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# Continued fractions with sequences of partial quotients over the field of Laurent series

by

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**1. Introduction.** Given a real number $x \in [0,1]$, let

$$x = \cfrac{1}{a_1(x) + \cfrac{1}{a_2(x) + \cdots}} = [a_1(x), a_2(x), \ldots],$$

be its simple continued fraction expansion, where $a_n(x) \in \mathbb{N}$, $n \geq 1$, are called the partial quotients of $x$. Let

$$\frac{p_n(x)}{q_n(x)} = [a_1(x), \ldots, a_n(x)], \qquad n \geq 1,$$

denote the convergents of $x$. Then with the conventions $p_{-1}(x) = q_0(x) = 1$ and $q_{-1}(x) = p_0(x) = 0$, for any $n \geq 0$ we have

$$p_{n+1}(x) = a_{n+1}(x)p_n(x) + p_{n-1}(x), \tag{1}$$

$$q_{n+1}(x) = a_{n+1}(x)q_n(x) + q_{n-1}(x). \tag{2}$$

For any $n \geq 1$ and $(a_1, \ldots, a_n) \in \mathbb{N}^n$, let $p_n(a_1, \ldots, a_n)$ and $q_n(a_1, \ldots, a_n)$ be defined by (1) and (2).

In [4], I. J. Good investigated the fractional dimensions of sets of continued fractions whose partial quotients $\{a_n(x) : n \geq 1\}$ obey various conditions, including the cases when $a_n(x)$, $n \geq 1$, are restricted to belong to some finite subset of $\mathbb{N}$. Let $D \subseteq \mathbb{N}$ be a finite or infinite nonempty set. Define

$$E_D = \{x \in [0,1) : a_n(x) \in D \text{ for } n \geq 1\}.$$

When $D$ is finite, I. J. Good showed that

$$\dim_{\mathrm{H}} E_D = \lim_{n\to\infty} \sigma_{D,n}, \tag{3}$$

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where $\dim_{\mathrm{H}}$ denotes the Hausdorff dimension and $\sigma_{D,n}$ is the real root of

$$\sum_{(a_1,\ldots,a_n)\in D^n} q_n(a_1,\ldots,a_n)^{-2\sigma_{D,n}} = 1.$$

Mauldin and Urbański [9], [10] generalized I. J. Good's result to all $D$. For any nonempty $D \subseteq \mathbb{N}$ and $s \in \mathbb{R}$, define the pressure function

$$P_D(s) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{(a_1,\ldots,a_n)\in D^n} q_n(a_1,\ldots,a_n)^{-2s}.$$

Mauldin and Urbański proved that

$$\dim_{\mathrm{H}} E_D = \sup\{\dim_{\mathrm{H}} E_J : J \subseteq D \text{ finite}\} = \inf\{s : P_D(s) \le 0\}. \tag{4}$$

In the case $a_n(x)$ tends to infinity, I. J. Good [4] proved that

$$\{x \in [0,1] : a_n(x) \to \infty \text{ as } n \to \infty\}$$

is of Hausdorff dimension $1/2$. Hirst [5] considered the case when $a_n(x)$ is further restricted to belong to some sequence of natural numbers. More precisely, let $\Lambda$ be an infinite sequence of positive integers $\lambda_1 < \lambda_2 < \cdots$ and $\tau(\Lambda)$ be the exponent of convergence of the series $\sum_{n=1}^{\infty} 1/\lambda_n$, i.e.,

$$\tau(\Lambda) = \inf\left\{s \ge 0 : \sum_{n=1}^{\infty} \frac{1}{\lambda_n^s} < \infty\right\}.$$

Define

$$E(\Lambda) = \{x \in [0,1] : a_n(x) \in \Lambda \ (n \ge 1) \text{ and } a_n(x) \to \infty \text{ as } n \to \infty\}.$$

