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Baker's explicit *abc*-conjecture and applications

by

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Dedicated to Professor Andrzej Schinzel on his 75th birthday

1. Introduction. The well known conjecture of Masser–Oesterlé is

Conjecture 1.1 (Oesterlé and Masser's *abc*-conjecture). *For any given $\epsilon > 0$ there exists a constant $\mathfrak{c}_\epsilon$ depending only on ϵ such that if*

$$a + b = c \tag{1}$$

where a, b and c are coprime positive integers, then

$$c \le \mathfrak{c}_\epsilon \Big(\prod_{p|abc} p\Big)^{1+\epsilon}.$$

It is known as *abc*-conjecture; the name derives from the usage of letters a, b, c in (1). For any positive integer $i > 1$, let $N = N(i) = \prod_{p|i} p$ be the radical of i, $P(i)$ be the greatest prime factor of i, and $\omega(i)$ be the number of distinct prime factors of i; moreover, we put $N(1) = 1, P(1) = 1$ and $\omega(1) = 0$. An explicit version of this conjecture due to Baker [Bak94] is the following:

Conjecture 1.2 (Explicit *abc*-conjecture). *Let a, b and c be pairwise coprime positive integers satisfying* (1). *Then*

$$c < \frac{6}{5} N \frac{(\log N)^\omega}{\omega!}$$

where $N = N(abc)$ and $\omega = \omega(N)$.

We observe that $N = N(abc) \ge 2$ whenever a, b, c satisfy (1). We shall refer to Conjecture 1.1 as *abc-conjecture* and Conjecture 1.2 as *explicit abc-conjecture*. Conjecture 1.2 implies the following explicit version of Conjecture 1.1.

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Theorem 1. *Assume Conjecture* 1.2. *Let a, b and c be pairwise coprime positive integers satisfying* (1) *and $N = N(abc)$. Then*

$$c < N^{1+3/4}. \tag{2}$$

Further for $0 < \epsilon \le 3/4$, there exists ω_ϵ depending only ϵ such that when $N = N(abc) \ge N_\epsilon = \prod_{p \le p_{\omega_\epsilon}} p$, we have

$$c < \kappa_\epsilon N^{1+\epsilon} \quad \text{where} \quad \kappa_\epsilon = \frac{6}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \le \frac{6}{5\sqrt{2\pi\omega_\epsilon}}$$

with $\omega = \omega(N)$. Here are some values of $\epsilon, \omega_\epsilon$ and N_ϵ.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6460
N_ϵ	$e^{37.1101}$	$e^{204.75}$	$e^{335.71}$	$e^{679.585}$	$e^{1004.763}$	$e^{3894.57}$	e^{63727}

Thus $c < N^2$, which was conjectured in Granville and Tucker [GrTu02]. We present here some consequences of Theorem 1.

The *Nagell–Ljunggren equation* is the equation

$$y^q = \frac{x^n - 1}{x - 1} \tag{3}$$

in integers $x > 1$, $y > 1$, $n > 2$, $q > 1$. It is known that

$$11^2 = \frac{3^5 - 1}{3 - 1}, \quad 20^2 = \frac{7^4 - 1}{7 - 1}, \quad 7^3 = \frac{18^3 - 1}{18 - 1},$$

which are called the *exceptional solutions*. Any other solution is termed *non-exceptional*. For an account of results on (3), see Shorey [Sho99] and Bugeaud and Mignotte [BuMi02]. It is conjectured that there are no non-exceptional solutions. We prove in Section 4 the following.

