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Geometric properties of the zeta function

by

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Dedicated to Andrzej Schinzel, on the occasion of his 75th birthday

0. Introduction. Our object is to create a coherent record of the state of knowledge concerning curves on which $|\zeta(s)|$ is constant, and also the curves on which $|\zeta(s)|$ has steepest ascent. The value of $|\zeta(s)|$ at critical points is also of interest, as it is at these values that lemniscates of constant value coalesce. The spirit of our inquiry is similar to that of Speiser [5]. Some of our results may already be known, and most of them should have been known for a very long time.

It is well-known that there is exactly one real zero of $\zeta'(s)$ between two consecutive trivial zeros, and that these are the only zeros of $\zeta'(s)$ with real part ≤ 0. Moreover, the Riemann Hypothesis (RH) is equivalent to the assertion that there are no further zeros of $\zeta'(s)$ in the half-plane $\sigma < 1/2$. The zero β_n of $\zeta'(s)$ lying between $-2n-2$ and $-2n$ satisfies

$$\beta_n = -2n - 2 + 1/\log(n/\pi) + O((\log n)^{-3}), \tag{0.1}$$

and the local maximum (on the real axis) of $|\zeta(s)|$ at β_n is

$$|\zeta(\beta_n)| = \frac{(2n+2)!}{e2^{2n+3}\pi^{2n+2}\log n}\left(1 + O\left(\frac{1}{\log n}\right)\right) \tag{0.2}$$

for $n \ge 2$. More explicitly, it is easy to show that if $n \ge 5$, then

$$|\zeta(s)| \le \frac{(2n)!}{2^{2n}\pi^{2n+1}} \tag{0.3}$$

for $-2n \le s \le 0$, and that

$$|\zeta(-2n-1)| > \frac{(2n+1)!}{2^{2n+1}\pi^{2n+2}}. \tag{0.4}$$

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Thus the local maximum of $|\zeta(s)|$ in the interval $(-2n-2, -2n)$ is strictly greater than the corresponding local maximum in $(-2n, -2n+2)$, for $n \geq 5$. By direct calculation (see Table 1), it can be further shown that actually this holds for all $n \geq 3$.

Table 1. The first twelve trivial zeros β_n of $\zeta'(s)$, and $\zeta(\beta_n)$

n	β_n	$\zeta(\beta_n)$	n	β_n	$\zeta(\beta_n)$
1	−2.71726283	0.00915989	7	−15.33872907	0.52058968
2	−4.93676211	−0.00398644	8	−17.37388334	−3.74356682
3	−7.07459714	0.00419400	9	−19.40313326	33.80830360
4	−9.17049316	−0.00785088	10	−21.42790225	−374.41885187
5	−11.24121232	0.02273075	11	−23.44918904	4988.00767609
6	−13.29557457	−0.09371731	12	−25.46771425	−78673.98339103

Of the various connected components on which $\zeta(s)$ is positive real, for each even integer k there is a unique one that we call $\mathcal{R}_k$ that has an asymptote $t = k\pi/(2\log 2)$ as $\sigma \to +\infty$. (Here, and in the following, we write $s = \sigma + it$, so that $\sigma = \Re s$ and $t = \Im s$.) If $4 \mid k$, then this is a curve of steepest ascent in the sense that $|\zeta(s)|$ is increasing as σ decreases from $+\infty$. If $k \equiv 2 \pmod 4$, then it is in the same sense a curve of steepest descent.

Of the various connected components on which $|\zeta(s)| = 1$, for each odd integer k there is a unique one we call $\mathcal{C}_k$ that has the asymptote $t = k\pi/(2\log 2)$ as $\sigma \to +\infty$. It will turn out in due course that the curves $\mathcal{R}_k$ do not intersect with each other, with the single exception that $\mathcal{R}_2$ and $\mathcal{R}_{-2}$ meet at $\beta_1 = -2.717262829$, the first trivial zero of ζ'. It is quite a different story with the $\mathcal{C}_k$, as $\mathcal{C}_1$ and $\mathcal{C}_{-1}$ meet on the real axis at 0.345372657. The inequality $|\zeta(s)| > 1$ holds for all points interior to $\mathcal{C}_{\pm 1}$. The zeta function has no zero nor any critical point inside this curve, and for each $c > 1$ there is a simple closed curve lying inside $\mathcal{C}_{\pm 1}$ on which $|\zeta(s)| = c$. Such curves form a nested family, each one encircling the pole at $s = 1$. Similarly, $\mathcal{C}_3$ and $\mathcal{C}_{-3}$ meet on the real axis at $\sigma_3 = -16.406143017$. The inequality $|\zeta(s)| < 1$ holds for all s between $\mathcal{C}_{\pm 1}$ and $\mathcal{C}_{\pm 3}$. The curve $\mathcal{C}_{\pm 3}$ encloses 8 trivial zeros and one pole of the zeta function, and 7 zeros of $\zeta'(s)$, namely the 7 trivial zeros of ζ' between the 8 trivial zeros of ζ. The curves $\mathcal{C}_{\pm 1}$ and $\mathcal{C}_{\pm 3}$ are depicted in Figure 1.

With the aid of the first part of Lemma 2.1 it is easy to show that $|\zeta(\sigma + 2\pi i/\log 2)| > 1$ for all σ, so the curve $\mathcal{C}_5$ does not cross this line. By direct computation it may be shown that $\mathcal{C}_5$ enters the critical strip, and emerges again as $\mathcal{C}_7$. The inequality $|\zeta(s)| < 1$ holds for all s inside $\mathcal{C}_{6\pm 1}$. The curve $\mathcal{R}_6$ lies in this domain, and thus must terminate at a nontrivial zero of $\zeta(s)$. It will be shown below that for every integer $k \geq 1$, the curve $\mathcal{C}_{4k+1}$

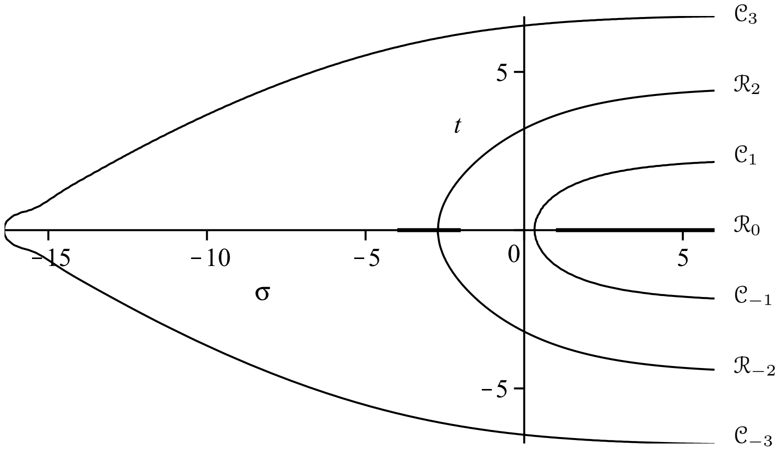

Fig. 1. The curves $\mathcal{C}_{-3}$, $\mathcal{R}_{-2}$, $\mathcal{C}_{-1}$, $\mathcal{R}_0$, $\mathcal{C}_1$, $\mathcal{R}_2$, $\mathcal{C}_3$

meets the curve $\mathcal{C}_{4k+3}$, and that the simple curve $\mathcal{C}_{4k+2\pm1}$ wraps around one or more nontrivial zeros of $\zeta(s)$, including the termination of $\mathcal{R}_{4k+2}$.

1. Notation

Symbol	Meaning
$\mathcal{C}_k$	Defined for odd k; the connected component on which $\lvert\zeta(s)\rvert = 1$ with asymptote $t = k\pi/(2\log 2)$ as $\sigma \to +\infty$.
$D(c)$	Defined for $c > 1$; the noncompact domain on which $\lvert\zeta(s)\rvert < c$.
$\mathcal{N}(c)$	Defined for $c > 1$; the noncompact connected component on which $\lvert\zeta(s)\rvert = c$.
$\mathcal{R}_k$	Defined for even k; the connected component on which $\zeta(s)$ is positive real, with asymptote $t = k\pi/(2\log 2)$ as $\sigma \to +\infty$.
β_n	The real zero of $\zeta'(s)$ lying between $-2n-2$ and $-2n$.
β_n'	The unique number $\sigma \in (\beta_{n+1}, -2n-2)$ for which $\zeta(\beta_n') = -\zeta(\beta_n)$. Defined for $n \ge 2$.
$\sigma_1(c)$	The largest number $\sigma < 0$ at which $\lvert\zeta(\sigma)\rvert = c$. Defined for $c > 1$.
$\sigma_2(c)$	The unique number $\sigma > 1$ at which $\zeta(\sigma) = c$.
σ_3	$= -16.406143017$ is the point at which $\mathcal{C}_{\pm 3}$ intersects the real axis.

