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Waring's problem for fields

by

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To "Ian" Cassels for his tenth decade

1. Introduction. The classic Waring problem for $\mathbb{Z}$ has a vast literature (for a glimpse see [7], [24], [25]). The solution of the corresponding problems associated with expressing the elements of a field $\mathbf{K}$ as the sum of kth powers is much less complete. For $k = 2$ we have Artin's criterion that $\alpha \in \mathbf{K}$ is a sum of squares if and only if $\alpha \succ 0$ for all orderings of $\mathbf{K}$. For k an enumber, Becker [2] extended Artin's ideas to characterise $\alpha \in \mathbf{K}$ as a sum of kth powers if and only if $\alpha \succ 0$ for all orderings of $\mathbf{K}$ and $k \mid \upsilon(\alpha)$ for all valuations υ of $\mathbf{K}$ with a formally real valuation field.

Denote by $P(\mathbf{K}, k)$ the $\alpha \in \mathbf{K}$ which are sums of kth powers of elements of $\mathbf{K}$, andby $P^{+}(\mathbf{K}, k)$ the set of $\alpha \in \mathbf{K}$ which are sums of kth powers of totally positive elements of $\mathbf{K}$. We are interested in deciding whether or not there exist integers $w(\mathbf{K}, k)$ and $g(\mathbf{K}, k)$ such that:

(i) $\alpha \in P(\mathbf{K}, k)$ implies that α is the sum of at most $w(\mathbf{K}, k)$ kth powers;
(ii) $\alpha \in P^{+}(\mathbf{K}, k)$ implies that α is the sum of at most $g(\mathbf{K}, k)$ totally positive kth powers.

Neither Artin's nor Becker's characterisations give any information about the existence of $w(\mathbf{K}, k)$ or $g(\mathbf{K}, k)$. This is to be expected since there are many fields with $w(\mathbf{K}, 2) = \infty$, which implies that $w(\mathbf{K}, 2k) = \infty$.

The integer $w(\mathbf{K}, 2)$ is called the *Pythagorean number* of $\mathbf{K}$. For a given integer s, one can construct fields such that $w(\mathbf{K}, 2) = s$ (see [13]). However, for a given field it is usually a difficult problem to determine its Pythagorean number. For an algebraic number field $\mathbf{K}$ it is a classic result that $w(\mathbf{K}, 2) \le 4$ and $w(\mathbf{K}(X), 2) \le 5$ ([21], [14]). For $n \ge 2$, it is known that $5 + n \le w(\mathbf{K}(X_1, \ldots, X_n), 2) \le 2^{n+1}$ ([3], [6]).

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If $\mathbf{R}$ is a real closed field, then $w(\mathbf{R}(X), 2) = 2$, $w(\mathbf{R}(X_1, X_2), 2) = 4$ (cf. [4]) and for $n \geq 3$, $1 + n \leq w(\mathbf{R}(X_1, \ldots, X_n), 2) \leq 2^n$ ([3], [19], [20]). If $\mathbf{K}$ is a real field that is finitely generated of transcendence degree $d \geq 0$ over a subfield, then $w(\mathbf{K}, 2) \geq d + 1$ (see [11]); when $\mathbf{K}$ is an algebraic function field over a discrete valuation field, upper bounds for $w(\mathbf{K}, 2)$ are given in [1]. The Pythagoras number for Laurent series fields in several variables has been investigated by Hu [15], who gives many references to earlier work.

The equivalent problems over more general structures such as rings also have a very extensive literature; see [22] for references.

We will show that if $w(\mathbf{K}, 2)$ is finite and if the kth powers are dense (in a sense described explicitly in Theorem 2.5) in $\mathbf{K}$, then $w(\mathbf{K}, k)$ is also finite for $k > 2$. The proofs are constructive, but the implied upper bounds for $w(\mathbf{K}, k)$ are large. This is to be expected since the proofs do not use any deep arithmetical or algebraic properties of the field $\mathbf{K}$.

2. The basic theorems. Our first theorem treats the case when k is odd. It is well known and is included here only for completeness. When k is even it simplifies the exposition if we separate the two cases: Theorem 2.4 ($\mathbf{K}$ a non-real field, i.e. -1 is a sum of squares in $\mathbf{K}$) and Theorem 2.5 ($\mathbf{K}$ a formally real field), though the proofs are very similar. As an application of Theorem 2.5 we give (Theorem 4.2) a characterisation of those rational functions over certain real fields $\mathbf{K}$ which can be written as sums of kth powers of rational functions. Choi et al. [5] and Hsia and Johnson [14] show that $g(\mathbb{R}(X), 2) = 2$. We extend this result to prove (Theorem 5.1) that $w(\mathbf{K}(X), 2) = g(\mathbf{K}(X), 2)$. Some of the results given here (Theorems 2.4, 2.5) were reported in informal seminars ([8], [9]) at the University of Bordeaux in 1971.

