1. Introduction and main results. Let $G$ be an additively written, finite cyclic group and $g \in G$ with $\text{ord}(g) = |G|$. For a sequence $S = (n_1 g) \cdot \ldots \cdot (n_l g)$ over $G$, where $l \in \mathbb{N}_0$ and $n_1, \ldots, n_l \in [1, n]$, we set
\[
\|S\|_g = n_1 + \cdots + n_l,
\]
and then
\[
\text{ind}(S) = \min\{\|S\|_h \mid h \in G \text{ with } \text{ord}(h) = |G|\} \in \mathbb{Q}_{\geq 0}
\]
denotes the index of $S$. The index of a sequence is a crucial invariant in the investigation of (minimal) zero-sum sequences (resp. of zero-sum free sequences) over cyclic groups. It was first considered by Lemke and Kleitman ([11]), used as key tool by Geroldinger ([6, p. 736]), and then investigated by Gao [3] in a systematical way. Since then it has found a lot of attention in recent years (see [1, 2, 5, 8, 12–16]). We briefly discuss some key results.

If $S$ is a minimal zero-sum sequence, then $|S| \leq 3$, as well as $|S| \geq \lfloor n/2 \rfloor + 2$, implies that $\text{ind}(S) = 1$ (see [1], [14], [16]). In contrast, it was shown that for every $k \in [5, \lfloor n/2 \rfloor + 1]$, there is a minimal zero-sum subsequence $T$ of length $|T| = k$ and with $\text{ind}(T) \geq 2$, and that the same is true for $k = 4$ and $\gcd(n, 6) \neq 1$. This leads to the conjecture that, in case $\gcd(n, 6) = 1$, every minimal zero-sum sequence $S$ over $G$ of length $|S| = 4$ has $\text{ind}(S) = 1$. Li, Plyley, Yuan and Zeng [12] recently proved that this holds true if $n$ is a prime power, but the general case is still open.

In 1989, Lemke and Kleitman [11, p. 344] stated the following conjecture, which we formulate in the present language.

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Conjecture 1.1. Let $G$ be a cyclic group of order $n$, $d$ a divisor of $n$, and $S$ a sequence over $G$ of length $|S| = n$. Then there exists a subsequence $T$ of $S$ and an element $g \in G$ with $\text{ord}(g) = n$ such that $d | n \| T \|_g | n$.

In the special case $d = n$, this is equivalent to the existence of a subsequence $T$ with $\text{ind}(T) = 1$.

Indeed, the above is the third of three interesting conjectures stated by Lemke and Kleitman in [11]. Their first conjecture has turned out to be true for all finite abelian groups (see [7]), and the second one is still open. In this paper we demonstrate that the above conjecture fails in general (see Theorem 1.2), but that it holds true under an additional assumption on the highest multiplicity of an element occurring in the sequence. Here are the main results of the present paper (for any undefined terminology or notation the reader is referred to the beginning of Section 2).

**Theorem 1.2.** Let $G$ be a cyclic group of order $n \geq 2$, where $n = 4k + 2$ for some $k \geq 5$, and let $g \in G$ with $\text{ord}(g) = n$. Then the sequence $S = g^{n/2 - 3} \left(\frac{n}{2} g\right) \left(\left(\frac{n}{2} + 1\right) g\right)^{n/2 - 1} \left(\left(\frac{n}{2} + 2\right) g\right)^{[n/4]^2 - 2}$ has no subsequence $T$ with $\text{ind}(T) = 1$.

**Theorem 1.3.** Let $G$ be a cyclic group of order $n \geq 2$ and $S$ be a sequence over $G$ of length $|S| = n$. If $h(S) < 4$ or $h(S) \geq n/2$, then $S$ has a subsequence $T$ with $\text{ind}(T) = 1$ and length $|T| \leq h(S)$.

**Theorem 1.4.** Let $G$ be a cyclic group of prime order $p > 24318$ and $S$ be a sequence over $G$ of length $|S| = p$. If $h(S) \geq (p - 2)/10$, then $S$ has a subsequence $T$ with $\text{ind}(T) = 1$.

In Section 2 we summarize our notation and prove Theorem 1.2. In the following two sections we provide the proofs of Theorems 1.3 and 1.4. We end the paper with a further conjecture and some open problems (see Section 5).

**2. Notation and proof of Theorem 1.2.** Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{P} \subset \mathbb{N}$ the set of prime numbers, and for rational numbers $a, b \in \mathbb{Q}$ we set $[a, b] = \{x \in \mathbb{Z} \mid a \leq x \leq b\}$. Let $G$ be an additively written abelian group and $G_0 \subset G$ a subset. We fix the notation concerning sequences over $G_0$ (which is consistent with [4] and [9]). Let $\mathcal{F}(G_0)$ be the free abelian monoid with basis $G_0$. The elements of $\mathcal{F}(G_0)$ are called sequences over $G_0$. We write sequences $S \in \mathcal{F}(G_0)$ in the form $S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\nu_g(S)}$, where $\nu_g(S)$ is the multiplicity of $g$ in $S$. The number $h(S) = \frac{1}{n} \sum_{g \in G} \nu_g(S)$ is called the height of $S$. If $\nu_g(S) = 0$ for all $g \in G$, then $S$ is said to have ind $1$ and $\text{ind}(S) = 1$. The sequence $S$ has ind $0$ if $h(S) = 0$. If $S \in \mathcal{F}(G_0)$ has ind $1$, then its length $|S|$ is uniquely determined by $h(S)$ and $\text{ind}(S) = 1$.

