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On a question of Schinzel about the length and Mahler's measure of polynomials that have a zero on the unit circle

by

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Dedicated to Professor Andrzej Schinzel
on the occasion of his 75th birthday

1. Introduction. Let $P(x) = \sum_{i=0}^{d} a_i x^i = a_0 \prod_{i=0}^{d}(x-\alpha_i)$ be a polynomial in $\mathbb{C}[x]$. Its *length* is defined by $L(P) = \sum_{i=0}^{d} |a_i|$, its *height* by $H(P) = \max\{|a_i| : 0 \le i \le d\}$, and (for $a_0 \ne 0$) its *Mahler's measure* by

$$M(P) = |a_0| \prod_{i=0}^{d} \max\{1, |\alpha_i|\} = \exp\Bigl(\int_0^1 \log|P(e(\theta)|\, d\theta\Bigr),$$

where $e(\theta) = e^{2\pi i\theta}$. The last equality follows from the well known Jensen formula. For a polynomial $P \in \mathbb{C}[x_1, \ldots, x_n]$ in several variables the length and height are defined in the same way, while its Mahler's measure is defined by

$$M(P) = \exp\Bigl(\int_{[0,1]^n} \log|P(e(\theta_1), \ldots, e(\theta_n))|\, d\theta\Bigr). \tag{1.1}$$

Several authors, e.g., [D, Sch08, Sch07a, Sch07b], studied the so called *reduced length* of a polynomial. For a polynomial P it is defined by

$$l(P) = \inf L(PG),$$

where G runs through all monic polynomials in $\mathbb{C}[x]$. In [Sch08], A. Schinzel stated one of the unresolved questions relating to reduced length as:

Does the inequality $L(P) \ge 2M(P)$ hold for every polynomial $P \in \mathbb{C}[x]$ that has a zero on the unit circle?

In the cited paper Schinzel proved this inequality for several particular cases and showed that in the general case $L(P) \ge \sqrt{2}\, M(P)$. The purpose of this paper is to prove that in the general case we have indeed $L(P) \ge 2M(P)$.

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2. Statement of the results. It is deceptively easy to establish the inequality for polynomials up to fourth degree; not much more is required than a skillful use of the triangle inequality. However, degree five appears to be a stumbling block. Our main theorem is

THEOREM 2.1. *$L(P) \geq 2M(P)$ for every $P \in \mathbb{C}[x]$ that has a zero on the unit circle.*

D. Boyd conjectured [B] and W. Lawton proved [L] that Mahler's measure of a polynomial in several variables is a limit of Mahler's measures of polynomials in one variable. Thus our theorem automatically generalizes to polynomials in several variables. The only requirement is that a polynomial in $\mathbf{z} = (z_1, \ldots, z_n)$ has a zero on the n-dimensional torus $\mathbb{T}^n = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_i| = 1,\ i = 1, \ldots, n\}$.

COROLLARY 2.2. *Suppose that $P \in \mathbb{C}[\mathbf{z}]$, where $\mathbf{z} = (z_1, \ldots, z_n)$, has a zero on $\mathbb{T}^n$. Then $L(P) \geq 2M(P)$.*

In particular, our result is valid for linear forms $\mathcal{L}_{\mathbf{a}}(\mathbf{z}) = a_1 z_1 + \cdots + a_n z_n$. Such forms have been extensively studied. For $n = 3$ an explicit formula for Mahler's measure of $\mathcal{L}_{\mathbf{a}}$ was established by Maillot and Cassaigne [M]. In the general case the authors of [RTV] give an estimate of $M(\mathcal{L}_{\mathbf{a}})$ in terms of the Euclidean norm of $\mathbf{a}$. However these results do not seem to be helpful in establishing our inequality. Corollary 2.2 immediately gives $L(\mathcal{L}_{\mathbf{a}}) \geq 2M(\mathcal{L}_{\mathbf{a}})$ if $\mathcal{L}_{\mathbf{a}}$ has a zero on $\mathbb{T}^n$. For $n = 3$ the last condition means that the lengths $|a_1|, |a_2|$ and $|a_3|$, in the formula for $\mathcal{L}_{\mathbf{a}}(\mathbf{z})$, can form a triangle. For such forms Maillot and Cassaigne expressed $M(\mathcal{L}_{\mathbf{a}})$ in terms of the Bloch–Wigner dilogarithm. Corollary 2.2 thus provides an interesting bound on the dilogarithm. However, we will not investigate this point in this paper.

