ACTA ARITHMETICA
155.4 (2012)

On some number-theoretic sums introduced by Jacobsthal

by

HELGE TVERBERG (Bergen)

1. Introduction. In 1957 E. Jacobsthal [3] defined and studied two types of number-theoretic functions, defined on $\mathbb{Z}$ by

$$f_{a,b;m}(k) = \lfloor (a+b+k)/m \rfloor - \lfloor (a+k)/m \rfloor - \lfloor (b+k)/m \rfloor + \lfloor k/m \rfloor$$

and on $\mathbb{N} \cup \{0\}$ by

$$F_{a,b;m}(K) = \sum_{k=0}^{K} f_{a,b;m}(k).$$

Here $\{a,b\} \subset \mathbb{Z}$, $m \in \mathbb{N}$, and $\lfloor x \rfloor$ is the largest integer at most equal to x, for any real x. Here a and b can be restricted to $[0, m-1]$ since $f_{a,b+m;m} = f_{a+m,b;m} = f_{a,b;m}$. Also, since $f_{a,b;m} = f_{b,a;m}$, one has at most $\binom{m+1}{2}$ different f's and F's for a given m. From now on we restrict the pair (a,b) as mentioned and we also write just f and F whenever the values of a and b are assumed to be given.

In [3], Jacobsthal proved the inequality

$$F \geq 0. \tag{1.1}$$

Later L. Carlitz [1] and R. C. Grimson [2] studied f and F more closely and also gave alternative proofs of (1.1). In this paper we first give a very simple proof of (1.1), and generalize the f's (and, correspondingly, the F's) to functions defined by $l+1$ parameters $a_1, \ldots, a_l; m$. Then we state three new theorems: Theorem 4.1 gives the analogue of (1.1) for the case $l = 3$, while Theorems 4.2 and 4.3 give upper bounds for the generalization of F in the cases $l = 2, 3$. All three theorems are sharp.

2. Proof of (1.1) in the case $l = 2$. Since $1 - 1 - 1 + 1 = 0$, f is clearly periodic with period m. In order to see that m is also a period for F, we make use of the classical identity $\sum_{k=0}^{m-1} \lfloor (x+k)/m \rfloor = x$, valid for any

---

2010 *Mathematics Subject Classification*: Primary 11A25.
Key words and phrases: integer values, arithmetic sums.

DOI: 10.4064/aa155-4-1

integer x. Applying it to $a+b$, a, b, and 0, we find that $F(m-1)=0$, which again, by the periodicity of f, yields that of F. It therefore suffices to prove (1.1) for $F|[0,m-1]$.

Assume this is false, so that $F(K)<0$ for some $K\in[0,m-1]$. Then $f(k)<0$ for some $k\in[0,K]$, and furthermore, since $F(m-1)=0$, $f(k')>0$ for some k' in $[K+1,m-1]$. But now, since $\lfloor(a+b+k)/m\rfloor-\lfloor(b+k)/m\rfloor\geq 0$ while $f(k)<0$, we have $\lfloor(a+k)/m\rfloor>0$. Similarly, as $\lfloor(a+b+k')/m\rfloor-\lfloor(b+k')/m\rfloor\leq 1$, while $f(k')>0$, we obtain $\lfloor(a+k')/m\rfloor\leq 0$. This gives the contradiction $k<k'$ and $\lfloor(a+k')/m\rfloor<\lfloor(a+k)/m\rfloor$.

3. The new sums. There is a natural way to generalize Jacobsthal's sums: Let m and l be in $\mathbb{N}$. Let furthermore S be a multiset of l integers $a_1,\dots,a_l$ (i.e. for some $i\neq j$, $a_i=a_j$ is allowed). Then one can define $f_{S;m}$ and $F_{S;m}$ on $\{0\}\cup\mathbb{N}$ by

$$f_{S;m}(k)=\sum_{T\subset[1,l]}(-1)^{l-|T|}\Big\lfloor\Big(k+\sum_{i\in T}a_i\Big)/m\Big\rfloor,$$

$$F_{S;m}(K)=\sum_{k=0}^{K}f_{S;m}(k).$$

Bounds on F are interesting only if $l>1$, so we assume from now on that $l>1$. For $l=2$ we have Jacobsthal's sums. For $l>2$, we observe that f and F are still periodic with period m, and that we may assume the a_i's to be in $[0,m-1]$; the arguments from the case $l=2$ are easily modified.

4. Theorems

THEOREM 4.1. *If* $l=3$, *then* $F>-2\lfloor m/2\rfloor$.

THEOREM 4.2. *If* $l=2$, *then* $F\leq\lfloor m/2\rfloor$.

THEOREM 4.3. *If* $l=3$, *then* $F\leq\lfloor m/3\rfloor$.

As the proofs are elementary and relatively simple we omit them. It would be interesting to see corresponding results for higher values of l, and whether the work by Grimson and Carlitz can be generalized to our general sums.

References

[1] L. Carlitz, *An arithmetic sum connected with the greatest integer function*, Norske Vid. Selsk. Forh. Trondheim 32 (1959), 24–30.

[2] R. C. Grimson, *The evaluation of a sum of Jacobsthal*, Norske Vid. Selsk. Skr. (Trondheim) 1974, no. 4, 6 pp.

[3] E. Jacobsthal, *Über eine zahlentheoretische Summe*, Norske Vid. Selsk. Forh. Trondheim 30 (1957), 35–41.

Helge Tverberg
Department of Mathematics
University of Bergen
Johs. Bruunsgt. 12
5008 Bergen, Norway
E-mail: tverberg@math.uib.no

Received on 2.9.2011
and in revised form on 20.4.2012 (6817)