Let $\mathcal{F}(G)$ be the free abelian monoid with basis $G$. The elements of $\mathcal{F}(G)$ are called sequences over $G$. We write sequences $S \in \mathcal{F}(G)$ in the form $S = g_1 \cdot \ldots \cdot g_l = \prod_{g \in G} g^{\nu_g(S)}$, where $\nu_g(S)$ is the multiplicity of $g$ in $S$. The number $h(S) = \frac{1}{n} \sum_{g \in G} \nu_g(S)$ is called the height of $S$. If $\nu_g(S) = 0$ for all $g \in G$, then $S$ is said to have ind $1$ and $\text{ind}(S) = 1$. The sequence $S$ has ind $0$ if $h(S) = 0$. If $S \in \mathcal{F}(G)$ has ind $1$, then its length $|S|$ is uniquely determined by $h(S)$ and $\text{ind}(S) = 1$.
where \( l \in \mathbb{N}_0, g_1, \ldots, g_l \in G_0, v_g(S) \in \mathbb{N}_0 \) and \( v_g(S) = 0 \) for almost all \( g \in G_0 \). We call \( |S| = l \) the length of \( S \), \( \sigma(S) = g_1 + \cdots + g_l \) the sum of \( S \), \( v_g(S) \) the multiplicity of \( g \) in \( S \), \( \text{supp}(S) = \{ g \in G \mid v_g(S) > 0 \} \) the support of \( S \), and we denote by

\[
h(S) = \max \{ v_g(S) \mid g \in G \} \in [0, |S|]
\]

the maximum of the multiplicities of \( S \). For every group homomorphism \( \varphi : G \to H \), we set \( \varphi(S) = \varphi(g_1) \cdot \cdots \cdot \varphi(g_l) \in \mathcal{F}(H) \), and if \( \varphi \) is multiplication by some \( m \in \mathbb{N} \), then we set \( mS = \varphi(S) \). We say that \( S \) is a zero-sum sequence if \( \sigma(S) = 0 \), and it is called a minimal zero-sum sequence if \( \sigma(S) = 0 \) but \( \sum_{i \in I} g_i \neq 0 \) for all \( \emptyset \neq I \subset [1, l] \). Suppose that \( G \) is finite cyclic. Then a simple calculation (see [8, Lemma 5.1.2]) shows that

\[
\text{ind}(S) = \min \{ ||S||_h \mid h \in G \text{ with supp}(S) \subset \langle h \rangle \} = \min \{ ||S||_h \mid h \in G \text{ with } \langle \text{supp}(S) \rangle = \langle h \rangle \}.
\]

**Proof of Theorem 1.2.** Assume to the contrary that \( S \) has a subsequence \( T \) with \( \text{ind}(T) = 1 \). Then there exists an element \( h \in G \) with \( \text{ord}(h) = n \) such that \( ||T||_h = 1 \). We set

\[
g = jh \quad \text{and} \quad T = g^x \left( \frac{n}{2}g \right)^y \left( \left( \frac{n}{2} + 1 \right)g \right)^z \left( \left( \frac{n}{2} + 2 \right)g \right)^w
\]

where \( j \in [1, n - 1] \) with \( \gcd(j, n) = 1 \), \( x \in [0, n/2 - 3] \), \( y \in [0, 1] \), \( z \in [0, n/2 - 1] \) and \( w \in [0, n/4 - 2] \). Then

\[
n||T||_g = (x + z + 2w) + \frac{n}{2}(y + z + w) \equiv 0 \pmod{n}.
\]

**CASE 1:** \( j < n/4 \). Then

\[
T = (jh)^x \left( \frac{n}{2}h \right)^y \left( \left( \frac{n}{2} + j \right)h \right)^z \left( \left( \frac{n}{2} + 2j \right)h \right)^w.
\]

Since \( ||T||_h = 1 \), we infer that \( y + z + w \leq 1 \), which implies that \( n||T||_g \leq x + (n/2 + 2) \leq n/2 - 3 + n/2 + 2 < n \), a contradiction.

**CASE 2:** \( n/4 < j < n/2 \). Then

\[
T = (jh)^x \left( \frac{n}{2}h \right)^y \left( \left( \frac{n}{2} + j \right)h \right)^z \left( \left( 2j - \frac{n}{2} \right)h \right)^w.
\]

Since \( ||T||_h = 1 \), we infer that \( x \leq 3 \) and \( z \leq 1 \), which implies that \( x + z + 2w \leq 3 + 1 + 2 (|n/4| - 2) < n/2 \). Since \( x + z + 2w > 0 \) and again by \( ||T||_h = 1 \), we derive that \( x + z + 2w \equiv 0 \pmod{n/2} \), a contradiction.

**CASE 3:** \( n/2 < j < 3n/4 \). Then

\[
T = (jh)^x \left( \frac{n}{2}h \right)^y \left( \left( j - \frac{n}{2} \right)h \right)^z \left( \left( 2j - \frac{n}{2} \right)h \right)^w.
\]
Since $\|T\|_h = 1$, we infer that $x + y + w \leq 1$. We assert that

$$x + y + w = 1. \tag{2}$$

Otherwise, $x = y = w = 0$ and $n\|T\|_g = z + (n/2)z \not\equiv 0 \pmod{n/2}$, a contradiction to $n\|T\|_g \equiv 0 \pmod{n}$. Note that $0 < x + z + 2w < n$. By (1), we have

$$x + z + 2w = n/2, \tag{3}$$

$$\quad y + z + w \equiv 1 \pmod{2}. \tag{4}$$

By (2) and (3), we have $y + z + w \equiv z + w - y = n/2 - 1 \equiv 0 \pmod{2}$, a contradiction to (4).

**Case 4:** $3n/4 < j < n$. Then

$$T = (jh)^z \left(\frac{n}{2}h\right)^y \left(\left(j - \frac{n}{2}\right)h\right)^z \left(\left(2j - \frac{3n}{2}\right)h\right)^w.$$ 

Since $\|T\|_h = 1$, we infer that $x \leq 1$ and $z \leq 3$, which implies that $x + z + 2w \leq 1 + 3 + 2(\lfloor n/4 \rfloor - 2) < n/2$. Clearly, $x + z + 2w > 0$. From (1), we derive a contradiction. □

3. **Proof of Theorem 1.3** We need the following two results. A simple proof of the first one can be found in [8, Proposition 4.2.6] (for historical comments see [10]), and a proof of Lemma 3.2 is given in [13].

