Difference sets and polynomials of prime variables

by

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1. Introduction. For a set $A$ of positive integers, define

$$\overline{d}(A) = \limsup_{x \to \infty} \frac{|A \cap [1, x]|}{x}.$$

Furstenberg [9, Theorem 1.2] and Sárközy [21] independently confirmed the following conjecture of Lovász:

**Theorem 1.1.** Suppose that $A$ is a set of positive integers with $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and a positive integer $z$ such that $x - y = z^2$.

In fact, the $z^2$ in Theorem 1.1 can be replaced by an arbitrary integral-valued polynomial $f(z)$ with $f(0) = 0$. On the other hand, Sárközy [22] also solved a problem of Erdős:

**Theorem 1.2.** Suppose that $A$ is a set of positive integers with $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and a prime $p$ such that $x - y = p - 1$.

For the further developments of Theorems 1.1 and 1.2, the readers are referred to [23], [18], [1], [11], [16], [17], [20]. In the present paper, we shall give a common generalization of Theorems 1.1 and 1.2. Define

$$\Lambda_{b,W} = \{x : Wx + b \text{ is prime}\}$$

for $1 \leq b \leq W$ with $(b, W) = 1$.

**Theorem 1.3.** Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and $z \in \Lambda_{1,W}$ such that $x - y = \psi(z)$.

**Corollary 1.1.** Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $A \subseteq \mathbb{Z}^+$ satisfies $\overline{d}(A) > 0$. Then there exist $x, y \in A$ and a prime $p$ such that $x - y = \psi(p - 1)$.

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Proof. Let $W$ be the least common multiple of the denominators of the coefficients of $\psi$. Then the coefficients of $\psi^*(x) = \psi(Wx)$ are all integers. Hence by Theorem 1.3, there exist $x, y \in A$ and $z \in A_{1,W}$ such that

$$x - y = \psi^*(z) = \psi(p - 1)$$

where $p = Wz + 1$. ■

About one month after the first version of this paper was put on the arXiv server, in [3] Bergelson and Lesigne proved that the set

$$\{(\psi_1(p-1), \ldots, \psi_m(p-1)) : p \text{ prime}\}$$

is an enhanced van der Corput set $\mathbb{Z}^m$, where $\psi_1, \ldots, \psi_m$ are polynomials with integral coefficients and zero constant term. Of course, their result can be extended to the set $\{(\psi_1(z), \ldots, \psi_m(z)) : z \in A_{1,W}\}$ without any special difficulty. On the other hand, Kamae and Mendès France [15] proved that any van der Corput set is also a set of 1-recurrence. Hence Bergelson and Lesigne’s result also implies our Theorem 1.3 and Corollary 1.1. In fact, they showed that the set $\{\psi(p-1) : p \text{ prime}\}$ is not only a set of 1-recurrence, but also a set of strong 1-recurrence.

For two sets $A, X$ of positive integers, define

$$d_X(A) = \limsup_{x \to \infty} \frac{|A \cap X \cap [1, x]|}{|X \cap [1, x]|}.$$

Let $P$ denote the set of all primes. In [12], Green established a Roth-type extension of a result of van der Corput [6] on 3-term arithmetic progressions in primes:

**Let $P$ be a set of primes with $d_P(P) > 0$. Then there exists a non-trivial 3-term arithmetic progression contained in $P$.**

The key to Green’s proof is a transference principle, which transfers a subset $P \subseteq P$ to a subset $A \subseteq \mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ with $|A|/N \geq d_P(P)/64$, where $N$ is a large prime. Using Green’s methods, we show:

**Theorem 1.4.** Let $\psi(x)$ be a polynomial with integral coefficients and zero constant term. Suppose that $P \subseteq P$ satisfies $d_P(P) > 0$. Then there exist $x, y \in P$ and $z \in A_{1,W}$ such that $x - y = \psi(z)$.

Similarly, we have

**Corollary 1.2.** Let $\psi(x)$ be a polynomial with rational coefficients and zero constant term. Suppose that $P \subseteq P$ satisfies $d_P(P) > 0$. Then there exist $x, y \in P$ and a prime $p$ such that $x - y = \psi(p - 1)$.

On the other hand, the well-known Szemerédi theorem [24] asserts that for any set $A$ of positive integers with $d(A) > 0$, there exist arbitrarily long arithmetic progressions contained in $A$. In [2], Bergelson and Leibman extended Theorem 1.1 and Szemerédi’s theorem:
Let $\psi_1(x), \ldots, \psi_m(x)$ be arbitrary integral-valued polynomials with $\psi_1(0) = \cdots = \psi_m(0) = 0$. Then for any set $A$ of positive integers with $\bar{d}(A) > 0$, there exist $x \in A$ and an integer $z$ such that $x + \psi_1(z), \ldots, x + \psi_m(z)$ are all contained in $A$.

Recently, Tao and Ziegler [26] proved the following:

Let $\psi_1(x), \ldots, \psi_m(x)$ be arbitrary integral-valued polynomials with $\psi_1(0) = \cdots = \psi_m(0) = 0$. Then for any set $P$ of primes with $\bar{d}_P(P) > 0$, there exist $x \in P$ and an integer $z$ such that $x + \psi_1(z), \ldots, x + \psi_m(z)$ are all contained in $P$.

This is a generalization of Green and Tao’s celebrated result [13] that the primes contain arbitrarily long arithmetic progressions. Furthermore, with the help of a very deep result due to Green and Tao [14] on the Gowers norms [10], Frantzikinakis, Host and Kra [8] proved that if $\bar{d}(A) > 0$ then $A$ contains a 3-term arithmetic progression with difference $p - 1$, where $p$ is a prime. In fact, using the methods of Green and Tao [14], it is not difficult to replace $A$ by $P$ with $\bar{d}_P(P) > 0$ in the result of Frantzikinakis, Host and Kra.

Motivated by the above results, here we propose two conjectures:

**Conjecture 1.1.** Let $\psi_1(x), \ldots, \psi_m(x)$ be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set $A$ of positive integers with $\bar{d}(A) > 0$, there exist $x \in A$ and a prime $p$ such that $x + \psi_1(p-1), \ldots, x + \psi_m(p-1)$ are all contained in $A$.

**Conjecture 1.2.** Let $\psi_1(x), \ldots, \psi_m(x)$ be arbitrary polynomials with rational coefficients and zero constant terms. Then for any set $P$ of primes with $\bar{d}_P(P) > 0$, there exist $x \in P$ and a prime $p$ such that $x + \psi_1(p-1), \ldots, x + \psi_m(p-1)$ are all contained in $P$.

The proofs of Theorems 1.3 and 1.4 will be given in Sections 3 and 4. Throughout this paper, without specific mention, the constants implied by $\ll$, $\gg$ and $O(\cdot)$ will only depend on the degree of $\psi$.

2. Some lemmas on exponential sums. Let $\mathbb{T}$ denote the torus $\mathbb{R}/\mathbb{Z}$. For any function $f$ over $\mathbb{Z}$, define $f^\Delta(x) = f(x+1) - f(x)$. Also, we abbreviate $e^{2\pi \sqrt{-1} x}$ to $e(x)$. Let

$$\psi(x) = a_1 x^k + \cdots + a_k x$$

be a polynomial with integral coefficients. In this section, we always assume that $W, |a_1|, \ldots, |a_k| \leq \log N$.

**Lemma 2.1.** Suppose that $h(x)$ is an arbitrary polynomial and $0 < \nu < 1$. Then for any $\alpha \in \mathbb{T}$,
\[
\sum_{x=1}^{N} h(x)e(\alpha \psi(x)) = \frac{1}{q} \sum_{r=1}^{q} e(\frac{a \psi(r)}{q}) \sum_{x=1}^{N} h(x)e((\alpha - \frac{a}{q}) \psi(x)) \\
+ O_{\deg h}(h(N)N^\nu)
\]

provided that \(|\alpha q - a| \leq N^\nu/\psi(N)\) with \(1 \leq a \leq q \leq N^\nu\).