The key result we need to prove Theorems 2.4 and 2.5 is the existence of 'Hilbert' identities:

LEMMA 2.1. *For all positive integers k and s there is an integer $M := M(k, s) = (2k + 3) \cdots (2k + 2 + s)/s!$, positive rational numbers λ_i with $0 \leq i \leq M$ and integers $\alpha_{i,j}$ with $0 \leq i \leq s$ and $0 \leq j \leq M$ such that we have an identity of the form:*

(i) $(X_0^2 + \cdots + X_s^2)^{k+1} = \sum_{j=1}^{M} \lambda_j (\alpha_{0j} X_0 + \cdots + \alpha_{sj} X_s)^{2k+2}$,

(ii) $(X_0^2 + \cdots + X_s^2)^k + 2k X_0^2 (X_0^2 + \cdots + X_s^2)^{k-1}$
$= (2k+1) \sum_{j=1}^{M} \lambda_j \alpha_{0j}^2 (\alpha_{0j} X_0 + \cdots + \alpha_{sj} X_s)^{2k}$.

Proof. For a short existence proof of (i), based upon the properties of convex sets, see [7]. For explicit constructive proofs see [23], [18], [17]. The identity (ii) is obtained from (i) by differentiating twice with respect to X_0. ∎

Corollary 2.2. *For any field* $\mathbf{K}$ *and positive integer* k, $P^+(\mathbf{K},k) = P(\mathbf{K},2k)$.

Proof. If $a \in P^+(\mathbf{K},k)$, then $a = b_1^k + \cdots + b_n^k$, where the b_i are totally positive. By Artin's theorem we have $b_i = c_1^2 + \cdots + c_r^2$ and we use the appropriate Hilbert identity to write b_i^k as a sum of $2k$th powers, so $a \in P(\mathbf{K},2k)$. Conversely, if $a \in P(\mathbf{K},2k)$, then

$$a = b_1^{2k} + \cdots + b_s^{2k} = (b_1^2)^k + \cdots + (b_n^2)^k \in P^+(\mathbf{K},k). \blacksquare$$

Theorem 2.3. *If* $\mathbf{K}$ *is a field of characteristic* 0, *then for every odd integer* k, $\mathbf{K} = P(\mathbf{K},k)$ *and* $w(\mathbf{K},k) < \infty$.

Proof. If $\Delta^1(x^k) = (x+1)^k - x^k$, $\Delta^r(x^k) = \Delta^1(\Delta^{r-1}(x^k))$ etc., then by calculating successively the differences we have:

$$\Delta^k(x^k) = k!x + \frac{(k-1)k!}{2},$$

$$\Delta^k(x^k) = \sum_{r=0}^{k-1}(-1)^{k-1-r}\frac{(k-1)(k-2)\cdots(k-r)}{r(r-1)\cdot 3\cdot 2\cdots 1}(x+r)^k.$$

We thus have the identity over $\mathbb{Q}[x]$:

$$x = -\frac{k-1}{2} + \sum_{r-1}^{k-1}(-1)^{k-1-r}\frac{(k-1)(k-2)\cdots(k-r)}{r(r-1)\cdots 3\cdot 2\cdot 1}(x+r)^k.$$

Since k is odd we can write the first term and each of the rational numbers in the summation as sums of kth powers. This proves the result. $\blacksquare$

Theorem 2.4. *If* $\mathbf{K}$ *is a non-real field of characteristic* 0, *then for any positive integer* k, $\mathbf{K} = P(\mathbf{K},k)$ *and* $w(\mathbf{K},k) = g(\mathbf{K},k) < \infty$.

Proof. If $-1 = a_1^2 + \cdots + a_s^2$ then any $\alpha \in \mathbf{K}$ can be written as

$$\alpha = \left(\frac{\alpha+1}{2}\right)^2 - \left(\frac{\alpha-1}{2}\right)^2 = \left(\frac{\alpha+1}{2}\right)^2 + \sum_{j=1}^{s} a_i^2\left(\frac{\alpha-1}{2}\right)^2.$$

Thus, every element of $\mathbf{K}$ is a sum of at most $s+1$ squares. If $a_i \in \mathbf{K}$ then we have $1 - a_i^2 = b_{1,i}^2 + \cdots + b_{s+1,i}^2$, where $b_{j,i} \in \mathbf{K}$ for $1 \le j \le (s+1)$. Substitute $X_0 = a_i$ and $X_j = b_{j,i}$ for $1 \le j \le s+1$ in the identity (ii) of Lemma 2.1 to obtain

$$1 + 2ka_i^2 = (2k+1)\sum_{j=1}^{M}\lambda_j\alpha_{0,j}^2(\alpha_{0,j}a_i + \alpha_{1,j}b_{1,i} + \cdots + \alpha_{s,j}b_{s,i})^{2k}.$$

