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Polynomial parametrizations of length 4 Büchi sequences

by

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1. Introduction. Since the projective surface with affine equations

$$x_4^2 - 2x_3^2 + x_2^2 = x_3^2 - 2x_2^2 + x_1^2 = 2 \tag{1.1}$$

is a Segre surface (a del Pezzo surface of degree 4), its $\mathbb{Q}$-rational points can be parametrized. O. Wittenberg showed us the following rational parametrization:

$$x_1(s,t) = \frac{2s^2t^2 - 2s^2t - 5st^2 + 8st - 6s + 2t^2 - 2t}{s^2t^2 - 3st^2 + 2st + 2s + 2t^2 - 4t + 2},$$

$$x_2(s,t) = \frac{s^2t^2 - 2s^2t - 2st^2 + 8st - 4s - 2t + 2}{s^2t^2 - 3st^2 + 2st + 2s + 2t^2 - 4t + 2},$$

$$x_3(s,t) = \frac{2s^2t + st^2 - 4st + 2s - 2t^2 + 6t - 4}{s^2t^2 - 3st^2 + 2st + 2s + 2t^2 - 4t + 2},$$

$$x_4(s,t) = \frac{s^2t^2 + 2s^2t - 4st^2 + 4st + 4t^2 - 10t + 6}{s^2t^2 - 3st^2 + 2st + 2s + 2t^2 - 4t + 2},$$

which gives a birational equivalence of the surface with $\mathbb{P}^2$ through

$$s = -\frac{x_2 + 2x_3 + x_4}{x_1 - 2x_2 - x_3} \quad \text{and} \quad t = -\frac{x_1 - 2x_2 - x_3}{1 - x_1 + x_2}.$$

In this work we are interested in characterizing the set of *integer* points on the affine surface with equations (1.1).

Let A be a commutative ring with unit and of characteristic 0. A sequence $(x_1, \dots, x_\ell)$ of elements of A is called a *Büchi sequence over* A if its second difference is the constant sequence (2): for each $i \in \{1, \dots, \ell - 2\}$ it satisfies

$$x_i^2 - 2x_{i+1}^2 + x_{i+2}^2 = 2.$$

A *trivial Büchi sequence* will be any sequence satisfying: there exists $x \in A$ such that $x_i^2 = (x + i)^2$ for all $i = 1, \dots, \ell$. *Büchi's problem* over A asks

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whether there exists an integer M such that no non-trivial Büchi sequence of length at least M exists. If such an M exists, let us write $M(A)$ for the smallest one, and $M_{\mathrm{f}}(A)$ for the least M such that there are only finitely many non-trivial Büchi sequences of length M. Hence, if one proves that $M_{\mathrm{f}}(A)$ exists, then one obtains automatically a positive answer to Büchi's problem *for some* $M \geq M_{\mathrm{f}}(A)$.

Let X_4 be the (affine) variety defined by (1.1) (*Büchi equations*). Having Büchi's problem in mind, we would like to characterize the set of integer points on X_4 (actually a cofinite subset of the set of integer points would be enough). There exists extensive literature about rational surfaces, but there seem to be few results about polynomial parametrizations over $\mathbb{Z}$.

Büchi got interested in this problem (for $A = \mathbb{Z}$) when he realized that from a positive answer to it he would be able to prove that there is no algorithm to decide whether or not an arbitrary system of quadratic diagonal forms over $\mathbb{Z}$ can represent an arbitrary given vector of integers (which, if true, would be one of the strongest forms of the negative answer to Hilbert's tenth problem—see [Mat] or [D], and [L]).

Büchi's problem remains open for the integers, but P. Vojta [Vo] showed that $M_{\mathrm{f}}(\mathbb{Q})$ would be 8 (actually H. Pasten noticed that the proof goes through for any number field) if the Bombieri conjecture were true for surfaces. It is striking that even though we cannot prove that Büchi's problem has a positive answer, no non-trivial Büchi sequence of length even just 5 over $\mathbb{Z}$ is known to exist. Indeed Büchi conjectured that $M(\mathbb{Z}) = 5$. See [PPV] and [BB] for a survey of results related to Büchi's problem, and Allison [A] and Bremner [B] for the analogous problem where the constant 2 is changed to another constant.

Büchi sequences of length 3 are not difficult to characterize over $\mathbb{Q}$, and with some divisibility conditions one obtains a complete characterization of sequences over $\mathbb{Z}$—the non-trivial ones are infinitely many—see [H, Theorem 2.1] or [PPV, Section 7]. We also know a characterization over $\mathbb{Z}$ that does not require any divisibility condition (i.e. without any reference to $\mathbb{Q}$)—see [SaV]. Obtaining a "good" characterization for (a cofinite subset of the set of all) length 4 sequences of integers could be a key step in solving Büchi's problem: proving that no sequences of length 4 (but finitely many) can be extended to length 5 could then be quite easier, and would prove that $M_{\mathrm{f}}(\mathbb{Z}) = 5$.

This work presents an effort to characterize all but finitely many Büchi sequences of length 4 over the integers. The idea comes from an unpublished paper by D. Hensley [H] from the early eighties, where a polynomial parametrization of degree 3 for length 4 integer sequences is described, and from a paper by R. G. E. Pinch [Pi] of 1993 where he lists (finitely) many length 4 non-trivial Büchi sequences and shows that none of them can be extended to a length 5 sequence.

In Section 2 we give an explicit birational map ζ on X_4, of infinite order, and we show in Section 3 that it preserves integrality on infinitely many integer points. In order to state our main theorem, let us introduce some notation.

Notation 1.1. Write

$$f(t) = 2t^2 + 10t + 10$$

and for all $n \in \mathbb{Z}$,

$$\xi(n,t) = (\xi_1(n,t), \xi_2(n,t), \xi_3(n,t), \xi_4(n,t)),$$

where ξ is defined by induction on n by

$$\xi(n+2,t) = f(t)\xi(n+1,t) - \xi(n,t), \tag{1.2}$$

with initial values

$$\xi(-1,t) = (t+4, -t-3, -t-2, t+1) \quad \text{and} \quad \xi(0,t) = (t+1, t+2, t+3, t+4).$$

Theorem 1.2. *For each $n, t \in \mathbb{Z}$, we have*

1. $\xi(n,t) = (\xi_4(-n-1,t), -\xi_3(-n-1,t), -\xi_2(-n-1,t), \xi_1(-n-1,t))$;
2. $\xi(n,t) = (-\xi_4(n,-t-5), -\xi_3(n,-t-5), -\xi_2(n,-t-5), -\xi_1(n,-t-5))$;
3. *$\xi(n,t)$ is a 4-tuple of polynomials of degree $|2n+1|$ in the variable t, and with leading coefficient $\pm 2^n$ if $n \geq 0$ and $\pm 2^{-n-1}$ if $n \leq -1$;*
4. *the sequence $\xi(n,t)$ is a Büchi sequence;*
5. *the sequence $\xi(n,t)$ is a trivial Büchi sequence if and only if $n \in \{-1, 0\}$ or $t \in \{-4, -3, -2, -1\}$; and*
6. *we have*

$$\zeta^{(n)}(t+1, t+2, t+3, t+4) = \xi(n,t)$$

where $\zeta^{(n)}$ stands for the nth iterate of ζ when n is non-negative and the $(-n)$th iterate of ζ^{-1} when n is negative.