**Lemma 3.1.** Let $G$ be a finite cyclic group and $S$ be a sequence over $G$ of length $|S| \geq |G|$. Then $S$ has a zero-sum subsequence $T$ of length $|T| \in [1, h(S)]$.

**Lemma 3.2.** Let $G$ be a finite cyclic group and $S$ be a minimal zero-sum sequence over $G$ of length $|S| \in [1, 3]$. Then $\text{ind}(S) = 1$.

**Proof of Theorem 1.3** We set $n = |G|$ and $h = h(S)$. If $h < 4$, then the assertion follows from Lemmas 3.1 and 3.2. Suppose that $h \geq n/2$. Let $g \in G$ with $v_g(S) = h$. If ord$(g) < n$, then ord$(g) \leq n/2 \leq h$, and $T = g^\text{ord}(g)$ has the required properties. If $0 \mid S$, then $T = 0$ has the required properties.

Suppose that ord$(g) = n$ and that $0 \nmid S$. Then we can write $S$ in the form

$$S = g^h(b_1g) \cdot \ldots \cdot (b_{n-h}g) \quad \text{where} \quad b_1, \ldots, b_{n-h} \in [2, n-1].$$

Assume to the contrary that $S$ has no subsequence $T$ with the required properties. We continue with the following assertion.

**A.** For every subset $I \subset [1, n-h]$ we have $\sum_{i \in I} b_i \leq n - h + |I| - 1.$
If $A$ holds, then we apply it with $I = [1, n - h]$ and obtain
\[
\sum_{i=1}^{n-h} b_i \leq 2(n - h) - 1,
\]
a contradiction to $b_1, \ldots, b_{n-h} \in [2, n - 1]$.

We prove $A$ by induction on $|I|$. If there were an $i \in [1, n - h]$ such that $b_i \geq n - h + 1$, then $T = g^{n-b_i}(b_ig)$ would be a subsequence of $S$ with $\text{ind}(T) = 1$ and length $|T| = n - b_i + 1 \leq h$, a contradiction. Let $I \subset [1, n - h]$ with $|I| = k + 1 \geq 2$, say $I = [1, k + 1]$, and suppose that $A$ holds for all proper subsets of $I$. We set $\beta = b_1 + \cdots + b_{k+1}$. By induction hypothesis we get $\beta - b_i \leq n - h + k - 1$ for every $i \in [1, k + 1]$, which implies that
\[
\beta = \frac{1}{k} (k\beta) = \frac{1}{k} \sum_{i=1}^{k+1} (\beta - b_i) \leq \frac{(k+1)(n - h + k - 1)}{k} \leq n
\]
(to get the last inequality, use that $h \geq n/2$ and $k \leq n - h - 1$). Thus, if $\beta \geq n - h + k + 1$, then $T = g^{n-\beta}(b_1g) \cdot \cdots \cdot (b_{k+1}g)$ is a subsequence of $S$ with $\text{ind}(T) = 1$ and length $|T| = n - \beta + k + 1 \leq h$. This is a contradiction, and thus $A$ is proved.

Note that the sequence $S$ given in Theorem 1.2 satisfies $h(S) = n/2 - 1$. Thus the assumption in Theorem 1.3, that $h(S) \geq n/2$, cannot be weakened for $n \equiv 2 \pmod{4}$.

4. Proof of Theorem 1.4. We fix our notation which remains valid throughout the whole section. Let $G$ be a prime cyclic group of order $|G| = p > 24318$, $G^* = G \setminus \{0\}$, and let $S$ be a sequence over $G^*$ of length $|S| = p$. If $g \in G^*$, $A \subset \mathbb{Z}$ and $S = (n_1g) \cdot \cdots \cdot (n_lg)$ with $n_1, \ldots, n_l \in [1, p - 1]$, then we set
\[
S(A, g) = \prod_{i \in [1, l], n_i \in A} (n_ig).
\]
For an element $g \in G^*$, we set
\[
\Sigma_g(S) = \{p\|T\|_g \mid T \text{ is a subsequence of } S \text{ with } \|T\|_g \leq 1\},
\]
and we denote by $m_g(S)$ the maximal $t \in [1, p]$ such that $\Sigma_g(T) = [1, t]$ for some subsequence $T$ of $S$. We define
\[
m(S) = \max\{m_g(S) \mid g \in G^*\}.
\]
From now on we fix an element $g \in G^*$ such that $m_g(S) = m(S)$.

**Lemma 4.1.** Let $T$ be a subsequence of $S$ such that $\Sigma_g(T) = [1, m(S)]$. Then $|T| \leq m(S)$, and if $x \in [1, p - 1]$ is such that $(xg)|ST^{-1}$, then $x \geq m(S) + 2$. Furthermore, if $m(S) = p$, or if there exists an $x \in [1, p - 1]$ such that $(xg)|ST^{-1}$ and $x \geq p - m(S)$, then $S$ has a subsequence with index 1.
Proof. By definition, we have \(|T| \leq \|T\|_g = m(S)\). If there is some \(x \in [1, p - 1]\) with \((xg) | ST^{-1}\) and \(x \leq m(S) + 1\), then \(\Sigma_g((xg)T) = [1, \min\{p, m(S) + x\}]\), a contradiction to the maximality of \(m(S)\). The second part of this lemma is clear.

From now on we suppose that \(S\) has no subsequence with index 1.