If $a \in \mathbf{K}$ we have $a = a_1^2 + \cdots + a_{s+1}^2$ with $a_i \in \mathbf{K}$. Substitute successively each of the a_i into the above identity and sum the resulting set of $s+1$ equations to obtain

$$s+1+2ka=\sum_{i=1}^{s+1}\sum_{j=1}^{M}(2k+1)\lambda_j\alpha_{0,j}^2(\alpha_{0,j}a_i+\alpha_{1,j}b_{1i}+\cdots+\alpha_{s,j}b_{s,i})^{2k}.$$

Let A_k be the least common multiple of the denominators of the λ_j for $0\le j\le M$, so that $A_k\lambda_j=\Lambda_j$, a positive integer. We now have

$$\begin{aligned}(s+1)A_k+2kA_ka&=\sum_{i=1}^{s+1}\sum_{j=1}^{M}(2k+1)\Lambda_j\alpha_{0j}^2(\alpha_{0j}a_i+\alpha_{1j}b_{1i}+\cdots+\alpha_{sj}b_{si})^{2k}\\&=\sum_{i=1}^{N}\zeta_i^{2k}\end{aligned}$$

where $N=(2k+1)(s+1)\sum_{j=1}^{M}\Lambda_j\alpha_{0,j}^2$ and $\zeta_i\in\mathbf{K}$, for $1\le i\le N$.

For any $\alpha\in\mathbf{K}$ we take $a=(\alpha-(s+1)A_k)/2kA_k$ and substitute in the above equation to obtain $\alpha=\sum_{i=1}^{N}\zeta_i^{2k}$, and the assertion follows with $w(\mathbf{K},k)=g(\mathbf{K},k)\le N$. ∎

Theorem 2.5. *Let* $\mathbf{K}$ *be a formally real field. Suppose that* $\alpha\in\mathbf{K}$ *has the following two properties:*

(a) *α can be written as a sum of at most s squares in* $\mathbf{K}$.
(b) *There exists $\beta\in\mathbf{K}$, depending upon α, that satisfies*

$$0\prec\frac{s}{s+2k}\alpha\prec\beta^k\prec\alpha\quad\text{for all orderings }\prec\text{ of }\mathbf{K}.$$

Then there exists a positive integer $\gamma(s,k)$, depending only on s and k, such that:

(i) *α can be written as a sum of $\gamma(s,k)$ kth powers of elements of* $\mathbf{K}$.
(ii) *If β can be chosen to be totally positive, then α can be written as a sum of $\gamma(s,k)$ kth powers of totally positive elements of* $\mathbf{K}$.

Proof. If $a_i\in\mathbf{K}$ and $1-a_i^2\succ 0$ for all orderings '$\succ$' of $\mathbf{K}$, then $1-a_i^2$ is totally positive and by hypothesis (a) we have $1-a_i^2=b_{1,i}^2+\cdots+b_{s,i}^2$, where $b_{j,i}\in\mathbf{K}$ for $1\le j\le s$. Substitute $X_0=a_i$ and $X_j=b_{j,i}$ for $1\le j\le s$ in the identity (ii) of Lemma 2.1 to obtain

$$s+2ka=\sum_{i=1}^{s}\sum_{j=1}^{M}(2k+1)\lambda_j\alpha_{0,j}^2(\alpha_{0,j}a_i+\alpha_{1,j}b_{1,i}+\cdots+\alpha_{s,j}b_{s,i})^{2k}.$$

If $a\in\mathbf{K}$ and $1\succ a\succ 0$ for all orderings '$\succ$' of $\mathbf{K}$, then by hypothesis (a) we have $a=a_1^2+\cdots+a_s^2$ with $a_i\in\mathbf{K}$. Substitute the a_i into the above identity and sum the resulting set of s equations to obtain

$$s+2ka=\sum_{i=1}^{s}\sum_{j=1}^{M}(2k+1)\lambda_j\alpha_{0,j}^2(\alpha_{0,j}a_i+\alpha_{1,j}b_{1,i}+\cdots+\alpha_{s,j}b_{s,i})^{2k}.$$

Let A_k be the least common multiple of the denominators of the λ_j for $0 \le j \le M$, so that $A_k\lambda_j = \Lambda_j$, a positive integer. We now have

$$sA_k + 2kA_k a = \sum_{i=1}^{s}\sum_{j=1}^{M}(2k+1)\Lambda_j\alpha_{0,j}^2(\alpha_{0,j}a_i + \alpha_{1,j}b_{1,i} + \cdots + \alpha_{s,j}b_{s,i})^{2k}$$
$$= \sum_{i=1}^{N}\zeta_i^{2k}$$

where $N = s(2k+1)\sum_{j=1}^{M}\Lambda_j\alpha_{0,j}^2$ and $\zeta_i \in \mathbf{K}$ for $1 \le i \le N$.