2. Basic lemmas. In this section we establish a variety of results, some of which are interesting in their own right.

LEMMA 2.1. *If $\sigma > 1$, then*

$$|\arg \zeta(s)| \le \sum_p \arcsin p^{-\sigma}.$$

Hence in particular,

$$\Re\,\zeta(s) > 0 \qquad (\sigma \ge 1.2) \tag{2.1}$$

and

$$\operatorname{sgn}\arg\zeta(s) = \operatorname{sgn}\Im\,\zeta(s) \qquad (\sigma \ge 1.034). \tag{2.2}$$

Proof of Lemma 2.1. If $0 \le r < 1$, then $|\arg(1+re^{i\phi})| \le \arcsin r$. Hence the stated bound follows from the Euler product for $\zeta(s)$. ∎

LEMMA 2.2. *Let k be an integer. Then*

$$\frac{\partial}{\partial t}\arg\zeta(\sigma+it) < 0, \qquad \frac{\partial}{\partial\sigma}|\zeta(\sigma+it)| < 0, \qquad |\zeta(\sigma+it)| > 1 \tag{2.3}$$

for s in the half-strip $\sigma \ge 4$, $|t - 4k\pi/(2\log 2)| \le \pi/(4\log 2)$;

$$\frac{\partial}{\partial t}|\zeta(\sigma+it)| < 0, \qquad \frac{\partial}{\partial\sigma}\arg\zeta(\sigma+it) > 0, \qquad \Im\,\zeta(\sigma+it) < 0 \tag{2.4}$$

for $\sigma \ge 4$, $|t - (4k+1)\pi/(2\log 2)| \le \pi/(4\log 2)$;

$$\frac{\partial}{\partial t}\arg\zeta(\sigma+it) > 0, \qquad \frac{\partial}{\partial\sigma}|\zeta(\sigma+it)| > 0, \qquad |\zeta(\sigma+it)| < 1 \tag{2.5}$$

for $\sigma \ge 4$, $|t - (4k+2)\pi/(2\log 2)| \le \pi/(4\log 2)$; and

$$\frac{\partial}{\partial t}|\zeta(\sigma+it)| > 0, \qquad \frac{\partial}{\partial\sigma}\arg\zeta(\sigma+it) < 0, \qquad \Im\,\zeta(\sigma+it) > 0 \tag{2.6}$$

for $\sigma \ge 4$, $|t - (4k+3)\pi/(2\log 2)| \le \pi/(4\log 2)$.

Suppose that $\sigma \ge 4$. From (2.3) and (2.5) we know that

$$\left|\zeta\left(\sigma + i\pi\frac{4k+1/2}{2\log 2}\right)\right| > 1, \qquad \left|\zeta\left(\sigma + i\pi\frac{4k+3/2}{2\log 2}\right)\right| < 1.$$

From (2.4) we also know that $|\zeta(\sigma+it)|$ is a decreasing function of t for $(4k+1/2)\pi/(2\log 2) \le t \le (4k+3/2)\pi/(2\log 2)$. Hence in this interval there is a unique $t = t(\sigma)$ such that $|\zeta(\sigma+it(\sigma))| = 1$. This $t(\sigma)$ is a continuous function of σ, and tends to $(4k+1)\pi/(2\log 2)$ as $\sigma \to +\infty$. Indeed, it is easy to see that

$$t(\sigma) = (4k+1)\pi/(2\log 2) + O((2/3)^\sigma).$$

These points $s + it(\sigma)$ form part of the curve we know as $\mathcal{C}_{4k+1}$. Similarly, from (2.4), (2.6), and (2.2) we see that

$$\arg\zeta\left(\sigma + i\pi\frac{4k+3/2}{2\log 2}\right) < 0, \qquad \arg\zeta\left(\sigma + i\pi\frac{4k+5/2}{2\log 2}\right) > 0.$$

From (2.5) we know that $\arg\zeta(\sigma+it)$ is an increasing function of t for $(4k+3/2)\pi/(2\log 2) \le t \le (4k+5/2)\pi/(2\log 2)$, so there is a unique $t = t(\sigma)$ in this interval for which $\zeta(\sigma+it(\sigma))$ is positive real. This is a portion of the curve $\mathcal{R}_{4k+2}$. Continuing in this manner, we see that for $\sigma \ge 4$, the curve $\mathcal{C}_{4k+3}$ is found in the half-strip $(4k+5/2)\pi/(2\log 2)$

$\leq t \leq (4k+7/2)\pi/(2\log 2)$, and that the curve $\mathcal{R}_{4k}$ is in the half-strip $(4k-1/2)\pi/(2\log 2) \leq t \leq (4k+1/2)\pi/(2\log 2)$.

In (2.3)–(2.6), the most favorable case is when t is in the center of the interval, and in this situation we can extend the range of validity somewhat: For $\sigma \geq 2$ we have

$$|\zeta(\sigma + i4k\pi/(2\log 2))| > 1, \tag{2.7}$$

$$\Im\,\zeta(\sigma + i(4k+1)\pi/(2\log 2)) < 0, \tag{2.8}$$

$$|\zeta(\sigma + i(4k+2)\pi/(2\log 2))| < 1, \tag{2.9}$$

$$\Im\,\zeta(\sigma + i(4k+3)\pi/(2\log 2)) > 0. \tag{2.10}$$

Proof of Lemma 2.2. For $\sigma > 1$, let

$$F(\sigma) = \frac{\log 2}{2^{\sigma+1/2}} - \sum_{n>2} \frac{\Lambda(n)}{n^\sigma} = (1+1/\sqrt{2})\frac{\log 2}{2^\sigma} + \frac{\zeta'}{\zeta}(\sigma). \tag{2.11}$$

By the Euler–Maclaurin summation formula we can calculate $\zeta(\sigma)$ and $\zeta'(\sigma)$. Thus we can compute $\frac{\zeta'}{\zeta}(\sigma)$ without needing estimates for the distribution of prime numbers. In this way we find that $F(4) = 0.010285 > 0$. From the first formula for $F(\sigma)$ in (2.11) it is clear that $2^\sigma F(\sigma)$ is an increasing function of σ. Thus

$$F(\sigma) > 0 \qquad (\sigma \geq 4). \tag{2.12}$$

If $(4k-1/2)\pi/(2\log 2) \leq t \leq (4k+1/2)\pi/(2\log 2)$, then $-\pi/4 \leq \arg 2^{-it} \leq \pi/4$, and hence $\Re\, 2^{-it} \geq 1/\sqrt{2}$. Hence if $\sigma \geq 4$, then

$$\Re\,\frac{\zeta'}{\zeta}(s) = -\Re \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \leq -\frac{\log 2}{2^{\sigma+1/2}} + \sum_{n>2} \frac{\Lambda(n)}{n^\sigma} = -F(\sigma) < 0$$

by (2.12). The first two parts of (2.3) follow by the Cauchy–Riemann equations.

If $(4k+1/2)\pi/(2\log 2) \leq t \leq (4k+3/2)\pi/(2\log 2)$, then $-3\pi/4 \leq \arg 2^{-it} \leq -\pi/4$, and so $\Im\, 2^{-it} \leq -1/\sqrt{2}$. Hence if $\sigma \geq 4$, then

$$\Im\,\frac{\zeta'}{\zeta}(s) = -\Im \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^s} \geq \frac{\log 2}{2^{\sigma+1/2}} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^\sigma} = F(\sigma) > 0$$

by (2.12). The first two parts of (2.4) follow by the Cauchy–Riemann equations.

The first two parts of (2.5) and (2.6) are derived similarly. To obtain the third part of (2.3)–(2.6), we observe that

$$\lim_{\sigma\to+\infty} |\zeta(\sigma+it)| = 1, \qquad \lim_{\sigma\to+\infty} \arg\zeta(\sigma+it) = 0,$$

uniformly in t. Thus the third part of (2.3)–(2.6) follows from the corresponding second part. In the case of $\Im\,\zeta(s)$ we are using (2.2). ∎

Lemma 2.3. *For even k let the curves $\mathcal{R}_k$ be defined as in §0. If $j < k$, then $\mathcal{R}_j$ and $\mathcal{R}_k$ intersect only in the case $j = -2$, $k = 2$.*

Proof. If $k \equiv 0 \pmod 4$, then $|\zeta(s)| > 1$ for all $s \in \mathcal{R}_k$, and if $k \equiv 2 \pmod 4$, then $|\zeta(s)| < 1$ for all $s \in \mathcal{R}_k$. Thus it is trivial that $\mathcal{R}_j$ and $\mathcal{R}_k$ do not intersect if $j \not\equiv k \pmod 4$. Suppose that $j \equiv k \equiv 2 \pmod 4$. Then there is an $\ell \equiv 0 \pmod 4$ with $j < \ell < k$, and so $\mathcal{R}_j$ is separated from $\mathcal{R}_k$ by $\mathcal{R}_\ell$. Here it is essential that $\ell \neq 0$, since $\mathcal{R}_0 = [1, +\infty)$ does not provide separation. There will be a nonzero ℓ between j and k unless $j = -2$ and $k = 2$. Now suppose that $j \equiv k \equiv 0 \pmod 4$. If $\mathcal{R}_j$ were to meet $\mathcal{R}_k$, then it would have to be at a critical point, say ρ_1. We may assume that this is the first point at which these two curves have met. Let $\mathcal{R}'_j$ denote that portion of $\mathcal{R}_j$ that connects ρ_1 to $\infty + ij\pi/(2\log 2)$, and let $\mathcal{R}'_k$ denote that portion of $\mathcal{R}_k$ that connects ρ_1 to $\infty + ik\pi/(2\log 2)$. Then $\mathcal{R}'_j \cup \mathcal{R}'_k$ forms the boundary of a domain D. By the maximum modulus principle, $|\zeta(s)| \le |\zeta(\rho_1)|$ for all $s \in D$. From the perspective of the point ρ_1, these curves are curves of steepest descent, and between these curves there must be a curve of steepest ascent. This contradicts the fact that in D, $|\zeta(s)|$ is largest at ρ_1. ∎

Lemma 2.4. *The inequality $\Re \frac{\zeta'}{\zeta}(s) < 0$ holds throughout the quarter-plane $\sigma \le -1$, $t \ge 6$, and*

$$\Re \frac{\zeta'}{\zeta}(s) = -\log|s| + O(1)$$

uniformly in this quarter-plane.