Consequently, there are infinitely many non-trivial parametrizations of Büchi sequences of length 4 over the integers. From items 1 and 2 of Theorem 1.2 we will deduce in Section 5 that both sequences $(\xi_i(n,t))_i$ (with $n, t \geq 0$ fixed) and $(\xi_i(n,t))_n$ (with i fixed and $t \geq 0$ fixed) are strictly increasing sequences of natural numbers.

The parametrization

$$\xi(1,t) = (2t^3+12t^2+19t+6, 2t^3+14t^2+31t+23, 2t^3+16t^2+41t+32, 2t^3+18t^2+49t+39)$$

is actually the one already appearing in Hensley's paper [H], and as far as we know, no other non-trivial polynomial parametrization has been known to exist.

In Section 4, we present two more polynomial parametrizations over $\mathbb{Z}$, of degree 4, and one polynomial parametrization over $\mathbb{Q}$, also of degree 4.

Computationally, it seems that there are no other polynomial parametrizations over the integers than the ones we have found, but we have not been able to prove it.

Nor can we prove the following, which seems to be true computationally: none of these parametrizations can represent an integer solution that extends to a length 5 Büchi sequence. Indeed, consider for example the sequence $\xi(1,t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ given above. Asking whether this sequence extends, for some fixed integer t, to a length 5 sequence is asking whether either $2x_4^2 - x_3^2 + 2$ (extension to the right) or $2x_1^2 - x_2^2 + 2$ (extension to the left) is a square. Namely: does one of the two curves

$$y^2 = 4t^6 + 80t^5 + 620t^4 + 2400t^3 + 4905t^2 + 5020t + 2020$$

or

$$y^2 = 4t^6 + 40t^5 + 120t^4 - 595t^2 - 970t - 455$$

have an integer point with $t \neq -4, -3, -2, -1$ (for $t \in \{-4, -3, -2, -1\}$ we obtain trivial Büchi sequences by item 5 of Theorem 1.2)? In Section 6, we will prove that it is enough for our purposes to work with extensions to the right. We will show that the polynomial $y_n(t) = 2\xi_4(n,t)^2 - \xi_3(n,t)^2 + 2$ satisfies a third order homogeneous linear recurrence. Indeed, the quantity

$$y_{n+2} - (f^2 - 2)y_{n+1} + y_n$$

does not depend on n. From this relation we can deduce that $y_n(t)$ cannot be a square when, for example, t is congruent to 0 modulo 5 and n is not congruent to 0 or -1 modulo 10 (see Lemmas 6.4 and 6.5). On the other hand, J. Browkin showed us a way to prove that the sequences $\xi(n,t)$ do not extend to a 5-term sequence, but unfortunately this needs a quantity of computations that increases together with the absolute value of n. Applying this method, we could verify that $\xi(n,t)$ is never a square for $0 \leq n \leq 6$ and any $t \neq -4, -3, -2, -1$ (we do not present this method in this paper).

In Section 7, we list all integer solutions that we found and that we have not been able to *parametrize* (i.e., they seem not to belong to the image of a polynomial parametrization over $\mathbb{Z}$). With the first term at most 1052749, they are 121 (counting only the strictly increasing sequences of positive integers) and we do not know whether or not we are missing finitely many. From the figure at the end of that section, it *seems* clear that the number of points that we are "missing" is decreasing exponentially with respect to the size of the points. None of these (non-parametrized) points can extend to a length 5 solution, as is easily verified with a computer software.

The symbol † in the text will mean that we are using a computer software for the formal computation (all the computations can actually be done by hand, but some are a bit tedious). We have used exclusively the open source software Xcas 0.8.6 and 0.9.0 for all our computations; see Giac/Xcas, Bernard Parisse et Renée De Graeve, version 0.8.6 (2010), http://www-fourier.ujf-grenoble.fr/~parisse/giac_fr.html.

2. Some birational maps on X_4

NOTATION 2.1.

1. Denote by $\mathrm{Bir}(X_4)$ the group of birational maps on X_4.
2. Let τ and μ_i, $i = 1, 2, 3, 4$, denote the following automorphisms of X_4:

$$\mu_1(a,b,c,d) = (-a,b,c,d), \quad \mu_2(a,b,c,d) = (a,-b,c,d),$$
$$\mu_3(a,b,c,d) = (a,b,-c,d), \quad \mu_4(a,b,c,d) = (a,b,c,-d),$$

   and

$$\tau(a,b,c,d) = (d,c,b,a).$$

   Observe that each μ_i is an odd function.
3. We will call any map from the subgroup Γ_1 of $\mathrm{Bir}(X_4)$ generated by the set $\{\mu_1, \mu_2, \mu_3, \mu_4, \tau\}$ a *trivial involution on* X_4.
4. Write Γ_0 for the group generated by $\{\mu_1, \mu_2, \mu_3, \mu_4\}$.
5. Write $\mu_{ij} = \mu_i\mu_j$ and $\mu_{ijk} = \mu_i\mu_j\mu_k$ for any $i, j, k \in \{1, 2, 3, 4\}$.

REMARK 2.2.

1. For all $i \neq j$ we have $\mu_i\mu_j = \mu_j\mu_i$, hence Γ_0 is isomorphic to $(\mathbb{Z}_2)^4$.
2. We have $\tau\mu_1 = \mu_4\tau$ and $\tau\mu_2 = \mu_3\tau$.
3. We have $\tau\mu_{14} = \mu_{14}\tau$ and $\tau\mu_{23} = \mu_{23}\tau$.
4. For each i, $\tau\mu_i$ has order 4.