If $\alpha \in \mathbf{K}$ and α is totally positive, then so is $A_k(s+2k)/\alpha$. Hence, by hypothesis (b) there is a $\beta \in \mathbf{K}$, which depends upon α, such that

$$\left(\frac{A_k(s+2k)}{\alpha}\right) \succ \beta^k \succ \frac{s}{s+2k}\left(\frac{A_k(s+2k)}{\alpha}\right) \succ 0$$

for all orderings '$\succ$' of $\mathbf{K}$. If we put $a = (\alpha\beta_k - sA_k)/2kA_k$, then $1 \succ a \succ 0$ for all orderings '$\succ$' of $\mathbf{K}$. Substitute this a in the above equation to obtain $\alpha\beta_k = \sum_{i=1}^{N}$, and the assertions of the theorem follow with $\gamma(s,k) = N$. ∎

Corollary 2.6. *If* $\mathbf{K} = \mathbb{Q}$, *then* $w(\mathbb{Q},k) \le g(\mathbb{Q},k) < \infty$ *for every positive integer* k.

Proof. Any positive rational number is the sum of at most 4 squares of positive rational numbers. There is only one ordering on $\mathbb{Q}$ and for any positive integer k and positive rational number α there is a positive rational number β such that $2\alpha/(2+k) < \beta^k < \alpha$. ∎

Corollary 2.7. *If* $\mathbf{K}$ *is an algebraic number field, then for every positive integer* k, $w(\mathbb{Q},k) \le g(\mathbb{Q},k) < \infty$.

Proof. If $\mathbf{K}$ is a totally imaginary number field, then it is a classic theorem that every $\alpha \in \mathbf{K}$ is the sum of at most 4 squares and the result follows from Theorem 2.4. If $\mathbf{K}$ is an algebraic number field that is not totally imaginary, then it is a classic result that every totally positive element of $\mathbf{K}$ can be written as a sum of at most 4 squares. Thus, hypothesis (a) is satisfied.

We now show that hypothesis (b) can be satisfied with β totally positive. If $\mathbf{K} = \mathbb{Q}(\theta)$, denote by $\mathbf{K}^{(1)} = \mathbb{Q}(\theta^{(1)}), \ldots, \mathbf{K}^{(r)} = \mathbb{Q}(\theta^{(r)})$ the real conjugate fields. We need the following trivial remark: If $\epsilon > 0$, $\eta_1, \ldots, \eta_r$ are given real numbers, then there is a $\beta \in \mathbf{K}$ such that $|\beta^{(i)} - \eta_i| < \epsilon$ for $1 \le i \le r$. The proof is simple: Let $f(x)$ be a polynomial of degree $r-1$ with real coefficients which takes the values $\eta_i + \epsilon/2$ at $x = \theta^{(i)}$ for $1 \le i \le r$. Let $g(x)$ be a polynomial of degree $r-1$ with rational coefficients such that $|f(x) - g(x)| < \epsilon/2$ for $x = \theta^{(1)}, \ldots, \theta^{(r)}$. Put $\beta = g(\theta)$; then $\beta^{(i)} = g(\theta^{(i)})$ for $1 \le i \le r$ and we have $|\beta^{(i)} - \eta_i| < \epsilon$ for $1 \le i \le r$.

We now return to the verification of hypothesis (b). If $\alpha \in \mathbf{K}$ and α is totally positive, then $\alpha^{(i)} > 0$ for $1 \le i \le r$. To satisfy hypothesis (b) we must find a $\beta \in \mathbf{K}$, depending upon α, such that

$$0 < \left(\frac{2}{2+k}\alpha^{(i)}\right)^{1/k} < \beta^{(i)} < (\alpha^{(i)})^{1/k} \quad \text{for } 1 \le i \le r.$$

The existence of such a β follows from the above remark by taking

$$\eta_i = \frac{1}{2}\left[\left(\frac{2}{2+k}\alpha^{(i)}\right)^{1/k} + (\alpha^{(i)})^{1/k}\right] \text{ for } 1 \le i \le r \quad \text{and} \quad \epsilon \text{ small enough.} \ \blacksquare$$

3. The upper bounds for $g(\mathbf{K}, k)$. The upper bounds for $\gamma(s,k)$ implied by the proofs of Theorems 2.4 and 2.5 are only of interest as an existence proof. We can produce a better upper bound by a trivial remark. From Corollary 2.6 we see that $g(\mathbb{Q}, k)$ is finite, so we can replace the integers $(2k+1)\Lambda_j\alpha_j^2$ by sums of at most $g(\mathbb{Q}, k)$ kth powers of rational numbers in the final sum. This slight change gives us an upper bound for $\gamma(s,k) \le g(\mathbb{Q},k)sM(s,k)$.

The precise value of $g(\mathbb{Q}, k)$ is only known for 3 values of k: $g(\mathbb{Q},2) = 4$, $g(\mathbb{Q},3) = 3$ and $g(\mathbb{Q},4) = 15$ (see [12]). For odd k, only $w(\mathbb{Q},3) = 3$ is known. It is obvious that, with the usual notation of the classic Waring problem for $\mathbb{Z}$, $g(\mathbb{Q},k) \le G(k)$. The best known estimate for $G(k)$ (see [25]) is: $G(k) < k(\log k + \log\log k + 2 + O(\log\log k/\log k))$. No general lower bound for $w(\mathbf{K}, k)$ is known.