**Proof.** Let \(\theta = \alpha - a/q\). Then by partial summation, we have

\[
\sum_{x=1}^{N} h(x)e(\frac{a \psi(x)}{q})e(\theta \psi(x)) = h(N)e(\theta \psi(N))F_N(a/q) \\
- \sum_{y=1}^{N-1} \left( h(y+1)e(\theta \psi(y+1)) - h(y)e(\theta \psi(y)) \right) F_y(a/q),
\]

where

\[
F_y(a/q) := \sum_{x=1}^{y} e(\frac{a \psi(x)}{q}) = \frac{y}{q} \sum_{r=1}^{q} e(\frac{a \psi(r)}{q}) + O(q).
\]

Clearly,

\[
\begin{align*}
\quad & h(y+1)e(\theta \psi(y+1)) - h(y)e(\theta \psi(y)) \\
& \quad = (h(y+1) - h(y))e(\theta \psi(y+1)) \\
& \quad \quad + h(y)e(\theta \psi(y))(e(\theta \psi^\Delta(y)) - 1) \\
& \quad = O(h^\Delta(y)) + O(h(y)\theta \psi^\Delta(y)).
\end{align*}
\]

This concludes that

\[
\sum_{x=1}^{N} h(x)e(\frac{a \psi(x)}{q})e(\theta \psi(x)) = \frac{1}{q} \sum_{r=1}^{q} e(\frac{a \psi(r)}{q}) \sum_{x=1}^{N} h(x)e(\theta \psi(x)) \\
+ O(\theta qN\psi^\Delta(N)h(N)) + O(qh^\Delta(N)N). \quad \blacksquare
\]

Define

\[
\lambda_{b,W}(x) = \begin{cases} 
\frac{\phi(W)}{W} \log(Wx + b) & \text{if } Wx + b \text{ is prime,} \\
0 & \text{otherwise,}
\end{cases}
\]

where \(\phi\) is the Euler totient function.
Lemma 2.2. Suppose that \( h(x) \) is an arbitrary polynomial and \( B > 1 \). Then for any \( \alpha \in \mathbb{T} \),

\[
\sum_{x=1}^{N} h(x) \lambda_{b,W}(x) e(\alpha \psi(x)) = \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr+b,q)=1} e(a \psi(r)/q) \sum_{x=1}^{N} h(x) e((\alpha - a/q) \psi(x)) \]

\[
+ O_{\text{deg } h}(h(N)N e^{-c\sqrt{\log N}})
\]

provided that

\[ |\alpha q - a| \leq (\log N)^B / \psi(N) \quad \text{with} \quad 1 \leq a \leq q \leq (\log N)^B, \]

where \( c \) is a positive constant.

Proof. Let

\[
F_y(a/q) = \sum_{x=1}^{y} \lambda_{b,W}(x) e(a \psi(x)/q)
\]

\[
= \sum_{1 \leq r \leq Wq \atop (r,q)=1 \atop r \equiv b \pmod{W}} e(a \psi((r-b)/W)/q) \sum_{x \in A_{r,Wq} \atop Wqx+r \leq Wy+b} \frac{\phi(Wq)}{\phi(W)} \lambda_{r,Wq}(x).
\]

The well-known Siegel–Walfisz theorem (cf. [7]) asserts that

\[
\sum_{p \leq y \text{ is prime} \atop p \equiv b \pmod{q}} \log p = \frac{y}{\phi(q)} + O(\frac{ye^{-c'\sqrt{\log y}}}{2})
\]

provided that \( q \leq (\log y)^{c_1} \), where \( c_1, c' \) are positive constants. Hence

\[
\sum_{x \in A_{r,Wq} \atop Wqx+r \leq Wy+b} \lambda_{r,Wq}(x) = \frac{y}{q} + O(W^2 ye^{-c'\sqrt{\log(Wy)}}).
\]

It follows that

\[
F_y(a/q) = \frac{\phi(W)y}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr+b,q)=1} e(a \psi(r)/q) + O(\frac{ye^{-c'\sqrt{\log y}}}{2}).
\]
Let $\theta = \alpha - a/q$. Then
\[
\sum_{x=1}^{N} h(x) \lambda_{b,W}(x)e(\alpha \psi(x)) \\
= h(N) e(\theta \psi(N)) F_N(a/q) \\
- \sum_{y=1}^{N-1} (h(y + 1)e(\theta \psi(y + 1)) - h(y)e(\theta \psi(y))) F_y(a/q) \\
= \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr + b,q) = 1} e(\frac{a \psi(r)}{q}) \sum_{y=1}^{N} h(y)e(\theta \psi(y)) \\
+ O_{\deg h}(h(N) Ne^{-c' \sqrt{\log N}/3})
\]

by noting that
\[
h(y + 1)e(\theta \psi(y + 1)) - h(y)e(\theta \psi(y)) = O(h(\Delta(y)) + O(h(y) \theta \psi(\Delta(y + 1))). \]

**Lemma 2.3.** For any $\theta \in \mathbb{T}$,
\[
\sum_{x=1}^{N} \psi^\Delta(x-1)e(\theta \psi(x)) = \sum_{x=1}^{\psi(N)} e(\theta x) + O(\theta \psi(N) \psi^\Delta(N)).
\]

**Proof.** Clearly
\[
\sum_{x=1}^{N} \psi^\Delta(x-1)e(\theta \psi(x)) - \sum_{x=1}^{\psi(N)} e(\theta x) = \sum_{x=1}^{N} e(\theta \psi(x)) \sum_{y=0}^{\psi(x-1) - 1} (1 - e(-\theta y)) \\
= O\left(\sum_{x=1}^{N} \sum_{y=0}^{\psi(x-1) - 1} \theta y\right) \\
= O(\theta \psi(N) \psi^\Delta(N)).
\]

**Lemma 2.4.** For any $\varepsilon > 0$,
\[
\sum_{x=1}^{N} e(\alpha \psi(x)) \ll_\varepsilon N^{1+\varepsilon} \left(\frac{a_1}{q} + \frac{a_1}{N} + \frac{q}{N^k}\right)^{2^{1-k}}
\]
provided that $|\alpha - a/q| \leq q^{-2}$.

**Proof.** We leave the proof as an exercise for the readers, since it is just a little modification of the proof of Weyl’s inequality [27, Lemma 2.4].
Lemma 2.5 (Hua). Suppose that \((q, a_1, \ldots, a_k) = 1\). Then
\[
\sum_{r=1}^{q} e(\psi(r)/q) \ll_{\varepsilon} q^{1-1/k+\varepsilon} \quad \text{for any } \varepsilon > 0.
\]

Proof. See [27, Theorem 7.1].

Lemma 2.6. \[
\left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha\psi(x)) \right|^\rho \, d\alpha \ll_{\rho} \gcd(\psi) \psi(N)^{\rho-1} \quad \text{for } \rho \geq k \cdot 2^{k+2},
\]
where \(\gcd(\psi)\) denotes the greatest common divisor of \(a_1, \ldots, a_k\).

Proof. Notice that
\[
\left| \sum_{x=1}^{N} (\alpha\psi)^\Delta(x-1)e(\alpha a\psi(x)) \right|^\rho \, d\alpha = a^{\rho-1} \left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha\psi(x)) \right|^\rho \, d\alpha \Rightarrow a^{\rho} \left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha\psi(x)) \right|^\rho \, d\alpha.
\]

So without loss of generality, we may assume that \(\gcd(\psi) = 1\). Let \(\nu = 1/5\) and \(\varepsilon = 2^{-k\nu} - k/(2\rho)\). Let
\[
\mathcal{M}_{a,q} = \{ \alpha \in \mathbb{T} : |\alpha q - a| \leq N^\nu/\psi(N) \}, \quad \mathcal{M} = \bigcup_{1 \leq a \leq q \leq N^\nu} \mathcal{M}_{a,q}
\]
and \(\mathfrak{m} = \mathbb{T} \setminus \mathcal{M}\). Clearly \(\text{mes}(\mathcal{M}) \leq 2N^{3\nu}/\psi(N)\), where \(\text{mes}\) denotes the Lebesgue measure.

If \(\alpha \in \mathfrak{m}\), then by Lemma 2.4 we have
\[
\sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha\psi(x)) \ll_{\varepsilon} \psi^\Delta(N) N^{1+\varepsilon-2^{1-k}\nu}.
\]

Hence
\[
\left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha\psi(x)) \right|^\rho \, d\alpha \ll_{\varepsilon} \psi(N)^\rho N^{\rho(\varepsilon-2^{1-k}\nu)} = o(\psi(N)^{\rho-1}).
\]
On the other hand, if $\alpha \in \mathfrak{M}$, then by Lemmas 2.1 and 2.3,

$$
\sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha \psi(x)) = \frac{1}{q} \sum_{r=1}^{q} e(\alpha \psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x) + O(\psi^{\Delta}(N)N^\nu).
$$

Let $L = \lfloor \rho/2 \rfloor$. Obviously

$$
\left| \sum_{x=1}^{\mathfrak{M}} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^\rho d\alpha \\
\ll \psi(N)^{\rho-2L} \int \left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^{2L} d\alpha.
$$

So it suffices to show that

$$
\int \left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^{2L} d\alpha \ll L \psi(N)^{2L-1}.
$$