LEMMA 2.3. *For all i, we have $\tau\mu_i\tau = \mu_{\sigma(i)}$, where σ stands for the permutation $(1\ 4)(2\ 3) \in S_4$. Hence the group Γ_0 is normal in Γ_1 and the group Γ_1 is a semi-direct product $\Gamma_0 \rtimes \langle\tau\rangle$.*

Proof. This is clear from the above remark. ∎

Next we define a rational map φ on X_4 that will turn out to be an involution, and the map ζ that will allow us to generate all our polynomial parametrizations.

NOTATION 2.4. We will consider the map φ from (a subset of) $\mathbb{Q}^4$ to $\mathbb{Q}^4$ defined by

$$(\varphi_1, \varphi_2, \varphi_3, \varphi_4) = \left(\frac{p_1}{q}, \frac{p_2}{q}, \frac{p_3}{q}, \frac{p_4}{q}\right)$$

where

$$q(a,b,c,d) = (b-c)^2(a-2b+c),$$

$$\begin{aligned}
p_1(a,b,c,d) = {} & -2ab^3+ab^2c+2ab^2d+4abc^2-5abcd+ab-2ac^3\\
& +2ac^2d+ac-ad+3b^4-2b^3c-3b^3d-6b^2c^2\\
& +8b^2cd+b^2+4bc^3-4bc^2d-5bc+bd+c^2+cd-2,\\
p_2(a,b,c,d) = {} & -2ab^2c+5abc^2-2abcd+2ab-2ac^3+ac^2d-ad+3b^3c\\
& -8b^2c^2+3b^2cd-2b^2+4bc^3-2bc^2d-bc+2bd-2,\\
p_3(a,b,c,d) = {} & -2ab^3+5ab^2c-2ab^2d-2abc^2+abcd\\
& +3ab-ac-ad+3b^4-8b^3c+3b^3d+4b^2c^2-2b^2cd\\
& -3b^2-bc+3bd+c^2-cd-2,\\
p_4(a,b,c,d) = {} & -3ab^3+8ab^2c-3ab^2d-4abc^2+2abcd+4ab-2ac-ad\\
& +4b^4-10b^3c+4b^3d+2b^2c^2-2b^2cd-4b^2\\
& +5bc^3-2bc^2d-bc+4bd-2c^4+c^3d+2c^2-2cd-2.
\end{aligned}$$

Observe that φ is an odd function (as q is odd and each p_i is even); by *odd* we mean that for all $(a,b,c,d)\in X_4$ we have

$$\begin{aligned}
&\varphi(-a,-b,-c,-d)\\
&\qquad = (-\varphi_1(a,b,c,d),-\varphi_2(a,b,c,d),-\varphi_3(a,b,c,d),-\varphi_4(a,b,c,d)).
\end{aligned}$$

NOTATION 2.5. Write $\zeta=\varphi\tau\mu_{14}$.

LEMMA 2.6. *The map φ is a rational map on X_4.*

Proof. This is a simple but tedious computation (note that one needs to replace formally (†) a^2 by $2b^2-c^2+2$ and d^2 by $2c^2-b^2+2$ in the expressions $(\varphi_1^2-2\varphi_2^2+\varphi_3^2)(a,b,c,d)$ and $(\varphi_2^2-2\varphi_3^2+\varphi_4^2)(a,b,c,d)$). The details are left to the reader. ∎

LEMMA 2.7. *The map φ is an involution.*

Proof. For each i, after substituting formally (†) x_4^2 by $2x_3^2-x_2^2+2$ and x_3^2 by $2x_2^2-x_1^2+2$ in $\varphi_i(\varphi(x_1,x_2,x_3,x_4))$ and doing the obvious simplifications (†), one obtains x_i. Note that it is not hard to prove this lemma without the help of a computer, by using the fact (verifiable by hand) that

$$\begin{aligned}
(\varphi_1-2\varphi_2+\varphi_3)(a,b,c,d) &= a-2b+c,\\
(\varphi_2-2\varphi_3+\varphi_4)(a,b,c,d) &= b-2c+d. \ ∎
\end{aligned}$$

REMARK 2.8. Observe that since φ is birational, so is $\zeta=\varphi\tau\mu_{14}$.

LEMMA 2.9. *We have $\tau\varphi=\varphi\tau$ and $\tau\zeta=\zeta\tau$.*

Proof. Verifying that $\tau\varphi-\varphi\tau=0$ needs replacing a^2 by $2b^2-c^2+2$ everywhere it occurs in the expression (†). Recalling the definition of $\zeta=$

$\varphi\tau\mu_{14}$, we have

$$\tau\zeta\tau = \tau(\varphi\tau\mu_{14})\tau = \varphi\mu_{14}\tau = \varphi\tau\mu_{14} = \zeta. \ \blacksquare$$

Unfortunately, we do not know the presentation of the group generated by Γ_1 and φ. We will prove later on that the map ζ has infinite order (see for example Corollary 5.3).

3. Büchi sequences of length 4 over $\mathbb{Z}[t]$. First we prove items 3, 4 and 6 of Theorem 1.2. Item 3 comes immediately from the inductive definition of ξ, by induction on n (to the left and to the right). Item 4 is easily verified (†) if one writes each $\xi_i(n,t)$ in the form

$$\frac{(g_i(n,t) + \alpha(t+i))\beta^n - (g_i(n,t) - \alpha(t+i))\bar{\beta}^n}{2\alpha}$$

where $\alpha = \sqrt{(t+1)(t+2)(t+3)(t+4)}$ and $\beta = t^2+5t+5+\alpha$ and $\bar{\beta} = t^2+5t+5-\alpha$.

We prove item 6 by induction on n. For $n=0$ it is trivial by the definition of $\xi(0,t)$. Suppose it is true up to $n \neq 0$ (negative or positive). One verifies (†) that

$$\xi(n+1,t) = \zeta(\xi(n,t))$$

for each $n \in \mathbb{Z}$, hence

$$\xi(n+1,t) = \zeta(\zeta^{(n)}(t+1,t+2,t+3,t+4)) = \zeta^{(n+1)}(t+1,t+2,t+3,t+4),$$

and since $\xi(n-1,t) = \zeta^{(-1)}(\xi(n,t))$ we also have

$$\xi(n-1,t) = \zeta^{(-1)}(\zeta^{(n)}(t+1,t+2,t+3,t+4)) = \zeta^{(n-1)}(t+1,t+2,t+3,t+4),$$

which finishes the induction.