3. Lemmas and proofs

LEMMA 3.1. *Let $P \in \mathbb{C}[x]$ with $\deg(P) = d$, and set $P^*(x) = \epsilon x^d P(x^{-1})$ or $P^*(x) = \epsilon x^d \bar{P}(x^{-1})$, $|\epsilon| = 1$. Then*

$$L(P) \geq 2M(P) \iff L(P^*) \geq 2M(P^*).$$

Proof. This is obvious, since $L(P) = L(P^*)$ and $M(P) = M(P^*)$. ∎

LEMMA 3.2 (Schinzel, [Sch08, Lemma 1]). *If $P \in \mathbb{C}[x]$ has a zero on the unit circle then $L(P) \geq 2H(P)$.*

The next lemma is crucial to the proof of the theorem.

LEMMA 3.3. *Let $P(z) = \sum_{j\in J} a_j z^{d-j} \in \mathbb{C}[z]$, where $|J| = k$, be a polynomial with exactly $k \geq 3$ nonzero terms, such that $P(0) \neq 0$, $P(1) = 0$ and P has no other zeros on the unit circle, P has at least one zero outside and at least one zero inside the unit circle. Then either there exists a polynomial Q*

of degree d for which $L(Q)/M(Q) < L(P)/M(P)$, or there is an open interval I containing 0 *and a continuous trajectory* $I \ni t \mapsto P_{[t]} = \sum_{j\in J} a_j(t)z^{d-j}$ *such that* $P_{[0]} = P$, $M(P_{[t]}) = M(P)$, $L(P_{[t]}) = L(P)$ *for all* $t \in I$, $a_j(t) \neq 0$ *for all* $j \in J$ *and* $t \in I$, *and the modulus of the leading coefficient of* $P_{[t]}$, $|a_0(t)|$, *is strictly decreasing on* I.

We leave the proof of this lemma to the last section.

3.1. Proof of Theorem 2.1. For a nonzero polynomial P define $\lambda(P) = L(P)/M(P)$. Observe that the definitions of the length and Mahler's measure of P immediately imply that $\lambda(cP) = \lambda(P)$ for any nonzero constant c. Thus, without loss of generality, we can assume that P is monic; later we will relax that assumption when convenient. Further, $\lambda(P(ux)) = \lambda(P(x))$ for any complex u on the unit circle, hence we can also assume without loss of generality that $P(1) = 0$, i.e., the zero of P on the unit circle is $z = 1$. Let

$$\mathfrak{P}_d = \{P \in \mathbb{C}[z] : \deg P = d,\ P(1) = 0,\ P \text{ is monic}\},$$

and $\mathfrak{P} = \bigcup_{d=1}^{\infty} \mathfrak{P}_d$. Further, let

$$\lambda_d = \inf_{P\in\mathfrak{P}_d} \lambda(P) \quad \text{and} \quad \lambda_0 = \inf_{P\in\mathfrak{P}} \lambda(P).$$

Jensen's formula for $M(P)$ implies that $L(P) \geq M(P)$ for any nonzero polynomial. Hence $\lambda(P) \geq 1$. On the other hand $\lambda(z^d - 1) = 2$, so $\lambda_d \in [1, 2]$ for all $d \in \mathbb{N}$; the same holds for λ_0. Clearly, $\lambda_0 = \inf\{\lambda_d : d \geq 1\}$, hence the conclusion of the theorem is equivalent to

$$\lambda_d = 2 \quad \text{for all } d \in \mathbb{N}. \tag{3.1}$$

In order to prove this equality we proceed by induction on d. For $d = 1$, $\mathfrak{P}_d$ consists of a single polynomial $z - 1$, so $\lambda_1 = 2$ trivially. Fix $d > 1$ and suppose that $\lambda_n = 2$ for all $n < d$. A priori two cases are possible:

CASE 1: The value of λ_d is not attained at any $P \in \mathfrak{P}_d$.
CASE 2: The value of λ_d is attained at some $P \in \mathfrak{P}_d$.

Proof of (3.1) *in Case 1.* For a polynomial $P(x) = \sum_{j=0}^{d} a_j z^{d-j}$, let $v(P) = (a_0, \ldots, a_d)$ denote the vector of its coefficients; conversely, for a vector v let $p(v)$ denote the corresponding polynomial, so that $p(v(P)) = P$. By definition of λ_d, there is a sequence $\{P_m\}$ of polynomials in $\mathfrak{P}_d$ such that $\lim_{m\to\infty} \lambda(P_m) = \lambda_d$. Further, $M(P)$ is a continuous function of the coefficients of P (see [L]), and so is $L(P)$ and $\lambda(P)$. Since λ_d is not attained and the set $v(\mathfrak{P}_d)$ is closed in $\mathbb{C}^{d+1}$, the sequence $\{P_m\}$ cannot contain any bounded subsequence. Hence

$$\lim_{m\to\infty} H(P_m) = \infty.$$

Let $\hat{P}_m = P_m/H(P_m)$. We have

$$\lim_{m\to\infty} \lambda(\hat{P}_m) = \lim_{m\to\infty} \lambda(P_m) = \lambda_d.$$