Now

$$
\left| \sum_{x=1}^{N} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^{2L} = \left| \frac{1}{q} \sum_{r=1}^{q} e(\alpha \psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x) \right|^{2L} \\
+ O(\psi(N)^{2L-1}\psi^\Delta(N)N^\nu).
$$

Hence

$$
\int \left| \sum_{x=1}^{\mathfrak{M}} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^{2L} d\alpha \\
= \sum_{1 \leq a \leq q \leq N^\nu, (a,q)=1} \int \left| \frac{1}{q} \sum_{r=1}^{q} e(\alpha \psi(r)/q) \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x) \right|^{2L} d\alpha \\
+ O(\psi(N)^{2L-1}\psi^\Delta(N)N^\nu \text{mes}(\mathfrak{M}))
$$

Clearly

$$
\int \left| \sum_{\mathfrak{M}_{a,q}} e((\alpha - a/q)x) \right|^{2L} d\alpha \leq \int \left| \sum_{x=1}^{\psi(N)} e((\alpha - a/q)x) \right|^{2L} d\alpha \\
= \sum_{1 \leq x_1, \ldots, x_{2L} \leq \psi(N)} 1 \leq \psi(N)^{2L-1}.
$$
And by Lemma 2.5,
\[
\sum_{1 \leq a \leq q \leq N^\nu \atop (a,q) = 1} \left| \frac{1}{q} \sum_{r=1}^q e(a \psi(r)/q) \right|^{2L} \ll \varepsilon \sum_{1 \leq a \leq q \leq N^\nu \atop (a,q) = 1} q^{-2L(1/k - \varepsilon)} \leq \sum_{1 \leq q \leq N^\nu} q^{1-2L(1/k - \varepsilon)} = O_L(1)
\]
since \( L > (1/k - \varepsilon)^{-1} \). We are done. \( \blacksquare \)

**Lemma 2.7.** Supposing that \((a,q) = 1\), we have
\[
\sum_{1 \leq r \leq q \atop (Wr+b,q) = 1} e(a \psi(r)/q) \ll \varepsilon \gcd(\psi)q^{1-1/k(k+1)+\varepsilon}.
\]

**Proof.** Clearly
\[
\sum_{1 \leq r \leq q \atop (Wr+b,q) = 1} e(a \psi(r)/q) = \sum_{r=1}^q e(a \psi(r)/q) \sum_{d \mid \gcd(Wr+b,q)} \mu(d),
\]
where \( \mu \) is the Möbius function. Note that \( d \mid (Wr+b) \Rightarrow (d,W) = 1 \) since \( (W,b) = 1 \). Hence
\[
\sum_{1 \leq r \leq q \atop (Wr+b,q) = 1} e(a \psi(r)/q) = \sum_{d \mid q} \mu(d) \sum_{b \equiv b_d \pmod{d}} \sum_{1 \leq r \leq q} e(a \psi(r)/q),
\]
where \( 1 \leq b_d \leq d \) is the integer such that \( Wb_d + b \equiv 0 \pmod{d} \).

For those \( d \leq q^{1/(k+1)} \) for which \( b_d \) exists, we have
\[
\sum_{1 \leq r \leq q \atop r \equiv b_d \pmod{d}} e(a \psi(r)/q) = \sum_{r=0}^{q/d-1} e(a \psi(dr+b_d)/q).
\]
Write
\[
\psi(dr+b_d) = \sum_{i=1}^{k} a_{k-i+1} \sum_{j=0}^{i} \binom{i}{j} d^j r^j b_d^{i-j} = \sum_{j=0}^{k} d^j r^j \sum_{i=j}^{k} \binom{i}{j} a_{k-i+1} b_d^{i-j} = a_1' r^k + a_2' r^{k-1} + \cdots + a_k' r + a_{k+1}'.
\]
Notice that
\[
(q, a_1', \ldots, a_k') = (q, d^k a_1, a_2', \ldots, a_k') \leq d^k (q, a_1, a_2', \ldots, a_k').
\]
Also
\[
a_2' = d^{k-1}(a_2 + ka_1 b_d).
\]
Therefore
\[
(q, a_1, a_2', \ldots, a_k') = (q, a_1, d^{k-1}a_2, a_3', \ldots, a_k') \leq d^{k-1}(q, a_1, a_2, a_3', \ldots, a_k').
\]
Similarly, we obtain
\[(q, a_1', \ldots, a_k') \leq d^{k(k+1)/2}(q, a_1, \ldots, a_k).\]
Thus by Lemma 2.5,
\[
\sum_{r=0}^{q/d-1} e(a\psi(dr + b_d)/q) \ll_{\epsilon} (q/d, a_1', \ldots, a_k') \left(\frac{q/d}{(q/d, a_1', \ldots, a_k')}\right)^{1-1/k+\epsilon/k}
\leq (q, a_1', \ldots, a_k') \frac{1-\epsilon}{k} d_{1-\epsilon/k}^{1-k} q^{1-1/k}
\leq (a_1, \ldots, a_k) \frac{1-\epsilon}{k} d_{1-\epsilon/k}^{1-k} q^{1-1/k}.
\]
On the other hand, clearly
\[
\left| \sum_{1 \leq r \leq q} e(a\psi(r)/q) \right| \leq \frac{q}{d} < q^{1-1/k(k+1)}
\]
when \(d > q^{1/k(k+1)}\). Thus
\[
\left| \sum_{1 \leq r \leq q} e(a\psi(r)/q) \right| \leq \sum_{d|q, d \leq q^{1/k(k+1)} \text{ and } b_d \text{ exists}} \sum_{1 \leq r \leq q} e(a\psi(r)/q)
\leq \frac{d(q)(\gcd(\psi))^1}{\rho d\alpha} \ll_{\epsilon} \psi(N)(\log N)^{-A}
\ll_{\epsilon} \gcd(\psi) q^{1-1/k(k+1)} + \epsilon,
\]
where \(d(q)\) is the divisor function. □

**Lemma 2.8.** For any \(A > 0\), there is a \(B = B(A, k) > 0\) such that
\[
\sum_{x=1}^{N} \lambda_{b,W}(x)e(\alpha\psi(x)) \ll_{B} N(\log N)^{-A}
\]
provided that \(|\alpha - a/q| \leq q^{-2}\) with \(1 \leq a \leq q\), \((a, q) = 1\) and \((\log N)^B \leq q \leq \psi(N)(\log N)^{-B}\).

**Proof.** Vinogradov dealt with the case \(\psi(x) = x^k\) and \(W = 1\) in [28]. The general proof is standard but long, so we omit it. □

**Lemma 2.9.**
\[
\int_{T} \left| \sum_{x=1}^{N} \psi^\Delta(x-1)\lambda_{b,W}(x)e(\alpha\psi(x)) \right|^\rho \, d\alpha \ll_{T} \gcd(\psi)\psi(N)^{\rho-1}
\]
for \(\rho \geq k2^{k+2} + 1\).
Proof. Without loss of generality, we assume that gcd\((\psi)\) = 1. Let \(B > 2\rho\) be a sufficiently large integer satisfying the requirement of Lemma 2.8 for \(A = 2\rho\). Let

\[
\mathcal{M}_{a,q} = \{ \alpha \in \mathbb{T} : |\alpha q - a| \leq (\log N)^{2B}/\psi(N) \},
\]

\[
\mathcal{M} = \bigcup_{1 \leq a \leq q \leq (\log N)^{2B}, (a,q) = 1} \mathcal{M}_{a,q}
\]

and \(m = \mathbb{T} \setminus \mathcal{M}\).

If \(\alpha \in m\), then there exist \((\log N)^{2B} \leq q \leq \psi(N)(\log N)^{-2B}\) and \(1 \leq a \leq q\) with \((a,q) = 1\) such that \(|\alpha - a/q| \leq q^{-2}\). By Lemma 2.8,

\[
\sum_{x=1}^{y} \lambda_{b,W}(x)e(\alpha\psi(x)) \ll_{B} y(\log y)^{-2\rho}
\]

for \(N(\log N)^{-B/k} \leq y \leq N\). Therefore

\[
\left| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x)) \right|
\]

\[
= \left| \psi^{\Delta}(N-1) \sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x) - \sum_{y=1}^{N-1} (\psi^{\Delta})^{\Delta}(y-1) \sum_{x=1}^{y} e(\alpha\psi(x))\lambda_{b,W}(x) \right|
\]

\[
\leq \psi^{\Delta}(N-1) \left| \sum_{x=1}^{N} e(\alpha\psi(x))\lambda_{b,W}(x) \right| + \sum_{1 \leq y < N(\log N)^{-B/k}} |y(\psi^{\Delta})^{\Delta}(y-1)|
\]

\[
\leq B \psi(N)(\log N)^{-2\rho}.
\]