Hirst [5] showed that $\dim_{\mathrm{H}} E(\Lambda) \le \tau(\Lambda)/2$, and conjectured that equality holds. In [3], Cusick proved Hirst's conjecture under a density assumption on $\Lambda$: if there exist constants $c$, $q$ and $r$ depending only on $\Lambda$ such that $r < \tau q$ and, for all real $p \ge q$, the sequence $\Lambda$ has at least $cn^{\tau p - r}$ members in every interval $[(n-1)^p,\ n^p]$, then $\dim_{\mathrm{H}} E(\Lambda) = \tau(\Lambda)/2$. In [16], Wang and the second author confirmed Hirst's conjecture without any assumption on $\Lambda$, that is,

$$\dim_{\mathrm{H}} E(\Lambda) = \tau(\Lambda)/2 \quad \text{for any } \Lambda \subseteq \mathbb{N}. \tag{5}$$

For continued fractions over the field of formal Laurent series, it is natural to ask if there are analogous results to (3), (4) and (5). In this paper, we obtain such analogues for continued fractions over the field of formal Laurent series. Continued fraction expansion and Diophantine approximation in the field of formal Laurent series have been studied in [2, 8, 15] and references therein. Kristensen [7] discussed analogues of the classical questions in Diophantine approximation (Hausdorff dimension and Khinchin type theorem) in the field of Laurent series. The Hausdorff dimensions of some other sets occurring in the continued fraction expansion of Laurent series have been discussed in [11, 12, 17, 18, 6].

This paper is organized as follows. In Section 2, we introduce some notations and present the main results of this paper. Section 3 is devoted to the proof of the main results.

**2. Statements of main results.** In this section, we present the main results of this paper. We first fix the notation and describe continued fractions over the field of formal Laurent series.

Let $p$ be a prime, $q$ be a power of $p$, and $\mathbb{F}_q$ be a finite field of $q$ elements. Let $\mathbb{F}_q((z^{-1}))$ denote the field of all formal Laurent series $C = \sum_{n=v}^{\infty} c_n z^{-n}$ in an indeterminate $z$, with coefficients $c_n$ all lying in the field $\mathbb{F}_q$. Recall that $\mathbb{F}_q[z]$ denotes the ring of polynomials in $z$ with coefficients in $\mathbb{F}_q$.

For a nonzero formal Laurent series $C$, we may assume that $c_v \neq 0$. Then the integer $v = v(C)$ is called the *order* of $C$. The *norm* (or *valuation*) of $C$ is defined to be $\|C\| = q^{-v(C)}$. It is well known that $\|\cdot\|$ is a non-Archimedean valuation on the field $\mathbb{F}_q((z^{-1}))$ and $\mathbb{F}_q((z^{-1}))$ is a complete metric space under the metric $\varrho$ defined by $\varrho(C_1, C_2) = \|C_1 - C_2\|$.

Remark 2.1. Since the valuation $\|\cdot\|$ is non-Archimedean, it follows that if two discs intersect, then one contains the other.

For $C = \sum_{n=v}^{\infty} c_n z^{-n} \in \mathbb{F}_q((z^{-1}))$, let $[C] = \sum_{v \le n \le 0} c_n z^{-n} \in \mathbb{F}_q[z]$. We call $[C]$ the *integral part* of $C$. It is evident that the integer $-v(C) := -v$ is equal to the degree $\deg[C]$ of the polynomial $[C]$ provided $v \le 0$, i.e., $[C] \neq 0$.

Let $I$ denote the valuation ideal of $\mathbb{F}_q((z^{-1}))$. It consists of all formal series $\sum_{n=1}^{\infty} c_n z^{-n}$. The ideal $I$ is compact. A natural measure on $I$ is the normalized Haar measure on $\prod_{n=1}^{\infty} \mathbb{F}_q$, which we denote by $\mathbf{P}$.

Consider the transformation from $I$ to $I$ defined by

$$Tx := \frac{1}{x} - \left[\frac{1}{x}\right], \qquad T0 := 0.$$

This map describes the regular continued fraction over the field of Laurent series and has been introduced by Artin [1]. As in the classical theory, every $x \in I$ has the following continued fraction expansion:

$$x = \cfrac{1}{A_1(x) + \cfrac{1}{A_2(x) + \cdots}} := [A_1(x), A_2(x), \ldots],$$

where the "digits" $A_i(x)$ are polynomials of strictly positive degree and are defined by

$$\forall n \ge 1, \qquad A_n(x) = \left[\frac{1}{T^{n-1}(x)}\right].$$

Let $\{P_n(x)/Q_n(x) : n \ge 0\}$ be the sequence of convergents in the expansion of $x$, i.e.,

$$\frac{P_n(x)}{Q_n(x)} = \cfrac{1}{A_1(x) + \cfrac{1}{A_2(x) + \cfrac{1}{\cdots + A_n(x)}}}.$$

The metric properties of continued fractions of Laurent series have been studied by Paysant-Leroux and Dubois [13], [14] (see also Niederreiter [11] and Berthé and Nakada [2]). In particular, we have

LEMMA 2.2. *The following results hold for all $x \in I$ outside a set of Haar measure* 0.