Proof. By taking the logarithmic derivative of the functional equation in the asymmetric form, we find that

$$\frac{\zeta'}{\zeta}(s) = -\frac{\zeta'}{\zeta}(1-s) + \log 2\pi - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\frac{\pi s}{2} \tag{2.13}$$

(cf. (10.27) of Montgomery–Vaughan [4]). For $\sigma \ge 2$,

$$\left|\frac{\zeta'}{\zeta}(s)\right| = \left|\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^s}\right| \le \sum_{n=2}^{\infty}\frac{\Lambda(n)}{n^2} = -\frac{\zeta'}{\zeta}(2) = 0.569960993. \tag{2.14}$$

Since

$$\cot\frac{\pi s}{2} = \frac{ie^{-\pi t} + 2\sin\pi\sigma - ie^{\pi t}}{e^{-\pi t} - 2\cos\pi\sigma + e^{\pi t}}, \tag{2.15}$$

it is easy to see that

$$\left|\Re\cot\frac{\pi s}{2}\right| \le 0.000000014 \tag{2.16}$$

when $t \geq 6$. Since

$$\frac{\Gamma'}{\Gamma}(s) = \log s + O(1/|s|) \tag{2.17}$$

when $\Re s \geq 2$ (cf. (C.17) of Montgomery–Vaughan [4]), we deduce that $\Re \frac{\zeta'}{\zeta}(s) = -\log|s| + O(1)$ in the quarter-plane. Since this is tending to $-\infty$ uniformly as $|s| \to \infty$, the maximum real part must occur on the boundary. By detailed numerical calculation one may show that $\Re \frac{\zeta'}{\zeta}(-1+it) \leq \Re \frac{\zeta'}{\zeta}(-1+6i)$ for $6 \leq t \leq 60$, and that $\Re \frac{\zeta'}{\zeta}(\sigma+6i) \leq \Re \frac{\zeta'}{\zeta}(-1+6i) < 0$ for $-60 \leq \sigma \leq -1$. In order to demonstrate that such a data comprises values sufficiently far for the asymptotics to apply, it is necessary to have (2.17) in a quantitative form, and in this connection we note that

$$\left|\frac{\Gamma'}{\Gamma}(s) - \log s\right| \leq \frac{0.55}{|s|} \tag{2.18}$$

uniformly for $\sigma \geq 2$. From these calculations we conclude that the maximum real part in the quarter-plane occurs at $-1 + 6i$, where the value is -0.008670053. ∎

Lemma 2.5. *The inequality*

$$\Im \frac{\zeta'}{\zeta}(s) < 0$$

holds throughout the half-strip $\sigma \leq -1$, $0 < t \leq 8$.

By the Cauchy–Riemann equations we deduce that if $\sigma \leq -1$ and $0 < t \leq 8$, then

$$\frac{\partial}{\partial t}\log|\zeta(\sigma+it)| = \frac{\partial}{\partial t}\Re \log \zeta(\sigma+it) = -\Im \frac{\zeta'}{\zeta}(s) > 0, \tag{2.19}$$

and thus $|\zeta(\sigma+it)|$ is a strictly increasing function of t for $0 \leq t \leq 8$. With more work, one could show that for each $\sigma \leq -1$ there is a $T(\sigma)$ such that $|\zeta(\sigma+it)|$ is increasing for $0 \leq t \leq T(\sigma)$. The function $T(\sigma)$ grows exponentially as $\sigma \to -\infty$, but is always a finite function of σ. In particular, $T(-1) = 11.441988$.

Proof of Lemma 2.5. The situation is a little delicate, since $\Im \frac{\zeta'}{\zeta}(\sigma) = 0$, and since $\frac{\zeta'}{\zeta}(s)$ has poles at negative even integers. For positive real r and positive integers n, let $D(n,r)$ denote the domain $-2n-1 < \sigma < -1$, $0 < t < 8$, with semidiscs $|s+2k| \leq r$ removed for $k = 1, \ldots, n$. Then $\Im \frac{\zeta'}{\zeta}(s)$ is a nonconstant harmonic function on $D(n,r)$, and the union as $n \to \infty$ and $r \to 0^+$ of these domains is the half-strip $-\infty < \sigma < -1$, $0 < t < 8$. Thus it suffices to show that $\Im \frac{\zeta'}{\zeta}(s) \leq 0$ on the boundary of $D(n,r)$ if r is sufficiently small and n is sufficiently large.

First we treat the interval from $-2n-1$ to $-1n-1+8i$ by means of (2.13). If $\sigma \ge 2$, then

$$\Im \frac{\zeta'}{\zeta}(s) = \int_0^t \frac{\partial}{\partial u} \Im \frac{\zeta'}{\zeta}(\sigma + iu)\, du = \int_0^t \Re \left(\frac{\zeta'}{\zeta}\right)'(\sigma + iu)\, du.$$

But

$$\left|\left(\frac{\zeta'}{\zeta}\right)'(s)\right| = \left|\sum_{n=2}^{\infty} \frac{\Lambda(n)\log n}{n^s}\right| \le 2^{-\sigma} \sum_{n=2}^{\infty} \frac{\Lambda(n)\log n}{(n/2)^\sigma}$$
$$\le 2^{2-\sigma}\left(\frac{\zeta'}{\zeta}\right)'(2) = 2^{2-\sigma}\left(\frac{\zeta''(2)}{\zeta(2)} - \left(\frac{\zeta'}{\zeta}(2)\right)^2\right) \le 2^{2-\sigma}.$$

Hence

$$\left|\Im \frac{\zeta'}{\zeta}(\sigma + it)\right| \le t2^{2-\sigma} \tag{2.20}$$

for $\sigma \ge 2$, $t \ge 0$.

Similarly, if $\sigma \ge 2$, then

$$\Im \frac{\Gamma'}{\Gamma}(s) = \int_0^t \frac{\partial}{\partial u} \Im \frac{\Gamma'}{\Gamma}(\sigma + iu)\, du = \int_0^t \Re \left(\frac{\Gamma'}{\Gamma}\right)'(\sigma + iu)\, du.$$

Now

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = \sum_{n=0}^{\infty} (n+s)^{-2},$$

and by the Euler–Maclaurin summation formula we know that

$$\left(\frac{\Gamma'}{\Gamma}\right)'(s) = s^{-1} + O(|s|^{-2})$$

for $\sigma \ge 2$. More specifically,

$$\left|\left(\frac{\Gamma'}{\Gamma}\right)'(s)\right| \le \frac{1.3}{|s|} \tag{2.21}$$

when $\sigma \ge 2$, and hence

$$\left|\Im \frac{\Gamma'}{\Gamma}(s)\right| \le \frac{1.3t}{\sigma} \tag{2.22}$$

for $\sigma \ge 2$.

From (2.15) we see that

$$\Im \cot \frac{\pi(-2n-1+it)}{2} = -\tanh\frac{\pi t}{2} \le -\frac{t}{9}$$

for $0 \le t \le 8$. On combining our estimates in (2.13), we see that $\Im \frac{\zeta'}{\zeta}(-2n-1+it) < 0$ for $0 < t \le 8$, provided that $n \ge 6$.

Next we consider $s = \sigma + 8i$. From (2.15) we see that

$$-1.000000000025 < \Im \cot \frac{\pi(\sigma + 8i)}{2} < -0.999999999975. \tag{2.23}$$

On combining this with our other estimates in (2.13), we deduce that $\Im \frac{\zeta'}{\zeta}(\sigma + 8i) < 0$ for $\sigma \le -10$. We calculate that $\Im \frac{\zeta'}{\zeta}(\sigma + 8i)$ is increasing for $-10 \le \sigma \le -1$, with the maximum value $\Im \frac{\zeta'}{\zeta}(-1+8i) = -0.257367$. We also find that $\Im \frac{\zeta'}{\zeta}(-1 + it) \le -t/32$ for $0 \le t \le 8$. As to the delicate behavior when t is near 0, we remark that

$$\frac{\partial}{\partial t} \Im \frac{\zeta'}{\zeta}(-1+it)\Big|_{t=0} = \left(\frac{\zeta'}{\zeta}\right)'(-1) = \frac{\zeta''(-1)}{\zeta(-1)} - \left(\frac{\zeta'}{\zeta}(-1)\right)^2 = -0.937985199.$$

It remains to consider $\Im \frac{\zeta'}{\zeta}(s)$ when s is on a small semicircle centered at a negative even integer. We recall that

$$\cot s = \frac{1}{s} + \sum_{n=1}^{\infty} \left(\frac{1}{s - \pi n} + \frac{1}{s + \pi n} \right).$$

Here each term has negative imaginary part when s is in the upper half-plane, and thus

$$\Im \cot re^{i\theta} \le \Im \frac{1}{r} e^{-i\theta} = -\frac{1}{r} \sin\theta = -\frac{t}{r^2}.$$

Hence

$$\Im \frac{\pi}{2} \cot \frac{\pi(-2n + re^{i\theta})}{2} = \Im \frac{\pi}{2} \cot \frac{\pi re^{i\theta}}{2} \le -\frac{2t}{\pi r^2}.$$

Thus in (2.13), the contribution of $\cot \pi s/2$ overwhelms those of the zeta function and gamma function, if r is sufficiently small, say $r \le 1/2$. Since $\Im \frac{\zeta'}{\zeta}(s) \le 0$ on the boundary of $D(n, r)$, it follows that this inequality holds throughout $D(n, r)$, and the proof is complete. ∎

LEMMA 2.6. *A curve of steepest ascent of $|\zeta(s)|$, once it enters the quarter-plane $\sigma \le -1$, $t \ge 6$, will never leave it. On such a curve, the real part will be monotonically decreasing to $-\infty$, and the imaginary part will tend to $+\infty$.*

By direct computation we find that $\mathcal{R}_4$ intersects the line $\sigma = -1$ at $-1 + 10.798685i$.