4. Other polynomial parametrizations. Note that by replacing t by t^2 in a polynomial parametrization of degree n, we obtain a polynomial parametrization of degree $2n$. Since there exist non-trivial polynomial parametrizations of any odd degree, there are non-trivial polynomial parametrizations of any degree.

The functions

$$\psi_1(t) = \frac{t^4 + 17t^3 + 104t^2 + 262t + 204}{4},$$
$$\psi_2(t) = \frac{t^4 + 19t^3 + 138t^2 + 458t + 592}{4},$$
$$\psi_3(t) = \frac{t^4 + 21t^3 + 168t^2 + 602t + 812}{4},$$
$$\psi_4(t) = \frac{t^4 + 23t^3 + 194t^2 + 718t + 984}{4}$$

give a polynomial parametrization $\psi = (\psi_1, \psi_2, \psi_3, \psi_4)$ over $\mathbb{Q}$ that takes an integer value for each integer t not congruent to 3 modulo 4. Hence $\psi(2t)$ and $\psi(4t+1)$ are polynomial parametrizations over $\mathbb{Z}$, of degree 4, which are *new* ([1]) in the sense that they generate Büchi sequences of integers that were not in the image of any of the $\xi(n,t)$.

The following is another polynomial parametrization over $\mathbb{Q}$, but it does not yield any integer solution:

$$\begin{aligned} x_1(t) &= \frac{1}{3}(4t^4 + 18t^3 + 14t^2 - 15t - 8),\\ x_2(t) &= \frac{1}{3}(4t^4 + 22t^3 + 36t^2 + 19t + 5),\\ x_3(t) &= \frac{1}{3}(4t^4 + 26t^3 + 54t^2 + 35t + 2),\\ x_4(t) &= \frac{1}{3}(4t^4 + 30t^3 + 68t^2 + 45t + 1). \end{aligned}$$

5. Some basic properties of the sequences $(\xi(n,t))_n$ and $(\xi_i)_i$. In this section we prove items 1, 2 and 5 of Theorem 1.2 and show that for fix $t \ge 0$, the sequences $(\xi_i(n,t))_i$ (with $n \ge 0$ fixed) and $(\xi_i(n,t))_n$ (with i fixed) are strictly increasing sequences of positive integers.

A straightforward computation shows that for all $n, t \in \mathbb{Z}$ we have

$$\xi_4(n,t) = -\xi_1(n,-t-5) \quad \text{and} \quad \xi_3(n,t) = -\xi_2(n,-t-5), \tag{5.1}$$

which easily implies the other two equalities of item 2 of Theorem 1.2.

Let us prove by induction on n that

$$\xi_1(n,t) = -\xi_1(-n-1,-t-5). \tag{5.2}$$

This is clear for $n=-1$ and for $n=0$ since $\xi_1(0,t) = t+1$ and $\xi_1(-1,t) = t+4$. Suppose that it is proven up to $n+1$ (the case with decreasing n is done similarly). Since $f(t) = f(-t-5)$, we have from (1.2)

$$\begin{aligned} \xi_1(n+2,t) &= f(t)\xi_1(n+1,t) - \xi_1(n,t)\\ &= -f(-t-5)\xi_1(-(n+1)-1,-t-5) + \xi_1(-n-1,-t-5)\\ &= -(f(-t-5)(\xi_1(-n-2,-t-5) - \xi_1(-n-1,-t-5))\\ &= -\xi_1(-n-3,-t-5), \end{aligned}$$

which was to be proved.

We conclude from (5.2) and (5.1) that $\xi_1(n,t) = \xi_4(-n-1,t)$, and replacing n by $-n-1$ in the latter equation we obtain $\xi_4(n,t) = \xi_1(-n-1,t)$.

([1]) I only know a tedious proof of this fact, using Hensley's parametrization of length 3 sequences—see [PPV, Section 7]. I do not include it as it is not really relevant to this work. Details are available upon request.

This proves two of the four equalities of item 1 of Theorem 1.2. The other two are obtained similarly.

From (5.1), we see that since

$$\xi(1,-2) = (0,1,-2,-3) \quad \text{and} \quad \xi(1,-1) = (-3,4,5,6)$$

are trivial sequences, also $\xi(1,-3)$ and $\xi(1,-4)$ are trivial sequences.

LEMMA 5.1. *For each $n \in \mathbb{Z}$, the Büchi sequences $\xi(n,-4)$, $\xi(n,-3)$, $\xi(n,-2)$ and $\xi(n,-1)$ are trivial sequences.*

Proof. By the definition of ξ, we have for $t = -1$

$$\xi_i(n+2,-1) = 2\xi_i(n+1,-1) - \xi_i(n,-1)$$

for each $i = 1,2,3,4$, with initial values for $n = 0,1$ (recalling that $\xi(0,t) = (t+1,t+2,t+3,t+4)$):

$$\begin{aligned} \xi_1(0,-1) &= 0, & \xi_1(1,-1) &= -3, \\ \xi_2(0,-1) &= 1, & \xi_2(1,-1) &= 4, \\ \xi_3(0,-1) &= 2, & \xi_3(1,-1) &= 5, \\ \xi_4(0,-1) &= 3, & \xi_4(1,-1) &= 6; \end{aligned}$$

and for $t = -2$,

$$\xi_i(n+2,-2) = -2\xi_i(n+1,-2) - \xi_i(n,-2)$$

for each $i = 1,2,3,4$, with initial values for $n = 0,1$

$$\begin{aligned} \xi_1(0,-2) &= -1, & \xi_1(1,-2) &= 0, \\ \xi_2(0,-2) &= 0, & \xi_2(1,-2) &= 1, \\ \xi_3(0,-2) &= 1, & \xi_3(1,-2) &= -2, \\ \xi_4(0,-2) &= 2, & \xi_4(1,-2) &= -3. \end{aligned}$$