(i) *For $b \in \mathbb{F}_q[z]$ and $\deg b \ge 1$, the digit $b$ has asymptotic frequency $q^{-2\deg b}$, i.e.*,

$$\lim_{n\to\infty} \frac{\operatorname{card}\{1 \le j \le n : a_j(x) = b\}}{n} = q^{-2\deg b}.$$

(ii) *There exists a Khinchin-type constant*, *namely $q/(q-1)$*, *such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \deg a_k(x) = \frac{q}{q-1}.$$

A good tool to describe the complexity and size of a set with null measure is Hausdorff dimension. We recall the definition of Hausdorff dimension on $I$ which is the same as on $\mathbb{R}^n$. Given $s > 0$ and a subset $E$ of $I$, the *$s$-dimensional Hausdorff measure* is given by

$$\mathcal{H}^s(E) = \lim_{\delta\to 0} \Big\{ \inf \sum_j |D_j|^s \Big\},$$

where the infimum is over all covers of $E$ by discs $D_j$ (in the metric $\varrho$) of diameter $|D_j|$ at most $\delta$. The *Hausdorff dimension* of $E$ is defined by

$$\dim_{\mathrm{H}} E = \inf\{s : \mathcal{H}^s(E) = 0\}.$$

Niederreiter and Vielhaber [12, Theorem 31] determined the Hausdorff dimension of the sets $\{x \in I : \deg A_i(x) \le d \text{ for all } i \ge 1\}$ for any $d \in \mathbb{N}$. The second author generalized Niederreiter and Vielhaber's result by proving

LEMMA 2.3. *Let $S$ be a nonempty finite set of polynomials with positive degree and coefficients lying in $\mathbb{F}_q$, say $S = \{a_1, \ldots, a_m\}$. Write*

$$E_S = \{x \in I : A_i(x) \in S \text{ for all } i \ge 1\}.$$

*Then $\dim_{\mathrm{H}} E_S = t$, where $t$ is given by*

$$\sum_{k=1}^{m} q^{-2t\deg a_k} = 1.$$

In this paper, we consider the following two kinds of sets:

$$E_B = \{x \in I : A_n(x) \in B \text{ for all } n \geq 1\},$$
$$E(B) = \{x : A_n(x) \in B \text{ for all } n \geq 1 \text{ and } \deg A_n(x) \to \infty\},$$

where $B$ is an infinite set of polynomials with positive degree and coefficients lying in $\mathbb{F}_q$. We calculate the Hausdorff dimensions of these two sets:

THEOREM 2.4. $\dim_{\mathrm{H}} E_B = t$, *where*

$$t = \inf\Big\{s : \sum_{b\in B} (q^{-2\deg b})^s \leq 1\Big\}.$$

THEOREM 2.5. $\dim_{\mathrm{H}} E(B) = \alpha$, *where*

$$\alpha = \inf\Big\{s : \sum_{b\in B} (q^{-2\deg b})^s < \infty\Big\}.$$

**3. Proofs of the results.** In this section, we give the proofs of Theorems 2.4 and 2.5. First, we collect some known results which we are going to use frequently (see Niederreiter [11] or Berthé and Nakada [2]).

LEMMA 3.1. *For any $x \in I$, let $P_n(x)/Q_n(x)$ denote the $n$th convergent of $x$. Then*:

(i) $(P_n(x), Q_n(x)) = 1$;
(ii) $1 = \|Q_0(x)\| < \|Q_1(x)\| < \|Q_2(x)\| < \cdots$;
(iii) $\|Q_n(x)\| = \prod_{k=1}^{n} \|A_k(x)\| = q^{\sum_{k=1}^{n} \deg A_k(x)}$;
(iv) $\left\|x - \dfrac{P_n(x)}{Q_n(x)}\right\| = \dfrac{1}{\|Q_n(x)\| \cdot \|Q_{n+1}(x)\|} < \dfrac{1}{\|Q_n(x)\|^2}$;
(v) $x = \sum_{n=0}^{\infty} \dfrac{(-1)^n}{Q_n(x)Q_{n+1}(x)}$.