Let \(L = \lfloor(a - 1)/2 \rfloor\). Then we have

\[
\int_{m} \left| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x)) \right|^{\rho} d\alpha
\]

\[
\ll_{B} (\psi(N)(\log N)^{-2\rho})^{\rho-2L} \int_{m} \left| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x)) \right|^{2L} d\alpha
\]

\[
\ll_{L} \psi(N)^{\rho-2L}(\log N)^{-2\rho} \int_{\mathbb{T}} \left| \sum_{x=1}^{N} \psi^{\Delta}(x-1)\lambda_{b,W}(x)e(\alpha\psi(x)) \right|^{2L} d\alpha.
\]
Noting that
\[
\int \left| \sum_{x=1}^{N} \psi^\Delta(x-1)\lambda_{b,W}(x)e(\alpha \psi(x)) \right|^2 \ d\alpha
\]
\[
= \sum_{1 \leq x_1, \ldots, x_{2L} \leq N} \prod_{j=1}^{2L} \psi^\Delta(x_j-1)\lambda_{b,W}(x_j)
\]
\[
\lesssim_L (\log (WN+b))^{2L} \int \left| \sum_{x \leq N} \psi^\Delta(x-1)e(\alpha \psi(x)) \right|^2 \ d\alpha,
\]
so using Lemma 2.6 we have
\[
\int m^{1 \leq x_1, \ldots, x_{2L} \leq N} \psi^\Delta(x-1)\lambda_{b,W}(x_j)e(\alpha \psi(x_j)) \right|^\rho \ d\alpha \ll_L \psi(N)^{\rho-1}(\log N)^{-\rho}.
\]

If \( \alpha \in \mathcal{M}_{a,q} \), then by Lemma 2.2,
\[
\left| \sum_{x \leq N} \psi^\Delta(x-1)\lambda_{b,W}(x)e(\alpha \psi(x)) \right|^\rho
\]
\[
= \left| \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q, (Wr+b,q)=1} e(\alpha q/(Wr+b,q)) \sum_{x \leq N} \psi^\Delta(x-1)e((\alpha-a/q)\psi(x)) \right|^\rho
\]
\[
+ O(\psi(N)^{\rho} (\log N)^{-7B}).
\]
In view of Lemma 2.7, letting \( \varepsilon = (k+2)^{-4} \), we have
\[
\sum_{1 \leq a \leq q \leq (\log N)^B} \left| \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q, (Wr+b,q)=1} e(\alpha q/(Wr+b,q)) \right|^\rho
\]
\[
\ll \varepsilon \sum_{1 \leq q \leq (\log N)^B} q^{1-\rho(\frac{1}{k(k+1)}-2\varepsilon)} = O_{\rho,\varepsilon}(1).
\]
Applying Lemma 2.6, we conclude that

\[
\int |\sum_{x \leq N} \psi^\Delta(x - 1) \lambda_{b,W}(x) e(\alpha \psi(x))|^\rho \, d\alpha
= \sum_{1 / \phi(Wq) \sum_{1 \leq r \leq q \atop (W, r) = 1} e(a \psi(r)/q) |^\rho \\
\times \int \left| \sum_{x \leq N} \psi^\Delta(x - 1) e((\alpha - a/q) \psi(x)) \right|^\rho \, d\alpha \\
+ O(\text{mes}(\mathfrak{M}) \psi(N)^\rho (\log N)^{-7B})
\leq \left( \sum_{1 \leq a \leq q \leq (\log N)^B \atop (a, q) = 1} \left| \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (W, r) = 1} e(a \psi(r)/q) \right|^\rho \right) \\
\times \int \left| \sum_{x \leq N} \psi^\Delta(x - 1) e(\alpha \psi(x)) \right|^\rho \, d\alpha + O(\psi(N)^{\rho - 1} (\log N)^{-B})
\ll_{\rho, \varepsilon} \psi(N)^{\rho - 1}. \quad \blacksquare
\]

**Lemma 2.10.** Suppose that \( \psi \) is positive and strictly increasing on \([1, N]\). Let \( p \geq \psi(N) \) be a prime. Then

\[
\frac{1}{p} \sum_{r=1}^{p} \left| \sum_{z=1}^{N} \psi^\Delta(z - 1) \lambda_{b,W}(z) e(-r \psi(z)/p) \right|^\rho \ll_{\rho} \gcd(\psi) \psi(N)^{\rho - 1}
\]

for \( \rho \geq k^2 + 2 + 1 \).

**Proof.** We require a well-known result of Marcinkiewicz and Zygmund (cf. [12, Lemma 6.5]):

\[
\sum_{r \in \mathbb{Z}_p} \left| \sum_{x=1}^{p} f(x) e(-xr/p) \right|^\rho \ll_{\rho} p \int_{\mathbb{T}} |\hat{f}(\theta)|^\rho \, d\theta
\]

for any function \( f : \mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z} \rightarrow \mathbb{C} \), where

\[
\hat{f}(\theta) = \sum_{x=1}^{p} f(x) e(-\theta x).
\]

Define

\[
f(x) = \begin{cases} 
\psi^\Delta(z - 1) \lambda_{b,W}(z) & \text{if } x = \psi(z) \text{ where } 1 \leq z \leq N, \\
0 & \text{otherwise}.
\end{cases}
\]
Then
\[
\sum_{r \in \mathbb{Z}_p} \left| \sum_{z=1}^N \psi^\Delta(z - 1) \lambda_{b,W}(z)e(-\psi(z)r/p) \right|^\rho = \sum_{r \in \mathbb{Z}_p} \left| \sum_{x=1}^p f(x)e(-xr/p) \right|^\rho \ll \rho \int_\mathbb{T} \left| f(x)e(-x\theta) \right|^\rho d\theta
\]
\[
= p \int_\mathbb{T} \left| \sum_{z=1}^N \psi^\Delta(z - 1) \lambda_{b,W}(z)e(-\psi(z)\theta) \right|^\rho d\theta \ll \rho \gcd(\psi)p\psi(N)^{\rho-1},
\]
where Lemma 2.9 is applied in the last inequality. □

### 3. Proof of Theorem 1.3

Clearly Theorem 1.3 is a consequence of the following theorem:

**Theorem 3.1.** Suppose that \(k \geq t \geq 1\) are integers, \(a_{k-t+1}\) is a non-zero integer and \(0 < \delta \leq 1\). Let \(\psi(x) = a_1x^k + a_2x^{k-1} + \cdots + a_{k-t+1}x^t\) be an arbitrary polynomial with integral coefficients and positive leading coefficient. Then for any positive integer \(W\), there exist \(N(\delta, W, \psi)\) and \(c(\delta, a_{k-t+1}) > 0\) satisfying
\[
\min_{A \subseteq \{1, \ldots, n\} \atop |A| \geq \delta n} |\{(x, y, z) : x, y \in A, z \in A_{1,W}, x - y = \psi(z)\}| \geq c(\delta, a_{k-t+1}) \frac{WN^{1+1/k}a_1^{-1/k}}{\phi(W) \log n}
\]
if \(n \geq N(\delta, W, \psi)\).

**Remark.** We emphasize that in Theorem 3.1 the constant \(c(\delta, a_{k-t+1})\) only depends on \(k, \delta, a_{k-t+1}\). As we will see later, this fact is important in the proof of Theorem 1.4.

**Proof.** Similarly to Tao’s arguments [25] on Roth’s theorem [19], we apply induction on \(\delta\). Suppose that \(P(\delta)\) is a proposition on \(0 < \delta \leq 1\). Assume that \(P(\delta)\) satisfies the following conditions:

(i) There exists \(0 < \delta_0 < 1\) such that \(P(\delta)\) holds for any \(\delta_0 \leq \delta \leq 1\).

(ii) There exists a continuous function \(\varepsilon(\delta) > 0\) such that \(\delta + \varepsilon(\delta) \leq 1\) for any \(0 < \delta \leq \delta_0\) and \(P(\delta + \varepsilon(\delta)) \Rightarrow P(\delta)\).

(iii) If \(0 < \delta' < \delta \leq 1\), then \(P(\delta') \Rightarrow P(\delta)\).