Solving the eight recurrence relations above, we obtain

$$\begin{aligned} \xi(n,-1) &= (-3n, 3n+1, 3n+2, 3n+3), \\ \xi(n,-2) &= (-1)^n(n-1, -n, n+1, n+2), \end{aligned}$$

which are clearly trivial sequences. From (5.1), we have

$$\begin{aligned} \xi_1(n,-3) &= -\xi_4(n,-2), & \xi_2(n,-3) &= -\xi_3(n,-2), \\ \xi_4(n,-3) &= -\xi_1(n,-2), & \xi_3(n,-3) &= -\xi_2(n,-2), \end{aligned}$$

hence

$$\xi(n,-3) = (-1)^n(-n-2, -n-1, n, -n+1),$$

and

$$\begin{aligned} \xi_1(n,-4) &= -\xi_4(n,-1), & \xi_2(n,-4) &= -\xi_3(n,-1), \\ \xi_4(n,-4) &= -\xi_1(n,-1), & \xi_3(n,-4) &= -\xi_2(n,-1), \end{aligned}$$

hence

$$\xi(n,-4) = (-3n-3, -3n-2, -3n-1, 3n),$$

which are also clearly trivial sequences. ∎

Lemma 5.2. *If (u_n) is a sequence of integers satisfying $u_1 > u_0 > 0$ and $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \geq 0$, with $\alpha \geq 2$, then $u_{n+1} > (\alpha-1)u_n > 0$ for all $n \geq 1$.*

Proof. We have

$$u_2 = \alpha u_1 - u_0 = (\alpha-1)u_1 + u_1 - u_0 > (\alpha-1)u_1 > 0.$$

Suppose that $u_{n+1} > (\alpha-1)u_n > 0$ for some $n \geq 1$. We have

$$u_{n+2} = \alpha u_{n+1} - u_n > \alpha u_{n+1} - \frac{u_{n+1}}{\alpha-1} \geq (\alpha-1)u_{n+1}.\ ∎$$

Corollary 5.3. *For each $i = 1, \ldots, 4$, we have*

$$\xi_i(n+1,t) > (2t^2+10t+9)\xi_i(n,t)$$

for each $t \geq 0$ and $n \geq 1$.

Proof. Fix $t \geq 0$. We apply Lemma 5.2 to the sequence $u_n = \xi_i(n,t)$ for each $i = 1, \ldots, 4$. By the definition of ξ, the u_n satisfy the recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$, with

$$\alpha = f(t) = 2t^2 + 10t + 10 \geq 2$$

and

$$u_1 = \begin{cases} \xi_1(1,t) = 2t^3+12t^2+19t+6 > t+1 = \xi_1(0,t) = u_0 > 0 & \text{if } i=1, \\ \xi_2(1,t) = 2t^3+14t^2+31t+23 > t+2 = \xi_2(0,t) = u_0 > 0 & \text{if } i=2, \\ \xi_3(1,t) = 2t^3+16t^2+41t+32 > t+3 = \xi_3(0,t) = u_0 > 0 & \text{if } i=3, \\ \xi_4(1,t) = 2t^3+18t^2+49t+39 > t+4 = \xi_4(0,t) = u_0 > 0 & \text{if } i=4, \end{cases}$$

so in each case we can apply Lemma 5.2. ∎

Lemma 5.4. *If (v_n) and (w_n) are sequences of integers both satisfying the same recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \geq 0$, with $\alpha \geq 2$, and $u_1 > u_0 > 0$, and moreover $w_0 \geq v_0$ and $w_1 - w_0 > v_1 - v_0$, then*

$$w_{n+1} - v_{n+1} > (\alpha-1)(w_n - v_n) > 0$$

for all $n \geq 1$.

Proof. We have

$$\begin{aligned} w_2 - v_2 &= \alpha w_1 - w_0 - (\alpha v_1 - v_0) \\ &= (\alpha-1)(w_1-v_1) + w_1 - w_0 - (v_1 - v_0) \\ &> (\alpha-1)(w_1-v_1) > (\alpha-1)(w_0-v_0) \geq 0. \end{aligned}$$

If for some $n \geq 1$ we have $w_{n+1} - v_{n+1} > (\alpha - 1)(w_n - v_n) > 0$ then

$$\begin{aligned} w_{n+2} - v_{n+2} &= \alpha w_{n+1} - w_n - (\alpha v_{n+1} - v_n) \\ &= \alpha(w_{n+1} - v_{n+1}) - (w_n - v_n) \\ &> \alpha(w_{n+1} - v_{n+1}) - \frac{w_{n+1} - v_{n+1}}{\alpha - 1} \\ &\geq (\alpha - 1)(w_{n+1} - v_{n+1}) > 0. \ \blacksquare \end{aligned}$$

COROLLARY 5.5. *For each $n \geq 1$ and each $t \geq 0$, the sequence $\xi(n,t)$ is a strictly increasing non-trivial Büchi sequence of positive integers. Moreover, for each $i = 1, 2, 3$ and for each $n \geq 1$ we have*

$$\xi_{i+1}(n+1,t) - \xi_i(n+1,t) > (2t^2 + 10t + 9)(\xi_{i+1}(n,t) - \xi_i(n,t)).$$

Proof. Fix $t \geq 0$. We will apply Lemma 5.4 to the sequences $v_n = \xi_1(n,t)$ and $w_n = \xi_2(n,t)$, for $n \geq 0$. By the definition of ξ, both v_n and w_n satisfy the recurrence relation $u_{n+2} = \alpha u_{n+1} - u_n$ with

$$\alpha = f(t) = 2t^2 + 10t + 10 \geq 2.$$

By the definition of ξ, we have

$$\begin{aligned} v_1 &= \xi_1(1,t) = 2t^3 + 12t^2 + 19t + 6 > t + 1 = \xi_1(0,t) = v_0 > 0, \\ w_1 &= \xi_2(1,t) = 2t^3 + 14t^2 + 31t + 23 > t + 2 = \xi_2(0,t) = w_0 > 0, \\ w_0 &= \xi_2(0,t) = t + 2 > t + 1 = \xi_1(0,t) = v_0, \end{aligned}$$

and

$$w_1 - w_0 = 2t^3 + 14t^2 + 30t + 21 > 2t^3 + 12t^2 + 18t + 5 = v_1 - v_0,$$

so all the hypotheses of Lemma 5.4 are satisfied and we deduce that $\xi_2(n,t) - \xi_1(n,t)$ is a positive integer for each $n \geq 0$, and for each $n \geq 1$ we have

$$\xi_2(n+1,t) - \xi_1(n+1,t) > (f(t) - 1)(\xi_2(n,t) - \xi_1(n,t)).$$

In particular, since $f(t) - 1 \geq 9$ and $\xi_2(1,t) - \xi_1(1,t) = 2t^2 + 12t + 17 \geq 17$, the difference $\xi_2(n,t) - \xi_1(n,t)$ is greater than 1 for each $n \geq 1$, so the sequence $\xi(n,t)$ is non-trivial.

The other two cases are verified similarly. ■

We conclude this section with a characterization of the trivial sequences among the sequences of the form $\xi(n,t)$, which proves item 5 of Theorem 1.2.