Proof of Lemma 2.6. If f is analytic at s_0 and $f(s_0) \ne 0$, then the ray $s_0 + re^{i\theta}$ points in the direction of the curve of steepest ascent through s_0 if $\theta = -\arg \frac{f'}{f}(s_0)$. For the zeta function we deduce from Lemma 2.4 that at $-1 + it$ with $t \ge 6$ we have $\theta \in (\pi/2, 3\pi/2)$, which is to say the curve is heading into the quarter-plane. Similarly, from Lemma 2.5 we see that at $\sigma + 6i$ with $\sigma \le -1$ we have $\theta \in (0, \pi)$, which again is pointing into the quarter-plane. In this quarter-plane we have $\Re\frac{\zeta'}{\zeta}(s) = -\log|s| + O(1)$ and $\Im\frac{\zeta'}{\zeta}(s) \ll 1$. Hence $\theta = \pi + O(1/\log|s|)$. Once this curve passes the ray $t = -\sigma$, we have $\Im\frac{\Gamma'}{\Gamma}(1-s) \le \pi/4 + o(1)$, while $\Im\cot\pi s/2$ is near -1, and so by (2.13) we see that there are positive constants $c_1 < c_2$ such that $-c_2 \le \Im\frac{\zeta'}{\zeta}(s) \le -c_1$. Thus θ will be less than π by an amount comparable to $1/\log|s|$, so t tends to infinity, with $t \asymp -\sigma/\log(-\sigma)$. Here the constants of proportionality depend on the particular curve being considered. ∎

The functional equation of the zeta function, in its asymmetric form, asserts that

$$\zeta(s) = \zeta(1-s)\Delta(s) \tag{2.24}$$

where

$$\Delta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2} = \frac{\Gamma((1-s)/2)}{\Gamma(s/2)}\pi^{s-1/2} \tag{2.25}$$

(cf. (10.5) and (10.9) of Montgomery–Vaughan [4]). Thus $\Delta(s)$ is a meromorphic function with simple zeros at the nonpositive even integers and simple poles at the positive odd integers. From the second formula for $\Delta(s)$ in (2.25) it is clear that

$$\Delta(s)\Delta(1-s) = 1. \tag{2.26}$$

Since $\Delta(1/2 - it) = \overline{\Delta(1/2 + it)}$, it follows that

$$|\Delta(1/2 + it)| = 1 \tag{2.27}$$

for all t. By taking logarithmic derivatives in (2.26), we deduce that

$$\frac{\Delta'}{\Delta}(s) = \frac{\Delta'}{\Delta}(1-s) \tag{2.28}$$

for all s. The sign of $\Im\frac{\Delta'}{\Delta}(s)$ is easily described, as follows.

Lemma 2.7. *Let $\Delta(s)$ be defined as in* (2.25). *Then*

$$\operatorname{sgn}\Im\frac{\Delta'}{\Delta}(s) = (\operatorname{sgn}(\sigma - 1/2))(\operatorname{sgn} t). \tag{2.29}$$

By the Cauchy–Riemann equations it follows that

$$\operatorname{sgn}\frac{\partial}{\partial t}|\Delta(s)| = -(\operatorname{sgn}(\sigma - 1/2))(\operatorname{sgn} t). \tag{2.30}$$

Proof of Lemma 2.7. By taking logarithmic derivatives in (2.25) we see that

$$\frac{\Delta'}{\Delta}(s) = -\frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{1-s}{2}\right) - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{s}{2}\right) + \log\pi.$$

By the partial fraction formula for $\frac{\Gamma'}{\Gamma}(s)$ (cf. (C.10) of Montgomery–Vaughan [4]) we deduce that

$$\frac{\Delta'}{\Delta}(s) = \frac{1}{s} - \frac{1}{s-1} + C_0 + \log\pi + \sum_{n=1}^{\infty}\left(\frac{1}{s+2n} - \frac{1}{s-1-2n} - \frac{1}{n}\right).$$

Hence

$$\Im\frac{\Delta'}{\Delta}(s) = t\sum_{n=0}^{\infty}\left(\frac{1}{(\sigma-1-2n)^2+t^2} - \frac{1}{(\sigma+2n)^2+t^2}\right).$$

If $\sigma > 1/2$, then $(\sigma-1-2n)^2 < (\sigma+2n)^2$ for all n, so all terms in the sum are positive. If $\sigma < 1/2$, then the signs are reversed. Hence the result. ∎

In view of our remarks following Lemma 2.5, the next lemma plays a useful role.

LEMMA 2.8. *Let $\sigma \le -1$ be fixed. Then $|\zeta(\sigma+it)| \ge |\zeta(\sigma+6i)|$ for $t \ge 6$.*

Proof. Suppose that $\sigma \le -1$. By (2.19) we know that $|\zeta(\sigma+it)| > |\zeta(\sigma+6i)|$ for $6 \le t \le 8$. Suppose that $t \ge 8$. By the functional equation in the asymmetric form (2.24) we see that

$$\frac{|\zeta(\sigma+it)|}{|\zeta(\sigma+6i)|} = \frac{|\zeta(1-\sigma+it)|}{|\zeta(1-\sigma+6i)|}\cdot\frac{|\Delta(\sigma+it)|}{|\Delta(\sigma+6i)|}. \tag{2.31}$$

Our first step is to show that

$$\frac{|\zeta(1-\sigma+it)|}{|\zeta(1-\sigma+6i)|} \ge 0.65797 \tag{2.32}$$

for $\sigma \le -1$, which is to say that

$$\frac{|\zeta(\sigma+it)|}{|\zeta(\sigma+6i)|} \ge 0.65797 \tag{2.33}$$

for $\sigma \ge 2$. By direct calculation we find that $|\zeta(\sigma+6i)| < 1$ for $2 \le \sigma \le 3$. By the familiar inequalities

$$\frac{\zeta(2\sigma)}{\zeta(\sigma)} \le |\zeta(\sigma+it)| \le \zeta(\sigma) \tag{2.34}$$

we deduce that

$$\frac{|\zeta(\sigma+it)|}{|\zeta(\sigma+6i)|} > |\zeta(\sigma+it)| \ge \frac{\zeta(2\sigma)}{\zeta(\sigma)} \ge \frac{\zeta(4)}{\zeta(2)} > 0.65797$$

for $2 \le \sigma \le 3$. This gives (2.33) in this range. If $\sigma \ge 3$, then by the inequalities (2.34) we find that

$$\frac{|\zeta(\sigma+it)|}{|\zeta(\sigma+6i)|} \ge \frac{\zeta(2\sigma)}{\zeta(\sigma)^2} \ge \frac{\zeta(6)}{\zeta(3)^2} > 0.7.$$

Thus we have (2.33) for all $\sigma \ge 2$, and hence (2.32) for $\sigma \le -1$.

Next we show that

$$\frac{|\Delta(\sigma+it)|}{|\Delta(\sigma+6i)|} \ge 1.527. \tag{2.35}$$

for $\sigma \le -1$ and $t \ge 8$. By Lemma 2.7 and the remarks following we know that $|\Delta(\sigma+it)|$ is an increasing function of t for $t > 0$, $\sigma \le -1$. Thus it suffices to show that

$$\frac{|\Delta(\sigma+8i)|}{|\Delta(\sigma+6i)|} \ge 1.527. \tag{2.36}$$

From the first formula for $\Delta(s)$ in (2.25) we see that

$$\frac{|\Delta(\sigma+8i)|}{|\Delta(\sigma+6i)|} = \frac{|\Gamma(1-\sigma+8i)|}{|\Gamma(1-\sigma+6i)|} \cdot \frac{|\sin\frac{\pi}{2}(\sigma+8i)|}{|\sin\frac{\pi}{2}(\sigma+6i)|}. \tag{2.37}$$

By the product formula for the gamma function (cf. (C.1) of Montgomery–Vaughan [4]) we see that

$$\frac{|\Gamma(\sigma+8i)|}{|\Gamma(\sigma+6i)|} < 1$$

for $\sigma \ge 2$, and that this quantity is monotonically increasing to 1 as $\sigma \to +\infty$. Thus

$$\frac{|\Gamma(1-\sigma+8i)|}{|\Gamma(1-\sigma+6i)|} \ge \frac{|\Gamma(2+8i)|}{|\Gamma(2+6i)|} > 0.066 \tag{2.38}$$

for $\sigma \le -1$. On the other hand,

$$\frac{|\sin\frac{\pi}{2}(\sigma+8i)|}{|\sin\frac{\pi}{2}(\sigma+6i)|} = e^{\pi}\left|\frac{1-e^{-8\pi}e^{i\pi\sigma}}{1-e^{-6\pi}e^{i\pi\sigma}}\right| \ge e^{\pi}\frac{1+e^{-8\pi}}{1+e^{-6\pi}} > 23.14.$$

On combining this and (2.38) in (2.37) we obtain (2.36) and hence (2.35). We insert (2.32) and (2.35) into (2.31) to see that $|\zeta(\sigma+it)| \ge 1.004|\zeta(\sigma+6i)|$ for $\sigma \le -1$, so the proof is complete. ■

In Lemma 2.7 we discussed $\Im\frac{\Delta'}{\Delta}(s)$ where $\Delta(s)$ is defined in (2.25). We now turn to $\Re\frac{\Delta'}{\Delta}(s)$.

Lemma 2.9. *Suppose that $t \ge 6.3$. Then $\Re\frac{\Delta'}{\Delta}(\sigma+it) < 0$ for all σ.*

From the above we see that $|\Delta(\sigma+it)|$ is a decreasing function of σ if $t \ge 6.3$. Thus from (2.27) it follows that $|\Delta(\sigma+it)| < 1$ if $\sigma > 1/2$ and $t \ge 6.3$. Hence

$$|\zeta(\sigma+it)| < |\zeta(1-\sigma+it)| \tag{2.39}$$

if $\sigma > 1/2$, $t \ge 6.3$, and $\zeta(\sigma+it) \ne 0$.