DEFINITION 3.2. Let $A_1, \ldots, A_n \in F_q[z]$ be of strictly positive degree. Call the set

$$I(A_1, \ldots, A_n) = \{x \in I : A_1(x) = A_1, \ldots, A_n(x) = A_n\}$$

a *fundamental $n$-cylinder.*

LEMMA 3.3. *Every fundamental $n$-cylinder $I(A_1, \ldots, A_n)$ is a disc with diameter*

$$|I(A_1, \ldots, A_n)| = q^{-2\sum_{k=1}^{n} \deg A_k - 1}$$

*and*

$$\mathbf{P}(I(A_1, \ldots, A_n)) = q^{-2\sum_{k=1}^{n} \deg A_k},$$

*where $\mathbf{P}$ is the Haar measure on $I$.*

Now we begin the proof of Theorem 2.4.

Write $B = \{b_1, b_2, \ldots\}$ and $B_n = \{b_1, \ldots, b_n\}$ for any $n \geq 1$. Let

$$E_{B,n} = \{x \in I : A_i(x) \in B_n \text{ for any } i \geq 1\} \quad \text{and} \quad t_n = \dim_{\mathrm{H}} E_{B,n}.$$

LEMMA 3.4. $\lim_{n\to\infty} t_n = t$, *where* $t$ *is given in Theorem* 2.4.

*Proof.* From Lemma 2.3, it is clear that $t_n$ is increasing and $t_n \leq t$ for any $n \geq 1$. Suppose $\lim_{n\to\infty} t_n = s$. For any $n \geq 1$, we have

$$\sum_{k=1}^{n} (q^{-2\deg b_k})^s \leq \sum_{k=1}^{n} (q^{-2\deg b_k})^{t_n} = 1.$$

Thus

$$\sum_{k=1}^{\infty} (q^{-2\deg b_k})^s \leq 1.$$

By the definition of $t$, we have $s \geq t$. ∎

*Proof of Theorem 2.4.* We divide the proof into two parts.

*Upper bound.* It is clear that

$$E_B = \bigcap_{n=1}^{\infty} \bigcup_{(A_1,\ldots,A_n)\in B^n} I(A_1, \ldots, A_n),$$

where $I(A_1, \ldots, A_n)$ is a fundamental $n$-cylinder as in Definition 3.2.

For any $s > t$, where $t$ is defined in Theorem 2.4, we have, on writing $s = (s+t)/2 + (s-t)/2$,

$$\sum_{k=1}^{\infty} (q^{-2\deg b_k})^s \leq q^{-(s-t)} \sum_{k=1}^{\infty} (q^{-2\deg b_k})^{(s+t)/2} \leq q^{-(s-t)}.$$

Therefore by Lemma 3.3,

$$\sum_{(A_1,\ldots,A_n)\in B^n} |I(A_1, \ldots, A_n)|^s \leq q^{-(s-t)} \sum_{(A_1,\ldots,A_{n-1})\in B^{n-1}} |I(A_1, \ldots, A_{n-1})|^s,$$

so by induction on $n$, the left hand side here is not larger than $q^{-n(s-t)}$, and

$$\mathcal{H}^s(E_B) \leq \liminf_{n\to\infty} \sum_{(A_1,\ldots,A_n)\in B^n} |I(A_1, \ldots, A_n)|^s = 0.$$

Thus $\dim_{\mathrm{H}} E_B \leq s$. Since $s > t$ is arbitrary, we have $\dim_{\mathrm{H}} E_B \leq t$.

*Lower bound.* It is clear that $E_{B,n} \subseteq E_B$ for any $n \geq 1$. Lemma 2.3 implies that $\dim_{\mathrm{H}} E_B \geq t_n$ for any $n \geq 1$. By Lemma 3.4, we have

$$\dim_{\mathrm{H}} E_B \geq \lim_{n\to\infty} t_n = t. \ ∎$$

Now we begin the proof of Theorem 2.5.