The definition of $\hat{P}_m$ implies that $v(\{\hat{P}_m\})$ lies in the compact set $\{(z_0, \ldots, z_d) \mid |z_j| \le 1,\ j = 0, \ldots, d\} \subset \mathbb{C}^{d+1}$. Therefore $\{\hat{P}_m\}$ contains a convergent subsequence. Let $\hat{P}_0$ be the limit of that subsequence. Then, by continuity, $H(\hat{P}_0) = 1$ and $\hat{P}_0(1) = 0$. However, since P_m is monic, the leading coefficient of $\hat{P}_m$ is $1/H(P_m)$, and $\lim_{m\to\infty} H(P_m) = \infty$; hence the coefficient of z^d in $\hat{P}_0$ must be 0. Hence $\deg(\hat{P}_0) < d$ and, by induction hypothesis, $\lambda_d = \lambda(\hat{P}_0) = 2$.

Proof of (3.1) *in Case 2.* Suppose that $\lambda_d = \lambda(P)$ for some $P \in \mathfrak{P}_d$. Write P as $P(z) = \sum_{i=0}^{k} c_i z^{n_i}$ where c_i, $i = 0, \ldots, k$, are not zero, and $n_0 > \cdots > n_k = 0$. Let $\gcd(n_0, \ldots, n_k) = m$. If $m \ne 1$ then the conclusion holds by induction hypothesis, since then $P(z) = P_1(z^m)$ for a polynomial P_1 of degree d/m with $\lambda(P) = \lambda(P_1)$. Consequently, in what follows we assume that $m = 1$.

Claim 3.1.1. *P has no other zeros on the unit circle besides $z = 1$, which is a zero of order* 1.

Proof of Claim 3.1.1. Suppose first to the contrary that P has another zero $z_0 \ne 1$ on the unit circle. The condition $\gcd(n_0, \ldots, n_k) = 1$ implies that for some $j \ne k$, the binomial $h(z) = z^{n_j} - 1$ does not vanish at z_0. Let θ_0 be the argument of z_0 and consider $z = e^{i\theta}$, $\theta \in \mathbb{R}$. Since h is continuous we can choose $\delta > 0$ and an open interval I_δ containing θ_0 such that $|h(z)| > \delta$ for θ in I_δ. Let $\rho > 0$ be a small real number to be determined later, and let $s = e^{i\psi}$. Define $P_{\rho s}(z) = P(z) + \rho s h(z)$. Let $\lambda_s = L(P_{\rho s})/M(P_{\rho s})$ and $\lambda_{\mathrm{ave}} = \exp\big(\frac{1}{2\pi}\int_0^{2\pi} \log|\lambda_s|\,d\psi\big)$. We have $\log\lambda_{\mathrm{ave}} = l_\rho(P) - m_\rho(P)$, where

$$m_\rho(P) = \frac{1}{2\pi}\int_0^{2\pi} \log|M(P_{\rho s})|\,d\psi \quad\text{and}\quad l_\rho(P) = \frac{1}{2\pi}\int_0^{2\pi} \log|L(P_{\rho s})|\,d\psi.$$

Let $P(z) = (z - z_0)P_1(z)$, $\kappa = \sup_{|z|=1}|P_1(z)|$, and $I_\rho = \{\theta : |\theta - \theta_0| \le \kappa^{-1}\rho\delta\}$. On I_ρ we have

$$|P(e^{i\theta})| = |(e^{i\theta} - e^{i\theta_0})P_1(e^{i\theta})| \le |\theta - \theta_0|\kappa \le \rho\delta,$$

while for sufficiently small ρ, $I_\rho \subset I_\delta$, so $|\rho h(e^{i\theta})| > \rho\delta$. Hence

$$\begin{aligned} m_\rho(P) &= \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi} \log|P(z) + \rho s h(z)|\,d\theta\,d\psi \\ &= \frac{1}{2\pi}\int_0^{2\pi} \max\{\log|P(z)|, \log|\rho h(z)|\}\,d\theta \end{aligned}$$

$$\geq \frac{1}{2\pi}\int_0^{2\pi} \log|P(z)|\,d\theta + \frac{1}{2\pi}\int_{I_\rho} (\log|\rho h(z)| - \log|P(z)|)\,d\theta$$
$$\geq m(P) + \frac{1}{2\pi}\int_{I_\rho} (\log|\rho\delta| - \log|(\theta-\theta_0)\kappa|)\,d\theta = m(P) + \frac{\delta}{\pi\kappa}\rho.$$

