sum_{2^{R_n}<p\le 2^{R_n r_n}} \frac{\log p}{p} \ll 2^{-n/2},$$

where

$$f(n) = \sum_{j=1}^{n} 100^{-2^{j/4}} 2^{j-n}.$$

Notice that

$$(10.11)\qquad f(n) < 2^{-(n+2)}$$

for all $n \geq 1$, since

$$\sum_{j=1}^{\infty} 100^{-2^{j/4}} 2^j < \frac{1}{4}.$$

Thus (10.7)–(10.10) together give the result.

If Y is large enough then, using (10.7), we have

$$1 < r_1 = \exp(a'/2) < \left(1 + \frac{\delta}{Y}\right)^{2/3} < 1 + \frac{7}{10}\frac{\delta}{Y} < 1 + \frac{\delta}{[2^{1/2}Y]} = 1 + \frac{\delta}{R_1},$$

and hence the bounds (10.8) hold for $n = 1$. Turning to (10.9), notice that, since $r_1 = \exp(a'/2)$, we have

$$b_1 = a'/2 + \mathrm{O}(\exp(-C_{20}2^{1/4}Y^{1/2}))$$

by (10.6), so (10.9) holds for $n = 1$ if Y is large enough. Moreover, (10.10) holds trivially for $n = 1$.

We assume now that X has been chosen large enough such that (10.7) holds, (10.9) holds for $n = 1$, and

$$a' \exp(C_{20}Y^{1/2}) > \frac{1}{5}\frac{\delta}{Y}\exp(C_{20}Y^{1/2}) > 100 \quad \text{if } Y > X. \tag{10.12}$$

In particular, the base step of the induction holds. Now assume that (10.8)–(10.10) hold for some $n = k - 1$, with $k \geq 2$. By (10.9) for $n = k - 1$, we have

$$a'\left(\frac{1}{2^k} - \frac{f(k-1)}{2}\right) < \log r_k < a'\left(\frac{1}{2^k} + \frac{f(k-1)}{2}\right). \tag{10.13}$$

Using (10.7), (10.11) and (10.13), we deduce that

$$\begin{aligned} 1 < r_k < \exp\left(\frac{5}{4}\frac{a'}{2^k}\right) &< \left(1 + \frac{\delta}{Y}\right)^{(5/3)\cdot 2^{-k}} < 1 + \frac{7}{4}\frac{1}{2^k}\frac{\delta}{Y} \\ &< 1 + \frac{\delta}{\left[2^{k/2}Y\right]} = 1 + \frac{\delta}{R_k}, \end{aligned}$$

and hence (10.8) holds for $n = k$. Using (10.6), (10.11), (10.13) and the definition of R_k, we have

$$\begin{aligned} \sum_{2^{R_k} < p \leq 2^{R_k r_k}} \frac{\log p}{p} &\ll \log 2^{R_k r_k} \sum_{2^{R_k} < p \leq 2^{R_k r_k}} \frac{1}{p} \\ &\ll R_k \sum_{2^{R_k} < p \leq 2^{R_k r_k}} \log\left(1 + \frac{1}{p}\right) \ll 2^{-k/2}. \end{aligned}$$

It follows that (10.10) holds for $n = k$. Moreover, by (10.6) and the definition of r_k, we have

$$b_1 + \cdots + b_k = \frac{a' + b_1 + \cdots + b_{k-1}}{2} + \mathrm{O}(\exp(-C_{20}2^{k/4}Y^{1/2})),$$

and so

$$\frac{a'+b_1+\cdots+b_{k-1}}{2} - a'\cdot 100^{-2^{k/4}} < b_1+\cdots+b_k < \frac{a'+b_1+\cdots+b_{k-1}}{2} + a'\cdot 100^{-2^{k/4}}, \tag{10.14}$$

using (10.12). From (10.9) for $n=k-1$ and (10.14), we deduce that

$$a'\left(1-\frac{1}{2^k}-\frac{f(k-1)}{2}-100^{-2^{k/4}}\right) < b_1+\cdots+b_k < a'\left(1-\frac{1}{2^k}+\frac{f(k-1)}{2}+100^{-2^{k/4}}\right).$$

This is equivalent to (10.9) for $n=k$, since

$$f(k) = f(k-1)/2 + 100^{-2^{k/4}},$$

completing the induction. ∎

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