Theorem 2. *Assume Conjecture* 1.2. *There are no non-exceptional solutions of equation* (3) *in integers $x > 1$, $y > 1$, $n > 2$, $q > 1$.*

Let $(p, q, r) \in \mathbb{Z}_{\ge 2}$ with $(p, q, r) \ne (2, 2, 2)$. The equation

$$x^p + y^q = z^r, \quad (x, y, z) = 1,\ x, y, z \in \mathbb{Z}, \tag{4}$$

is called the *generalized Fermat equation* or *Fermat–Catalan equation* with signature (p, q, r). An integer solution (x, y, z) is said to be *non-trivial* if $xyz \ne 0$, and *primitive* if x, y, z are coprime. We are interested in finding non-trivial primitive integer solutions of (4). The case $p = q = r$ is the famous *Fermat equation*, which was completely solved by Wiles [Wil95]. One of known solution $1^p + 2^3 = 3^2$ of (4) comes from *Catalan's equation*. Let $\chi = 1/p + 1/q + 1/r - 1$. A complete parametrization of non-trivial primitive integer solutions for (p, q, r) with $\chi \ge 0$ has been found ([Beu04], [Coh07]). It was shown by Darmon and Granville [DaGr95] that (4) has only finitely many solutions in x, y, z if $\chi < 0$. When $2 \in \{p, q, r\}$, there are some known solutions. So, we consider $p \geq 3$, $q \geq 3$, $r \geq 3$. An open problem in this direction is the following.

CONJECTURE 1.3 (Tijdeman, Zagier). *There are no non-trivial solutions to* (4) *in positive integers* x, y, z, p, q, r *with* $p \geq 3$, $q \geq 3$ *and* $r \geq 3$.

This is also referred to as *Beal's Conjecture* or *Fermat–Catalan Conjecture*. This conjecture has been established for many signatures (p, q, r), including for several infinite families of signatures. For exhaustive surveys, see [Beu04], [Coh07, Chapter 14], [Kra99] and [PSS07]. Let $[p, q, r]$ denote all permutations of the ordered triple (p, q, r) and let

$$Q = \{[3, 5, p] : 7 \leq p \leq 23, p \, \text{prime}\} \cup \{[3, 4, p] : p \, \text{prime}\}.$$

We prove the following in Section 5.

THEOREM 3. *Assume Conjecture* 1.2. *There are no non-trivial solutions to* (4) *in positive integers* x, y, z, p, q, r *with* $p \geq 3$, $q \geq 3$ *and* $r \geq 3$ *with* $(p, q, r) \notin Q$. *Further for* $(p, q, r) \in Q$, *we have* $\max(x^p, y^q, z^r) < e^{1758.3353}$.

Another equation which we will be considering is the *equation of Goormaghtigh*

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1}, \quad \text{integers } x > 1, y > 1, m > 2, n > 2 \text{ with } x \neq y. \tag{5}$$

We may assume without loss of generality that $x > y > 1$ and $2 < m < n$. It is known that

$$31 = \frac{5^3 - 1}{5 - 1} = \frac{2^5 - 1}{2 - 1} \quad \text{and} \quad 8191 = \frac{90^3 - 1}{90 - 1} = \frac{2^{13} - 1}{2 - 1} \tag{6}$$

are solutions of (5) and it is conjectured that there are no other solutions. A weaker conjecture states that there are only finitely many solutions x, y, m, n of (5). We refer to [Sho99] for a survey of results on (5). In Section 6 we prove

THEOREM 4. *Assume Conjecture* 1.2. *Then equation* (5) *in integers* $x > 1$, $y > 1$, $m > 2$, $n > 3$ *with* $x > y$ *implies that* $m \leq 6$ *and further* $7 \leq n \leq 17$, $n \notin \{11, 16\}$ *if* $m = 6$; *moreover there exists an effectively computable absolute constant* C *such that*

$$\max(x, y, n) \leq C.$$

Thus, assuming Conjecture 1.2, equation (5) has only finitely many solutions in integers $x > 1$, $y > 1$, $m > 2$, $n > 3$ with $x \neq y$, which considerably improves Saradha's result [Sar12, Theorem 1.4].

2. Notation and preliminaries. For an integer $i > 0$, let p_i denote the ith prime. For a real $x > 0$, let $\Theta(x) = \prod_{p \leq x} p$ and $\theta(x) = \log(\Theta(x))$. We write $\log_2 i$ for $\log(\log i)$.

LEMMA 2.1. *We have*

(i) $\pi(x) \le \frac{x}{\log x}\left(1 + \frac{1.2762}{\log x}\right)$ *for* $x > 1$.