On the other hand

$$L(P_{\rho s}) = L(P) + |c_j + \rho s| - |c_j| + |c_k - \rho s| - |c_k|.$$

For $l \in \{j,k\}$, let $c_l = |c_l|e^{i\theta_l}$. Then

$$|c_j + \rho s| - |c_j| = \sqrt{|c_j|^2 + \rho^2 + 2\rho\Re(\bar{c}_j s)} - |c_j| = \rho\cos(\psi - \theta_j) + \mathcal{O}(\rho^2).$$

Similarly

$$|c_k - \rho s| - |c_k| = -\rho\cos(\psi - \theta_k) + \mathcal{O}(\rho^2).$$

Hence

$$|L(P_{\rho s})| = |L(P)| + \rho\cos(\psi-\theta_j) - \rho\cos(\psi-\theta_k) + \mathcal{O}(\rho^2).$$

Consequently,

$$l_\rho(P) = \frac{1}{2\pi}\int_0^{2\pi} \log|L(P_{\rho s})|\,d\psi = \log|L(P)| + \mathcal{O}(\rho^2).$$

Thus, for sufficiently small ρ, $\log\lambda_{\mathrm{ave}} = l_\rho(P) - m_\rho(P) < \log\lambda(P)$. Therefore for some $\rho > 0$ and s on the unit circle, $\lambda(P_{\rho s}) < \lambda(P) = \lambda_d$, contrary to the choice of P.

Now suppose that P has a multiple zero at $z = 1$. In the notation of the previous case consider again $P_{\rho s}(z) = P(z) + \rho s h(z)$. Then again $l_\rho(P) = \log|L(P)| + \mathcal{O}(\rho^2)$. However, in order to obtain a lower bound for $m_\rho(P_{\rho s})$, we apply the previous argument to $\hat{P}(z) = P(z)/(z-1)$ and $\hat{h}(z) = h(z)/(z-1)$ (both quotients are polynomials in $\mathbb{C}[z]$), instead of P and h. Then $\hat{P}(1) = 0$, $M(\hat{P} + \rho s\hat{h}) = M(P_{\rho s})$ and $\hat{h}(1) \neq 0$. Thus the same argument can be applied in the case of $z_0 = 1$, and we get the same estimate $m_\rho(P_{\rho s}) \geq m(P) + \frac{\delta}{\pi\kappa}\rho$, with $\kappa = \max_{|z|=1}|P(z)/(z-1)^2|$ and δ determined by $\hat{h}$. Again, this contradicts the choice of P. ∎

Consequently, in what follows, we assume that P has exactly one zero, $z = 1$, on the unit circle. Further, we can also assume that P has at least one zero outside as well as at least one zero inside the unit circle. For if P has no zeros inside the unit circle then by Lemma 3.2,

$$L(P) \geq 2H(P) \geq 2|P(0)| = 2M(P), \tag{3.2}$$

provided $P(0) \neq 0$. If P has no zeros outside the unit circle then P^* has no zeros inside it and by Lemma 3.1 we are reduced to the previous case. Finally, if $P(0) = 0$ the induction hypothesis applies to $P(z)/z$, so we also assume that $P(0) \neq 0$. The assumptions about P imply that it has at least three nonzero terms. Thus P satisfies all the hypotheses of Lemma 3.3.

Suppose that $\deg(P) = d$. Let $\hat{\mathfrak{P}}_d$ be the set of all polynomials of degree d satisfying all hypotheses of that lemma. Further, let $\hat{\mathfrak{P}}_d(P)$ be the set of all polynomials in $\hat{\mathfrak{P}}_d$ that have the same length and Mahler's measure as P. Let $\mathfrak{A}_0$ be the set of the absolute values of the leading coefficients of all polynomials in $\hat{\mathfrak{P}}_d(P)$. By Lemma 3.3 for every polynomial in $\hat{\mathfrak{P}}_d(P)$ we can always find another one in this set with a smaller leading coefficient. Hence $\inf \mathfrak{A}_0 \notin \mathfrak{A}_0$. Since the length of each polynomial in $\hat{\mathfrak{P}}_d(P)$ equals $L(P)$, this set is bounded and $v(\hat{\mathfrak{P}}_d(P))$ is contained in a compact subset of $\mathbb{C}^{d+1}$. Therefore $\inf \mathfrak{A}_0$ is attained at some point Q of the closure $\overline{v(\hat{\mathfrak{P}}_d(P))}$. Clearly

$$Q \in \overline{v(\hat{\mathfrak{P}}_d(P))} \setminus v(\hat{\mathfrak{P}}_d(P)).$$