Let \(k \geq 2\) be a positive integer, and let \(F[1/k, (k - 1)/k]\) be the set of all irreducible fractions between 1/k and (k - 1)/k and with denominators in \([2, k]\), i.e.,

\[
F\left[\frac{1}{k}, \frac{k - 1}{k}\right] = \left\{ \frac{a}{b} \mid a \in \mathbb{N}, b \in [2, k] \text{ with } \gcd(a, b) = 1 \text{ and } \frac{1}{k} \leq \frac{a}{b} \leq \frac{k - 1}{k} \right\}.
\]

Lemma 4.2. Let \(a/b\) and \(c/d\) be two adjacent fractions in \(F[1/k, (k - 1)/k]\) with \(a/b < c/d\). Then

(i) \(b + d \geq k + 1\).
(ii) \(bc - ad = 1\).

Proof. (i) Note that \(\frac{a}{b} < \frac{a + c}{b + d} < \frac{c}{d}\). Since \(a/b\) and \(c/d\) are adjacent, it follows that the irreducible fraction with value \(\frac{a + c}{b + d}\) is not in \(F[1/k, (k - 1)/k]\). This forces that \(b + d \geq k + 1\).

(ii) Since \(\gcd(a, b) = 1\), there are two integers \(u\) and \(v\) such that \(bu + av = 1\). Note that \(b(u + ma) + a(v - mb) = 1\) for any integer \(m\). Let \(x = u + ma\) and \(y = mb - v\). Then \(bx - ay = 1\). By choosing \(m\) suitably we may assume that \(y \leq k\) and \(y + b \geq k + 1\). It follows that \(y \geq k + 1 - b > 0\) and \(x > 0\).

From \(bx - ay = 1\) we get

\[
\frac{x}{y} - \frac{a}{b} = \frac{1}{by}.
\]

If \(y > 1\), then \(x/y\) is a fraction in \(F[1/k, (k - 1)/k]\). So, either \(c/d = x/y\) and we are done, or \(c/d < x/y\). For the latter case we have

\[
\frac{1}{by} = \frac{x}{y} - \frac{a}{b} = \left(\frac{x}{y} - \frac{c}{d}\right) + \left(\frac{c}{d} - \frac{a}{b}\right) = \frac{b(dx - cy) + y(cb - ad)}{byd} \geq \frac{b + y}{byd}.
\]

This implies that \(d \geq b + y \geq k + 1\), a contradiction.

Now assume that \(y = 1\) and we must have \(b = k\). It follows from \(bx - ay = 1\) that \(a = kx - 1\). Therefore, \(x = 1\) and \(a = k - 1\). So, \(a/b = (k - 1)/k\) is the largest fraction in \(F[1/k, (k - 1)/k]\), a contradiction.

We set

\[
k = \left\lfloor \frac{p}{m(S)} \right\rfloor, \quad f = \left| F\left[\frac{1}{k}, \frac{k - 1}{k}\right]\right|,
\]
and we arrange all fractions in $F[1/k, (k - 1)/k]$ increasingly; so let
\[ \frac{a_1}{b_1} < \cdots < \frac{a_f}{b_f} \]
denote the elements of $F[1/k, (k - 1)/k]$. Furthermore, we set
\[ S_1 = S([1, m(S)], g), \quad S_2 = S\left(\left[ m(S) + 2, \frac{p - 1}{b_1} \right], g \right) \]
and, for every $i \in [1, f]$, we set
\[ S_{2i+1} = S\left(\left[ \frac{a_ip + 1}{b_i}, \frac{a_ip + m(S)}{b_i} \right], g \right), \]
\[ S_{2i+2} = S\left(\left[ \frac{a_ip + m(S) + 1}{b_i}, \frac{a_{i+1}p - 1}{b_{i+1}} \right], g \right). \]
Furthermore, for every $i \in [2, k]$, we define
\[ R_i = S(\{x \in [1, p] \mid If \ x_i \in [1, p] with \ p \mid (x_i - ix), \ then \ x_i \in [1, m(S)] and \ gcd(x_i, i) = 1\}, g). \]

**Lemma 4.3.** We have $S = \prod_{j=1}^{2f+1} S_j$.  

**Proof.** This is clear by construction. \( \blacksquare \)

**Lemma 4.4.** Suppose that
\[ 4 \leq m(S) \leq \frac{p - 3}{2} \quad \text{and} \quad \max \left\{ \frac{p - m(S) - 2}{m(S)}, \frac{p - m(S)}{m(S) + 1} \right\} \leq k \leq \frac{p + 1}{m(S)}. \]
(i) $|S_{2i+2}| \leq b_{i+1} - 1$ for every $i \in [0, f - 1]$.  
(ii) $p = |S| \leq m(S) + \sum_{i=2}^{k} \sum_{j \in [1, i-1]} with gcd(i,j)=1(i - 1) + \sum_{i=2}^{k} |R_i|$.

**Proof.** (i) Suppose that $i = 0$. Then $S_2 = S([m(S) + 2, (p - 1)/b_1], g)$ and $b_1 = k$. If $|S_2| \geq b_1 = k$, then we can take a $k$-term subsequence $U$ of $S_2$. Note that $p - 1 \geq p \|U\|_g \geq k(m(S) + 2) \geq p - m(S)$ and one can find a subsequence $V$ of $S_1$ such that $UV$ has index 1, a contradiction.

Now suppose that $i \in [1, f - 1]$, and assume to the contrary that $|S_{2i+2}| \geq b_{i+1}$. We choose an arbitrary $b_{i+1}$-term subsequence $X$ of $S_{2i+2}$, and write $b_i S$ in the form
\[ b_i S = (x_1 g) \cdot \cdots \cdot (x_p g) \quad \text{with} \quad x_1, \ldots, x_p \in [1, p - 1]. \]
It follows from Lemma 4.2 that $a_{i+1}b_i - a_i b_{i+1} = 1$, and so
\[ b_i \left( \frac{a_{i+1}p - 1}{b_{i+1}} \right) - a_ip = \frac{p - b_i}{b_{i+1}}. \]
Thus for every $\nu \in [1, p]$ with $(x_{\nu} g) \mid S_{2i+2}$, we infer that $x_{\nu} \in [m(S) + 1$, $(p - b_i)/b_{i+1}]$ and $x_{\nu} \equiv -a_ip \mod b_i$. Therefore, since $b_i + b_{i+1} \geq k + 1$ by

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Lemma 4.2, we get
\[ p - b_i \geq p\|b_iX\|_g \geq b_{i+1}(m(S) + 1) \geq p - b_i m(S) \]
and
\[ p\|b_iX\|_g \equiv -b_{i+1}a_ip = (1 - a_{i+1}b_i)p \equiv p \pmod{b_i}. \]
Therefore there exists a subsequence \( Y \) of \( S_1 \) such that \( p\|b_i(XY)\|_g = p \), a contradiction.