Proof of Lemma 2.9. In view of (2.28), we may restrict our attention to $\sigma \le 1/2$. By taking logarithmic derivatives in (2.25) we find that

$$\frac{\Delta'}{\Delta}(s) = \log 2\pi - \frac{\Gamma'}{\Gamma}(1-s) + \frac{\pi}{2}\cot\frac{\pi s}{2}. \tag{2.40}$$

We calculate that $\Re \frac{\Delta'}{\Delta}(\sigma + 6.3i)$ is increasing for $-2 \le \sigma \le 1/2$ with the maximum value $\Re \frac{\Delta'}{\Delta}(1/2 + 6.3i) = -0.001618$. From (2.15) we deduce that

$$\left| \frac{\pi}{2} \Re \cot \frac{\pi}{2}(\sigma + it) \right| < 10^{-8}$$

for $t \ge 6.3$. Hence by (2.18) we know that

$$\Re \frac{\Delta'}{\Delta}(\sigma + 6.3i) \le 1.8378771 - \log|1 - \sigma + 6.3i| + \frac{0.55}{|1 - \sigma + 6.3i|}$$

for $\sigma \le -2$. This upper bound is an increasing function of σ in this interval, and has the value -0.0260386 at $\sigma = -2$, and so $\Re \frac{\Delta'}{\Delta}(\sigma + 6.3i) < 0$ for all real σ. From (2.17) and (2.28) we know that $\Re \frac{\Delta'}{\Delta}(s)$ is large and negative on a semicircle $|s| = R$, if R is large. Since the largest value of this harmonic function must occur on the boundary, we conclude that

$$\Re \frac{\Delta'}{\Delta}(\sigma + it) \le \Re \frac{\Delta'}{\Delta}(1/2 + 6.3i) = -0.001618$$

if $t \ge 6.3$. ∎

Lemma 2.10. *If* $\zeta(1/2 + it) \ne 0$, *then*

$$\frac{d}{dt} \arg \zeta(1/2 + it) = \Re \frac{\zeta'}{\zeta}(1/2 + it) = -\frac{1}{2} \log t + O(1) \tag{2.41}$$

for $t \ge 1$, *and*

$$\Re \frac{\zeta'}{\zeta}(1/2 + it) < 0 \tag{2.42}$$

for $t \ge 6.3$.

A notable consequence of (2.42) is that a curve of steepest ascent of $|\zeta(s)|$ crosses the critical line only from right to left for $|t| \ge 6.3$.

Proof of Lemma 2.10. The first identity is a Cauchy–Riemann equation, so we turn our attention to the estimate and inequality. By taking logarithmic derivatives in (2.24) we find that

$$\frac{\zeta'}{\zeta}(s) + \frac{\zeta'}{\zeta}(1 - s) = \frac{\Delta'}{\Delta}(s).$$

On taking $s = 1/2 + it$ we obtain

$$2\Re \frac{\zeta'}{\zeta}(1/2 + it) = \frac{\Delta'}{\Delta}(1/2 + it) = \Re \frac{\Delta'}{\Delta}(1/2 + it).$$

By Lemma 2.9 we know that this last quantity is negative for $t \ge 6.3$. The stated estimate follows by combining (2.16) and (2.17) in (2.40). ∎

Lemma 2.11. *Let* $A(\sigma, T, \theta)$ *denote the number of* t, $0 \le t \le T$, *such that*

$$\arg \zeta(\sigma + it) \equiv \theta \pmod{2\pi}.$$

Then

$$\int_0^{2\pi} A(\sigma,T,\theta)\,d\theta = \int_0^T \left|\Re\frac{\zeta'}{\zeta}(\sigma+it)\right| dt. \tag{2.43}$$

Proof. The identity clearly holds when $T=0$. Thus it suffices to show that the two sides have the same derivative. When we pass from T to $T+\delta$, $\arg\zeta(\sigma+it)$ changes by approximately $\delta\Re\frac{\zeta'}{\zeta}(\sigma+iT)$. Thus $A(\sigma,T+\delta,\theta)=A(\sigma,T,\theta)$ for most θ, but for θ lying in an interval of length approximately $\delta\left|\Re\frac{\zeta'}{\zeta}(\sigma+iT)\right|$ we have $A(\sigma,T+\delta,\theta)=A(\sigma,T,\theta)+1$. Thus

$$\int_0^{2\pi} A(\sigma,T+\delta,\theta)\,d\theta - \int_0^{2\pi} A(\sigma,T,\theta)\,d\theta \sim \delta\left|\Re\frac{\zeta'}{\zeta}(\sigma+it)\right|. \blacksquare$$

In the special case $\sigma=1$, we know that $\Re\frac{\zeta'}{\zeta}(1+it)$ has a limiting distribution with an absolutely continuous everywhere positive density that tends rapidly to 0 at infinity, and thus there is a constant $c>0$ such that

$$\int_0^T \left|\Re\frac{\zeta'}{\zeta}(1+it)\right| dt \sim cT \tag{2.44}$$

as $T\to\infty$. With less sophistication now we show that the above integral is $\gg T$, and indeed that a proportionate lower bound holds even for intervals of bounded length.

LEMMA 2.12. *There is an A_0 such that*

$$\int_{T-A}^{T+A} \left|\Re\frac{\zeta'}{\zeta}(1+it)\right| dt \gg A$$

uniformly for $A\ge A_0$.

Proof. We note first that we may assume that $T\ge A$. Let $K(u)=\max(0,1-|u|/A)$. For $\sigma>1$,

$$\begin{aligned} &-\int_{T-A}^{T+A} \Re\frac{\zeta'}{\zeta}(\sigma+it)(1+\cos(t\log 2))K(t-T)\,dt \\ &\quad= \sum_n \frac{\Lambda(n)}{n^\sigma}\int_{-A}^{A}(1+\cos((T+u)\log 2))(\cos((T+u)\log n))K(u)\,du. \end{aligned} \tag{2.45}$$

For $f\in L^1(\mathbb{R})$ we define the Fourier transform of f to be

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x)e(-tx)\,dx$$

where $e(\theta) = e^{2\pi i\theta}$ is Vinogradov's notation for the complex exponential with period 1. In this notation the integral on the right in (2.45) is

$$\frac{1}{2}n^{iT}\widehat{K}\Big(\frac{-\log n}{2\pi}\Big)+\frac{1}{2}n^{-iT}\widehat{K}\Big(\frac{\log n}{2\pi}\Big)+\frac{1}{4}(2n)^{iT}\widehat{K}\Big(\frac{-\log 2n}{2\pi}\Big)$$
$$+\frac{1}{4}(2n)^{-iT}\widehat{K}\Big(\frac{\log 2n}{2\pi}\Big)+\frac{1}{4}(n/2)^{iT}\widehat{K}\Big(\frac{-\log n/2}{2\pi}\Big)+\frac{1}{4}(n/2)^{-iT}\widehat{K}\Big(\frac{\log n/2}{2\pi}\Big).$$

Now $\widehat{K}(0) = A$, and

$$\widehat{K}(t) = \frac{1}{A}\Big(\frac{\sin A\pi t}{\pi t}\Big)^2$$

for $t \neq 0$, so $\widehat{K}(t) \ll \min(A, 1/(At^2))$. Hence the sum of the six terms displayed above is

$$\begin{cases} \dfrac{A}{2} + O\Big(\dfrac{1}{A}\Big) & (n = 2), \\ O\Big(\dfrac{1}{A(\log n)^2}\Big) & (n > 2). \end{cases}$$

Since

$$\sum_{n=2}^{\infty}\frac{\Lambda(n)}{n(\log n)^2} < \infty,$$

the expression (2.45) is

$$\frac{A\log 2}{2^{\sigma+1}} + O\Big(\frac{1}{A}\Big)$$

uniformly for $\sigma > 1$. We let $\sigma \to 1^+$ and note that

$$\Big|\int_{T-A}^{T+A}\Re\frac{\zeta'}{\zeta}(1+it)(1+\cos(t\log 2))K(t-T)\,dt\Big| \le 2\int_{T-A}^{T+A}\Big|\Re\frac{\zeta'}{\zeta}(1+it)\Big|\,dt$$

to complete the proof. ■

3. Main results

THEOREM 3.1. *There exist positive constants k_0 and C such that if $k > k_0$, $k \equiv 0 \pmod 4$, and $s = \sigma + it \in \mathcal{R}_k$ with $\sigma \ge -1$, then*

$$|t - \pi k/(2\log 2)| < C\log k.$$

This has many consequences, not the least of which is that $\mathcal{R}_k$ must intersect the interval from $-1 + i(k\pi/(2\log 2) - C\log k)$ to $-1 + i(k\pi/(2\log 2) + C\log k)$ when $k \equiv 0 \pmod 4$.

Proof of Theorem 3.1. Suppose that $j \ge 2$, and put $t_0 = (4j-1)\pi/(2\log 2)$. By a method due to Backlund [1], we show that the number of $\sigma \in [-1, 4]$

such that $\zeta(\sigma + it_0)) \in \mathbb{R}$ is $O(\log j)$. To this end, let

$$f(s) = \zeta(s + it_0) - \zeta(s - it_0).$$

By the reflection principle, $f(\sigma) = 2i\Im\zeta(\sigma + it_0)$. Thus $(f(\sigma) = 0$ whenever $\zeta(\sigma + it_0))$ is real. Our first task is to show that

$$|f(4)| \geq 0.09. \tag{3.1}$$

Since $2^{-it_0} = i$ we see that

$$\begin{aligned}\Im \log \zeta(4 + it_0) &= \frac{1}{16} + \Im \sum_{n>2} \frac{\Lambda(n)}{n^{4+it_0}\log n} \geq \frac{1}{16} - \sum_{n>2} \frac{\Lambda(n)}{n^4 \log n} \\ &= \frac{1}{8} - \sum_{n=2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} = \frac{1}{8} - \log \zeta(4) > 0.04589.\end{aligned}$$

Similarly,

$$\begin{aligned}\Re \log \zeta(4 + it_0) &= \Re \sum_{n>2} \frac{\Lambda(n)}{n^{4+it_0}\log n} \geq -\sum_{n>2}^{\infty} \frac{\Lambda(n)}{n^4 \log n} \\ &= \frac{1}{16} - \log \zeta(4) > -0.01661.\end{aligned}$$

Thus if we write $\zeta(4 + it_0) = re^{i\phi}$, then $\Im\,\zeta(4 + it_0) = r \sin\phi > 0.045$. This gives (3.1).