Let $p(Q)$ be the corresponding polynomial. Since length and Mahler's measure are continuous functions of the coefficients of a polynomial, we have $L(p(Q)) = L(P)$ and $M(p(Q)) = M(P)$. Consequently, $\lambda(p(Q)) = \lambda_d$. Also, by continuity, $p(Q)(1) = 0$. However, $p(Q) \notin \hat{\mathfrak{P}}_d(P)$, hence it must violate some of the properties of this set. Therefore, either $\deg p(Q) = d$ and $p(Q)$ has more than one zero on the unit circle or has no zeros outside or no zeros inside the unit circle or $P(0) = 0$, or else $\deg p(Q) < d$. Since $\lambda(p(Q)) = \lambda_d$ is minimal, the first possibility is ruled out by Claim 3.1.1. By (3.2), the next two possibilities of the case $\deg(p(Q)) = d$ give $\lambda(p(Q)) = \lambda_d = 2$. Finally, if $p(Q)(0) = 0$ then we can lower its degree by taking $p(Q)(z)/z$. Thus we are reduced to the case $\deg(p(Q)) < d$, and $\lambda_d = 2$ follows by induction hypothesis. ■

Proof of Corollary 2.2. Suppose that $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{T}^n$ is a zero of P. Then $\hat{P}(x) = P(x^{m_1} z_1, \ldots, x^{m_n} z_n)$ has a zero at $x = 1$. The conclusion follows immediately from Theorem 2.1 and [L], with an appropriate choice of varying exponents $m_1, \ldots, m_n$. ■

3.2. Proof of Lemma 3.3. Under the conditions of the lemma, P factors as

$$P = P_{\text{in}} P_0 P_{\text{out}},$$

where

- $P_0(z) = z - 1$,
- P_{in} is monic and has all zeros inside the unit circle, and $\deg(P_{\text{in}}) = n_1 \geq 1$,
- P_{out} has all zeros outside the unit circle and is not necessarily monic, and $\deg(P_{\text{out}}) = n_2 \geq 1$.

Further, we have

$$M(P) = M(P_{\text{out}}) = |P_{\text{out}}(0)|. \tag{3.3}$$

We emphasize that this equality also holds when P_{out} is not monic.

Let g be any polynomial of degree no greater than $n_2 - 1$. Consider $\hat{P}_{\text{out}}(z) = P_{\text{out}}(z) + \epsilon z g(z)$. For sufficiently small complex ϵ, $\deg \hat{P}_{\text{out}} = \deg P_{\text{out}}$, and since P_{out} does not vanish on the unit circle, we have

$$|P_{\text{out}}(z)| > |\epsilon z g(z)| \quad \text{for } |z| = 1.$$

By the Rouché theorem $\hat{P}_{\text{out}}$ has no zeros inside the unit circle, and since it has the same degree as P_{out}, all its zeros are outside the unit circle. Similarly, for any polynomial h of degree not greater than $n_1 - 1$, and any sufficiently small ϵ, $\hat{P}_{\text{in}}(z) = P_{\text{in}}(z) + \epsilon h(z)$ is monic, has all its zeros inside the unit circle, and does not vanish at 0. Let

$$Q(z) = (P_{\text{in}}(z) + \epsilon h(z))P_0(z)(P_{\text{out}}(z) + \epsilon z g(z)).$$

The zeros of Q outside the unit circle coincide with the zeros of $\hat{P}_{\text{out}}$. Hence $M(Q) = M(\hat{P}_{\text{out}})$. The definition of $\hat{P}_{\text{out}}$ guarantees that $\hat{P}_{\text{out}}(0) = P_{\text{out}}(0)$. Hence, by (3.3),

$$M(Q) = M(\hat{P}_{\text{out}}) = |\hat{P}_{\text{out}}(0)| = |P_{\text{out}}(0)| = M(P).$$

The construction of Q allows us to slightly modify P while preserving its Mahler's measure. We have

$$Q = P + \epsilon(zgP_{\text{in}} + hP_{\text{out}})P_0 + \epsilon^2 zghP_0. \tag{3.4}$$

We initially ignore the term of smaller magnitude, $\epsilon^2 zghP_0$, and examine what kind of modification of P can be obtained from the term

$$\epsilon(zgP_{\text{in}} + hP_{\text{out}})P_0. \tag{3.5}$$

CLAIM 3.2.1. *With suitable h and g,* (3.5) *is a nonzero polynomial with*

$$(zgP_{\text{in}} + hP_{\text{out}})P_0 = \sum_{j\in J} v_j z^{d-j}.$$

Note. Recall that $P(z) = \sum_{j\in J} a_j z^{d-j}$. Claim 3.2.1 asserts that we can construct the polynomial (3.5) in a way that modifies only the nonzero coefficients of P and does not create new terms.