(ii) For every \( \ell \in [2, k] \), we have \( R_\ell = \prod_{b_i=\ell} S_{2i+1} \), and hence
\[ S = S_1 \prod_{i=0}^{f-1} S_{2i+2} \prod_{\ell=2}^{k} R_\ell. \]
Now (ii) follows from (i).

**Lemma 4.5.** Let \( \ell \in \mathbb{N}_{\geq 2} \) and \( S \in F(\mathbb{Z}) \) be a sequence of length \( |S| = \ell \). Suppose that every element from \( S \) is coprime to \( \ell \). Then for every \( m \in \mathbb{Z} \) there exists a subsequence \( S_m \) such that \( \sigma(S_m) \equiv m \pmod{\ell} \). Moreover, if \( m \notin \ell\mathbb{Z} \), then \( S_m \neq S \).

**Proof.** Let \( \varphi: \mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \) be the canonical epimorphism and \( \varphi(S) = a_1 \cdots a_l \). We denote by \( A = \{a_1, 0\} + \cdots + \{a_{\ell-1}, 0\} \subset \mathbb{Z}/\ell\mathbb{Z} \) the subset, and by \( H = \text{Stab}(A) \) the stabilizer of \( A \). Clearly, it suffices to verify that \( A = \mathbb{Z}/\ell\mathbb{Z} \). If \( H \) were a proper subgroup of \( \mathbb{Z}/\ell\mathbb{Z} \), then Kneser’s Theorem would imply that
\[ |A| \geq \sum_{i=1}^{\ell-1} |\{a_i, 0\} + H| - (\ell - 2)|H| = (\ell - 1)2|H| - (\ell - 2)|H| \geq \ell, \]
whence \( A = H = \mathbb{Z}/\ell\mathbb{Z} \). Thus \( H = \mathbb{Z}/\ell\mathbb{Z} \), which implies that \( A = \mathbb{Z}/\ell\mathbb{Z} \), and we are done.

**Lemma 4.6.** Let \( t, \ell \in [2, k-1] \) with \( t < \ell \) and \( d = \gcd(t, \ell) < t \), and let \( u \in [2, m(S)] \). If
\[ \frac{(t-d)p - \ell}{t\ell} \leq m(S) \leq \frac{dp - t(u-1)}{\ell}, \]
then
\[ |R_t| = 0 \quad \text{or} \quad |R_\ell| \leq \frac{p - \ell m(S) - 2\ell + 1}{u} + 2\ell - 1. \]

**Proof.** Suppose that \( |R_t| > 0 \). Let \( x \in [1, p-1] \) be such that \( (xg) \mid R_t \), and let \( x_\ell \in [1, p-1] \) be such that \( p \mid (\ell x - x_\ell) \). By the definition of \( R_t \), we get
\[ x_\ell \in \bigcup_{i \in [1, t-1] \text{ with } \gcd(i, t) = 1} \left[ \frac{lip + \ell}{t}, \frac{lip + \ell m(S)}{t} \right], \]
and thus 
\[ x_\ell \in \bigcup_{i \in [1, t-1]} \left[ \frac{ip + \ell}{t}, \frac{ip + \ell m(S)}{t} \right] \subset \left[ \frac{dp + \ell}{t}, \frac{(t-d)p + \ell m(S)}{t} \right] \subset [p - \ell m(S), p - \ell(u-1)]. \]

If \(|(\ell R_\ell)([1, u-1], g)| \geq \ell\), then, by Lemma 4.5 and the definition of \( R_\ell \), we may choose a subsequence \( W \) of \( R_\ell \) of length at most \( \ell \) with \((\ell W)([1, u-1], g) = \ell W \) and \( x_\ell + p\|\ell W\|_g \equiv p \pmod{\ell} \). Since \( p\|\ell W\|_g \leq \ell(u-1) \), we have \( x_\ell + p\|\ell W\|_g \in [p - \ell m(S), p] \). Thus, we can construct a subsequence of \((gx)W_1S_1\) of index 1, a contradiction. Therefore,
\[ |(\ell R_\ell)([1, u-1], g)| \leq \ell - 1. \]

If \(|R_\ell| < \ell\) then we are done. Otherwise, by Lemma 4.5 we get a subsequence \( R_0 \) of \( R_\ell \) with \( p\|\ell R_0\|_g \equiv p \pmod{\ell} \) and
\[ |R_0| \geq |R_\ell| - \ell. \]

We assert that
\[ p\|\ell R_0\|_g \leq p - \ell m(S) - \ell. \]
Assume to the contrary that \( p\|\ell R_0\|_g \geq p - \ell m(S) \), and choose \( T \) to be the minimal subsequence of \( R_0 \) such that \( p\|\ell T\|_g \geq p - \ell m(S) \) and \( p\|\ell T\|_g \equiv p \pmod{\ell} \). If \( p\|\ell T\|_g \leq p \), then we can construct a subsequence of \( TS_1 \) with index 1, a contradiction. Now suppose that \( p\|\ell T\|_g > p \). If \( y \in [1, p-1] \) is such that \((yg) | R_\ell \) and \( y_\ell \in [1, p-1] \) such that \( p | (\ell y - y_\ell) \), then \( y_\ell \in [1, m(S)] \) and \( \gcd(y_\ell, \ell) = 1 \). By Lemma 4.5 by dropping at most \( \ell \) terms from \( T \), we get a proper subsequence \( T' \) such that \( p\|\ell T'\|_g \geq p - \ell m(S) \) and \( p\|\ell T'\|_g \equiv p \pmod{\ell} \), a contradiction to the minimality of \( T \). Therefore, (7) holds.