4. Sums of kth powers in $\mathbf{K}(X)$. If $\mathbf{K}$ is a formally real field, we need to know that $w(\mathbf{K}(X), 2)$ is finite and that hypothesis (ii) of Theorem 2.5 is satisfied for $\mathbf{K}(X)$. This involves showing that certain field quantities are totally positive. In order to simplify the enunciations and proofs of our theorems we need a result of the type: *$f(X) \in \mathbf{K}(X)$ is totally positive if and only if $f(x) \ge 0$ for all $x \in \mathbf{K}$.* Unfortunately such a statement is false in general without some restrictions upon $\mathbf{K}$. One such restriction, due to Artin, is that $\mathbf{K}$ is a formally real field with precisely one ordering and this ordering is Archimedean. For example, we can take $\mathbf{K} = \mathbb{Q}$ or $\mathbb{R}$ or a finite algebraic extension of $\mathbb{Q}$ with precisely one real conjugate field. From now on $\mathbf{K}$ will be such a field and we will consider it as a subfield of $\mathbb{R}$. With this restriction upon $\mathbf{K}$, hypothesis (b) of Theorem 2.5 becomes: *If $f(X) \in \mathbf{K}(X)$ is such that $f(x) \ge \mu > 0$ for all $x \in \mathbf{K}$, there is a rational function $b(X) \in \mathbf{K}(X)$ such that*

$$1 < \frac{f(x)}{b(x)^k} < 1 + \frac{2k}{s} \quad \textit{for all } x \in \mathbf{K}.$$

The existence of such a rational function will be inferred from the following lemma. The proof is, in principle, constructive, since one can use the Bernstein polynomials to construct the function, but it is of no real use for finding a suitable $b(X)$ in practice.

LEMMA 4.1. *Let $F(x)$ be a strictly positive definite, everywhere defined continuous real-valued function on $\mathbb{R}$. Suppose that there exist real numbers a, b, δ, C and a positive definite rational function $h(X) \in \mathbb{Q}(X)$, defined for all $x \in \mathbb{R}$, such that*

(1) $0 < a < F(x)/h(x)^k < b < \infty$,
(2) $0 < \delta \leq F(x)$

for all x with $x^2 \geq C > 1$. Then, given $\epsilon > 0$, there exists $\gamma(X) \in \mathbb{Q}(X)$ such that, for all $x \in \mathbb{R}$,

$$0 < a - \epsilon < \frac{F(x)}{\gamma(x)^k} < b + \epsilon.$$

Proof. The idea of the proof is simple: we use the Weierstrass approximation theorem to construct a rational function $\gamma(X) \in \mathbb{Q}(X)$ which is very close to $h(x)$ for all $x \in \mathbb{R}$ with $x^2 > C$ and is sandwiched between

$$\left(\frac{F(x)}{b+\epsilon}\right)^{1/k} \quad \text{and} \quad \left(\frac{F(x)}{a-\epsilon}\right)^{1/k} \quad \text{for all } x \in \mathbb{R} \text{ with } x^2 \leq C.$$

However, the details can get a little confusing. We note that hypothesis (2) on $F(x)$, together with the fact that $F(x)$ is strictly positive definite, implies that there is a $\Delta > 0$ such that $F(x) \geq \Delta > 0$ for all $x \in \mathbb{R}$.

We will use the following function, where $C > 1$ and $m > 0$ is an integer:

$$\alpha_m(X) = \left[1 + \left(\frac{X^2}{C+1}\right)^m\right]^{-1}.$$

The principal properties of $\alpha_m(X)$ are:

(a) For each positive integer m and all $x \in \mathbb{R}$ we have $0 < \alpha_m(x) \leq 1$.
(b) If $x^2 > C + 1$, then $\alpha_m(x) \to 0$ as $m \to \infty$.
(c) If $x^2 = C + 1$, then $\alpha_m(x) = 1/2$ for all m.
(d) If $x^2 < C + 1$, then $\alpha_m(x) \to 1$ as $m \to \infty$.

The proof is divided into five steps; each step constructs a successive approximation to the function $h(X)$.