Then we claim that \(P(\delta)\) holds for any \(0 < \delta \leq 1\). In fact, suppose on the contrary that there exists \(0 < \delta \leq 1\) such that \(P(\delta)\) does not hold. Let
\[
\delta^* = \limsup_{0 < \delta \leq 1} \delta.
\]

Since \(\delta^* \neq 1\), there exists \(0 < \delta^* < 1\) such that \(P(\delta^*)\) does not hold. If \(\delta^* \neq 0\), then \(\delta^* \neq 0\) and \(P(\delta^*) \Rightarrow P(\delta^*)\), which contradicts the choice of \(\delta^*\). Therefore, \(\delta^* = 0\), and the proof is complete.
From (i), we know that $\delta^* \leq \delta_0$. Since $\delta + \varepsilon(\delta)$ is continuous, there exists $0 < \delta_1 < \delta^*$ such that

$$|\delta^* + \varepsilon(\delta^*) - (\delta_1 + \varepsilon(\delta_1))| < \frac{1}{2} \varepsilon(\delta^*),$$

i.e., $0 < \delta_1 < \delta^* < \delta_1 + \varepsilon(\delta_1) \leq 1$. Hence $P(\delta_1 + \varepsilon(\delta_1))$ holds but $P(\delta_1)$ does not by the definition of $\delta^*$. This obviously contradicts (ii) and (iii).

Suppose that $A \subset \{1, \ldots, n\}$ with $|A| \geq \delta n$. Firstly, we shall show that the conclusion of Theorem 3.1 holds for $\delta \geq 3/4$. Define

$$r_{W,\psi}(A) = |\{(x, y, z) : x, y \in A, z \in A_1, W, x - y = \psi(z)\}|.$$

Clearly

$$|\{z \in A_1, W : 1 \leq \psi(z) \leq n/3\}| \geq \frac{1}{4k} \frac{W n^{1/k} a_1^{-1/k}}{\phi(W) \log n},$$

whenever $n$ is sufficiently large (depending on the coefficients of $\psi$). Moreover, for any $1 \leq z \leq n/3$,

$$|\{(x, y) : x, y \in A, x - y = z\}| = |A \cap (z + A)| - |A \cup (z + A)| \geq \frac{2 \cdot 3n}{4} - \frac{4n}{3} = \frac{n}{6}.$$

Hence

$$r_{W,\psi}(A) \geq \frac{1}{24k} \frac{W n^{1+1/k} a_1^{-1/k}}{\phi(W) \log n}.$$

Now we assume that $\delta < 3/4$. Let $\varepsilon = \varepsilon(\delta, a_{k-t+1})$ be a small positive real number and $Q = Q(\delta, a_{k-t+1})$ be a large integer to be chosen later. We shall show that if the assertion of Theorem 3.1 holds for $\delta + \varepsilon$, it also holds for $\delta$. Define

$$\psi_q(x) = \psi(qx)/q^t = a_1 q^{k-t} x^k + \cdots + a_{k-t+1} x^t.$$

By the induction hypothesis on $\delta + \varepsilon$, for any $1 \leq q \leq Q$,

$$\min_{A \subseteq \{1, \ldots, n\} \atop |A| \geq (\delta + \varepsilon)n} r_{W_q,\psi_q}(A) \geq \frac{c(\delta + \varepsilon, a_{k-t+1})}{2} \frac{W_q}{\phi(W_q)} \frac{n^{1+1/k}(a_1 q^{k-t})^{-1/k}}{\log n},$$

provided that

$$n \geq \max_{1 \leq q \leq Q} N(\delta + \varepsilon, W_q, \psi_q).$$

Let $A_m(b, d)$ denote the arithmetic progression $\{b, b+d, \ldots, b+(m-1)d\}$. Suppose that

$$n \geq \max\{e^{k(|a_1| + \cdots + |a_{k-t+1}|)Q^{k-t}}, 10^4 \varepsilon^{-1} Q^t \max_{1 \leq q \leq Q} N(\delta + \varepsilon, W_q, \psi_q)\}$$
and $A \subseteq \{1, \ldots, n\}$ with $|A| = \delta n$. Let $m = \lfloor 10^{-2} \varepsilon Q^{-t} n \rfloor$. Observe that $|\{b : x, y \in \mathbb{A}_m(b, q^t)\}| \leq m$ for every pair $(x, y)$. Let

$$A_{b, q^t} = \{1 + (x - b)/q^t : x \in A \cap \mathbb{A}_m(b, q^t)\} \subseteq \{1, \ldots, m\}.$$  

Clearly if $x', y' \in A_{b, q^t}$ and $z' \in A_{1, W_q}$ satisfy that $x' - y' = \psi_q(z')$, then $x = b + (x' - 1)q^t$, $y = b + (y' - 1)q^t \in A$, $z = z'q \in A_{1, W}$ and $x - y = \psi(z)$. So if there exists $1 \leq q \leq Q$ such that

$$|\{1 \leq b \leq n - mq^t : |A_{b, q^t}| \geq (\delta + \varepsilon)m\}| \geq \varepsilon n,$$

then

$$r_{W, \psi}(A) \geq \frac{1}{m} \sum_{1 \leq b \leq n - mq^t} r_{W_q, \psi_q}(A_{b, q^t}) \geq \varepsilon n \frac{c(\delta + \varepsilon, a_{k-t+1})}{2} \frac{W_q}{\phi(W_q)} \frac{m^{1/k} (a_1 q^{k-t} - 1/k)}{\log m} \geq \varepsilon n \frac{c(\delta + \varepsilon, a_{k-t+1}) \varepsilon^{1+1/k}}{400Q} \frac{W n^{1+1/k} a_1^{-1/k}}{\phi(W) \log n}.$$  

So we may assume that

$$|\{1 \leq b \leq n - mq^t : |A \cap \mathbb{A}_m(b, q^t)| \geq (\delta + \varepsilon)m\}| < \varepsilon n \quad (3.1)$$

for each $1 \leq q \leq Q$. Let

$$M = \max\{x \in \mathbb{Z} : \psi(x) \leq n\}.$$  

Clearly $M = n^{1/k} a_1^{-1/k} (1 + o(1))$. We shall show that

$$\int_{T} \left( \left| \sum_{x \in A \cap [1, n]} e(\alpha x) \right|^2 - \delta^2 \left| \sum_{x \leq n} e(\alpha x) \right|^2 \right) \left( \sum_{z \leq M} \psi(z - 1) \lambda_{1, W}(z) e(\alpha \psi(z)) \right) d\alpha$$

is relatively small.

For $1 \leq q \leq Q$, define

$$\mathcal{M}_{a, q} = \left\{ \alpha : |\alpha - a/q| \leq \frac{1}{2} q^{-t} m^{-1} \right\}.$$  

Let

$$M = \bigcup_{1 \leq a \leq q \leq Q \atop (a, q) = 1} \mathcal{M}_{a, q}, \quad m = T \setminus M.$$  

Let $B$ be a sufficiently large integer. For $1 \leq q \leq (\log M)^B$, define

$$\mathcal{M}_{a, q}^* = \{ \alpha : |\alpha q - a| \leq (\log M)^B / \psi(M) \}.$$  

Let

$$\mathcal{M}^* = \bigcup_{1 \leq a \leq q \leq (\log M)^B \atop (a, q) = 1} \mathcal{M}_{a, q}^*,$$  

$$m^* = T \setminus \mathcal{M}^*.$$
Suppose that $\alpha \in m$. We know

$$|\alpha q - a| \leq (\log M)^B / \psi(M)$$

for some $1 \leq a \leq q < \psi(M)(\log M)^{-B}$ with $(a, q) = 1$. If $\alpha \in m^*$, i.e., $q > (\log M)^B$, then

$$|\alpha - a/q| \leq q^{-2} \quad \text{and} \quad (\log y)^{B/2} \leq \psi(y)(\log y)^{-B/2}$$

for any $M(\log M)^{-B/(2k)} \leq y \leq M$. So applying Lemma 2.8 and partial summation, we have

$$\sum_{z \leq M} \psi^\Delta(z - 1) \lambda_{1,W}(z)e(\alpha\psi(z)) \ll_B \psi(M)(\log M)^{-1} \leq n(\log M)^{-1}$$

whenever $B$ is sufficiently large.