By (5), we have \( p\|\ell R_0\|_g \geq (\ell - 1) + u(|R_0| - \ell + 1) \). This together with (7) gives
\[ |R_0| \leq \frac{p - \ell m(S) - 2\ell + 1}{u} + \ell - 1. \]

Now the lemma follows from (6).

**LEMMA 4.7.** Let \( t \in [2, k] \), and let \( 1 = \alpha_1 < \alpha_2 < \cdots \) denote all positive integers coprime to \( t \). If
\[ m(S) \leq \frac{p - 2t + w\alpha_{u+1} + 2}{t + \sum_{i=2}^{u} \alpha_i} \quad \text{for some } w, u \in \mathbb{N}_0, \]
then
\[ |R_\ell| \leq \frac{p - (t + \sum_{i=2}^{u} \alpha_i)m(S) - 2t + 2}{\alpha_{u+1}} + \delta_u(u-1)m(S) + 2t + w \]
where
\[ \delta_u = \begin{cases} 0 & \text{for } u = 0, \\ 1 & \text{for } u \geq 1. \end{cases} \]
Proof. Assume to the contrary that $|R_t|$ is strictly larger than the above bound. Since

$$m(S) \leq \frac{p - 2t + w\alpha_{u+1} + 2}{t + \sum_{i=2}^{u} \alpha_i},$$

it follows that $|R_t| \geq 2t + 1$. By Lemma 4.5 there exists a nonempty subsequence $R_0$ of $R_t$ with

$$p\|tR_0\|_g \equiv p \pmod{t} \quad \text{and} \quad |R_0| \geq |R_t| - t.$$  \hspace{1cm} (8)

Similarly to Lemma 4.6, we can prove that

$$p\|tR_0\|_g \leq p - tm(S) - t.$$ \hspace{1cm} (9)

Note that $tR_0$ contains $\alpha_1g = g$ at most $t - 2$ times, because otherwise we would get

$$m(S) \geq m_g(tS) \geq tm_g(S) + t - 1 > m_g(S) = m(S),$$

a contradiction. Since $\nu_{\alpha_i} g(S) \leq h(S) \leq m(S)$ for all $i \geq 2$, it follows that

$$p\|tR_0\|_g \geq \alpha_1(t - 2) + \left(\sum_{i=2}^{u} \alpha_i\right)m(S) + \alpha_{u+1}(|R_0| - (u - 1)m(S) - (t - 2)).$$

By (9), we have

$$|R_0| \leq \frac{p - (t + \sum_{i=2}^{u} \alpha_i)m(S) - 2t + 2}{\alpha_{u+1}} + \delta(u - 1)m(S) + t - 2.$$  \hspace{1cm} (10)

By (8), we derive a contradiction. \blacksquare

Proof of Theorem 1.4. We use the notation introduced at the beginning of this section. In particular, we assume to the contrary that there exists a sequence $S \in F(G^*)$ of length $|S| = p$ which has no subsequence with index 1. We have to derive a contradiction.

Clearly, $h(S) \leq m(S) \leq p - 1$. Lemma 4.1 implies that, for every $x \in [1, p - 1]$ with $(xg) \mid ST^{-1}$, we have $m(S) + 2 \leq x \leq p - m(S) - 1$. Thus it follows that

$$\frac{p - 2}{10} \leq h(S) \leq m(S) \leq \frac{p - 3}{2}.$$  \hspace{1cm} (11)

We distinguish several cases.

CASE 1: $(p - 2)/3 \leq m(S) \leq (p - 3)/2$. With $k = 2$ in Lemma 4.4 we have

$$p \leq m(S) + 1 + |R_2|.$$  \hspace{1cm} (12)

Applying Lemma 4.7 with $u = 0$ and $w = 6$, we infer that

$$|R_2| \leq p - 2m(S) + 8.$$  \hspace{1cm} (13)

It follows that $p \leq m(S) + 1 + |R_2| = m(S) + 1 + p - 2m(S) + 8 < p$, a contradiction.
Case 2: \((p + 3)/4 \leq m(S) \leq (p - 4)/3\). With \(k = 3\) in Lemma 4.4 we have

\[ p \leq m(S) + 1 + 2 + 2 + |R_2| + |R_3|. \]

Applying Lemma 4.7 with \(u = 1\) and \(w = 6\), we infer that

\[ |R_2| \leq \frac{p - 2m(S) + 28}{3}, \quad |R_3| \leq \frac{p - 3m(S) + 20}{2}. \]

It follows that

\[ p \leq m(S) + 5 + \sum_{i=2}^{3} |R_i| = m(S) + 5 + \frac{p - 2m(S) + 28}{3} + \frac{p - 3m(S) + 20}{2} < p, \]

a contradiction.

Case 3: \((p - 2)/5 \leq m(S) \leq (p + 1)/4\). With \(k = 4\) in Lemma 4.4 we have

\[ p \leq m(S) + 1 + 2 \cdot 2 + 3 \cdot 2 + |R_2| + |R_3| + |R_4|. \]

Applying Lemma 4.7 with \(u = 1\) and \(w = 6\), we infer that

\[ |R_2| \leq \frac{p - 2m(S) + 28}{3}, \quad |R_3| \leq \frac{p - 3m(S) + 20}{2}, \quad |R_4| \leq \frac{p - 4m(S) + 36}{3}. \]

It follows that

\[ p \leq m(S) + 11 + \frac{p - 2m(S) + 28}{3} + \frac{p - 3m(S) + 20}{2} + \frac{p - 4m(S) + 36}{3} < p, \]

a contradiction.

