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The digamma function, Euler–Lehmer constants and their p-adic counterparts

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1. Introduction. For a real number $x \neq 0, -1, \ldots$, the digamma function $\psi(x)$ is the logarithmic derivative of the gamma function defined by

$$-\psi(x) = \gamma + \frac{1}{x} + \sum_{n=1}^{\infty}\left(\frac{1}{n+x} - \frac{1}{n}\right),$$

where γ is Euler's constant. Just like the case of the gamma function, the nature of the values of the digamma function at algebraic or even rational arguments is shrouded in mystery.

In the rather difficult subject of irrationality or transcendence, sometimes it is more pragmatic to look at a family of special values as opposed to a single specific value and derive something meaningful. An apt instance here is the result of Rivoal [10] about irrationality of infinitude of odd zeta values as opposed to that of a single specific odd zeta value.

In this context, Murty and Saradha [7] have made some breakthroughs about transcendence of a certain family of digamma values. In particular, they proved the following.

Theorem 1.1 (Murty and Saradha). *For any positive integer $n > 1$, at most one of the $\phi(n) + 1$ numbers in the set*

$$\{\gamma\} \cup \{\psi(r/n) : 1 \leq r \leq n,\ (r, n) = 1\}$$

is algebraic.

In this article, we extend their result and prove the following.

Theorem 1.2. *At most one element in the infinite set*

$$\{\gamma\} \cup \{\psi(r/n) : n > 1,\ 1 \leq r < n,\ (r, n) = 1\}$$

is algebraic.

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In a recent work [3], the question of linear independence of these numbers is studied.

In another context, Lehmer [6] defined generalized Euler constants $\gamma(r,n)$ for $r,n \in \mathbb{N}$ with $r \le n$ by the formula

$$\gamma(r,n) = \lim_{x\to\infty}\Bigg(\sum_{\substack{m\le x\\ m\equiv r \,(\mathrm{mod}\, n)}} \frac{1}{m} - \frac{\log x}{n}\Bigg).$$

Murty and Saradha, in their papers [7, 9], investigated the nature of Euler–Lehmer constants $\gamma(r,n)$ and proved results similar to Theorems 1.1 and 1.2. For an exhaustive account of Euler's constant, see the recent article of Lagarias [5].

From now onwards p and q will always denote prime numbers. In another work [8], Murty and Saradha investigated the p-adic analog γ_p of Euler's constant as well as the generalized p-adic Euler–Lehmer constants $\gamma_p(r,q)$. Here $r \in \mathbb{N}$ with $1 \le r < q$. They also studied the values of the p-adic digamma function $\psi_p(r/p)$ for $1 \le r < p$.

In the next section we shall give the definitions of γ_p, $\gamma_p(r,q)$ and $\psi_p(x)$ following Diamond [2]. Here are the results of Murty and Saradha [8].

Theorem 1.3 (Murty and Saradha). *Let q be prime. Then at most one of*

$$\gamma_p,\ \gamma_p(r,q), \qquad 1 \le r < q,$$

is algebraic.

Theorem 1.4 (Murty and Saradha). *The numbers $\psi_p(r/p) + \gamma_p$ are transcendental for $1 \le r < p$.*

In this paper, we generalize these results. Let $\mathcal{P}$ denote the set of prime numbers in $\mathbb{N}$. Here we prove the following:

Theorem 1.5. *At most one number in the set*

$$\{\gamma_p\} \cup \{\gamma_p(r,q) : q \in \mathcal{P},\ 1 \le r < q/2\}$$

is algebraic.

If we normalize the p-adic Euler–Lehmer constants by setting

$$\gamma_p^*(r,q) = q\gamma_p(r,q),$$

then we have the following result:

Theorem 1.6. *All the numbers in the list*

$$\gamma_p,\ \gamma_p^*(r,q), \qquad q \in \mathcal{P},\ 1 \le r < q/2,$$

are distinct.

Using this theorem, we prove:

THEOREM 1.7. *As before, let q run through the set of all prime numbers. Then there is at most one pair of repetition among the numbers*

$$\gamma_p,\ \gamma_p(r,q), \qquad 1 \le r < q/2.$$

Also if there is such a repetition, then γ_p is transcendental.