COROLLARY 5.6. *The sequence $\xi(n,t)$ is trivial if and only if $t \in \{-4, -3, -2, -1\}$ or $n \in \{-1, 0\}$.*

Proof. By Lemma 5.1 we need only prove that if $t \notin \{-4, -3, -2, -1\}$ and $n \notin \{-1, 0\}$ then $\xi(n,t)$ is not a trivial Büchi sequence. By Corollary 5.5, we may suppose that moreover we have $n \ngeq 1$ or $t \ngeq 0$.

Case $n \leq -2$ and $t \geq 0$. Since $-n-1 \geq 1$, by Corollary 5.5 the sequence

$$(\xi_1(-n-1,t), \xi_2(-n-1,t), \xi_3(-n-1,t), \xi_4(-n-1,t))$$

is non-trivial, hence so is $(\xi_4(-n-1,t), -\xi_3(-n-1,t), -\xi_2(-n-1,t), \xi_1(-n-1,t))$, which is $\xi(n,t)$ by item 1 of Theorem 1.2.

Case $n \geq 0$ and $t \leq -5$. Since $-t-5 \geq 0$, by Corollary 5.5 the sequence

$$(\xi_1(n,-t-5), \xi_2(n,-t-5), \xi_3(n,-t-5), \xi_4(n,-t-5))$$

is non-trivial, hence so is $(-\xi_4(n,-t-5), -\xi_3(n,-t-5), -\xi_2(n,-t-5), -\xi_1(n,-t-5))$, which is $\xi(n,t)$ by item 2 of Theorem 1.2.

Case $n \leq -2$ and $t \leq -5$. Since $-n-1 \geq 1$ and $-t-5 \geq 0$, by Corollary 5.5 the sequence

$$(\xi_1(-n-1,-t-5), \xi_2(-n-1,-t-5), \xi_3(-n-1,-t-5), \xi_4(-n-1,-t-5))$$

is non-trivial, hence so is

$$(-\xi_1(-n-1,-t-5), \xi_2(-n-1,-t-5), \xi_3(-n-1,-t-5), -\xi_4(-n-1,-t-5)),$$

which is $\xi(n,t)$ by combining items 1 and 2 of Theorem 1.2. ■

6. A family of curves associated to length 5 sequences.

Each length 4 integer Büchi sequence (x_1, x_2, x_3, x_4) might extend to the right or to the left. For given integers n and t, a Büchi sequence $\xi(n,t)$ extends to the right if and only if the quantity

$$y_5(n,t) := 2\xi_4^2(n,t) - \xi_3^2(n,t) + 2$$

is a square, and it extends to the left if and only if

$$y_0(n,t) := 2\xi_1^2(n,t) - \xi_2^2(n,t) + 2$$

is a square. So for each integer $n \notin \{-1, 0\}$, we want to know whether or not the curves

$$y^2 = y_5(n,t) \qquad (C_n^{\mathrm{r}})$$

and

$$y^2 = y_0(n,t) \qquad (C_n^{\ell})$$

have integer points at all with $t \notin \{-4,-3,-2,-1\}$ (otherwise we have trivial sequences by Corollary 5.6). Note that by Corollary 5.6, any integer point with $t \notin \{-4,-3,-2,-1\}$ on one of the curves C_n^{r} or C_n^{ℓ} would give a non-trivial Büchi sequence of length 5. Note also that the polynomials on the right-hand sides have degree $2|2n+1| = |4n+2|$ by item 3 of Theorem 1.2.

Definition 6.1. We will say that an integer point (t,y) on C_n^{r} or C_n^{ℓ} is *non-trivial* if $t \notin \{-4,-3,-2,-1\}$.

From items 1 and 2 of Theorem 1.2, we have

$$\begin{aligned} y_0(n,t) &= 2\xi_1^2(n,t) - \xi_2^2(n,t) + 2 = 2\xi_4^2(-n-1,t) - \xi_3^2(-n-1,t) + 2 \\ &= y_5(-n-1,t) \end{aligned}$$

for each $n \in \mathbb{Z}$ and $t \in \mathbb{Z}$. Therefore, given $n \neq -1, 0$, there is a non-trivial integer point (t, y) on C_n^ℓ if and only if there is one on C_{-n-1}^{r}. Since this is true for *any* integer $n \neq -1, 0$, we deduce that there is a non-trivial point on C_n^ℓ *for some* $n \neq -1, 0$ if and only if there is a non-trivial point on C_n^{r} *for some* $n \neq -1, 0$. Hence in particular, in order to show that none of the sequences $\xi(n,t)$ extends to a non-trivial length 5 Büchi sequence, it is enough to show that the polynomial $y_5(n,t)$ cannot be a square if $n \notin \{-1, 0\}$ and $t \notin \{-4, -3, -2, -1\}$. So from now on we will write for simplicity

$$y_n(t) = 2\xi_4^2(n,t) - \xi_3^2(n,t) + 2$$

for each $n \in \mathbb{Z}$.

In the rest of this section we will show that the sequence of polynomials $y_n(t)$ satisfies a third order linear recurrence, and then show that for some infinite families of pairs (n,t), the quantity $y_n(t)$ is not a square. Unfortunately we have not been able to cover all cases.

LEMMA 6.2. *If (u_n) is a sequence of integers with $u_{n+2} = \alpha u_{n+1} - u_n$ for each $n \in \mathbb{Z}$, then the quantity*

$$\nu_u(n) = u_{n+2}^2 - (\alpha^2 - 2)u_{n+1}^2 + u_n^2$$

does not depend on n.