Case 4: \((p - 1)/6 \leq m(S) \leq (p - 3)/5\). With \(k = 5\) in Lemma 4.4 we have

\[ p \leq m(S) + 27 + \sum_{i=2}^{5} |R_i|. \]

Applying Lemma 4.7 with \(u = 1\) and \(w = 6\), we infer that

\[ |R_2| \leq \frac{p - 2m(S) + 28}{3}, \quad |R_3| \leq \frac{p - 3m(S) + 20}{2}, \quad |R_4| \leq \frac{p - 4m(S) + 36}{3}, \quad |R_5| \leq \frac{p - 5m(S) + 24}{2}. \]

Applying Lemma 4.6 with \(t = 2\), \(\ell = 3\) and \(u = 12\), we obtain that either

\[ |R_2| = 0 \quad \text{or} \quad |R_3| \leq \frac{p - 3m(S) + 55}{12}, \]

and therefore

\[ |R_2| + |R_3| \leq \max \left\{ \frac{p - 2m(S) + 28}{3} + \frac{p - 3m(S) + 55}{12}, \frac{p - 3m(S) + 20}{2} \right\} \]

\[ = \frac{5p - 11m(S) + 167}{12}. \]
Summing up we obtain
\[ p \leq m(S) + 27 + \sum_{i=2}^{5} |R_i| = m(S) + 27 + (|R_2| + |R_3|) + |R_4| + |R_5| \]
\[ \leq \frac{5p - 11m(S) + 167}{12} + \frac{p - 4m(S) + 36}{3} + \frac{p - 5m(S) + 24}{2} + 27 < p, \]
a contradiction.

**Case 5**: \((p - 5)/7 \leq m(S) \leq (p - 5)/6\). With \(k = 6\) in Lemma 4.4 we have
\[ p \leq m(S) + 37 + \sum_{i=2}^{6} |R_i|. \]
Applying Lemma 4.7 with \(u = 2\) and \(w = 0\), we infer that
\[ |R_2| \leq \frac{p + 18}{5}, \quad |R_3| \leq \frac{p - m(S) + 20}{4}. \]
Applying Lemma 4.7 with \(u = 1\) and \(w = 6\), we infer that
\[ |R_4| \leq \frac{p - 4m(S) + 36}{3}, \quad |R_5| \leq \frac{p - 5m(S) + 24}{2}, \quad |R_6| \leq \frac{p - 6m(S) + 80}{5}. \]
Summing up we obtain
\[ p \leq m(S) + 37 + \sum_{i=2}^{6} |R_i| \]
\[ = m(S) + 37 + \frac{p + 18}{5} + \frac{p - m(S) + 20}{4} + \frac{p - 4m(S) + 36}{3} \]
\[ + \frac{p - 5m(S) + 24}{2} + \frac{p - 6m(S) + 80}{5} < p, \]
a contradiction.

**Case 6**: \((p - 2)/8 \leq m(S) \leq (p - 3)/7\). With \(k = 7\) in Lemma 4.4 we have
\[ p \leq m(S) + 73 + \sum_{i=2}^{7} |R_i|. \]
Applying Lemma 4.7 with \(u = 2\) and \(w = 0\), we infer that
\[ |R_2| \leq \frac{p + 18}{5}, \quad |R_3| \leq \frac{p - m(S) + 20}{4}. \]
Applying Lemma 4.7 with \(u = 1\) and \(w = 6\), we infer that
\[ |R_4| \leq \frac{p - 4m(S) + 36}{3}, \quad |R_5| \leq \frac{p - 5m(S) + 24}{2}, \]
\[ |R_6| \leq \frac{p - 6m(S) + 80}{5}, \quad |R_7| \leq \frac{p - 7m(S) + 28}{2}. \]
Applying Lemma 4.6 with $t = 2$, $\ell = 5$ and $u = 10$, we infer that
\[
|R_2| + |R_5| \leq \max \left\{ \frac{p - 5m(S) + 4}{2}, \frac{p + 18}{5}, \frac{p - 5m(S) - 9}{10} + 9 \right\}
\]
\[
= \frac{3p - 5m(S) + 117}{10}.
\]
Summing up we obtain
\[
p \leq m(S) + 73 + \sum_{i=2}^{7} |R_i|
\]
\[
= m(S) + 73 + (|R_2| + |R_5|) + |R_3| + |R_4| + |R_6| + |R_7|
\]
\[
\leq m(S) + 73 + \frac{3p - 5m(S) + 117}{10} + \frac{p - m(S) + 20}{4} + \frac{p - 4m(S) + 36}{3}
\]
\[
+ \frac{p - 6m(S) + 80}{5} + \frac{p - 7m(S) + 28}{2} < p,
\]
a contradiction.