(ii) $p_i \ge i(\log i + \log_2 i - 1)$ *for* $i \ge 1$.

(iii) $\theta(p_i) \ge i(\log i + \log_2 i - 1.076869)$ *for* $i \ge 1$.

(iv) $\theta(x) < 1.000081x$ *for* $x > 0$.

(v) $\sqrt{2\pi k}(k/e)^k e^{1/(12k+1)} \le k! \le \sqrt{2\pi k}\,(k/e)^k e^{1/12k}$.

Here we understand that $\log_2 1 = -\infty$. The estimates (i) and (ii) are due to Dusart (see [Dus99b] and [Dus99a], respectively). The estimate (iii) is [Rob83, Theorem 6]. For the estimate (iv), see [Dus99b]. The estimate (v) is [Rob55, Theorem 6].

3. Proof of Theorem 1. Let $\epsilon > 0$, and let $N \ge 1$ be an integer with $\omega(N) = \omega$. Then $N \ge \Theta(p_\omega)$ or $\log N \ge \theta(p_\omega)$. Given i, we observe that $M^\epsilon/(\log M)^i$ is an increasing function for $\log M \ge i/\epsilon$. Let

$$X_0(i) = \log i + \log_2 i - 1.076869.$$

Then $\theta(p_i) \ge iX_0(i)$ by Lemma 2.1(iii). Observe that $X_0(i) > 1$ for $i \ge 5$. Let $\omega_1 \ge 5$ be the smallest integer such that

$$\epsilon X_0(i) - \log X_0(i) \ge 1 \quad \text{for all } i \ge \omega_1. \tag{7}$$

Note that $\epsilon X_0(i) \ge 1$ for $i \ge \omega_1$, implying $\log N \ge \theta(p_\omega) \ge \omega X_0(\omega) \ge \omega/\epsilon$ when $\omega \ge \omega_1$ by Lemma 2.1(iii). Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \ge \frac{\omega!\Theta(p_\omega)^\epsilon}{(\theta(p_\omega))^\omega} \ge \frac{\omega! e^{\epsilon\omega X_0(\omega)}}{\omega X_0(\omega)^\omega} > \sqrt{2\pi\omega}\left(\frac{\omega}{e}\right)^\omega \frac{e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega} \text{ when } \omega \ge \omega_1.$$

Thus for $\omega \ge \omega_1$, from (7) we have

$$\begin{aligned}\log\left(\frac{\omega! e^{\epsilon\omega X_0(\omega)}}{(\omega X_0(\omega))^\omega}\right) &> \log\sqrt{2\pi\omega} + \omega(\log(\omega) - 1) + \epsilon\omega X_0(\omega) \\ &\quad - \omega(\log\omega + \log X_0(\omega)) \\ &> \log\sqrt{2\pi\omega} + \omega(\epsilon X_0(\omega) - \log X_0(\omega) - 1) \ge \log\sqrt{2\pi\omega},\end{aligned}$$

implying

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \ge \frac{\omega!\Theta(p_\omega)^\epsilon}{\theta(p_\omega)^\omega} > \sqrt{2\pi\omega} \quad \text{when } \omega \ge \omega_1.$$

Define ω_ϵ to be the smallest integer $\le \omega_1$ such that

$$\theta(p_i) \ge \frac{i}{\epsilon} \quad \text{and} \quad \frac{i!\Theta(p_i)^\epsilon}{\theta(p_i)^i} > \sqrt{2\pi i} \quad \text{for all } \omega_\epsilon \le i \le \omega_1 \tag{8}$$

by taking the exact values of i and θ. Then clearly

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \ge \frac{\omega!\Theta(p_\omega)^\epsilon}{\theta(p_\omega)^\omega} > \sqrt{2\pi\omega} \quad \text{when } \omega \ge \omega_\epsilon. \tag{9}$$

Here are the values of ω_ϵ for some ϵ values.