Proof of Claim 3.2.1. Recall that $\deg(g) \le n_2 - 1$ and $\deg(h) \le n_1 - 1$. Let

$$zg(z) = \sum_{i=1}^{n_2} x_i z^{n_2-i+1} \quad \text{and} \quad h(z) = \sum_{i=1}^{n_1} x_{n_2+i} z^{n_1-i}, \tag{3.6}$$

$$P_{\text{out}}(z) = \sum_{i=0}^{n_2} c_i z^{n_2-i} \quad \text{and} \quad P_{\text{in}}(z) = \sum_{i=0}^{n_1} b_i z^{n_1-i}. \tag{3.7}$$

Then

$$v(zgP_{\rm in} + hP_{\rm out}) = MX, \tag{3.8}$$

where $X = [x_1, \ldots, x_{n_1+n_2}]^T$ and M is an $(n_1+n_2+1)\times(n_1+n_2) = d\times(d-1)$ matrix whose columns consist of 'shifted' vectors $v(P_{\rm in})$ and $v(P_{\rm out})$ and zeros. More precisely, the first n_2 columns are

$$[b_0\ \ldots\ b_{n_1}\ 0\ \ldots\ 0]^T, [0\ b_0\ \ldots\ b_{n_1}\ 0\ \ldots\ 0]^T, \ldots, [0\ \ldots\ 0\ b_0\ \ldots\ b_{n_1}\ 0]^T,$$

while the remaining n_1 are

$$[0\ c_0\ \ldots\ c_{n_2}\ 0\ \ldots\ 0]^T, [0\ 0\ c_0\ \ldots\ c_{n_2}\ 0\ \ldots\ 0]^T, [0\ \ldots\ 0\ c_0\ \ldots\ c_{n_2}]^T.$$

We notice that M^T is a submatrix of a $d \times d$ Sylvester matrix $\mathbb{S}_{zP_{\rm in},P_{\rm out}}$. Recall that $\det(\mathbb{S}_{zP_{\rm in},P_{\rm out}}) = \operatorname{Res}(zP_{\rm in}, P_{\rm out})$. Since $zP_{\rm in}$ and $P_{\rm out}$ have no common zero, $\operatorname{rank} \mathbb{S}_{zP_{\rm in},P_{\rm out}} = d$; consequently, $\operatorname{rank} M = d-1$.

The required conditions on the polynomial (3.5) mean that it is a nonzero polynomial with no nonzero terms of the form $v_j z^{d-j}$ for $j \notin J$. In order to verify that this can be achieved by suitable choice of h and g we let

$$\begin{aligned} V_0 &= \{(y_0, \ldots, y_d) \in \mathbb{C}^{d+1} : \forall_{j\notin J}\ y_j = 0\} \\ U_1 &= \{zgP_{\rm in} + hP_{\rm out} : g \in \mathbb{P}_{n_2-1},\ h \in \mathbb{P}_{n_1-1}\} \subset \mathbb{P}_{d-1}, \\ U_2 &= P_0U_1 = \{P_0Q : Q \in U_1\} \subset \mathbb{P}_d, \end{aligned}$$

where $\mathbb{P}_n$ denotes the space of all complex polynomials of degree at most n. Since $|J| = k$ we have $\dim V_0 = k$, and by (3.6),

$$\dim U_2 = \dim U_1 = \operatorname{rank} M = d-1.$$

Put $V_1 = v(U_1)$ and $V_2 = v(U_2)$. We have

$$\dim(V_2 \cap V_0) \ge d - 1 + k - (d+1) = k-2.$$

Put $V_m = V_2 \cap V_0$. Since $k \ge 3$, $\dim V_m \ge 1$. By (3.8) and (3.6) there are g and h such that $v((zgP_{\rm in}+hP_{\rm out})P_0) \in V_m$. Let $\mathbf{v}_m = v((zgP_{\rm in}+hP_{\rm out})P_0) = (v_0, \ldots, v_d)$. Then $v_j = 0$ for $j \notin J$, and

$$p(v((zgP_{\rm in} + hP_{\rm out})P_0)) = \sum_{j\in J} v_j z^{d-j},$$

which proves the claim. ∎

Now fix h and g satisfying Claim 3.2.1. By (3.4) we have

$$L(Q) = L\Big(\sum_{j\in J}(a_j + \epsilon v_j)z^{d-j}\Big) + \mathcal{O}(\epsilon^2). \tag{3.9}$$

Write $\epsilon = tu$, where t is a positive real number and $|u| = 1$. Suppose that $\frac{d}{dt}L(Q)|_{t=0} \neq 0$ for some u on the unit circle. Then by choosing an appropriate sufficiently small t we can make $L(Q) < L(P)$, while still having $M(Q) = M(P)$ and $\deg Q = \deg P$. Then $L(Q)/M(Q) < L(P)/M(P)$ and the first case of the conclusion of the lemma occurs. Otherwise, we have

$\frac{d}{dt}L(Q)|_{t=0} = 0$ for any u on the unit circle and all $\mathbf{v}_m \in V_m$. This is equivalent to

$$\sum_{j\in J} v_j \frac{\bar{a}_j}{|a_j|} = 0 \tag{3.10}$$

for every $\mathbf{v}_m \in V_m$.