Our final result is:

THEOREM 1.8. *Fix an integer $n > 1$. At most one element of the set*

$$\{\psi_p(r/p^n) + \gamma_p : 1 \le r < p^n,\ (r,p) = 1\}$$

is algebraic. Moreover, $\psi_p(r/p) + \gamma_p$ are distinct when $1 \le r < p/2$.

2. Preliminaries. For all discussions in this section, let us fix a prime p. Let $\overline{\mathbb{Q}}_p$ be a fixed algebraic closure of $\mathbb{Q}_p$ and $\mathbb{C}_p$ be its completion. We fix an embedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}_p$. Thus the elements in the set $\mathbb{C}_p \setminus \overline{\mathbb{Q}}$ are the transcendental numbers.

We begin by recalling the notion of p-adic logarithms, which are of primary importance in our context. For the elements in the open unit ball around 1, that is,

$$D := \{\alpha \in \mathbb{C}_p : |\alpha - 1|_p < 1\},$$

$\log_p \alpha$ is defined using the formal power series

$$\log(1+X) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}X^n}{n},$$

which has radius of convergence 1. To extend this to all of $\mathbb{C}_p^\times$, note that every element $\beta \in \mathbb{C}_p^\times$ is uniquely expressible as

$$\beta = p^r w \alpha$$

where $\alpha \in D$, $r \in \mathbb{Q}$ and w is a root of unity of order prime to p. Here p^r is the positive real rth power of p in $\overline{\mathbb{Q}}$, embedded in $\overline{\mathbb{Q}}_p$ from the beginning. With this, one defines

$$\log_p \beta := \log_p \alpha.$$

Note that $\log_p \beta = 0$ if and only if β is p^r times a root of unity. We refer to Washington [11, Chapter 5] for details.

We shall now define the p-adic analog of the digamma function following the strategy of Diamond. The idea is to first define a suitable analog of the classical log-gamma function. This is defined, for $x \in \mathbb{C}_p^\times$, as

$$G_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} (x+n)\log_p(x+n) - (x+n).$$

This function has properties analogous to the classical log-gamma function, for instance,

$$G_p(x+1) = G_p(x) + \log_p x.$$

It also satisfies an analog of the classical Gauss' identity (up to a term $\log\sqrt{2\pi}$), namely

$$G_p(x) = \left(x - \frac{1}{2}\right)\log_p m + \sum_{a=0}^{m-1} G_p\left(\frac{x+a}{m}\right)$$

for a positive integer m when the right side is defined.

The p-adic digamma function $\psi_p(x)$ is defined as the derivative of $G_p(x)$ and hence is given by (for $-x \notin \mathbb{N}$)

$$\psi_p(x) = \lim_{k\to\infty} \frac{1}{p^k} \sum_{n=0}^{p^k-1} \log_p(x+n).$$

Recall that the classical generalized Euler constant $\gamma(r,f)$ defined by Lehmer satisfies

$$\psi(r/f) = \log f - f\gamma(r,f)$$

for $1 \le r \le f$.

In the p-adic set up, one also defines $\gamma_p(r,f)$ for integers r, f with $f \ge 1$ as follows. If the p-adic valuation $\nu(r/f)$ of r/f is negative, then

$$\gamma_p(r,f) = -\lim_{k\to\infty} \frac{1}{fp^k} \sum_{\substack{m=1 \\ m\equiv r \ (\mathrm{mod}\ f)}}^{fp^k-1} \log_p m.$$

On the other hand, when $\nu(r/f) \ge 0$, we first write f as $f = p^k f_1$ with $(p, f_1) = 1$ and then define

$$\gamma_p(r,f) = \frac{p^{\varphi(f_1)}}{p^{\varphi(f_1)} - 1} \sum_{n\in N(r,f)} \gamma_p(r + nf, p^{\varphi(f_1)} f)$$

where

$$N(r,f) = \{n : 0 \le n < p^{\varphi(f_1)},\ nf + r \not\equiv 0 \ (\mathrm{mod}\ p^{\varphi(f_1)+k})\}.$$

Finally, we set

$$\gamma_p = \gamma_p(0,1) = -\frac{p}{p-1} \lim_{k\to\infty} \frac{1}{p^k} \sum_{\substack{m=1 \\ (m,p)=1}}^{p^k-1} \log_p m.$$

We shall need the following identity of Diamond (see [2, p. 334]).