Now suppose that $q < (\log M)^B$, i.e., $\alpha \in \mathfrak{m}^*$. Applying Lemmas 2.2 and 2.3, we have

$$\sum_{z \leq M} \psi^\Delta(z - 1) \lambda_{1,W}(z)e(\alpha\psi(z))$$

$$= \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr+1,q)=1} e(a\psi(r)/q) \sum_{z \leq M} \psi^\Delta(z - 1)e((\alpha - a/q)\psi(z))$$

$$+ O(\psi^\Delta(M)M(\log M)^{-4B})$$

$$= \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr+1,q)=1} e(a\psi(r)/q) \sum_{z \leq n} e((\alpha - a/q)z) + O(\psi^\Delta(M)M(\log M)^{-4B}).$$

Since $\alpha \in m$, either $q > Q$ or $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$. If $q > Q$, then in light of Lemma 2.7,

$$\left| \frac{\phi(W)}{\phi(Wq)} \sum_{1 \leq r \leq q \atop (Wr+1,q)=1} e(a\psi(r)/q) \right| \leq \frac{1}{\phi(q)} \sum_{1 \leq r \leq q \atop (Wr+1,q)=1} e(a\psi(r)/q)$$

$$\leq C_1 |a_k - t+1| q^{-1/k(k+2)}.$$

And if $|\alpha - a/q| > \frac{1}{2}q^{-t}m^{-1}$, then

$$\left| \sum_{z = 1}^n e((\alpha - a/q)z) \right| = \left| \frac{1 - e((\alpha - a/q)n)}{1 - e(\alpha - a/q)} \right| \leq 4\pi q^t m.$$

Hence for $\alpha \in m$,

$$\sum_{z \leq M} \psi^\Delta(z - 1) \lambda_{1,W}(z)e(\alpha\psi(z)) \leq C_1 |a_k - t+1| Q^{-1/k(k+2)}n + 4\pi mQ^t$$

$$+ O(n(\log n)^{-1}).$$
Suppose that $\alpha \in \mathcal{M}$. Let $\tau = 1_A - \delta$ where $1_A(x) = 1$ or 0 according to whether $x \in A$ or not. Let

$$S(\alpha) = \sum_{c=0}^{m-1} e(\alpha c) \quad \text{and} \quad T(\alpha) = \sum_{b=1}^{n} \tau(b)e(\alpha b).$$

Then

$$S(\alpha q^t)T(\alpha) = \sum_{b=1}^{n} \tau(b) \sum_{c=0}^{m-1} e(\alpha(b + cq^t)) = \sum_{b=1}^{n-mq^t} e(\alpha(b + (m-1)q^t)) \sum_{c=0}^{m-1} \tau(b + cq^t) + R(\alpha),$$

where $|R(\alpha)| \leq 2m^2q^t$. When $|\alpha q^t - aq^{t-1}| \leq \frac{1}{2}m^{-1}$,

$$|S(\alpha q^t)| = |S(\alpha q^t - aq^{t-1})| = \left| \frac{1 - e(m(\alpha q^t - aq^{t-1}))}{1 - e(\alpha q^t - aq^{t-1})} \right| \geq \frac{m}{\pi}.$$

Hence for $\alpha \in \mathcal{M}_{a,q}$,

$$m|T(\alpha)| \leq \pi|S(\alpha q^t)T(\alpha)|$$

$$\leq \pi \left| \sum_{b=1}^{n-mq^t} e(\alpha(b + (m-1)q^t)) \sum_{c=0}^{m-1} \tau(b + cq^t) \right| + \pi|R(\alpha)|.$$

Notice that $|\{1 \leq b \leq n - mq^t : x \in \mathbb{A}_m(b, q^t)\}| \leq m$, and the equality holds if $1 + (m-1)q^t \leq x \leq n - mq^t$. It follows that

$$m|A| \geq \sum_{b=1}^{n-mq^t} |A \cap \mathbb{A}_m(b, q^t)| = \sum_{x \in A} \sum_{b=1}^{n-mq^t} 1_{\mathbb{A}_m(b, q^t)}(x)$$

$$\geq m|A| - 2m^2q^t,$$

whence

$$\sum_{1 \leq b \leq n-mq^t \atop |A \cap \mathbb{A}_m(b, q^t)| \geq (\delta + \varepsilon)m} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta + \varepsilon)m) \leq \varepsilon n(1 - \delta)m.$$
It follows that
\[
\sum_{b=1}^{n-mq^t} \left| |A \cap \mathbb{A}_m(b, q^t)| - \delta m \right| 
\leq \sum_{b=1}^{n-mq^t} \left| |A \cap \mathbb{A}_m(b, q^t)| - (\delta + \varepsilon)m \right| + \varepsilon nm
\leq 2 \sum_{1 \leq b \leq n-mq^t} (|A \cap \mathbb{A}_m(b, q^t)| - (\delta + \varepsilon)m) + \varepsilon nm
\leq 4\varepsilon nm + 4m^2q^t.
\]

Thus for any \( \alpha \in \mathfrak{M} \),
\[
|T(\alpha)| \leq \frac{\pi}{m} \left( \sum_{b=1}^{n-mq^t} e(\alpha(b + (m - 1)q^t)) \sum_{c=0}^{m-1} \tau(b + cq^t) \right) + 2m^2q^t
\leq \frac{\pi}{m} \left( \sum_{b=1}^{n-mq^t} |A \cap \mathbb{A}_m(b, q^t)| - \delta m \right) + 2m^2q^t
\leq 4\pi\varepsilon n + 6\pi mQ^t,
\]
i.e.,
\[
\left| \sum_{x=1}^{n} 1_A(x)e(\alpha x) - \delta \sum_{x=1}^{n} e(\alpha x) \right| \leq 16\varepsilon n.
\]

It is easy to see that
\[
||x||^2 - |y|^2 \leq |x - y|^{2/\rho}(|x| + |y|)^{2-2/\rho} + 4|x - y|^{2-2/\rho}(|x|^{2-2/\rho} + |y|^{2-2/\rho})
\]
for any \( \rho \geq 2 \). Let \( \rho = k2^{k+3} \). Then
\[
\left| \int_{\mathfrak{M}} \left( \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right)^2 - \delta^2 \right| \sum_{x=1}^{n} e(\alpha x)^2 \left( \sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right) d\alpha \right|
\leq 4(16\varepsilon n)^{2/\rho} \int_{\mathfrak{M}} \left( \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right)^{2-2/\rho} + \delta^{2-2/\rho} \left| \sum_{x=1}^{n} e(\alpha x) \right|^{2-2/\rho}
\times \left| \sum_{z=1}^{M} \psi^{\Delta}(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right| d\alpha .
\]
By the Hölder inequality,
\[
\left| \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right|^{2-2/\rho} \left| \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right| \, d\alpha \\
\leq \left( \int_{T} \left| \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right|^{2} \, d\alpha \right)^{1-1/\rho} \left( \int_{T} \left| \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right|^\rho \, d\alpha \right)^{1/\rho}.
\]

Lemma 2.9 yields
\[
\int_{T} \left| \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right|^\rho \, d\alpha \leq C_2 |a_{k-t+1}|^{1/\rho} (\delta n)^{1-1/\rho} n^{1-1/\rho}.
\]

Therefore
\[
\int_{T} \left| \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right|^{2-2/\rho} \left| \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right| \, d\alpha \\
\leq C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} (\delta n)^{1-1/\rho} n^{1-2/\rho}.
\]

Similarly,
\[
\left| \int_{\mathbb{R}} \left( \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right)^{2} - \delta^2 \sum_{x=1}^{n} e(\alpha x)^2 \right) \left( \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right) \, d\alpha \\
\leq 8C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} \varepsilon^2/\rho (\delta^{1-1/\rho} + \delta^{2-2/\rho}) n^2.
\]

We conclude that
\[
\left| \int_{\mathbb{R}} \left( \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right)^{2} - \delta^2 \sum_{x=1}^{n} e(\alpha x)^2 \right) \left( \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right) \, d\alpha \\
\times \left| \sum_{x=1}^{n} 1_A(x)e(\alpha x) \right|^{2} \left| \sum_{x=1}^{n} e(\alpha x) \right|^{2} \left( \sum_{z=1}^{M} \psi^\Delta(z-1)\lambda_{1,W}(z)e(\alpha\psi(z)) \right) \, d\alpha \\
+ 8C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} \varepsilon^2/\rho (\delta^{1-1/\rho} + \delta^{2-2/\rho}) n^2 \\
\leq 4C_1 |a_{k-t+1}| Q^{-1/k(k+2)} \delta n^2 + \varepsilon \delta n^2 + 16C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} \varepsilon^2/\rho \delta^{1-1/\rho} n^2.
\]
On the other hand, we have

\[
\int \left| \sum_{x=1}^{n} e(\alpha x) \right|^2 \left( \sum_{z=1}^{M} \psi^\Delta(z-1) \lambda_{1,W}(z) e(\alpha \psi(z)) \right) d\alpha \\
= \sum_{1 \leq x, y \leq n, \frac{1}{M} \leq z \leq M} \psi^\Delta(z-1) \lambda_{1,W}(z) \geq \sum_{1 \leq x, y \leq n, M/4+1 \leq z \leq M/2} \psi^\Delta(z-1) \lambda_{1,W}(z) \\
\geq \frac{M}{8} (n - \psi(M/2)) \psi^\Delta(M/4).
\]