CLAIM 3.2.2. *If* (3.10) *holds for a given* P, *then there are distinct nonzero indices* $i, j \in J$ *and unique polynomials* h *and* g *such that* $\mathbf{v}_m = v((zgP_{\text{in}} + hP_{\text{out}})P_0) \in V_m$ *has all components other than* v_0, v_i, v_j *equal to* 0. *Further,* $v_0 = -a_0$, *where* a_0 *is the leading coefficient of* P.

Proof of Claim 3.2.2. Let $\bar{\varepsilon}_j = \bar{a}_j/|a_j|$, or equivalently $a_j = \varepsilon_j|a_j|$, for $j \in J$. Define a vector $\boldsymbol{\varepsilon} \in \mathbb{C}^{d+1}$ by letting its jth component be ε_j if $j \in J$, and 0 otherwise. Similarly let $\mathbf{1}$ have components 1 for $j \in J$ and zero outside J. Then (3.10) and the definition of V_m imply that V_m is orthogonal to both $\boldsymbol{\varepsilon}$ and $\mathbf{1}$. Further, the vectors $\boldsymbol{\varepsilon}$ and $\mathbf{1}$ are linearly independent, since by definition of $\boldsymbol{\varepsilon}$ and the fact that $P(1) = 0$, we have $\langle v(P), \boldsymbol{\varepsilon}\rangle = L(P) \neq 0$, while $\langle v(P), \mathbf{1}\rangle = 0$, where the inner product is the usual hermitian product on $\mathbb{C}^{d+1}$. Recall that $\dim V_m \geq k-2$, so in fact we must have $\dim V_m = k-2$, and

$$V_0 = V_m \oplus^{\perp} \operatorname{span}\{\mathbf{1}, \boldsymbol{\varepsilon}\}.$$

Since $\mathbf{1}$ and $\boldsymbol{\varepsilon}$ are linearly independent, there is a pair of indices $(i, j) \in J\times J$ for which the vectors $(\varepsilon_i, \varepsilon_j)$ and $(1, 1)$ are linearly independent. Further, there is such a pair for which neither i or j is 0. Indeed, otherwise we would have $\varepsilon_i = \varepsilon_j$ for all nonzero i and j. Hence $P(1) = \varepsilon_0|a_0| + \varepsilon_i \sum_{j\in J,\, j\neq 0} |a_j| = 0$, so $|a_0| = \sum_{j\in J,\, j\neq 0} |a_j|$. The last equality however implies that P has no zeros outside the unit circle, contrary to our assumption. Therefore we can fix a pair (i, j) such that $(\varepsilon_i, \varepsilon_j)$ and $(1, 1)$ are linearly independent, and $i, j \neq 0$. The system

$$\bar{\varepsilon}_i v_i + \bar{\varepsilon}_j v_j = \bar{\varepsilon}_0 a_0 v_i + v_j = a_0$$

has a unique solution (v_i, v_j). Define a vector $\mathbf{v}_m = (v_0, \ldots, v_d)$ by letting $v_0 = -a_0$, v_i and v_j be the solutions of the above system, and all other components be 0. Thus $\mathbf{v}_m \in V_m$ and this vector is uniquely determined by P for the fixed pair (i, j). Hence, there are polynomials g and h as in (3.5) such that $v((zgP_{\text{in}} + hP_{\text{out}})P_0) = \mathbf{v}_m$. By (3.8) we have

$$p(\mathbf{v}_m) = (zgP_{\text{in}} + hP_{\text{out}})P_0 = p(MX)P_0. \tag{3.11}$$

The polynomials g and h are in one-to-one correspondence with the vector X through formula (3.6). Since a vector X satisfying (3.11) exists, we have an explicit formula

$$X = (M^T M)^{-1} v(p(\mathbf{v}_m)/P_0). \tag{3.12}$$

Note that the matrix $M^T M$ has size $(d-1)\times(d-1)$ and is invertible because $\operatorname{rank} M = d-1$. Thus we have a uniquely determined sequence of mappings

$$P \mapsto \boldsymbol{\varepsilon} \mapsto \mathbf{v}_m \mapsto X \mapsto (h,g). \ \blacksquare$$

Formula (3.4) and the particular form of $\mathbf{v}_m$ with sufficiently small real positive ϵ allow us to decrease the magnitude of the leading coefficient of P while preserving its Mahler's measure. Unfortunately, in the process, the term $\epsilon^2 zghP_0$ can create new nonzero coefficients and slightly increase the length of P. Fortunately, we shall see that this can be avoided if we change dynamically P and $\mathbf{v}_m$ together in an appropriate way. We can achieve this by creating a special system of differential equations whose solution generates a trajectory of polynomials $P_{[t]}$ satisfying the conclusion of the lemma.