Theorem 2.1. *If $q > 1$ and ζ_q is a primitive qth root of unity, then*

$$q\gamma_p(r,q) = \gamma_p - \sum_{a=1}^{q-1} \zeta_q^{-ar} \log_p(1-\zeta_q^a).$$

Let us now state the prerequisites from transcendence theory. We shall need the following result of Baker (see [1, p. 11]) involving classical logarithms of complex numbers.

Theorem 2.2 (Baker). *If $\alpha_1,\dots,\alpha_n$ are non-zero algebraic numbers and $\beta_1,\dots,\beta_n$ are algebraic numbers, then*

$$\beta_1 \log\alpha_1 + \cdots + \beta_n \log\alpha_n$$

is either zero or transcendental.

We shall need analogous results for linear forms in p-adic logarithms. More precisely, we shall need the following consequence of a theorem of Kaufman [4] as noticed by Murty and Saradha (see [8, p. 357]).

Theorem 2.3. *Suppose that $\alpha_1,\dots,\alpha_m$ are non-zero algebraic numbers that are multiplicatively independent over $\mathbb{Q}$ and that $\beta_1,\dots,\beta_m$ are arbitrary algebraic numbers (not all zero). Further suppose that*

$$|\alpha_i - 1| < p^{-c} \quad \text{for } 1 \le i \le m,$$

where c is a constant which depends only on the degree of the number field generated by $\alpha_1,\dots,\alpha_m,\beta_1,\dots,\beta_m$. Then

$$\beta_1 \log_p \alpha_1 + \cdots + \beta_m \log_p \alpha_m$$

is transcendental.

3. Proof of Theorem 1.2. We need the following lemmas.

Lemma 3.1. *For all $n > 1$ and for all $r \in \mathbb{N}$ with $(r,n) = 1$ and $1 \le r < n$, all the numbers in the list*

$$\gamma,\ \psi(r/n)$$

are distinct.

Proof. For a real number $x > 0$, we have

$$\psi'(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^2} > 0.$$

Hence $\psi(x)$ is strictly increasing function for $x > 0$. ∎

Lemma 3.2. *For $q > 1$ and $1 \le a < q$ with $(a,q) = 1$, one has*

$$-\psi(a/q) - \gamma = \log q - \sum_{b=1}^{q-1} e^{-2\pi iba/q} \log(1 - e^{2\pi ib/q}).$$

For a proof of this lemma, see [7, p. 311].

Proof of Theorem 1.2. Suppose that the assertion is not true. By the work of Murty and Saradha [7], it follows that γ and $\psi(a/q)$ for some $1 \le a < q$ with $(a,q)=1$ cannot be both algebraic. So assume that there exist $1 \le a_1 < q_1$ with $(a_1,q_1)=1$ and $1 \le a_2 < q_2$ with $(a_2,q_2)=1$ such that both $\psi(a_1/q_1)$ and $\psi(a_2/q_2)$ are algebraic numbers. Note that by Lemma 3.2, we have

$$(1)\qquad \psi(a_1/q_1) - \psi(a_2/q_2) = \log\frac{q_2}{q_1} - \sum_{b=1}^{q_2-1} e^{-2\pi i b a_2/q_2}\log(1-e^{2\pi i b/q_2}) + \sum_{c=1}^{q_1-1} e^{-2\pi i b a_1/q_1}\log(1-e^{2\pi i b/q_1}).$$

The right hand side is a algebraic linear combination of linear forms of logarithms of algebraic numbers. Also, by Lemma 3.1, it is non-zero. Hence by Baker's theorem it is transcendental, a contradiction. ∎

4. Proofs of other theorems. Next, we prove a proposition which will play a pivotal role in proving the rest of the theorems.

PROPOSITION 4.1. *For $p_i \in \mathcal{P}$, let $q_i = p_i^{m_i}$, where $m_i \in \mathbb{N}$, and let ζ_{q_i} be a primitive q_ith root of unity. Then for any finite subset* J *of $\mathcal{P}$, the numbers*

$$1-\zeta_{q_i},\ \frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}, \qquad \textit{where} \qquad 1 < a_i < q_i/2,\ (a_i,q_i)=1 \ \textit{and}\ p_i \in \mathrm{J},$$

are multiplicatively independent.