We apply Jensen's inequality to $f(s)$ in the disc $|s-4| \leq 6$. Since $f(s) \ll j^{5/2}$ in this disc, the number of zeros of f in the smaller disc $|s - 4| \leq 5$ is $O(\log j)$. In particular, the number of $\sigma \in [-1, 4]$ for which $\zeta(\sigma + t_0)$ is real is $O(\log j)$.

Now let m be an integer, $m \geq 3$, and put $t_1 = (k - 4m - 1)\pi/(2\log 2)$, $t_2 = (k+4m-1)\pi/(2\log 2)$. Suppose that $\mathcal{R}_k$ intersects the interval $[-1+it_1, 4 + it_1]$. Since $\mathcal{R}_k$ and $\mathcal{R}_{k-4}$ do not intersect, it follows that $\mathcal{R}_{k-4}$ must also intersect the interval $[-1 + it_1, 4 + it_1]$. This same reasoning applies to $\mathcal{R}k - 4j$ for $1 \leq j \leq m$. Thus there are at least m points on the interval $[-1 + it_1, 4 + it_1]$ for which $\zeta(s)$ is real, so $m = O(\log k)$. Similarly, suppose that $\mathcal{R}_k$ intersects the interval $[-1 + it_2, 4 + it_2]$. Then $\mathcal{R}_{k+4j}$ must also intersect this interval for $1 \leq j < m$. It again follows that $m = O(\log k)$, so the proof is complete. ∎

Corollary 3.2. *Any curve of steepest ascent of* $|\zeta(s)|$, *other than those tending toward* $s = 1$, *and the curves* $\mathcal{R}_k$ *with* $k \equiv 2 \pmod 4$, *must have real part tending to minus infinity.*

Proof. Let $s_0 = \sigma_0 + it_0$ be given with $\zeta(s_0) \neq 0$, and consider the curve of steepest ascent ascending from s_0. For s_0 inside $\mathcal{R}_{\pm 2}$, curves of steepest ascent spring from the trivial zero at -2 and tend to the pole at $s = 1$.

Now suppose that s_0 is not enclosed by $\mathcal{R}_{\pm 2}$. On the curve through s_0 of steepest ascent, the zeta function has constant argument, say $\theta \in (-\pi, \pi]$. Suppose first that $\theta \neq 0$, and choose ε so that $\varepsilon < |\theta|$. Now $\arg \zeta(s) \ll 2^{-\sigma}$ uniformly as $\sigma \to +\infty$. Thus there is a σ_1 such that $|\arg \zeta(s)| < \varepsilon$ uniformly for $\sigma \geq \sigma_1$. Choose a $k_+ \equiv 0 \pmod 4$ sufficiently large so that $\mathcal{R}_{k_+}$ lies above s_0 for $-1 \leq \sigma < +\infty$. Similarly, choose a $k_- \equiv 0 \pmod 4$ so that s_0 lies above $\mathcal{R}_{k_-}$ for $-1 \leq \sigma < +\infty$. We can form a path from $\sigma = -1$ to $\sigma = \sigma_1$ along $\mathcal{R}_{k_+}$, followed by a path on $\sigma = \sigma_1$ to $\mathcal{R}_{k_-}$, and then along $\mathcal{R}_{k_-}$ to $\sigma = -1$. The zeta function is nonzero on this path, and has argument different from θ, so the curve of steepest ascent from s_0 does not intersect this path. Therefore it must cross the abscissa $\sigma = -1$. If $0 < k_- < k_+$, then we are done, since $\mathcal{R}_2$ reaches the abscissa -1 at $t = 10.079868532$, and so the curve in question is forced to enter the quarter-plane discussed in Lemma 2.6. If $k_- < 0 < k_+$, then as we come up from $\mathcal{R}_{k_-}$ on the abscissa σ_1 we should take $\mathcal{R}_{-1}$ to $\beta_1 = -2.717262829$, then $\mathcal{R}_1$ back to the abscissa σ_1, and then continue up this abscissa. Thus the curve is again forced to reach the line $\sigma = -1$. If it does so at height $t \geq 6$ or ≤ -6, then we are done. In case it reaches the abscissa -1 at $-1 + it$ with $|t| < 6$, we observe that the zeta function is positive real on a curve starting at $-8.700603531 - 6i$, running through $\beta_3 = -7.074597145$, and then ending at $-8.700603531 + 6i$. Thus the curve in question is still forced to enter the quarter-plane of Lemma 2.6.

Finally, we consider the case in which s_0 lies outside $\mathcal{R}_{\pm 2}$, but $\theta = 0$, so the zeta function is positive real on our curve. If the real part σ becomes large on the curve, then the curve is one of the $\mathcal{R}_k$ with $k \equiv 2 \pmod 4$. Otherwise, the curve is still blocked by the same barriers as before, but now we have to consider the possibility that it meets the barrier at a critical point. Suppose the curve meets $\mathcal{R}_{k_+}$ at a critical point ρ' of the zeta function. The two curves arrive from opposite directions, and so we turn by $-\pi/2$ and continue. The curve cannot contact $\mathcal{R}_{k_+}$ between ρ' and $+\infty + ik\pi/\log 2$, because the value on the curve is now larger than on this part of $\mathcal{R}_{k_+}$. If it were to contact $\mathcal{R}_{k_+}$ again farther to the left, then we would again turn by $-\pi/2$ and then we would be inside a simple closed curve on which the zeta function is taking real values smaller than those on the curve, which is something that the harmonic function $\Re\zeta(s)$ cannot do. If after making a first contact with $\mathcal{R}_{k_+}$, the curve were then to contact $\mathcal{R}_{k_-}$, then we would turn by $\pi/2$, and we would have the same contradiction as before. Of course we can argue similarly in the case that the zero of ζ' that is encountered is more than a simple zero. ∎

Corollary 3.3. *For each positive integer k, the curves $\mathcal{C}_{4k+2\pm 1}$ are in fact opposite ends of the same simple curve. The curve $\mathcal{R}_{4k+2}$ lies entirely within this curve, and terminates at a nontrivial zero enclosed by $\mathcal{C}_{4k+2\pm 1}$.*

Proof. The region between $\mathcal{C}_{4k+1}$ and $\mathcal{C}_{4k+3}$ is one in which $|\zeta(s)| < 1$. If these curves do not meet to bound the region, then it must continue off to infinity in some direction. However, $|\zeta(s)| > 1$ on $\mathcal{R}_{4k}$ and on $\mathcal{R}_{4k+4}$, and these curves enter the quarter-plane of Lemma 2.6, and will each reach the abscissa $\sigma = -4$, where they can be linked, since $|\zeta(-4+it)| > 1$ for $t \ge 6$. Thus the region in which $|\zeta(s)| < 1$ is contained, so the curves $\mathcal{C}_{4k+1}$ and $\mathcal{C}_{4k+3}$ must meet. ∎

COROLLARY 3.4. *Let k be a positive integer, and let r denote the number of zeros of $\zeta(s)$ inside $\mathcal{C}_{4k+2\pm1}$. Then $r > 0$, and the number of zeros of ζ' inside $\mathcal{C}_{4k+2\pm1}$ is exactly $r-1$.*

That there must be at least one zero inside $\mathcal{C}_{4k+2\pm1}$ can be seen in a variety of ways. For example, $|\zeta(s)| < 1$ inside the curve, and the minimum modulus cannot be positive. Alternatively, we may observe that $|\zeta(s)|$ is decreasing as one comes in from $\sigma = +\infty$ on $\mathcal{R}_{4k+2}$. This curve of steepest descent must terminate at a zero, which must be inside $\mathcal{C}_{4k+2\pm1}$. Suppose that c is a very small positive number. Then the connected components of the level set $|\zeta(s)| = c$ inside $\mathcal{C}_{4k+2\pm1}$ consist of r small ovals, one around each of the zeros. As c increases, these ovals enlarge, and occasionally two of them coalesce. The point at which they touch is a zero of $\zeta'(s)$. When c is only slightly less than 1, the level set is a simple closed curve only slightly inside $\mathcal{C}_{4k+2\pm1}$ until σ is large. Since we started with r components and end with 1, there have been $r-1$ mergers, and hence $r-1$ zeros of $\zeta'(s)$. While this is an instructive way to view things, we supplement these comments with a rigorous proof utilizing ideas found in §3.55 of Titchmarsh's *Theory of Functions* [6].