ϵ	$\frac{3}{4}$	$\frac{7}{12}$	$\frac{6}{11}$	$\frac{1}{2}$	$\frac{34}{71}$	$\frac{5}{12}$	$\frac{1}{3}$
ω_ϵ	14	49	72	127	175	548	6458

Let $\omega < \omega_\epsilon$ and $N \geq \Theta(p_{\omega_\epsilon})$. Then $\log N \geq \theta(p_{\omega_\epsilon}) \geq \omega_\epsilon/\epsilon$. Therefore

$$\frac{\omega! N^\epsilon}{(\log N)^\omega} \geq \frac{\omega! \Theta(p_{\omega_\epsilon})^\epsilon}{\theta(p_{\omega_\epsilon})^\omega} = \frac{\omega_\epsilon! \Theta(p_{\omega_\epsilon})^\epsilon}{\theta(p_{\omega_\epsilon})^{\omega_\epsilon}} \cdot \frac{\omega!}{\omega_\epsilon!} \theta(p_{\omega_\epsilon})^{\omega_\epsilon - \omega}$$
$$> \sqrt{2\pi\omega_\epsilon}\, \frac{\omega! \omega_\epsilon^{\omega_\epsilon - \omega}}{\omega_\epsilon!} \geq \sqrt{2\pi\omega_\epsilon}.$$

Combining this with (9), we obtain

$$(10) \qquad \frac{(\log N)^\omega}{\omega!} < \frac{N^\epsilon}{\sqrt{2\pi \max(\omega, \omega_\epsilon)}} \leq \frac{N^\epsilon}{\sqrt{2\pi\omega_\epsilon}} \quad \text{when } N \geq \Theta(p_{\omega_\epsilon}).$$

Further we now prove

$$(11) \qquad \frac{(\log N)^\omega}{\omega!} < \frac{5N^{3/4}}{6} \quad \text{for } N \geq 1.$$

For that we take $\epsilon = 3/4$. Then $\omega_\epsilon = 14$ and we may assume that $N < \Theta(p_{14})$. Then $\omega < 14$. Observe that $N \geq \Theta(p_\omega)$ and $N^{3/4}/(\log N)^\omega$ is increasing for $\log N \geq 4\omega/3$. For $4 \leq \omega < 14$, we check that

$$\theta(p_\omega) \geq \frac{4\omega}{3} \quad \text{and} \quad \frac{\omega! \Theta(p_\omega)^{3/4}}{\theta(p_\omega)^\omega} > \frac{6}{5},$$

implying (11) when $4 \leq \omega < 14$. Thus we may assume $\omega < 4$. We check that

$$(12) \qquad \frac{\omega! N^{3/4}}{(\log N)^\omega} > \frac{6}{5} \quad \text{at } N = e^{4\omega/3}$$

for $1 \leq \omega < 4$, implying (11) for $N \geq e^{4\omega/3}$. Thus we may assume that $N < e^{4\omega/3}$. Then $N \in \{2, 3\}$ if $\omega = 1$, $N \in \{6, 10, 12, 14\}$ if $\omega = 2$, and $N \in \{30, 42\}$ if $\omega = 3$. For these values of N too, we find that (12) is valid, implying (11). Clearly (11) is valid when $N = 1$.

We now prove Theorem 1. Assume Conjecture 1.2. Let $\epsilon > 0$ be given. Let a, b, c be positive integers such that $a + b = c$ and $\gcd(a, b) = 1$. By Conjecture 1.2, $c \leq \frac{6}{5} N (\log N)^\omega / \omega!$ where $N = N(abc)$. Now assertion (2) follows from (11). Let $0 < \epsilon \leq 3/4$ and $N_\epsilon = \Theta(p_{\omega_\epsilon})$. By (10), we have

$$c < \frac{6N^{1+\epsilon}}{5\sqrt{2\pi \max(\omega, \omega_\epsilon)}}.$$

The table is obtained by taking the table values of $\epsilon, \omega_\epsilon$ given after (9) and computing N_ϵ for those ϵ given in the table. Hence the theorem follows. ∎