Proof. Write $\mathrm{I} = \{i : p_i \in \mathrm{J}\}$. We will prove this proposition by induction on $|\mathrm{I}|$. First suppose that $|\mathrm{I}|=1$. Then the proposition is true by the work of Murty and Saradha (see [8, p. 357]). Next suppose that the proposition is true for all I with $|\mathrm{I}|<n$. Now suppose that $|\mathrm{I}|=n$. Note that for any i, the numbers

$$(2)\qquad \frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}, \qquad \text{where } 1 < a_i < q_i/2,\ (a_i,q_i)=1,$$

are multiplicatively independent units in $\mathbb{Q}(\zeta_{q_i})$ (see [11, p. 144]). Suppose that there exist integers α_i, β_{a_i} for $i \in \mathrm{I}$ such that, with a_i as in the lemma,

$$(3)\qquad \prod_{i\in \mathrm{I}}\Bigg\{(1-\zeta_{q_i})^{\alpha_i}\prod_{\substack{1<a_i<q_i/2\\(a_i,q_i)=1}}\left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{\beta_{a_i}}\Bigg\} = 1.$$

Taking norm on both sides, we get

$$\prod_{i\in \mathrm{I}} p_i^{\alpha_i A_i} = 1, \qquad \text{where } A_i \neq 0,\ A_i \in \mathbb{N}.$$

Since p_i's are distinct primes, we have $\alpha_i = 0$ for all $i \in \mathrm{I}$. Thus (3) reduces to

$$\prod_{i\in \mathrm{I}} \prod_{\substack{1<a_i<q_i/2\\ (a_i,q_i)=1}} \left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{\beta_{a_i}} = 1. \tag{4}$$

Since $|\mathrm{I}| > 1$, there exist $i_1, i_2 \in \mathrm{I}$ with $i_1 \neq i_2$ such that

$$\prod_{\substack{i\in \mathrm{I},\\ i\neq i_1}} \prod_{\substack{1<a_i<q_i/2\\ (a_i,q_i)=1}} \left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{\beta_{a_i}} = \prod_{\substack{1<a_{i_1}<q_{i_1}/2\\ (a_{i_1},q_{i_1})=1}} \left(\frac{1-\zeta_{q_{i_1}}^{a_{i_1}}}{1-\zeta_{q_{i_1}}}\right)^{-\beta_{a_{i_1}}}.$$

Note that the left hand side of the above equation belongs to the number field $\mathbb{Q}(\zeta_\delta)$, where $\delta = \prod_{i\in \mathrm{I}\setminus\{i_1\}} q_i$, whereas the right hand side belongs to $\mathbb{Q}(\zeta_{q_{i_1}})$. Since

$$\mathbb{Q}(\zeta_\delta) \cap \mathbb{Q}(\zeta_{q_{i_1}}) = \mathbb{Q},$$

we see that both sides of the above equation are a rational number having norm 1. Thus we have

$$\prod_{i\in \mathrm{I}\setminus\{i_1\}} \prod_{\substack{1<a_i<q_i/2\\ (a_i,q_i)=1}} \left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{\beta_{a_i}} = \prod_{\substack{1<a_{i_1}<q_{i_1}/2\\ (a_{i_1},q_{i_1})=1}} \left(\frac{1-\zeta_{q_{i_1}}^{a_{i_1}}}{1-\zeta_{q_{i_1}}}\right)^{-\beta_{a_{i_1}}} = \pm 1.$$

Squaring both sides, we get

$$\prod_{i\in \mathrm{I}\setminus\{i_1\}} \prod_{\substack{1<a_i<q_i/2\\ (a_i,q_i)=1}} \left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{2\beta_{a_i}} = \prod_{\substack{1<a_1<q_{i_1}/2\\ (a_{i_1},q_{i_1})=1}} \left(\frac{1-\zeta_{q_{i_1}}^{a_{i_1}}}{1-\zeta_{q_{i_1}}}\right)^{-2\beta_{a_{i_1}}} = 1.$$

Using (2), we see that $\beta_{a_{i_1}} = 0$ for all $1 < a_{i_1} < q_{i_1}/2$ and $(a_{i_1}, q_{i_1}) = 1$. Then (4) reduces to

$$\prod_{i\in \mathrm{I}\setminus i_1} \prod_{\substack{1<a_i<q_i/2\\ (a_i,q_i)=1}} \left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right)^{\beta_{a_i}} = 1.$$

Now by induction hypothesis, we have $\beta_{a_i} = 0$ for all a_i and $i \in \mathrm{I} \setminus \{i_1\}$. ∎

Using Proposition 4.1 and Theorem 2.3, we can prove the following statement.