Proof of Corollary 3.4. Let s_1 be a point on $\mathcal{C}_{4k+1}$ with σ_1 very large, and let s_2 be a point on $\mathcal{C}_{4k+3}$ with $\sigma_2 = \sigma_1$. For $s \in \mathcal{C}_{4k+2\pm1}$, the outward unit normal at s points in the direction of steepest ascent. At s_1 the argument of this outward normal is approximately $\pi/2$. As s moves along the curve from s_1 to s_0, the argument varies, and has increased from $\pi/2$ to approximately $3\pi/2$ by the time s reaches s_0. But the direction of steepest ascent is the negative of the argument of $\frac{\zeta'}{\zeta}(s)$. Thus the argument of the logarithmic derivative has decreased by π. On the abscissa σ_1, the logarithmic derivative is very close to $c2^{-s}$ where $c = -\log 2$. Since t_1 is near $(4k+1)\pi/(2\log 2)$ and t_2 is near $(4k+3)\pi/(2\log 2)$, as s passes from s_1 to s_2 along the abscissa σ_1, the argument of $\frac{\zeta'}{\zeta}(s)$ decreases by approximately π. Since the total change of argument around this simple closed curve must be an integral multiple of 2π, we deduce that it is -2π. But this is the difference between the change of argument of $\zeta'(s)$ and the change of argument of $\zeta(s)$. Thus $\zeta'(s)$ has one fewer zero inside $\mathcal{C}_{4k+2\pm1}$ than does $\zeta(s)$. Since the number of zeros of $\zeta'(s)$ is at least 0, it follows that the number of zeros of $\zeta(s)$ is at least 1. ∎

COROLLARY 3.5. *Suppose that $0 < c < 1$. Then every connected component of the level curve $|\zeta(s)| = c$ is compact.*

Proof. We know that

$$\inf_t |\zeta(\sigma + it)| = \frac{\zeta(2\sigma)}{\zeta(\sigma)}$$

when $\sigma > 1$. This tends to 1 as $\sigma \to +\infty$, so $\zeta(2\sigma_0)/\zeta(\sigma_0) > c$ if σ_0 is sufficiently large. Thus we may ignore the half-plane $\sigma \geq \sigma_0$. We can also ignore the half-plane $\sigma \leq -17$, since $|\zeta(s)| > 1$ in the quarter-planes $\sigma \leq -17$, $|t| \geq 6$ and also on the rectangular paths with vertices $-2n + 1 + 6i$, $-2n - 1 + 6i$, $-2n - 1 - 6i$, $-2n + 1 - 6i$. Since $|\zeta(s)| > 1$ for $s \in \mathcal{R}_{4k}$, any connected component in the strip $-17 \leq \sigma \leq \sigma_0$ will lie between $\mathcal{R}_{4k}$ and $\mathcal{R}_{4k+4}$ for some k, and must therefore be compact. ■

COROLLARY 3.6. *All connected components on which $|\zeta(s)| = 1$ are compact, except for the curves $\mathcal{C}_{\pm 1}$, $\mathcal{C}_{\pm 3}$, and $\mathcal{C}_{4k+2\pm 1}$ with $k = 1, 2, 3, \ldots$ or $k = -2, -3, -4, \ldots$.*

Proof. Let s_0 be a point for which $|\zeta(s_0)| = 1$. Suppose first that s_0 lies between $\mathcal{R}_{-4}$ and $\mathcal{R}_4$. Choose an odd integer $k \leq -17$ such that $k < \sigma_0$. Let t_1 be chosen so that $4 + it_1 \in \mathcal{R}_4$, put $s_1 = 4 + it_1$, let t_2 be chosen so that $k + it_2 \in \mathcal{R}_4$, and put $s_2 = k + it_2$. Let $\mathcal{C}$ be the simple closed curve that runs along $\mathcal{R}_4$ from s_1 to s_2, along the line joining s_2 to $\overline{s_2}$, along $\mathcal{R}_{-2}$ from $\overline{s_2}$ to $\overline{s_1}$, and then along the straight line connecting $\overline{s_1}$ to s_1. Then s_0 lies inside this curve. However, $|\zeta(s)| > 1$ along the first three legs of $\mathcal{C}$, and by the remarks made after Lemma 2.2 we know that the only curves crossing the final segment are $\mathcal{C}_{\pm 1}$ and $\mathcal{C}_{\pm 3}$. Thus either s_0 is on one of these two noncompact curves, or else s_0 lies on a compact component. The first eight trivial zeros of the zeta function lie between $\mathcal{C}_{\pm 1}$ and $\mathcal{C}_{\pm 3}$, a region in which $|\zeta(s)| < 1$. However, around each of $-18, -20, -22, \ldots$ there is a simple closed curve on which $|\zeta(s)| = 1$. Each such curve contains exactly one trivial zero $-2k$, since $|\zeta(s)| > 1$ on the rectangular path from $-2k+1+6i$ to $-2k-1+6i$ to $-2k-2-6i$ to $-2k+1-6i$ to $-2k+1+6i$.

Suppose now that s_0 lies above $\mathcal{R}_4$. Let n be an even integer chosen to be so large that all points of $\mathcal{R}_n$ have imaginary part $> t_0$. Choose t_1 so that $4 + it_1 \in \mathcal{R}_n$, t_2 so that $17 + it_2 \in \mathcal{R}_n$, t_3 so that $-17 + it_3 \in \mathcal{R}_2$, and t_4 so that $4 + it_4 \in \mathcal{R}_2$. Consider the simple closed curve $\mathcal{C}$ that starts at $4 + it_1$, runs along $\mathcal{R}_k$ to $-17 + it_2$, and along the line to $-17 + it_3$, then along $\mathcal{R}_2$ to $4 + it_4$, and then along the abscissa $\sigma = 4$ to $4 + it_1$. Then s_0 lies in the interior of this curve, and we have $|\zeta(s)| > 1$ on the first three portions of $\mathcal{C}$. The only points on $\sigma = 4$ for which $|\zeta(s)|$ are points on $\mathcal{C}_k$ for some k. Thus the connected component containing s_0 is compact, or else is one of the $\mathcal{C}_k$.

If s_0 lies below $\mathcal{R}_{-2}$, then we can reflect about the real axis and appeal to the case just considered. Thus the proof is complete. ■

Corollary 3.7. *For each $c > 1$, the level curve on which $|\zeta(s)| = c$ has exactly one noncompact component, say $\mathcal{N}(c)$. Let $\sigma_1(c)$ denote the largest real number < 0 for which $|\zeta(\sigma)| = c$. Then $\sigma_1(c) \in \mathcal{N}(c)$, and this is the only intersection of $\mathcal{N}(c)$ with the real axis, unless $c = |\zeta(\beta_n)|$ for some trivial zero β_n of ζ', in which case there is exactly one more intersection β_n' with $\beta_{n+1} < \beta_n' < -2n - 2$.*

The number $\sigma_1(c)$ described above must in fact be < -16, since it must lie outside the curve $\mathcal{C}_{\pm 3}$ on which $|\zeta(s)| = 1$.

Proof of Corollary 3.7. First we demonstrate the existence of such a curve. Let $\sigma_2(c)$ be the unique real number $\sigma > 1$ for which $\zeta(\sigma) = c$. Then $|\zeta(s)| < c$ when $\sigma > \sigma_2(c)$. Let $D(c)$ denote the largest pathwise connected domain containing the half-plane $\sigma > \sigma_2(c)$ in which $|\zeta(s)| < c$. Inside $\mathcal{C}_{\pm 1}$ there is a compact oval on which $|\zeta(s)| = c$; thus $D(c)$ has one hole. It does not have any further hole, since any such hole must contain a pole, and the zeta function has only the one pole. Consider a path that follows $\mathcal{C}_3$ to the negative real axis at $\sigma_0 = -16.406143017$, and then continues on the negative real axis to $\sigma_1(c)$. This path lies in $D(c)$ up to $\sigma_1(c)$, so $\sigma_1(c)$ is on the boundary of $D(c)$.

Let $\mathcal{N}(c)$ denote the connected component of the level set $|\zeta(s)| = c$ that passes through $\sigma_1(c)$. This curve is symmetric with respect to the real axis, and could be compact only by returning to the real axis. We show that this does not happen, with the exception already noted. Certainly $\mathcal{N}(c)$ does not intersect the segment from $\sigma_1(c)$ to σ_0, nor does it cross the curve $\mathcal{C}_3$. Hence any further possible intersection with the real axis must occur at a point $\sigma < \sigma_1(c)$. Let n be determined so that $|\zeta(\beta_{n-1})| < c \le |\zeta(\beta_n)|$. Suppose first that $c < |\zeta(\beta_n)|$. Then by Lemma 2.5 we see that $|\zeta(s)| > c$ on the rectangular paths with vertices $\beta_k + 6i$, $\beta_{k+1} + 6i$, $\beta_{k+1} - 6i$, $\beta_k - 6i$ for all $k \ge n$. Each of these rectangles contains a simple closed curve on which $|\zeta(s)| = c$, surrounding the trivial zero at $-2k - 2$. Thus the curve $\mathcal{N}(c)$ cannot enter the half-strip $\sigma \le \sigma_1(c)$, $|t| \le 6$. Now suppose that $c = |\zeta(\beta_n)|$. By arguing as above we deduce that $N(c)$ loops around the trivial zero $-2n - 2$, and leaves the nth rectangle where it entered. See Figure 2 for an example of this. The curve $\mathcal{N}(c)$ clearly does not enter the half-strip $\sigma \le \beta_{n+1}$, $|t| \le 6$. Thus $\mathcal{N}(c)$ is noncompact in this case also.