For any $n \geq 1$, let $C_n$ be a set of polynomials with strictly positive degree. Let

$$C = \{x \in I : A_n(x) \in C_n \text{ for all } n \geq 1\},$$
$$C_N = \{x \in I : A_n(x) \in C_n \text{ for all } n \geq N\}, \qquad N \in \mathbb{N}.$$

The following lemma is essentially due to I. J. Good [4].

Lemma 3.5. *For any $N \in \mathbb{N}$, $\dim_{\mathrm{H}} C = \dim_{\mathrm{H}} C_N$.*

Lemma 3.6. *For any $n \geq 1$, let $D_n = \operatorname{card}\{b \in B : \deg b = n\}$. Then*

$$\alpha = \limsup_{n\to\infty} \frac{\log D_n}{2n \log q},$$

*where $\alpha$ is given in Theorem 2.5.*

*Proof.* For any $\eta < \alpha$, since $\sum_{A\in B}(q^{-2\deg A})^{\eta} = \sum_{n=1}^{\infty} D_n q^{-2n\eta}$ diverges, there exist infinitely many $n$, say $\{n_k : k \geq 1\}$, such that

$$D_{n_k} q^{-2n_k\eta} \geq \frac{1}{n_k^2},$$

which implies

$$\limsup_{n\to\infty} \frac{\log D_n}{2n \log q} \geq \eta.$$

On the other hand, for any $\beta > \alpha$, since $\sum_{A\in B}(q^{-2\deg A})^{\beta} = \sum_{n=1}^{\infty} D_n q^{-2n\beta} < \infty$, we have $D_n q^{-2n\beta} \leq 1$ when $n$ is large enough. Thus

$$\limsup_{n\to\infty} \frac{\log D_n}{2n \log q} \leq \beta. \ \blacksquare$$

*Proof of Theorem 2.5.* We divide the proof into two parts.

*Upper bound.* For any $\beta > \alpha$, since $\sum_{A\in B}(q^{-2\deg A})^{\beta} < \infty$, there exists $M_\beta \in \mathbb{N}$ such that

$$\sum_{A\in B,\, \deg A\geq M_\beta} (q^{-2\deg A})^{\beta} \leq 1.$$

Notice that

$$E(B) = \bigcap_{m=1}^{\infty} \bigcup_{N=1}^{\infty} E(B,N,m),$$

where

$$E(B,N,m) = \{x \in I : A_n(x) \in B \text{ for all } n \geq 1, \text{ and } \deg A_n(x) \geq m \text{ for all } n \geq N\}.$$

From Lemma 3.5, we have

$$\dim_{\mathrm{H}} E(B) \leq \inf_{m\geq 1} \dim_{\mathrm{H}} E(B,1,m) \leq \dim_{\mathrm{H}} E(B,1,M_\beta).$$

Since

$$E(B,1,M_\beta)=\{x\in I: A_n(x)\in B \text{ and } \deg A_n(x)\geq M_\beta \text{ for all } n\geq 1\}$$
$$=\bigcap_{n=1}^{\infty}\bigcup_{\substack{(A_1,\ldots,A_n)\in B^n\\ \deg A_i\geq M_\beta,\, i=1,\ldots,n}} I(A_1,\ldots,A_n),$$

we have

$$\begin{aligned}\mathcal{H}^\beta(E(B,1,M_\beta)) &\leq \liminf_{n\to\infty}\sum_{\substack{(A_1,\ldots,A_n)\in B^n\\ \deg A_i\geq M_\beta,\, i=1,\ldots,n}} |I(A_1,\ldots,A_n)|^\beta\\ &\leq \liminf_{n\to\infty}\sum_{\substack{(A_1,\ldots,A_{n-1})\in B^{n-1}\\ \deg A_i\geq M_\beta,\, i=1,\ldots,n-1}}\ \sum_{\substack{A_n\in B\\ \deg A_n\geq M_\beta}} |I(A_1,\ldots,A_n)|^\beta\\ &\leq \liminf_{n\to\infty}\sum_{\substack{(A_1,\ldots,A_{n-1})\in B^{n-1}\\ \deg A_i\geq M_\beta,\, i=1,\ldots,n-1}}\Big(\sum_{\substack{A\in B\\ \deg A\geq M_\beta}} q^{-2\beta\deg A}\Big)|I(A_1,\ldots,A_{n-1})|^\beta\\ &\leq \liminf_{n\to\infty}\Big(\sum_{\substack{A\in B\\ \deg A\geq M_\beta}} q^{-2\beta\deg A}\Big)^n\leq 1.\end{aligned}$$

Thus $\dim_{\mathrm H} E(B)\leq \dim_{\mathrm H} E(B,1,M_\beta)\leq\beta$. Since $\beta>\alpha$ is arbitrary, we have $\dim_{\mathrm H} E(B)\leq\alpha$.