It follows that

\[
\int \left| \sum_{x=1}^{n} 1_A(x) e(\alpha x) \right|^2 \left( \sum_{z=1}^{M} \psi^\Delta(z-1) \lambda_{1,W}(z) e(\alpha \psi(z)) \right) d\alpha \\
\geq \delta^2 \int \left| \sum_{x=1}^{n} e(\alpha x) \right|^2 \left( \sum_{z=1}^{M} \psi^\Delta(z-1) \lambda_{1,W}(z) e(\alpha \psi(z)) \right) d\alpha \\
- 4C_1 |a_{k-t+1}| Q^{-1/k(k+2)} \delta n^2 - \varepsilon \delta n^2 - 16C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} \epsilon^2 / \rho \delta^{1-1/\rho} n^2 \\
\geq \frac{k \delta^2 n^2}{4k+1} - 4C_1 |a_{k-t+1}| Q^{-1/k(k+2)} \delta n^2 - \varepsilon \delta n^2 \\
- 16C_2^{1/\rho} |a_{k-t+1}|^{1/\rho} \epsilon^2 / \rho \delta^{1-1/\rho} n^2.
\]

Let \( \varepsilon = 4^{-(k+2) \rho \delta^{(\rho+1)/2}} C_2^{-1/2} |a_{k-t+1}|^{-1/2} \) and

\[ Q = 4^{(k+1)^4} \delta^{-2(k+2)} C_1^{k(k+2)} |a_{k-t+1}|^{k(k+2)}. \]

Therefore

\[
\{|(x, y, z) : x, y \in A, z \in \Lambda_{1,W}, x - y = \psi(z)\} \\
\geq \frac{W/\phi(W)}{\psi^\Delta(M) \log(WM + 1)} \\
\times \left\| \sum_{x=1}^{n} 1_A(x) e(\alpha x) \right\|^2 \left( \sum_{z=1}^{M} \psi^\Delta(z-1) \lambda_{1,W}(z) e(\alpha \psi(z)) \right) d\alpha \\
\geq \frac{W \delta^2}{4k+2k\phi(W)} \frac{n^{1+1/k} a_{1}^{-1/k}}{\log n}.
\]

This yields the desired result.

Finally, let us briefly discuss the bound in Theorem 1.3. Let \( R_{W,\psi}(\delta) \) be the least integer \( n \) such that for any \( A \subseteq \{1, \ldots, n\} \), there exist \( x,y \in A \) and \( z \in \Lambda_{1,W} \) satisfying \( x - y = \psi(z) \). In our proof, we choose \( \varepsilon = \varepsilon(\delta) = O_{|a_{k-t}|}(\delta O_k^{(1)}) \) and \( Q = Q(\delta) = O_{|a_{k-t}|}(\delta^{-O_k^{(1)}}) \). So the iteration process
\[ \delta \rightarrow \delta + \varepsilon(\delta) \] will end after \( O_{|a_{k-t}|}(\delta^{-O_k(1)}) \) steps. Also, clearly for \( \delta > 3/4 \),
\[ R_{W,\psi}(\delta) \ll (|a_1| + \cdots + |a_{k-t}|)(\min\{p : p \in A_1, W\})^k. \]

Notice that when the iteration process ends, \( W \) becomes \( WQ^O_{|a_{k-t}|}(\delta^{-O_k(1)}) \) and \( a_i \) becomes \( a_i Q^O_{|a_{k-t}|}(\delta^{-O_k(1)}) \). Hence we have
\[ R_{W,\psi}(\delta) \leq \exp(O_{W,a_1,\ldots,a_{k-t}}(\delta^{-O_k(1)})), \]
since \( \min\{p : p \in A_1, W\} \leq e^{O(W)} \). In other words, if a subset \( A \subseteq \{1, \ldots, n\} \) satisfies \( |A| \geq O_{W,a_1,\ldots,a_{k-t}}(n/\log \log \log n) \), then there exist \( x, y \in A \) and \( z \in A_1, W \) such that \( x - y = \psi(z) \). Of course, this bound is very rough. We believe that it could be improved using some more refined estimations (e.g. \[ H. Z. Li and H. Pan \]
\[ 4. \text{Proof of Theorem 1.4.} \] Write \( \psi(x) = a_1x^k + a_2x^{k-1} + \cdots + a_{k-t+1}x^t \) where \( a_{k-t+1} \neq 0 \). Let \( \delta = \bar{d}_P(P) \). Since \( \bar{d}_P(P) > 0 \), there exist infinitely many \( n \) such that
\[ |P \cap [1, n]| \geq \frac{4\delta}{5} \frac{n}{\log n}. \]

Define
\[ w(n) = \max\{w \leq \log \log \log n : n \geq 16W(w)N(\delta, W(w), \psi_{W(w)})\}, \]
where \( N(\delta, W, \psi) \) is as defined in Theorem 3.1 and \( W(w) = \prod_{p \leq \psi, p \text{ prime}} p \). Clearly \( \lim_{n \to \infty} w(n) = \infty \). Let \( w = w(n) \) and \( W = W(w) \). Then
\[ \sum_{\substack{x \in P \cap [1, n] \quad \text{(mod } W^t)} \log x \geq \frac{2\log n}{3} (|P \cap [1, n]| - n^{2/3}) \geq \frac{\delta}{2} n. \]

Hence there exists \( 1 \leq b \leq W^t \) with \( (b, W) = 1 \) such that
\[ \sum_{x \in P \cap [1, n] \quad x \equiv b \text{ (mod } W^t)} \log x \geq \frac{\delta}{2\phi(W^t)} n. \]

Let
\[ A = \{(x - b)/W^t : x \in P \cap [1, n], x \equiv b \text{ (mod } W^t)\}. \]

Let \( N \) be a prime in the interval \((2n/W^t, 4n/W^t)\). Define \( \lambda_{b,W^t,N} = \lambda_{b,W^t}/N \) and \( a = 1_A \lambda_{b,W^t,N} \). Then
\[ \sum_{x} a(x) \geq \frac{\phi(W^t)}{W^t N} \frac{\delta n}{2\phi(W^t)} \geq \frac{\delta}{8}. \]

Let \( \psi_W(x) = \psi(Wx)/W^t = a_1W^{k-t}x^k + \cdots + a_{k-t+1}x^t \).
Clearly $\psi_W(z)$ is positive and strictly increasing for $z \geq 1$, whenever $W$ is sufficiently large.

Below we consider $A$ as a subset of $\mathbb{Z}_N$. Let

$$M = \max\{z \in \mathbb{N} : \psi_W(z) < N/2\}.$$ 

If $x, y \in A$ and $1 \leq z \leq M$ satisfy $x - y = \psi_W(z)$ in $\mathbb{Z}_N$, then we also have $x - y = \psi_W(z)$ in $\mathbb{Z}$. In fact, since $1 \leq x, y < N/2$ and $1 \leq z \leq M$, it is impossible that $x - y = \psi_W(z) - N$ in $\mathbb{Z}$. For a function $f : \mathbb{Z}_N \to \mathbb{C}$, define

$$\hat{f}(r) = \sum_{x \in \mathbb{Z}_N} f(x)e(-xr/N).$$

**Lemma 4.1** (Bourgain [4], [5] and Green [12]). Suppose that $\rho > 2$. Then

$$\sum_r |\hat{a}(r)|^\rho \leq C(\rho),$$

where $C(\rho)$ is a constant only depending on $\rho$.

**Proof.** See [12, Lemma 6.6].

**Lemma 4.2.**

$$\sum_{r \in \mathbb{Z}_N} \left| \sum_{z=1}^M \psi_W^A(z - 1)\lambda_{1,WW}(z)e(-\psi_W(z)r/N) \right|^{\rho} \leq C'(\rho)|a_{k-t+1}|N^\rho$$

provided that $\rho \geq k2^{k+3}$, where $C'(\rho)$ is a constant only depending on $\rho$.

**Proof.** This is an immediate consequence of Lemma 2.10 since $\gcd(\psi_W) \leq |a_{k-t+1}|$.

Let $\eta$ and $\varepsilon$ be two positive real numbers to be chosen later. Let

$$R = \{r \in \mathbb{Z}_N : |\hat{a}(r)| \geq \eta\}, \quad B = \{x \in \mathbb{Z}_N : \|xr/N\| \leq \varepsilon \text{ for all } r \in R\},$$

where $\|x\| = \min\{|x - z| : z \in \mathbb{Z}\}$. Define $\beta = 1_B/|B|$ and $a' = a \ast \beta \ast \beta$, where

$$f \ast g(x) = \sum_{y \in \mathbb{Z}_N} f(y)g(x - y).$$

Let $\varrho = k2^{k+3}$.