4. Nagell–Ljungrenn equation: Proof of Theorem 2. Let $x > 1$, $y > 1$, $n > 2$ and $q > 1$ be a non-exceptional solution of (3). It was proved by Ljunggren [Lju43] that there are no further solutions of (3) when $q = 2$. Thus we may suppose that $q \geq 3$. Further it has been proved that $4 \nmid n$ by Nagell [Nag20], $3 \nmid n$ by Ljunggren [Lju43] and $5 \nmid n$, $7 \nmid n$ by Bugeaud, Hanrot and Mignotte [BHM02]. Therefore $n \geq 11$. From (3), we get

$$1 + (x-1)y^q = x^n.$$

Then $y < x^{n/q} \leq x^{n/3}$ since $q \geq 3$, implying $N = N(x(x-1)y) < x^2y < x^{2+n/3}$. From (2) in Theorem 1, we obtain

$$x^n < N^{7/4} < x^{7/2+7n/12}, \quad \text{implying} \quad n < \frac{7}{2} + \frac{7n}{12}.$$

This gives $n \leq 8$, which is a contradiction.

5. Fermat–Catalan equation. We may assume that each of p, q, r is either 4 or an odd prime. Let $[p, q, r]$ denote all permutations of the ordered triple (p, q, r). Fermat's Last Theorem, the case (p, p, p), was proved by Wiles [Wil95]; $[3, p, p], [4, p, p]$ for $p \geq 7$ by Darmon and Merel [DaGr95] and $[3, 5, 5], [4, 5, 5]$ by Poonen; $[4, 4, p]$ by Bennett, Ellenberg, Ng [BEN10]. The signatures $[3, 3, p]$ for $p \leq 10^9$ were solved by Chen and Siksek [ChSi09], $[3, 4, 5]$ by Siksek and Stoll [SiSt12] and $[3, 4, 7]$ by Poonen, Schefer and Stoll [PSS07]. Hence we may suppose (p, q, r) is different from those values.

We may assume that $x > 1, y > 1, z > 1$. Then

$$x < z^{r/p}, \qquad y < z^{r/q}.$$

Given $\epsilon > 0$, by Theorem 1, we have

$$z^r < \begin{cases} N_\epsilon^{7/4} & \text{if } N(xyz) < N_\epsilon, \\ N(xyz)^{1+\epsilon} \leq (xyz)^{1+\epsilon} & \text{if } N(xyz) \geq N_\epsilon. \end{cases} \tag{13}$$

In particular, taking $\epsilon = 3/4$, we get

$$z^r < (xyz)^{7/4} < z^{\frac{7}{4}(1+r/p+r/q)},$$

implying

$$\frac{4}{7} < \frac{1}{p} + \frac{1}{q} + \frac{1}{r}. \tag{14}$$

Thus we need to consider $[3, 3, p]$ for $p > 10^9$ and $(p, q, r) \in Q$. Let $\epsilon = 34/71$. First assume that $N(xyz) \geq N_\epsilon$. Then

$$z^r < (xyz)^{1+\epsilon} < z^{(1+\epsilon)(1+r/p+r/q)},$$

implying

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} > \frac{1}{1+\epsilon} = \frac{71}{105} = \frac{1}{3} + \frac{1}{5} + \frac{1}{7}.$$

Therefore we may suppose that $N(xyz) < N_{34/71}$. Then from (13) it follows that $\max(x^p, y^q, z^r) < N_{34/71}^{7/4} \le e^{1758.3353}$, implying x, y, z, p, q, r are all bounded. This will imply that $[3, 3, p]$ with $p > 10^9$ does not have any solution. Hence the assertion. ■

6. Goormaghtigh equation. Let $d = \gcd(x, y)$. From (5), we have

$$x^{m-1} + \cdots + x = y^{n-1} + \cdots + y,$$

implying $\mathrm{ord}_p(x) = \mathrm{ord}_p(y)$ for all primes $p \mid d$. Further

$$\sum_{i=1}^{m-1}(x^i - y^i) = (x-y)\left\{1 + \sum_{i=2}^{m-1}\frac{x^i - y^i}{x-y}\right\} = y^{n-1} + \cdots + y^m,$$

which is

$$1 + \sum_{i=2}^{m-1}\frac{x^i - y^i}{x-y} = \frac{y^m}{x-y}\,\frac{y^{n-m}-1}{y-1}.$$