STEP (i). *Given any $\epsilon > 0$, if $m > m_0(\epsilon)$ then the function*

$$H_m(x) = \{1 - \alpha_m(x)\}h(x) \in \mathbb{Q}$$

satisfies the inequalities

$$\frac{F(x)}{b+\epsilon} < H_m^k(x) < \frac{F(x)}{a} \quad \text{for all } x \in \mathbb{R} \text{ with } x^2 \geq C + \frac{3}{2}.$$

Proof of Step (i). We have

$$H_m^k(x) = \{1-\alpha_m(x)\}^k h^k(x) \le h^k(x) < \frac{F(x)}{a}$$

and if we take $\epsilon_1 > 0$ such that $b(1-\epsilon_1)^{-k} < b+\epsilon$, then, by taking m sufficiently large, depending only upon ϵ_1, we have $0 < \alpha_m(x) < \epsilon_1$ for all $x \in \mathbb{R}$ which satisfy $x^2 > C + 3/2$. It then follows that

$$H_m^k(x) = \{1-\alpha_m(x)\}^k h^k(x) \ge (1-\epsilon_1)^k h^k(x) \ge \frac{(1-\epsilon_1)^k}{b} F(x) > \frac{F(x)}{b+\epsilon}.$$

We now define a continuous function $G(x)$ on the compact set $x^2 \le C + 3/2$ as follows:

(1) $G(x) = \frac{1}{2}(1/a^{1/k} + (1/a^{1/k})F(x)^{1/k}$ for $x \in \mathbb{R}$ and $x^2 \le C$.
(2) $G(x) = h(x)$ for $x \in \mathbb{R}$ and $C + 1/2 \le x^2 \le C + 3/2$.
(3) For $x \in \mathbb{R}$ that satisfy $C \le x^2 \le C+1/2$, $G(x)$ can be any continuous function satisfying:

(a) $G(x) = \frac{1}{2}(1/a^{1/k} + (1/a^{1/k})F(x)^{1/k}$ whenever $x^2 = C$.
(b) $G(x) = h(x)$ whenever $x^2 = C + 1/2$.
(c) $[F(x)/b]^{1/k} < G(x) < [F(x)/a]^{1/k}$ for $C < x^2 < C + 1/2$.

By the Weierstrass polynomial approximation theorem, given any $\epsilon_2 > 0$ there is a polynomial $P(X) \in \mathbb{Q}(X)$ such that $|G(x) - P(x)| < \epsilon_2$ for all $x \in \mathbb{R}$ with $x^2 \le C + 3/2$.

Consider the following rational function, with rational coefficients, defined for all $x \in \mathbb{R}$: $\gamma_m(X) = \alpha_m(X) \cdot P(X) + H_m(X)$. We will now show, in the next four steps, that if m is sufficiently large, then $\gamma_m(x)$ satisfies the inequalities of the lemma.

Step (ii). *Given $\epsilon_3 > 0$ there exists $m_3(\epsilon_3)$ such that for all $m > m_3$ we have $0 < |\alpha_m(x) \cdot P(x)| < \epsilon_3$ for all $x \in \mathbb{R}$ with $x^2 \ge C + 3/2$.*

Proof of Step (ii). We have $\alpha_m(X) \cdot P(X) = P(X)[1 + X^{2m}/(C+1)^m]^{-1}$. If the degree of $P(X)$ is r and if $m > r$, then it is a bounded function of x for $x^2 \ge C + 3/2$ and, as $m \to \infty$, this maximum value tends to zero.

Step (iii). *If ϵ_2 satisfies the inequalities*

$$0 < \epsilon_2 < \Delta^{1/k} \cdot \min\left(\left[\left(\frac{1}{a-\epsilon}\right)^{1/k} - \left(\frac{1}{a}\right)^{1/k}\right], \left[\left(\frac{1}{b+\epsilon}\right)^{1/k} - \left(\frac{1}{b+2\epsilon}\right)^{1/k}\right]\right),$$

then if $m > m_2(\epsilon_2)$, for all $x \in \mathbb{R}$ with $x^2 \ge C + 3/2$ we have

$$\left[\frac{F(x)}{b+2\epsilon}\right]^{1/k} < \gamma_m(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k}.$$

Proof of Step (iii). If $m > m_3(\epsilon_2)$, we deduce by Step (i) that

$$\gamma_m(x) = (1-\alpha_m(x))\cdot h(x) + \alpha_m(x)\cdot P(x) \le (1-\alpha_m(x))\cdot h(x) + \epsilon_2$$
$$\le \left[\frac{F(x)}{a}\right]^{1/k} + \epsilon_2 \le \left[\frac{F(x)}{a-\epsilon}\right]^{1/k}.$$

And if $m > m_3(\epsilon_2)$ we have

$$\left[\frac{F(x)}{b+2\epsilon}\right]^{1/k} + \epsilon_2 \le \left[\frac{F(x)}{b+\epsilon}\right]^{1/k} < (1-\alpha_m(x))\cdot h(x).$$

Hence

$$\left[\frac{F(x)}{b+\epsilon}\right]^{1/k} < (1-\alpha_m(x))\cdot h(x) + \alpha_m(x)\cdot P(x) = \gamma_m(x).$$

STEP (iv). *If $m > m_1(\epsilon)$ then for all $x \in \mathbb{R}$ with $x^2 \le C + 1/2$ we have*

$$\left[\frac{F(x)}{b+2\epsilon}\right]^{1/k} < \gamma_m(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k}.$$

Proof of Step (iv). Let ϵ_3 satisfy the inequality

$$0 < 2\epsilon_3 < \min\left(\inf\left[G(x) - \left(\frac{F(x)}{b+\epsilon}\right)^{1/k}\right], \inf\left[\left(\frac{F(x)}{a-\epsilon}\right]^{1/k} - G(x)\right]\right)$$

where the inf's are over the $x \in \mathbb{R}$ with $x^2 \le C + 1/2$.