*Lower bound.* Now we prove $\dim_{\mathrm H} E(B)\geq\alpha$. If $\alpha=0$, we have the desired result. Assume $\alpha>0$. Let $n_0=\min\{\deg b: b\in B\}$.

For any $\varepsilon>0$ satisfying $\alpha-\varepsilon>0$, from Lemma 3.6, there exist $n_0<n_1<n_2<\cdots$ such that for any $k\geq 1$,

$$D_{n_k}\geq q^{2n_k(\alpha-\varepsilon)}. \tag{6}$$

Choose an integer sequence $\{t_k: k\geq 0\}$ satisfying

$$t_0=0,\quad \text{and}\quad t_k=n_{k+1}^2 \quad \text{for any } k\geq 1. \tag{7}$$

Let

$$E^*(B)=\Big\{x\in I: A_n(x)\in B \text{ for any } n\geq 1, \text{ and if } \sum_{i=0}^{k} t_i\leq n<\sum_{i=0}^{k+1} t_i$$
$$\text{for some } k\geq 0, \text{ then } n_k\leq \deg A_n(x)\leq n_{k+1}\Big\}.$$

It is clear that $E^*(B)$ is compact and $E^*(B)\subseteq E(B)$.

For any $s\geq 0$ and any covering system $W=\{w_1,w_2,\ldots\}$ of $E^*(B)$, write $\Lambda_s(W)=\sum_{i=1}^{\infty}|w_i|^s$.

CLAIM. *If there exists $\eta > 0$ such that $\varLambda_s(W) \geq \eta$ for any finite covering system $W$ which consists of fundamental cylinders, then* $\dim_{\mathrm{H}} E^*(B) \geq s$.

*Proof of Claim.* For any $\delta > 0$, by (7), there exists $K_0 \in \mathbb{N}$ such that for any $k \geq K_0$,

$$\frac{sn_{k+1}}{\sum_{i=1}^{k} t_i n_{i-1}} < \delta. \tag{8}$$

Let $U = \{u_1, u_2, \ldots\}$ be any disc covering system of $E^*(B)$ such that

$$|u_i| < q^{-2\sum_{j=1}^{K_0+1} t_j n_j} \quad \text{for any } i \geq 1. \tag{9}$$

Since $E^*(B)$ is compact, we can find a finite subsystem which also covers $E^*(B)$ and each disc of this subsystem intersects $E^*(B)$. It is clear that this subsystem, say $V = \{v_1, v_2, \ldots\}$, satisfies

$$\varLambda_{s-\delta}(U) \geq \varLambda_{s-\delta}(V).$$

For any $J \in V$, choose $x \in E^*(B) \cap J$. Suppose the continued fraction expansion of $x$ is $[A_1, A_{2,}, \ldots]$. From Remark 2.1, there exists a unique $n = n(J)$ such that

$$I(A_1, \ldots, A_n) \subseteq J \subseteq I(A_1, \ldots, A_{n-1}).$$

Suppose $\sum_{i=0}^{k} t_i \leq n < \sum_{i=0}^{k+1} t_i$ for some $k \geq 0$. Since $x \in E^*(B)$, by the definition of $E^*(B)$ and Lemma 3.3 we have

$$|J| \geq |I(A_1, \ldots, A_n)| = q^{-2\sum_{i=1}^{n} \deg A_i - 1} \geq q^{-2\sum_{i=1}^{k+1} t_i n_i}.$$