LEMMA 4.2. *Let* J *be any finite subset of* $\mathcal{P}$. *For* $q \in \mathrm{J}$ *and* $1 < a < q/2$, *let* s_q, t_q^a *be arbitrary algebraic numbers, not all zero. Further, let* t_q^a *be not all zero when* $p \in \mathrm{J}$. *Then*

$$\sum_{q\in \mathrm{J}} s_q \log_p(1-\zeta_q) + \sum_{\substack{q\in \mathrm{J}\\ 1<a<q/2}} t_q^a \log_p\left(\frac{1-\zeta_q^a}{1-\zeta_q}\right)$$

is transcendental.

Proof. Write $\delta = \prod_{q \in \mathrm{J}} q$. For any $\alpha \in \mathbb{Z}[\zeta_\delta]$ with $p \nmid \alpha$ and $M \in \mathbb{N}$, one has

$$|\alpha^A - 1| < p^{-M}$$

for some $A \in \mathbb{N}$. By choosing M sufficiently large and using Theorem 2.3 and Proposition 4.1, we get the lemma. ∎

LEMMA 4.3. *Let q_1, q_2 be two distinct prime numbers and $1 \le r_i < q_i$ for $i = 1, 2$. Then*

$$\sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p(1 - \zeta_{q_2}^b) - \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p(1 - \zeta_{q_1}^a) \tag{5}$$

is transcendental.

Proof. For any $q > 1$ and $(r, q) = 1$, we know that

$$\sum_{a=1}^{q-1} \zeta_q^{-ar} = -1.$$

Hence (5) can be written as

$$\log_p\left(\frac{1-\zeta_{q_1}}{1-\zeta_{q_2}}\right) - \sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1} \log_p\left(\frac{1-\zeta_{q_1}^a}{1-\zeta_{q_1}}\right) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2} \log_p\left(\frac{1-\zeta_{q_2}^b}{1-\zeta_{q_2}}\right). \tag{6}$$

It is because

$$(1 - \zeta_q^{-t}) = -\zeta_q^{-t}(1 - \zeta_q^t)$$

for any $t \in \mathbb{N}$ and the p-adic logarithm is zero on roots of unity, we have

$$\log_p(1 - \zeta_q^{-t}) = \log_p(1 - \zeta_q^t).$$

Note that the summands in (6) for $a = 1$, $b = 1$ and $a = q_1 - 1$, $b = q_2 - 1$ are zero. Now pairing up a with $-a$ and b with $-b$ in (6), we get

$$\log_p\left(\frac{1-\zeta_{q_1}}{1-\zeta_{q_2}}\right) - \sum_{1<a<q_1/2} \alpha_a \log_p\left(\frac{1-\zeta_{q_1}^a}{1-\zeta_{q_1}}\right) + \sum_{1<b<q_2/2} \beta_b \log_p\left(\frac{1-\zeta_{q_2}^b}{1-\zeta_{q_2}}\right),$$

where $\alpha_a = (\zeta_{q_1}^{-ar_1} + \zeta_{q_1}^{ar_1})$, $\beta_b = (\zeta_{q_2}^{-br_2} + \zeta_{q_2}^{br_2})$ are non-zero algebraic numbers. Hence using Lemma 4.2, we deduce that (6) is transcendental. ∎

4.1. Proof of Theorem 1.5. Suppose that two of the numbers from the set under study are algebraic. It follows from the work of Murty and Saradha (see [8, p. 351]) that one of them cannot be equal to γ_p or both of them cannot be of the form $\gamma_p(r_1, q)$ and $\gamma_p(r_2, q)$.

Without loss of generality, we can assume that these two numbers are of the form $\gamma_p(r_1, q_1)$, where $1 \le r_1 < q_1$, and $\gamma_p(r_2, q_2)$, where $1 \le r_2 < q_2$ and $q_1 \ne q_2$. Then $q_1\gamma_p(r_1, q_1) - q_2\gamma_p(r_2, q_2)$ is algebraic. Now by Diamond's theorem (see [2, Theorem 18]), we have

$$q_1\gamma_p(r_1,q_1) - q_2\gamma_p(r_2,q_2) = -\sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1}\log_p(1-\zeta_{q_1}^a) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2}\log_p(1-\zeta_{q_2}^b). \tag{7}$$

The left hand side is algebraic by assumption, whereas the right hand side is transcendental by Lemma 4.3, a contradiction. ∎