Proof. We have $u_{n+2}^2 = \alpha^2 u_{n+1}^2 + u_n^2 - 2\alpha u_{n+1}u_n$. One can then prove the lemma by solving the induction and using some telescoping argument. We thank J. Browkin for showing us the following more elegant proof. We have

$$\begin{aligned} \nu_u(n) &= u_{n+2}^2 - (\alpha^2 - 2)u_{n+1}^2 + u_n^2 \\ &= (\alpha^2 u_{n+1}^2 + u_n^2 - 2\alpha u_{n+1}u_n) - (\alpha^2 - 2)u_{n+1}^2 + u_n^2 \\ &= 2u_{n+1}^2 + 2u_n^2 - 2\alpha u_{n+1}u_n \end{aligned}$$

so it is sufficient to show that the quantity $u_{n+1}^2 + u_n^2 - \alpha u_{n+1}u_n$ does not depend on n. We have

$$\begin{aligned} \frac{1}{2}(\nu_u(n+1) - \nu_u(n)) &= u_{n+2}^2 + u_{n+1}^2 - \alpha u_{n+2}u_{n+1} - (u_{n+1}^2 + u_n^2 - \alpha u_{n+1}u_n) \\ &= u_{n+2}^2 - u_n^2 - \alpha u_{n+1}(u_{n+2} - u_n) \\ &= (u_{n+2} - u_n)(u_{n+2} + u_n - \alpha u_{n+1}) = 0, \end{aligned}$$

which proves the lemma. ∎

COROLLARY 6.3. *The quantity*

$$\nu_y = y_{n+2} - (f^2 - 2)y_{n+1} + y_n \tag{6.1}$$

does not depend on $n \in \mathbb{Z}$. *Moreover, since*

$$f(t)^2 - 2 = 2(2t^4 + 20t^3 + 70t^2 + 100t + 49),$$
$$y_{-1}(t) = t^2, \qquad y_0(t) = (t+5)^2,$$

and

$$y_1(t) = 4t^6 + 80t^5 + 620t^4 + 2400t^3 + 4905t^2 + 5020t + 2020,$$

we have

$$\nu_y = \nu_y(-1) = -2(10t^4 + 100t^3 + 346t^2 + 480t + 215).$$

Proof. Applying Lemma 6.2 to the sequences $(u_n)_n = (\xi_3(n,t))_n$ and $(v_n)_n = (\xi_4(n,t))_n$ (taking $\alpha = f(t) = 2t^2 + 10t + 10$), we obtain

$$\begin{aligned}
y_{n+2} &= 2v_{n+2}^2 - u_{n+2}^2 + 2 \\
&= 2((\alpha^2-2)v_{n+1}^2 - v_n^2 + \nu_v) - ((\alpha^2-2)u_{n+1}^2 - u_n^2 + \nu_u) + 2 \\
&= (\alpha^2-2)(2v_{n+1}^2 - u_{n+1}^2) - (2v_n^2 - u_n^2) + 2\nu_v - \nu_u + 2 \\
&= (\alpha^2-2)(y_{n+1}-2) - (y_n-2) + 2(v_2^2 - (\alpha^2-2)v_1^2 + v_0^2) \\
&\quad - (u_2^2 - (\alpha^2-2)u_1^2 + u_0^2) + 2 \\
&= (\alpha^2-2)(y_{n+1}-2) - (y_n-2) + (y_2-2) - (\alpha^2-2)(y_1-2) \\
&\quad + (y_0-2) + 2 \\
&= (\alpha^2-2)y_{n+1} - y_n + y_2 - (\alpha^2-2)y_1 + y_0,
\end{aligned}$$

which proves the corollary. ∎

LEMMA 6.4. *If* $t \in 5\mathbb{Z}$ *and* n *is not congruent to* 0 *or* -1 *modulo* 10 *then* $y_n(t)$ *is not a square.*

Proof. If $t \in 5\mathbb{Z}$ then ν_y, $y_{-1}(t)$ and $y_0(t)$ are multiples of 5, hence $y_n(t)$ is a multiple of 5 for each $n \in \mathbb{Z}$ (by (6.1)). Therefore, if $y_n(t)$ is a square then it must be a multiple of 25. Let $\equiv$ denote congruence modulo 25. Since (6.1) becomes

$$y_{n+2} + 2y_{n+1} + y_n + 5 \equiv 0$$

and y_{-1} and y_0 are multiples of 25, we have

$$\begin{aligned}
y_1 + 5 &\equiv 0 \quad \text{hence} \quad y_1 \equiv -5, \\
y_2 + 2y_1 + 5 &\equiv 0 \quad \text{hence} \quad y_2 \equiv 5, \\
y_3 + 2y_2 + y_1 + 5 &\equiv 0 \quad \text{hence} \quad y_3 \equiv -10,
\end{aligned}$$

and going on like that, one finds $y_4 \equiv 10$, $y_5 \equiv 10$, $y_6 \equiv -10$, $y_7 \equiv 5$, $y_8 \equiv -5$, $y_9 \equiv 0$ and finally $y_{10} \equiv 0$. So we are back to the situation of having two consecutive multiples of 25, and the lemma is proven. ∎

Lemma 6.5. *Assume $t \notin 5\mathbb{Z}$. We have*

1. $y_n(t) \in 5\mathbb{Z}$ *if and only if n is congruent to* 2 *modulo* 5*;*
2. *assuming that n is congruent to* 2 *modulo* 5*, $y_n(t) \in 5^2\mathbb{Z}$ if and only if t is congruent to* 21*,* 22*,* 23 *or* 24 *modulo* 25*.*

Therefore, if $t \notin 5\mathbb{Z}$ is not congruent to 21*,* 22*,* 23 *or* 24 *modulo* 25 *and n is congruent to* 2 *modulo* 5 *then $y_n(t)$ is not a square.*

Proof. Assume $t \notin 5\mathbb{Z}$. We have $y_{-1}(t) = t^2 \equiv_5 y_0(t)$ and $\nu_y(t) \equiv_5 -2t^2 \equiv_5 -2y_0(t)$. Note that since t^2 is congruent to either 1 or -1 modulo 5, we have $f(t)^2 - 2 \equiv_5 4t^4 - 2 \equiv_5 2$, hence (6.1) gives

$$y_{n+2} - 2y_{n+1} + y_n + 2t^2 \equiv_5 0.$$

So we have $y_1 - 2y_0 + y_{-1} + 2t^2 \equiv_5 0$, hence $y_1 \equiv_5 -t^2$. Similarly, we find $y_2 \equiv_5 0$, $y_3 \equiv_5 -t^2$, $y_4 \equiv_5 t^2$ and $y_5 \equiv_5 t^2$. So the first item is proven.

We have

$$f^2 - 2 \equiv_{25} 4t^4 - 10t^3 - 10t^2 - 2 \quad \text{and} \quad \nu_y \equiv_{25} 5t^4 + 8t^2 - 10t - 5,$$

so that (6.1) gives

$$y_{n+2} - (4t^4 - 10t^3 - 10t^2 - 2)y_{n+1} + y_n - (5t^4 + 8t^2 - 10t - 5) \equiv_{25} 0.$$

There are many congruences to verify in order to prove item 2, but with the help of a computer program, one can use the recurrence relation above and compute $y_1(t)$ up to $y_{25}(t)$ modulo 25, for t of the form $5m + a$, where $a \in \{1, 2, 3, 4\}$. One sees that when n is congruent to 2 modulo 5 then

$$y_n(5m + a) \equiv_{25} a(5m + 5)$$

and that the sequence $(y_n(5m + a))_n$ has period 25. We can conclude since $a(5m + 5)$ is congruent to 0 modulo 25 if and only if m is congruent to 4 modulo 5, if and only if t is congruent to 21, 22, 23 or 24 modulo 25. ∎

Note that one can easily derive many results in the same flavour as Lemmas 6.4 and 6.5, by studying other congruences.