**Lemma 4.3.**

$$\sum_{\substack{x, y \in \mathbb{Z}_N \\ 1 \leq z \leq M \\ x - y = \psi_W(z)}} (a'(x)a'(y) - a(x)a(y))\psi_W^A(z - 1)\lambda_{1,WW}(z) \leq C(\varepsilon^2\eta^{-5/2} + \eta^{1/\varrho}).$$
Proof. It is not difficult to check that
\[
\sum_{x,y \in \mathbb{Z}_N, 1 \leq z \leq M} a(x) a(y) \psi_W^\Delta(z-1) \lambda_{1,wW}(z)
\]
\[
= \frac{1}{N} \sum_{r \in \mathbb{Z}_N} \tilde{a}(r) \tilde{a}(-r) \left( \sum_{z=1}^M \psi_W^\Delta(z-1) \lambda_{1,wW}(z) e(-\psi_W(z)r/N) \right).
\]

Also, it is easy to see that \((f * g)^- = \tilde{f} \tilde{g}\). Then
\[
\sum_{x,y \in \mathbb{Z}_N, 1 \leq z \leq M} a'(x) a'(y) \psi_W^\Delta(z-1) \lambda_{1,wW}(z) - \sum_{x,y \in \mathbb{Z}_N, x-y=\psi_W(z)} a(x) a(y) \psi_W^\Delta(z-1) \lambda_{1,wW}(z)
\]
\[
= \frac{1}{N} \sum_{r \in \mathbb{Z}_N} \tilde{a}(r) \tilde{a}(-r) (\tilde{\beta}(r)^2 \tilde{\beta}(-r)^2 - 1)
\]
\[
\times \left( \sum_{z=1}^M \psi_W^\Delta(z-1) \lambda_{1,wW}(z) e(-\psi_W(z)r/N) \right).
\]

If \(r \in R\), then by the proof of Lemma 6.7 of [12], we know that
\[
|\tilde{\beta}(r)^2 \tilde{\beta}(-r)^2 - 1| \leq 2^{16} \varepsilon^2.
\]

And applying Lemma 2.2 with \(\alpha = a = q = 1\),
\[
\sum_{z=1}^M \psi_W^\Delta(z-1) \lambda_{1,wW}(z) = \sum_{z=1}^M \psi_W^\Delta(z-1) + O(\psi_W^\Delta(M) Me^{-c\sqrt{\log M}})
\]
\[
\leq 2\psi_W(M).
\]

Therefore
\[
\left| \sum_{r \in R} \tilde{a}(r) \tilde{a}(-r) (\tilde{\beta}(r)^2 \tilde{\beta}(-r)^2 - 1) \left( \sum_{z=1}^M \psi_W^\Delta(z-1) \lambda_{1,wW}(z) e(-\psi_W(z)r/N) \right) \right|
\]
\[
\leq 2^{16} \varepsilon^2 \sum_{r \in R} |\tilde{a}(r)|^2 \left| \sum_{z=1}^M \psi_W^\Delta(z-1) \lambda_{1,wW}(z) e(-\psi_W(z)r/N) \right|
\]
\[
\leq 2^{17} \varepsilon^2 \psi_W(M) |R|.
\]

In view of Lemma 4.1 with \(\rho = 5/2\), we have \(|R| \leq C'' \eta^{-5/2}\). On the other hand, by the Hölder inequality,
\[ \left| \sum_{r \not\in R} \bar{a}(r) \tilde{a}(-r)(\bar{\beta}(r)^2 - 1) \right| \]
\[ \times \left( \sum_{z=1}^{M} \psi_{WW}(z-1)\lambda_{1,WW}(z)e(-\psi_{WW}(z)r/N) \right) \]
\[ \leq 2 \sup_{r \not\in R} |\bar{a}(r)|^{1/\rho} \left( \sum_{r \not\in R} |\bar{a}(r)|^{2\rho-1} \right)^{\frac{\rho-1}{\rho}} \]
\[ \times \left( \sum_{r \not\in R} \sum_{z=1}^{M} \psi_{WW}(z-1)\lambda_{1,WW}(z)e(-\psi_{WW}(z)r/N) \right)^{1/\rho} \]
\[ \leq 2\eta^{1/\rho} C((2\rho - 1)/(\rho - 1))^{1-1/\rho} (|a_{k-t+1}| C'(\varrho))^{1/\rho} N, \]
where in the last step we apply Lemma 4.1 with \( \rho = (2\varrho - 1)/(\varrho - 1) \) and Lemma 4.2 with \( \rho = \varrho \).

**Lemma 4.4.** If \( \varepsilon |R| \geq 2 \log \log w/w \), then \( |a'(x)| \leq 2/N \) for any \( x \in \mathbb{Z}_N \).

**Proof.** See [12, Lemma 6.3].

Let \( A' = \{ x \in \mathbb{Z}_N : a'(x) \geq \frac{1}{16} \delta N^{-1} \} \). Then
\[ \frac{2}{N} |A'| + \frac{\delta}{16N} (N - |A'|) \geq \sum_{x \in \mathbb{Z}_N} a'(x) = \sum_{x \in \mathbb{Z}_N} a(x) \geq \frac{\delta}{8}, \]
whence \( |A'|/N \geq \delta/32 \). Let \( A'_1 = A' \cap [1, (N - 1)/2] \) and \( A'_2 = \{ x - (N - 1)/2 : x \in A' \cap [(N + 1)/2, N - 1] \} \). Clearly there exists \( i \in \{1, 2\} \) such that \( |A'_i|/N \geq \delta/64 \), say \( |A'_i|/N \geq \delta/64 \).
Applying Theorem 3.1, we know that
\[ |\{(x, y, z) : x, y \in A'_1, z \in A_{1,WW} \cap [1, M], x - y = \psi_{WW}(z)\}| \]
\[ \geq \frac{c(\delta/64, a_{k-t+1})}{8} WW(N/2)^{1+k} (a_1 WW^{k-t})^{-1/k} \phi(WW) \log N. \]

Let \( c' = \frac{1}{10k} c(\delta/64, a_{k-t+1}) \). Clearly
\[ |\{(x, y, z) : x, y \in A'_1, z \in A_{1,WW} \cap [1, c'M], x - y = \psi_{WW}(z)\}| \]
\[ \leq \frac{WW(c'M)}{\phi(WW) \log M} N. \]

Therefore
\[ |\{(x, y, z) : x, y \in A'_1, z \in A_{1,WW} \cap (c'M, M), x - y = \psi_{WW}(z)\}| \]
\[ \geq \frac{c(\delta/64, a_{k-t+1})}{8} WW N^{1+k} (a_1 WW^{k-t})^{-1/k} \phi(WW) \log N. \]
It follows that
\[
\sum_{\substack{x,y\in A_1' \\ 1\leq z\leq M \\ x-y=\psi_W(z) \atop x,y\in Z_N}} \psi_1^\Delta_W(z-1)\lambda_{1,WW}(z)
\geq \frac{c(\delta/64, a_{k-t+1})}{8} \frac{WW.N^{1+1/k}(a_1W^{k-t})^{-1/k}}{\phi(WW)\log N} \frac{\psi_1^\Delta_W(c'M)\phi(WW)\log M}{2WW}
\geq \frac{c(\delta/64, a_{k-t+1})c'^{k-1}}{64} N^2.
\]
So
\[
\sum_{\substack{x,y\in Z_N \\ 1\leq z\leq M \\ x-y=\psi_W(z) \atop x,y\in A_1'}} a(x)a(y)\psi_1^\Delta_W(z-1)\lambda_{1,WW}(z)
\geq \sum_{\substack{x,y\in Z_N \\ 1\leq z\leq M \\ x-y=\psi_W(z) \atop x,y\in A_1'}} a'(x)a'(y)\psi_1^\Delta_W(z-1)\lambda_{1,WW}(z) - C(\varepsilon^2\eta^{-5/2} + \eta^{1/2})
\geq \frac{\delta^2}{28N^2} \sum_{\substack{x,y\in A_1' \\ 1\leq z\leq M \\ x-y=\psi_W(z) \atop x,y\in Z_N}} \psi_1^\Delta_W(z-1)\lambda_{1,WW}(z) - C(\varepsilon^2\eta^{-5/2} + \eta^{1/2})
\geq c''(\delta, a_{k-t+1}) - C(\varepsilon^2\eta^{-5/2} + \eta^{1/2}).
\]
Finally, we may choose \(\eta, \varepsilon > 0\) satisfying \(\varepsilon C''\eta^{-5/2} \geq 2\log\log w/w\) such that
\(C(\varepsilon^2\eta^{-5/2} + \eta^{1/2}) < c''(\delta, a_{k-t+1})/2\)
whenever \(w\) is sufficiently large. Hence
\[
\sum_{\substack{x,y\in Z_N \\ 1\leq z\leq M \\ x-y=\psi_W(z) \atop x,y\in A_1'}} a(x)a(y)\psi_1^\Delta_W(z-1)\lambda_{1,WW}(z) \geq \frac{c''(\delta, a_{k-t+1})}{2} > 0
\]
for sufficiently large \(N\). ■

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