**Case 7:** $(p - 2)/9 \leq m(S) \leq (p - 3)/8$. With $k = 8$ in Lemma 4.4 we have
\[
p \leq m(S) + 111 + \sum_{i=2}^{8} |R_i|.
\]
Applying Lemma 4.7 with $u = 2$ and $w = 0$, we infer that
\[
|R_2| \leq \frac{p + 18}{5}, \quad |R_3| \leq \frac{p - m(S) + 20}{4},
\]
\[
|R_4| \leq \frac{p - 2m(S) + 34}{5}, \quad |R_5| \leq \frac{p - 4m(S) + 22}{3}.
\]
Applying Lemma 4.7 with $u = 1$ and $w = 6$, we infer that
\[
|R_6| \leq \frac{p - 6m(S) + 80}{5}, \quad |R_7| \leq \frac{p - 7m(S) + 28}{2}, \quad |R_8| \leq \frac{p - 8m(S) + 52}{3}.
\]
Applying Lemma 4.6 with $t = 2$, $\ell \in \{5, 7\}$ and $u = 20$, we can prove that either
\[
|R_2| = 0 \quad \text{or} \quad |R_i| \leq \frac{p - im(S) - 2i + 1}{20} + 2i - 1 \quad \text{for} \quad i \in \{5, 7\},
\]
and therefore
\[
|R_2| + |R_5| + |R_7| \leq \max \left\{ \frac{p - 4m(S) + 22}{3} + \frac{p - 7m(S) + 28}{2},
\right.
\]
\[
\left. \frac{p - m(S) + 20}{4} + \frac{p - 5m(S) - 9}{20} + 9 + \frac{p - 7m(S) - 13}{20} + 13 \right\}
\]
\[
= \frac{5p - 29m(S) + 128}{6}.
\]
Applying Lemma 4.6 with \( t = 4, \ell = 6 \) and \( u = 10 \), we find that either
\[
|R_4| = 0 \quad \text{or} \quad |R_6| \leq \frac{p - 6m(S) - 11}{10} + 11,
\]
and therefore
\[
|R_4| + |R_6| \leq \max \left\{ \frac{p - 2m(S) + 34}{5} \right. + \frac{p - 6m(S) - 11}{10} + 11, \frac{p - 6m(S) + 80}{5} \right\}
= \frac{3p - 10m(S) + 167}{10}.
\]
Summing up we obtain
\[
p \leq m(S) + 111 + \sum_{i=2}^{8} |R_i|
= m(S) + 111 + (|R_2| + |R_5| + |R_7|) + (|R_4| + |R_6|) + |R_3| + |R_8|
\leq m(S) + 111 + \frac{5p - 29m(S) + 128}{6} + \frac{3p - 10m(S) + 167}{10}
+ \frac{p - m(S) + 20}{4} + \frac{p - 8m(S) + 52}{3} < p,
\]
a contradiction.

**Case 8:** \((p - 2)/10 \leq m(S) \leq (p - 4)/9\). With \( k = 9 \) in Lemma 4.4 we have
\[
p \leq m(S) + 159 + \sum_{i=2}^{9} |R_i|.
\]
Applying Lemma 4.7 with \( u = 2 \) and \( w = 0 \), we infer that
\[
|R_2| \leq \frac{p + 18}{5}, \quad |R_3| \leq \frac{p - m(S) + 20}{4},
|R_4| \leq \frac{p - 2m(S) + 34}{5}, \quad |R_5| \leq \frac{p - 4m(S) + 22}{3}.
\]
Applying Lemma 4.7 with \( u = 1 \) and \( w = 6 \), we infer that
\[
|R_6| \leq \frac{p - 6m(S) + 80}{5}, \quad |R_7| \leq \frac{p - 7m(S) + 28}{2},
|R_8| \leq \frac{p - 8m(S) + 52}{3}, \quad |R_9| \leq \frac{p - 9m(S) + 32}{2}.
\]
Applying Lemma 4.6 with \( t = 2, \ell \in \{5, 7\} \) and \( u = 10 \), we deduce that either
\[
|R_2| = 0 \quad \text{or} \quad |R_i| \leq \frac{p - im(S) - 2i + 1}{10} + 2i - 1 \quad \text{for} \ i \in \{5, 7\},
\]
and therefore
\[ |R_2| + |R_5| + |R_7| \leq \max \left\{ \frac{p - 4m(S) + 22}{3} + \frac{p - 7m(S) + 28}{2}, \right. \\
\left. \frac{p + 18}{5} + \frac{p - 5m(S) - 9}{10} + 9 + \frac{p - 7m(S) - 13}{10} + 13 \right\} = \frac{5p - 29m(S) + 128}{6}. \]

Applying Lemma 4.6 with \( t = 3, \ell = 8 \) and \( u = 5 \), we deduce that either
\[ |R_3| = 0 \text{ or } |R_8| \leq \frac{p - 8m(S) - 15}{8} + 15, \]
and therefore
\[ |R_3| + |R_8| \leq \max \left\{ \frac{p - m(S) + 20}{4} + \frac{p - 8m(S) - 15}{8} + 15, \frac{p - 8m(S) + 52}{3} \right\} = \frac{3p - 10m(S)}{8} + 20. \]

Summing up we obtain
\[ p \leq m(S) + 159 + \sum_{i=2}^{9} |R_i| \]
\[ = M + 159 + (|R_2| + |R_5| + |R_7|) + (|R_3| + |R_8|) + |R_4| + |R_6| + |R_9| \]
\[ \leq m(S) + 159 + \frac{5p - 29m(S) + 128}{6} + \left( \frac{3p - 10m(S)}{8} + 20 \right) \]
\[ + \frac{p - 2m(S) + 34}{5} + \frac{p - 6m(S) + 80}{5} + \frac{p - 9m(S) + 32}{2} < p, \]
a contradiction. \( \blacksquare \)

5. A conjecture and an open problem. In spite of Theorem 1.2 and in view of Lemma 3.1, we formulate a conjecture which sharpens the original Lemke–Kleitman Conjecture for prime cyclic groups.

Conjecture 5.1. Let \( G \) be a cyclic group of prime order and \( S \) be a sequence over \( G \) of length \( |S| = |G| \). Then \( S \) has a subsequence \( T \) with \( \text{ind}(T) = 1 \) and length \( |T| \in [1, h(S)] \).

Let \( G \) be a cyclic group of order \( n \geq 2 \). We denote by
- \( t(n) \) the smallest integer \( \ell \in \mathbb{N} \) such that every sequence \( S \) over \( G \) of length \( |S| \geq \ell \) has a subsequence \( T \) with \( \text{ind}(T) = 1 \),
- \( T(n) \) the smallest integer \( \ell \in \mathbb{N} \) such that every squarefree sequence \( S \) over \( G \) of length \( |S| \geq \ell \) has a subsequence \( T \) with \( \text{ind}(T) = 1 \).

By Theorem 1.2 it follows that \( t(n) \geq n + \lfloor n/4 \rfloor - 4 \) for \( n = 4k + 2 \geq 22 \).

Problem. Determine \( t(n) \) and \( T(n) \) for all \( n \geq 2 \).
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References


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