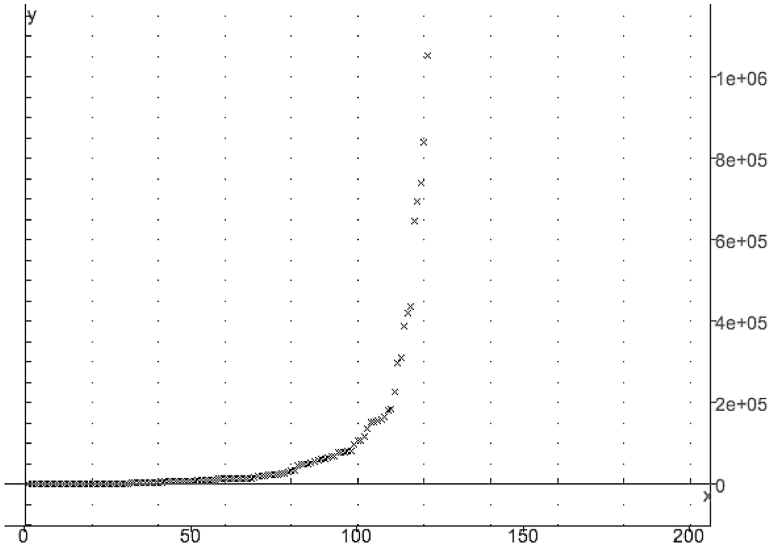
7. A list of non-parametrized integer points on X_4. In this section we list the strictly increasing sequences that we found and that are not obtained from any of the polynomial parametrizations presented in this paper. The first column is just the number of the row of the matrix. The graph is a plot of the first two columns. The number of points that we are not able to parametrize seems to go exponentially to zero.

1	59	630	889	1088
2	83	516	725	886
3	108	6643	9394	11505
4	108	707	994	1215
5	177	878	1229	1500
6	240	839	1162	1413
7	287	11838	16739	20500
8	311	752	1017	1226
9	334	3693	5212	6379
10	386	6237	8812	10789
11	419	11020	15579	19078
12	430	801	1048	1247
13	477	3572	5029	6150
14	510	1699	2348	2853
15	514	1537	2112	2561
16	570	7879	11128	13623
17	601	4832	6807	8326
18	862	1713	2264	2705
19	883	25566	36145	44264
20	916	26605	37614	46063
21	1346	20353	28752	35201
22	1546	5257	7272	8839
23	1574	2693	3468	4099
24	1616	3353	4458	5339
25	1674	2695	3424	4023
26	1766	8837	12372	15101
27	1812	11587	16286	19905
28	2066	6963	9628	11701
29	2437	13062	18311	22360
30	2477	15876	22315	27274
31	2636	20685	29134	35633
32	3048	5047	6454	7605
33	3051	11578	16087	19584
34	3247	9746	13395	16244
35	3333	36682	51769	63360
36	3673	5478	6821	7940
37	4090	5701	6948	8003
38	4743	36806	51835	63396
39	5148	12253	16546	19935
40	5331	15988	21973	26646
41	5781	22342	31063	37824
42	6449	25358	35277	42964
43	6504	18065	24706	29907
44	6756	33773	47282	57711

45	7104	9823	11938	13731
46	7234	24447	33808	41089
47	7386	17033	22928	27591
48	7414	16875	22684	27283
49	7594	10997	13572	15731
50	7871	12162	15293	17884
51	8562	17089	22600	27009
52	9343	26408	36159	43790
53	9741	19460	25739	30762
54	9752	25249	34350	41501
55	10888	25561	34470	41509
56	11358	47107	65644	79995
57	12129	18232	22753	26514
58	12539	21430	27591	32608
59	12710	46491	64508	78493
60	13305	44986	62213	75612
61	13500	29971	40178	48273
62	13811	38380	52491	63542
63	13835	33596	45453	54802
64	13836	25693	33598	39969
65	14416	40737	55778	67549
66	14843	26758	34809	41320
67	15369	52022	71947	87444
68	15451	47988	66083	80194
69	18793	33744	43865	52054
70	20476	44445	59426	71327
71	21648	38497	49954	59235
72	21924	32243	39982	46449
73	22377	45328	60071	71850
74	23173	49926	66695	80024
75	23174	56283	76148	91811
76	25079	34122	41227	47276
77	27283	57918	77231	92600
78	27699	38828	47413	54666
79	31659	51412	65453	76974
80	33426	58483	75652	89589
81	34030	59119	76368	90383
82	45007	85256	111855	133246
83	49040	61729	72222	81373
84	50430	70781	86468	99717
85	51077	89226	115385	136624
86	53119	70562	84477	96404
87	55506	72097	85528	97119
88	58599	87328	108713	126534

89	62429	86532	105253	121114
90	63626	118165	154524	183827
91	64776	98815	123826	144573
92	68986	106617	134072	156791
93	70143	94792	114241	130830
94	77391	92440	105361	116862
95	78741	128278	163433	192264
96	79292	91693	102606	112465
97	80251	100090	116601	131048
98	81770	131541	167092	196307
99	98804	118755	135806	150943
100	107366	169275	213964	250813
101	108523	139124	164115	185774
102	117178	144071	166680	186569
103	138004	167365	192294	214343
104	154097	200846	238605	271156
105	154097	200846	238605	271156

106	155730	226399	279752	324447
107	158435	195324	226277	253478
108	165267	222418	267631	306240
109	183122	235379	277980	314869
110	186101	246132	294157	335374
111	225341	270018	308287	342304
112	297422	352179	399500	441781
113	311680	401551	474702	537997
114	388048	447801	500470	548101
115	421884	499235	566114	625887
116	435682	484931	529620	570821
117	646914	739327	821408	896001
118	695001	761728	823063	880134
119	740566	869223	981152	1081559
120	839833	974682	1093019	1199740
121	1052749	1157218	1253007	1341976



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