4.2. Proof of Theorem 1.6. It follows from Diamond's theorem [2] that

$$\begin{aligned}\gamma_p^*(r_1,q) - \gamma_p &= -\sum_{a=1}^{q-1} \zeta_q^{-ar}\log_p(1-\zeta_q^a),\\ \gamma_p^*(r_1,q) - \gamma_p^*(r_2,q) &= -\sum_{a=1}^{q-1}(\zeta_q^{-ar_1} - \zeta_q^{-ar_2})\log_p(1-\zeta_q^a),\end{aligned} \tag{8}$$

where $1 \le r_1, r_2 < q/2$ and $r_1 \ne r_2$. Transcendence of the first number follows from the work of Murty and Saradha (see [8, p. 358]) while that of the second one follows from the fact that $\zeta^{-ar_1} + \zeta^{ar_1} \ne \zeta^{-ar_2} + \zeta^{-ar_2}$ when $1 \le a, r_1, r_2 < q/2$ with $r_1 \ne r_2$. Again by Diamond's theorem, we have

$$\gamma_p^*(r_1,q_1) - \gamma_p^*(r_2,q_2) = -\sum_{a=1}^{q_1-1} \zeta_{q_1}^{-ar_1}\log_p(1-\zeta_{q_1}^a) + \sum_{b=1}^{q_2-1} \zeta_{q_2}^{-br_2}\log_p(1-\zeta_{q_2}^b),$$

where $q_1 \ne q_2$. By Lemma 4.3, we know that this number is transcendental and hence non-zero. ∎

4.3. Proof of Theorem 1.7. We will prove this theorem by contradiction. First note that it is impossible to have

$$\gamma_p(r_1,q) = \gamma_p = \gamma_p(r_2,q),$$

as otherwise $\gamma_p^*(r_1,q) = \gamma_p^*(r_2,q)$, a contradiction to Theorem 1.6. Next assume that

$$\gamma_p(r_1,q_1) = \gamma_p = \gamma_p(r_2,q_2),$$

where $q_1 \ne q_2$ and $1 \le r_i < q_i/2$. Using Diamond's theorem, we can write

$$(q_i - 1)\gamma_p = \log_p(1-\zeta_{q_i}) - \sum_{1<a<q_i/2} (\zeta_{q_i}^{ar_i} + \zeta_{q_i}^{-ar_i})\log_p\left(\frac{1-\zeta_{q_i}^a}{1-\zeta_{q_i}}\right), \tag{9}$$

where $i = 1, 2$. From this, we get

$$\begin{aligned} 0 = {} & (q_2 - 1)\log_p(1-\zeta_{q_1}) \; - \; (q_1 - 1)\log_p(1-\zeta_{q_2}) \\ & - \sum_{1<a<q_1/2} (q_2-1)(\zeta_{q_1}^{ar_1} + \zeta_{q_1}^{-ar_1})\log_p\left(\frac{1-\zeta_{q_1}^a}{1-\zeta_{q_1}}\right) \\ & + \sum_{1<a<q_2/2} (q_1-1)(\zeta_{q_2}^{ar_2} + \zeta_{q_2}^{-ar_2})\log_p\left(\frac{1-\zeta_{q_2}^a}{1-\zeta_{q_2}}\right), \end{aligned}$$

which is transcendental by Lemma 4.2, a contradiction.

Now suppose that

$$\gamma_p(r_1, q_1) = \gamma_p(r_2, q_2) \quad \text{and} \quad \gamma_p(r_3, q_3) = \gamma_p(r_4, q_4)$$

for some $1 \le r_i < q_i/2, 1 \le i \le 4$. We may assume that $q_1 \neq q_2$. For if $q_1 = q_2 = q$ (say), then $\gamma_p(r_1, q) = \gamma_p(r_2, q)$ implies $\gamma_p^*(r_1, q) = \gamma_p^*(r_2, q)$, a contradiction to Theorem 1.6. Similarly, we may assume that $q_3 \neq q_4$. Now, from equality $\gamma_p(r_1, q_1) = \gamma_p(r_2, q_2)$, we deduce that

$$\begin{aligned} (q_1 - q_2)\gamma_p = {} & q_2 \log_p(1-\zeta_{q_1}) - q_1\log_p(1-\zeta_{q_2}) \\ & - q_2 \sum_{1<a_1<q_1/2} (\zeta_{q_1}^{a_1r_1} + \zeta_{q_1}^{-a_1r_1})\log_p\left(\frac{1-\zeta_{q_1}^{a_1}}{1-\zeta_{q_1}}\right) \\ & + q_1 \sum_{1<a_2<q_2/2} (\zeta_{q_2}^{a_2r_2} + \zeta_{q_2}^{-a_2r_2})\log_p\left(\frac{1-\zeta_{q_2}^{a_2}}{1-\zeta_{q_2}}\right). \end{aligned} \tag{10}$$