We observe that d is coprime to $\frac{y^{n-m}-1}{y-1}$ and also to the left hand side. Therefore

$$\mathrm{ord}_p(x-y) = m \cdot \mathrm{ord}_p(x) = m \cdot \mathrm{ord}_p(y) = m \cdot \mathrm{ord}_p(d)$$

for every prime $p \mid d$. Let $d_2 = \gcd(y-1, x-1, x-y)$ and let d_3 be given by $x - y = d^m d_2 d_3$. We observe that $d_2 d_3 = 1$ if $n = m+1$ and $d_2 d_3 \mid (y+1)$ if $n = m+2$. We now rewrite (5) as

$$\frac{(y-1)x^m}{d^m d_2} + d_3 = \frac{(x-1)y^n}{d^m d_2}. \tag{15}$$

Let

$$N = N\left(\frac{x^m y^n (x-1)(y-1)d_3}{d^{2m} d_2^2}\right) \le N(xy(x-1)(y-1)d_3) \le \frac{xy(x-1)(y-1)d_3}{2^\delta d d_2}$$

where $\delta = 0$ if $2 \mid dd_2$ and 1 otherwise. Recall that $d = \gcd(x, y)$ and $d_2 \mid (x-1)$. Let $\epsilon < 3/4$. From (15) and Theorem 1 and $x - y = d^m d_2 d_3$ we obtain

$$\max\left\{\frac{(y-1)x^m d_3}{(x-y)}, \frac{(x-1)y^n d_3}{x-y}\right\} < \begin{cases} N_\epsilon^{7/4} & \text{if } N < N_\epsilon, \\ N^{1+\epsilon} & \text{if } N \ge N_\epsilon. \end{cases} \tag{16}$$

Assume that $N \ge N_\epsilon$. Then using (16) we obtain

$$x^m < x^{2+2\epsilon} y^{1+2\epsilon}(x-y)\frac{d_3^\epsilon}{(2^\delta d d_2)^{1+\epsilon}} < x^{4+5\epsilon}, \tag{17}$$

$$y^n < x^{1+2\epsilon} y^{1+\epsilon}(y-1)^{1+\epsilon}(x-y)\frac{d_3^\epsilon}{(2^\delta d d_2)^{1+\epsilon}}, \tag{18}$$

since $y < x$ and $d_3 \le x - y < x$. We observe that (5) yields $x^{m-1} < 2y^{n-1}$, implying $x < 2^{\frac{1}{m-1}} y^{\frac{n-1}{m-1}}$. This together with (18), $d_3 \le x - y < x$ and $2^\delta d d_2 \ge 2$ gives

$$y^n < 2^{\frac{2+3\epsilon}{m-1}-1-\epsilon} y^{2+2\epsilon+\frac{n-1}{m-1}(2+3\epsilon)}. \tag{19}$$

From (17), we obtain $m < 4 + 5\epsilon$, and further from (19), we get

$$n < 2 + 2\epsilon + \frac{n-1}{m-1}(2+3\epsilon)$$

if $m > 3$.

Let $\epsilon = 3/4$ and $N_\epsilon = 1$. Then $m \le 7$, and further $7 \le n \le 17$ if $m = 6$, and $n \in \{8, 9\}$ if $m = 7$.

Let $m = 7$ and $n = m + 1 = 8$. Then $d_2 d_3 = 1$ and from the first inequality of (17) and $y < x$ we get $x^m < x^{4+4\epsilon} = x^7$, implying $7 = m < 7$, a contradiction.

Let $m = 7$ and $n = m + 2 = 9$. Then $d_2 d_3 \le y + 1$ and from (18) with $x < 2^{\frac{1}{m-1}} y^{\frac{n-1}{m-1}}$, $d_3(y-1) < y^2$ and $2^\delta d d_2 \ge 2$ we get

$$y^n < 2^{\frac{2+2\epsilon}{m-1}-1-\epsilon} y^{2+3\epsilon+\frac{n-1}{m-1}(2+2\epsilon)} < y^9,$$

which is a contradiction again.