For this consider the coefficients c_i and b_i in (3.7) as functions of an independent real variable t, except for $b_0 = 1$ and c_{n_2} which we will keep constant. Form the vector function $Y(t) = [c_0(t) \ \ldots \ c_{n_2-1}(t)\ b_1(t) \ \ldots \ b_{n_1}(t)]^T$ and consider the initial value problem

$$(3.13)\qquad Y' = (M^T M)^{-1} v(p(\mathbf{v}_m)/P_0)), \quad Y(0) = [c_0 \ \ldots \ c_{n_2-1}\ b_1 \ \ldots \ b_{n_1}]^T.$$

The vector $Y(t)$ determines the polynomials

$$P_{\mathrm{out}[t]}(z) = c_{n_2} + \sum_{i=0}^{n_2-1} c_i(t) z^{n_2-i} \quad \text{and} \quad P_{\mathrm{in}[t]}(z) = z^{n_1} + \sum_{i=1}^{n_1} b_i(t) z^{n_1-i}$$

such that for $P_{[t]} = P_{\mathrm{out}[t]} P_{\mathrm{in}[t]} P_0$ we have $P_{[0]} = P$. The matrix $M = M(Y)$ is determined by Y via $P_{\mathrm{out}[t]}$ and $P_{\mathrm{in}[t]}$ in the same way as the matrix M described by the formulas following (3.8). Similarly, $\mathbf{v}_m = \mathbf{v}_m(t)$ is determined by $P_{[t]}$ in the same way as $\mathbf{v}_m$ by P. Thus we have a sequence of mappings

$$Y(t) \mapsto P_{[t]} \mapsto \boldsymbol{\varepsilon}(Y(t)) \mapsto \mathbf{v}_m(Y(t)),$$

and also

$$Y \mapsto M(Y).$$

Let $Y = \Re(Y) + i\Im(Y)$. The system (3.13) corresponds to a pair of systems in real variables

$$\Re(Y') = \Re((M^T M)^{-1} v(p(\mathbf{v}_m/P_0))), \quad \Im(Y') = \Im((M^T M)^{-1} v(p(\mathbf{v}_m/P_0))).$$

By examining the mappings listed above we conclude that the right-hand side functions of these systems are rational functions of the components of $\Re(Y)$ and of $\Im(Y)$, and of the absolute values of the coefficients of $P_{[t]}$, which in turn are polynomial functions of the components of Y. The coefficients of $P_{[0]} = P$ correspond to $Y(0)$ and are not zero. Therefore the coefficients of $P_{[t]}$ corresponding to Y are not zero for Y in some open ball containing $Y(0)$. Thus on a sufficiently small open ball containing $(\Re(Y_0), \Im(Y_0))$ the right-hand sides of the systems are continuously differentiable functions of the vector $(\Re(Y), \Im(Y))$. Consequently, the initial value problem (3.13) has a unique solution in some open interval I containing 0. Further, by (3.6), (3.12), and the definition of $Y(t)$ we have

$$\frac{d}{dt}P_{\mathrm{out}[t]}\Big|_{t=0} = zg(z) \quad \text{and} \quad \frac{d}{dt}P_{\mathrm{in}[t]}\Big|_{t=0} = h(z).$$

Thus

$$\frac{d}{dt}P_{[t]} = \Big(P_{\mathrm{in}[t]}\frac{d}{dt}P_{\mathrm{out}[t]} + P_{\mathrm{out}[t]}\frac{d}{dt}P_{\mathrm{in}[t]}\Big)P_0.$$

Hence

$$\frac{d}{dt}P_{[t]}\Big|_{t=0} = (P_{\mathrm{in}}zg(z) + P_{\mathrm{out}}h(z))P_0.$$

Notice that this modification of P corresponds to (3.4), but we have managed to eliminate the remainder term with ϵ^2. Further, the vectors $\mathbf{v}_m(t)$ and $\boldsymbol{\varepsilon}(t)$ determined by $P_{[t]}$ are orthogonal for $t \in I$. Thus condition (3.10) is satisfied and $\frac{d}{dt}L(P_{[t]}) = 0$. Hence

$$L(P_{[t]}) = L(P), \qquad M(P_{[t]}) = M(P), \qquad a_j(t) \neq 0 \text{ for } j \in J$$

for sufficiently small t. Moreover $v(\frac{d}{dt}P_{[t]}) = \mathbf{v}_m(t)$ and the first component of $\mathbf{v}_m(t)$ is $v_0 = -a_0(t)$, where $a_0(t)$ is the leading coefficient of $P_{[t]}$. Hence $\frac{d}{dt}a_0(t) = -a_0(t)$, so that $|a_0(t)| = |a_0|e^{-t}$ is decreasing. ∎

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