Now we establish the uniqueness of $\mathcal{N}(c)$. Any connected noncompact component of $|\zeta(s)| = c$ is confined to the strip $\beta_{n+1} \le \sigma \le \sigma_2(c)$. On a curve $\mathcal{R}_{4k}$ there is exactly one point at which $|\zeta(s)| = c$. Since we arrive there on a curve within $D(c)$, this is a point on $\mathcal{N}(c)$. However, no other connected component of $|\zeta(s)| = c$ can intersect $\mathcal{R}_{4k}$. Thus all other connected components are compact. ∎

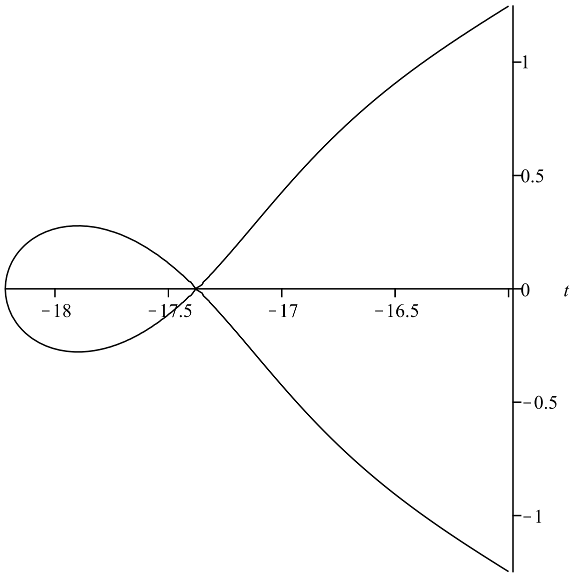

Fig. 2. Part of the noncompact curve $\mathcal{N}(c)$ where $c = |\zeta(\beta_8)|$

Suppose that $c > 1$ and t are fixed, with $|t| \ge \pi/(2\log 2)$. On the line $\sigma + it$, we have $|\zeta(s)| = c$ at abscissæ $\sigma_1(c,t) \ge \cdots \ge \sigma_R(c,t)$. Since

$$\lim_{\sigma\to-\infty} |\zeta(\sigma+it)| = +\infty, \qquad \lim_{\sigma\to+\infty} |\zeta(\sigma+it)| = 1,$$

it follows that R is an odd positive number. Also, since the ray $(\sigma_1(c,t)+it, +\infty+it)$ lies entirely within $D(c)$, it follows that $\sigma_1(c,t)+it \in \mathcal{N}(c)$. Suppose that $k \equiv 2 \pmod 4$, and let $\rho = \beta + i\gamma$ denote the point at which $\mathcal{R}_k$ terminates. Clearly $\rho \in D(c)$, and so $\sigma_R(c,\gamma) < \beta$.

Theorem 3.8. *Let $c > 1$, and let $\mathcal{N}(c)$ denote the noncompact component described in Corollary* 3.7. *As s tends to infinity along $\mathcal{N}(c)$, the limsup of the real part of s is $\sigma_2(c)$, and the liminf is $\le 1/2$. If RH is true, then there exist arbitrarily large t for which $\sigma + it \in \mathcal{N}(c)$ and $\sigma < 1/2$, but the liminf of σ is $= 1/2$.*

Proof. By Dirichlet's theorem on simultaneous approximation, there exist arbitrarily large t for which $\|t(\log p)/(2\pi)\| < \varepsilon$ for all primes $p \le y$. Here $\|\theta\| = \min_{n\in\mathbb{Z}} |\theta - n|$ is the distance from θ to the nearest integer. (This is the natural metric on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$.) Let δ be given, with $0 < \delta < \sigma_2(c) - 1$. If ε is sufficiently small, and y is sufficiently large, then $|\zeta(\sigma_2(c) - \delta + it)| > c$. On the other hand, $\sigma_2(c) + it \in D(c)$, so $\mathcal{N}(c)$ intersects the segment from $\sigma_2(c) - \delta + it$ to $\sigma_2(c) + it$. Hence the limsup of the real part of $s \in \mathcal{N}(c)$ is $\sigma(c)$.

Let $N(\sigma, T)$ denote the number of zeros $\rho = \beta + i\gamma$ of $\zeta(s)$ such that $\beta \ge \sigma$ and $0 < \gamma \le T$. Ingham [3] showed that

$$N(\sigma, T) \ll T^{\frac{3(1-\sigma)}{2-\sigma}} (\log T)^5.$$

If we take $\sigma = 1/2 + C(\log\log T)/\log T$ with C sufficiently large, then this upper bound is $o(T)$. However, there are $\gg T$ zeros $\rho = \beta + i\gamma$ that mark the termination of the curves $\mathcal{R}_k$ with $k \equiv 2 \pmod 4$, so $\gg T$ of these 'special zeros' have real part $< 1/2 + C(\log\log T)/\log T$. Since $\sigma_R(c, \gamma) < \beta$ for such a zero, we have $\sigma_R(c, t) < 1/2 + C(\log\log t)/\log t$ for arbitrarily large t. Assuming RH, we have $\beta = 1/2$, so $\sigma_R(c, \gamma) < 1/2$. However, on RH we know that if $0 < \sigma < 1/2$, then $|\zeta(\sigma + it)| > t^{1/2-\sigma+o(1)}$ as $t \to \infty$, so $\mathcal{N}(c)$ crosses the abscissa σ at most finitely many times. ∎

From the above it is clear that $\mathcal{N}(c)$ crosses the abscissa $\sigma = 1$ infinitely many times. However, since $|\zeta(it)| \gg t^{1/2}/\log t$, it crosses the imaginary axis only finitely many times (but at least once).

Corollary 3.9. *Suppose that $c > 1$. There exists a $t_0 > 0$, and a t_1 such that there is a curve joining it_0 to $1 + it_1$, lying entirely in the open strip $0 < \sigma < 1$ apart from the endpoints, such that $|\zeta(s)| = c$ on this curve.*

If c is only slightly larger than 1, say $1 < c < 3$, then there is more than one curve with this property, but for all sufficiently large c there is only one such curve in the upper half-plane.

Proof of Corollary 3.9. Let the point it_0 be the last departure of $\mathcal{N}(c)$ from the imaginary axis, and the point $1 + it_1$ be its first arrival on the 1-line. ∎

4. Open questions. 1. How many zeros does the typical loop $\mathcal{C}_{4k+2\pm1}$ contain? It must contain at least one. For t of moderate size, it seems that it generally contains $\sim (\log t)/(2\log 2)$ zeros, but it may be that at greater heights it contains fewer.

2. Let $c > 1$ be fixed, and assume RH. Then $\Re\frac{\zeta'}{\zeta}(\sigma + it) < 0$ for $-\infty < \sigma \le 1/2$, $t \ge 7$. Thus for any fixed $t \ge 7$, the quantity $|\zeta(\sigma + it)|$ is a monotonically decreasing function of σ on the interval $-\infty < \sigma \le 1/2$. Let $\rho = 1/2 + i\gamma$ be a zero. Then there is a unique $\sigma < 1/2$ such that $|\zeta(\sigma + i\gamma)| = c$. There are two possibilities: Either $\sigma + i\gamma \in \mathcal{N}(c)$, or $\sigma + i\gamma$ is on a compact lemniscate surrounding ρ. It should be possible to show that the latter possibility holds for at least half the zeros (asymptotically), but even showing that this happens infinitely often seems difficult. Perhaps it holds for almost all zeros (in the sense of density). The former possibility occurs infinitely often, namely for the zeros that mark the termination of the curve of steepest descent $\mathcal{R}_k$ ($k \equiv 2 \pmod 4$). Does the former possibility

occur for a positive proportion of the zeros? It seems to, in numerical studies of t of modest size, but perhaps it happens less frequently when t is large. Possibly the eventual asymptotic behavior is revealed only when $\log\log t$ is large, which means beyond the possibility of computation.

3. Baker and Montgomery [2] showed that most quadratic L-functions have a large number of critical points in the interval $(1/2, 1)$. Let t be given, and let $\sigma_1 \ge \sigma_2 \ge \cdots \ge \sigma_R$ denote those abscissæ at which $|\zeta(\sigma_r + it)| = c$. Certainly R is odd, and $\sigma_1 + it \in \mathcal{N}(c)$. In case t is the ordinate of a zero, $\sigma_R < 1/2 < \sigma_{R-1}$ (assuming there is an $R-1$). By borrowing the Baker–Montgomery ideas, it should be possible to show that R is large for most t, possibly even that R is of the order of $\sqrt{\log\log t}$ for most t.

4. Let ρ' range over the nontrivial zeros of ζ'. How is $|\zeta(\rho')|$ distributed? Are these values unbounded? If so, then there exist infinitely many zeros ρ that are contained in compact lemniscates.

5. Let c be a value attained by $|\zeta(\rho')|$ at some critical point. At this level, two lemniscates coalesce, generally two compact curves becoming a larger compact curve, but also occasionally $\mathcal{N}(c)$ swallowing a compact lemniscate. Fix $m > 0$ and $n > 0$, and consider the distribution of the real parts of the zeros ρ' at which lemniscates containing m and n zeros, respectively, join to form a lemniscate containing $m+n$ zeros. What is the $(1, 1)$ distribution, at least experimentally? $(1, 2)$? It has been observed (experimentally) that the distribution of the real parts has a density with a first hump, and then a second one. If for each (m, n) one has a distribution function, then the overall distribution function is the sum of all of these over m and n. Possibly $(m, n) = (1, 1)$ accounts for the first hump, $(1, 2)$ the second, and then after that it all gets blurred together.

6. Consider a curve of steepest ascent starting at $1 + it$ and ending on the imaginary axis. What is its typical arc length? Probably not bounded. Is the liminf of the arc length bounded? This would be somewhat analogous to the existence of infinitely many quadratic discriminants for which the L-function is monotonic in $[1/2, +\infty)$.

7. By (2.8) and (2.10) we see that the curve of steepest ascent $\mathcal{R}_{4k}$ reaches the 2-line between $(4k-1)\pi/(2\log 2)$ and $(4k+1)\pi/(2\log 2)$, say at t. Is there a constant $c > 1$ such that necessarily $|\zeta(2 + it)| \ge c$? Perhaps $c = \pi^2/9$. When $\mathcal{R}_{4k}$ reaches a given abscissa σ, $1 < \sigma < 2$, is the ordinate still only a bounded distance from $2k\pi/\log 2$?

8. What does random matrix theory tell us? The nature of the level curves of the characteristic function of a random matrix might be informative. Insights as to question 5 above might be gleaned through numerical studies of the absolute values of critical points of various types.

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