Let $m = 6$ and $n \in \{11, 16\}$. From Nesterenko and Shorey [NeSh98], we get $y \le 8, 15$ when $n = 11, 16$, respectively. For $2 \le y \le 15$ and $y + 1 \le x \le \left(\frac{y^n-1}{y-1}\right)^{\frac{1}{m-1}}$, we check that (5) does not hold. Therefore $n \notin \{11, 16\}$ when $m = 6$. Hence we have the first assertion of Theorem 4.

Now we take $\epsilon = 1/18$. Since $m \le 7$ and $G < x$, we get an explicit bound of x, y, m, n from (16) if $N < N_{1/18}$, implying Theorem 4 in that case. Thus we may suppose that $N \ge N_{1/18}$. Then we deduce from (17) with $\epsilon = 1/18$ that $m < 4 + 5\epsilon$, implying $m \in \{3, 4\}$, and further from (19) that $n < 5$ if $m = 4$. This is a contradiction for $m = 4$ since $n > m$ and $n \in \mathbb{Z}$.

Let $m = 3$. We rewrite (5) as

$$(2x+1)^2 = 4(y^{n-1} + \cdots + y) + 1. \tag{20}$$

By [NeSh98], we may assume that $n \ne 5$. Let $n = 4$ and denote by $f(y)$ the polynomial on the right hand side of (20). Let $f'(\alpha) = 0$. Then $\alpha = (-1 \pm \sqrt{2}\, i)/3$ and we check that $f(\alpha) \ne 0$. Therefore the roots of f are simple. Now we apply [Bak69] to conclude that y and hence x are bounded by effectively computable absolute constants. Let $n \ge 6$. Now we rewrite (5) as

$$4y^n = (y-1)(2x+1)^2 + (3y+1). \tag{21}$$

Let $G = \gcd(4y^n, (y-1)(2x+1)^2, 3y+1)$. Then $G = 4, 2, 1$ according as

$4 \mid (y-1)$, $4 \mid (y-3)$ and $2 \mid y$, and we infer from (21) that

$$\frac{4}{G} y^n = \frac{y-1}{G}(2x+1)^2 + \frac{3y+1}{G}. \tag{22}$$

Let

$$N = N\left(\frac{4y(y-1)(2x+1)(3y+1)}{G^3}\right) \le \frac{y(y-1)(2x+1)(3y+1)}{G} < \frac{6xy^3}{G_1}.$$

Let $\epsilon = 1/12$. We see from Theorem 1 with $\epsilon = 1/12$ that

$$\frac{4y^n}{G} < \begin{cases} N_{1/12}^{7/4} & \text{if } N < N_{1/12}, \\ N^{1+1/12} & \text{if } N \ge N_{1/12}. \end{cases} \tag{23}$$

If $N < N_{1/12}$, then $y^n < N_{1/12}^{7/4}$, implying the assertion of Theorem 4. Hence we may suppose that $N \ge N_{1/12}$ and further that y is sufficiently large. Then we conclude from $x^2 < 2y^{n-1}$ that

$$4y^n < (6\sqrt{2}\, y^{(n+5)/2})^{1+1/12}.$$

Therefore

$$n - \frac{13(n+5)}{24} < \frac{\frac{13}{12}\log(6\sqrt{2}) - \log 4}{\log y} < \frac{1}{24}$$

since y is sufficiently large. This is not possible since $n \ge 6$. Hence the assertion follows. ∎

Remarks. The examples in this paper show that in applications of the *abc*-conjecture to diophantine equations, it is sufficient to assume that ϵ is not very near to 0. Sometimes it is sufficient to use *abc* with $\epsilon = 1/2$ or $3/4$ or even larger. See also the paper of Browkin [Bro08], where the minimal sufficient values of ϵ are discussed for some diophantine equations. In general they are large. From this point of view it is probably irrelevant what the *abc*-conjecture says in the case of ϵ near to 0.

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