If $m > m_1(\epsilon_3)$ then $|1-\alpha_m(x)\cdot h(x)| < \epsilon_3/4$ and $|1-\alpha_m(x)\cdot P(x)| < \epsilon_3/4$ for all $x \in \mathbb{R}$ with $x^2 \le C + 1/2$, since $\alpha_m(w) \to 1$ uniformly as $m \to \infty$ and $h(x)$ and $P(x)$ are bounded. Thus we have $|\gamma_m(x) - P(x)| < \epsilon_3/2$ and $|\gamma_m(x) - F(x)| < \epsilon_3$ for all $x \in \mathbb{R}$ with $x^2 \le C + 1/2$.

Since we have the inequalities

$$\left[\frac{F(x)}{b+\epsilon}\right]^{1/k} + 2\epsilon_3 < G(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k},$$

we can conclude that, for all $x \in \mathbb{R}$ with $x^2 \le C + 1/2$,

$$\left[\frac{F(x)}{b+\epsilon}\right]^{1/k} < \gamma_m(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k} - 2\epsilon_3.$$

STEP (v). *If $m > m_3(\epsilon_2)$ then for all $x \in \mathbb{R}$ which satisfy $C \le x^2 \le C + 1/2$ we have*

$$\left[\frac{F(x)}{b+\epsilon}\right]^{1/k} < \gamma_m(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k}.$$

Proof of Step (v). For values of x in the above range we have $G(x) = h(x)$, and so $|P(x) - h(x)| < \epsilon_2/2$.

Thus, $|\gamma_m(x) - h(x)| = |\gamma_m(x) \cdot (P(x) - h(x))| < \epsilon_2/2$ and by hypothesis

$$\left[\frac{F(x)}{b}\right]^{1/k} < h(x) < \left[\frac{F(x)}{a}\right]^{1/k}, \quad \left[\frac{F(x)}{b+\epsilon}\right]^{1/k} < \gamma_m(x) < \left[\frac{F(x)}{a-\epsilon}\right]^{1/k}.$$

To complete the proof of Lemma 4.1, take $m > \max[m_0, m_1, m_2, m_3]$ and $\gamma(x) = \gamma_m(x)$. ■

THEOREM 4.2. *Let $k > 2$ be a positive integer. If $f(X) \in \mathbf{K}(X)$ is positive definite, then a necessary and sufficient condition that $f(X)$ can be written as a sum of $g(\mathbf{K}(X), k)$ kth powers of totally positive elements of $\mathbf{K}(X)$ is that $2k \mid \partial(f)$.*

Proof. The condition on the degree of $f(X)$ is obviously necessary. There is no loss in generality if we suppose that $f(X) \in \mathbf{K}[X]$ and that $f(X)$ is strictly positive definite, i.e. there is a $\mu > 0$ such that $f(x) \geq \mu$ for all $x \in \mathbf{K}$. Indeed, $f = p/q$ is a sum of n kth powers in $\mathbf{K}(X)$ if and only if $p \cdot q^{k-1}$ is a sum of n kth powers and if k is even or, whenever k is odd and all the summands are positive definite, then if $f(\theta) = 0$, with $\theta \in \mathbb{R}$, the irreducible polynomial satisfied by θ is a kth power factor of $f(X)$.

For the fields $\mathbf{K}$ under consideration we note that $w(\mathbf{K}(X), 2) \leq \infty$. To prove the theorem it suffices to show that the modified hypothesis (ii) of Theorem 2.5 is satisfied. That is, we can find a rational function $b(X) \in \mathbf{K}(X)$ such that $1 < f(x)/b(x)^k < 1 + 2k/s$ for all $x \in \mathbf{K}$. We use Lemma 4.1 with ϵ in the range $0 < \epsilon < k/(2s + 2k)$.

If $f(X) = \alpha X^{2m} + \cdots$ with $\alpha > 0$, we take $\beta \in \mathbb{Q}$ such that

$$1 - \frac{\epsilon}{2} < \frac{\alpha s}{(s+k)\beta^k} < 1 + \frac{\epsilon}{2} \quad \text{and} \quad h(X) = \beta X^{2m} + 1.$$

Then, for all $x \in \mathbf{K}$ with $x^2 \geq C(\epsilon)$ we have

$$1 - \epsilon < \frac{sf(x)}{(s+k)h(x)^k} < 1 + \epsilon.$$

From Lemma 4.1 there is a $\gamma(X) \in \mathbb{Q}(X)$ such that

$$1 - 2\epsilon < \frac{sf(x)}{(s+k)\gamma(x)^k} < 1 + 2\epsilon$$

and since $\epsilon < k/(2s + 2k)$ we have $1 < (1-2\epsilon)(s+k)/s$ and $(1+2\epsilon)(s+k)/s < 1 + k/s$. ■