Similarly, from $\gamma_p(r_3, q_3) = \gamma_p(r_4, q_4)$, we find that

$$\begin{aligned} (q_3 - q_4)\gamma_p = {} & q_4 \log_p(1-\zeta_{q_3}) - q_4 \sum_{1<a_3<q_3/2} (\zeta_{q_3}^{a_3r_3} + \zeta_{q_3}^{-a_3r_3})\log_p\left(\frac{1-\zeta_{q_3}^{a_3}}{1-\zeta_{q_3}}\right) \\ & - q_3\log_p(1-\zeta_{q_4}) + q_3 \sum_{1<a_4<q_4/2} (\zeta_{q_4}^{a_4r_4} + \zeta_{q_4}^{-a_4r_4})\log_p\left(\frac{1-\zeta_{q_4}^{a_4}}{1-\zeta_{q_4}}\right). \end{aligned}$$

Eliminating γ_p from the above two equations, we get

$$\begin{aligned} 0 = {} & (q_3 - q_4)q_2\log_p(1-\zeta_{q_1}) - (q_3-q_4)q_1\log_p(1-\zeta_{q_2}) \\ & - (q_1-q_2)q_4\log_p(1-\zeta_{q_3}) + (q_1-q_2)q_3\log_p(1-\zeta_{q_4}) \\ & + \sum_{i=1}^{4} \sum_{1<a_i<q_i/2} c_{a_i}\log_p\left(\frac{1-\zeta_{q_i}^{a_i}}{1-\zeta_{q_i}}\right), \end{aligned}$$

where c_{a_i} are algebraic numbers for all $1 \le i \le 4$ and $1 < a_i < q_i/2$. Again by Lemma 4.2, the number is transcendental.

To prove the second part of the theorem, suppose that

$$\gamma_p(r_1, q_1) = \gamma_p \quad \text{or} \quad \gamma_p(r_1, q_1) = \gamma_p(r_2, q_2),$$

where $q_1 \neq q_2$ and $1 \leq r_i < q_i/2$ for $i = 1, 2$. As above, we deduce the result from (9) or (10) using Lemma 4.2. ■

4.4. Proof of Theorem 1.8. Write

$$\mathrm{S} = \{\psi_p(r/p^n) + \gamma_p : 1 \leq r < p^n,\ (r,p) = 1\}.$$

Suppose that $a, b \in \mathrm{S}$ are distinct and algebraic. Suppose further that

$$(a,b) = (\psi_p(r_1/p^n) + \gamma_p, \psi_p(r_2/p^n) + \gamma_p).$$

Using Diamond's theorem, we have

$$\begin{aligned}\psi_p(r_1/p^n) - \psi_p(r_2/p^n) &= \sum_{a=1}^{p^n-1} \zeta^{-ar_1} \log_p(1-\zeta_{p^n}^a) - \sum_{a=1}^{p^n-1} \zeta^{-ar_2} \log_p(1-\zeta_{p^n}^a) \\ &= \sum_{\substack{1<a<p^n/2 \\ (a,p)=1}} \alpha_a \log_p\left(\frac{1-\zeta_{p^n}^a}{1-\zeta_{p^n}}\right),\end{aligned}$$

where α_a's are algebraic numbers. But by Lemma 4.2, this is necessarily transcendental, a contradiction.

Moreover, when $n = 1$, we have

$$\psi_p(r_1/p) - \psi_p(r_2/p) = \sum_{1<a<p/2} (\zeta^{-ar_1} + \zeta^{ar_1} - \zeta^{-ar_2} - \zeta^{ar_2}) \log_p\left(\frac{1-\zeta_p^a}{1-\zeta_p}\right).$$

Since $1 \leq r_1, r_2 < p/2$, the above linear form in logarithm is transcendental by Lemma 4.2 and hence non-zero. ■

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