From (9), we have

$$k \geq K_0. \tag{10}$$

Write $I_J = I(A_1, \ldots, A_{n-1})$. Since

$$|J|^s \geq |I(A_1, \ldots, A_n)|^s = q^{-2s \deg A_n} |I(A_1, \ldots, A_{n-1})|^s = q^{-2s \deg A_n} |I_J|^s,$$

by (8) we have

$$\begin{aligned} |I_J|^s &\leq q^{2s \deg A_n} |J|^s \leq q^{2s \deg A_n - 2\delta(\deg A_1 + \cdots + \deg A_{n-1})} |J|^{s-\delta} \\ &\leq q^{2sn_{k+1} - 2\delta(t_1 n_0 + t_2 n_1 + \cdots + t_k n_{k-1})} |J|^{s-\delta} \leq |J|^{s-\delta}. \end{aligned} \tag{11}$$

Let $\widetilde{W} = \{I_J : J \in V\}$. We select all those discs in $\widetilde{W}$ which are maximal ($I_J$ is maximal if there is no $J' \in V$ such that $I_J \subsetneq I_{J'}$ and $J \neq J'$). Let $W$ be the set consisting of all maximal discs in $\widetilde{W}$. It is obvious that $W$ is a covering system of $E^*(B)$ by fundamental cylinders. By (11), we have

$$\varLambda_{s-\delta}(U) \geq \varLambda_{s-\delta}(V) \geq \varLambda_s(W) \geq \eta.$$

Since $\delta > 0$ is arbitrary, we have $\dim_{\mathrm{H}} E^*(B) \geq s$, and the proof of the Claim is finished.

From the Claim, we need only verify that $\varLambda_{\alpha-\varepsilon}(W) \geq \eta > 0$ for any finite covering system of $E^*(B)$ made up of fundamental cylinders, where

$\eta > 0$ is some fixed constant. Without loss of generality, we can assume that each element of $W$ is maximal.

Suppose the largest order of the fundamental cylinders in $W$ is $\kappa$. Then there exists $I(A_1, \dots, A_\kappa) \in W$. Suppose $\sum_{i=0}^{k} t_i \le \kappa < \sum_{i=0}^{k+1} t_i$ for some $k \ge 0$. If $\kappa > 0$, since each fundamental cylinder $I(A_1, \dots, A_{\kappa-1}, T)$, where $T \in B$ and $n_k \le \deg T \le n_{k+1}$, contains infinitely many points in $E^*(B)$, the fundamental cylinders $I(A_1, \dots, A_{\kappa-1}, R)$ with $R \in B$ and $n_k \le \deg R \le n_{k+1}$ must all be elements of $W$. By (6), we have

$$\begin{aligned}\sum_{R\in B,\, n_k \le \deg R < n_{k+1}} & |I(A_1, \dots, A_{\kappa-1}, R)|^{\alpha-\varepsilon} \\ &= \sum_{R\in B,\, n_k \le \deg R < n_{k+1}} q^{-2(\alpha-\varepsilon)\deg R} |I(A_1, \dots, A_{\kappa-1})|^{\alpha-\varepsilon} \\ &\ge D_{n_k} q^{-2n_k(\alpha-\varepsilon)} |I(A_1, \dots, A_{\kappa-1})|^{\alpha-\varepsilon} \ge |I(A_1, \dots, A_{\kappa-1})|^{\alpha-\varepsilon}.\end{aligned}$$

Denote by $L$ the new covering system of $E^*(B)$ obtained by just replacing all fundamental cylinders $I(A_1, \dots, A_{\kappa-1}, A_\kappa) \in W$ by the fundamental cylinder $I(A_1, \dots, A_{\kappa-1})$. Then

$$\Lambda_{\alpha-\varepsilon}(W) \ge \Lambda_{\alpha-\varepsilon}(L).$$

Proceeding in this manner, after a finite number of steps we reach a system whose largest order is zero, thus

$$\Lambda_{\alpha-\varepsilon}(W) \ge \Lambda_{\alpha-\varepsilon}(L) \ge \dots \ge |I|^{\alpha-\varepsilon} = q^{-(\alpha-\varepsilon)}.$$

By the Claim, we have

$$\dim_{\mathrm{H}} E^*(B) \ge \alpha - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and $E^*(B) \subseteq E(B)$, we have

$$\dim_{\mathrm{H}} E(B) \ge \alpha.$$

The proof of Theorem 2.5 is finished. ∎

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