COROLLARY 4.3. *If $f(X) \in \mathbf{K}[X]$ is positive definite, k odd and $k \mid \partial(f)$, then $f(X)$ is the sum of at most $g(\mathbf{K}, k)$ kth powers of positive definite rational functions.*

Proof. Since $f(X)$ is strictly positive definite, $2 \mid \partial(f)$ and k odd, we see that $k \mid \partial(f)$ implies that $2k \mid \partial(f)$. ■

COROLLARY 4.4. *For any integer $k > 0$, a positive definite polynomial $f(X) \in \mathbf{K}[X]$, with degree divisible by k, can be written as a sum of at most $w(\mathbf{K}, k)$ kth powers of $\mathbf{K}(X)$.*

Proof. We need only treat the case of k even. Suppose $k = 2^r k_1$, where $r > 0$ and k_1 is odd. By hypothesis $2 \cdot 2^{r-1} k_1 \mid \partial(f)$, so from the theorem we can write $f(X)$ as a sum of at most $g(\mathbf{K}, k/2)$ $(k/2)$th powers of positive definite rational functions, say $f(X) = \sum_{i=1}^{n} q_i^{k/2}(X)$ with $n \leq g(\mathbf{K}, k/2)$. Each $a_i(X)$ is positive definite and so can be written as a sum of s squares of rational functions: $a_i^{k/2} = (b_{1,i}^2 + \cdots + b_{s,i}^2)^{k/2}$.

We now use the corresponding Hilbert identity of Lemma 2.1 with $k+1$ replaced by $k/2$ to get a representation of $f(X)$ as a sum of kth powers of rational functions. ∎

COROLLARY 4.5. *The positive definite rational functions $f(X) = a(X)/b(X)$, $a(X), b(X) \in \mathbf{K}[X]$, that can be expressed as sums of at most $g(\mathbf{K}, k)$ kth powers, for every positive integer k, are precisely the set of positive definite rational functions with $\partial(a) = \partial(b)$.*

Proof. If $\partial(a) = m$ and $\partial(b) = n$ then $f(X) = a(X)/b(X)$ is a sum of kth powers if and only if $a(X)b^{k-1}(X)/b^k(X)$ is a sum of kth powers, $a(X)b^{k-1}(X)$ is a sum of kth powers if and only if $k \mid (m + (k-1)n)$. This latter condition holds for all k if and only if $m = n$. ∎

5. Upper bounds for $w(\mathbf{K}(X), k)$ and $g(\mathbf{K}(X), k)$. There is little precise information available. For $k = 2$, $w(\mathbf{K}(X), 2) = s(\mathbf{K}) + 1$ (see [14] when $\mathbf{K}$ is an algebraic number field, and it is classic when $\mathbf{K} = \mathbb{R}$). Choi et al. [5] and Hsia and Johnson [14] showed that $g(\mathbb{R}(X), 2) = 2$. We extend this result in Theorem 5.1 by showing that $w(\mathbf{K}(X), 2) = g(\mathbf{K}(X), 2)$.

THEOREM 5.1. *Let $\mathbf{K}$ be a real field satisfying Artin's criterion, and let $f(X) \in \mathbf{K}[X]$.*

(1) *If $4 \mid \partial f$ and $f(X) = a_1^2(X) + \cdots + a_n^2(X)$ where $a_i(X) \in \mathbf{K}(X)$, then $f(X)$ can be represented as $f(X) = g_1^2(X) + \cdots + g_n^2(X)$ where $g_i(X) \in \mathbf{K}(X)$ and $g_i(x) \geq 0$ for all $x \in \mathbf{K}$.*
(2) *If $4 \mid \partial f$, $f(x) \geq \mu_0 > 0$ for all $x \in \mathbf{K}$ and $f(X) = a_1^2(X) + \cdots + a_n^2(X)$ where $a_i(X) \in \mathbf{K}(X)$, then $f(X)$ can be represented as $f(X) = g_1^2(X) + \cdots + g_n^2(X)$ where $g_i(X) \in \mathbf{K}(X)$ and $g_i(x) \geq \mu_i > 0$ for all $x \in \mathbf{K}$.*
(3) *If $w(\mathbf{K}, 2)$ is finite, then $w(\mathbf{K}(X), 2) = g(\mathbf{K}(X), 2)$.*

Proof. The third conclusion is an obvious consequence of the first two assertions. We first show that the truth of (2) implies the truth of (1).

Recall Corollary 2.2 of Lemma 2.1: $P^+(\mathbf{K},k) = P(\mathbf{K},4k)$. Suppose that $f(x) \geq 0$ for all $x \in \mathbf{K}$ and that $f(X) \in P(\mathbf{K}(X),4)$. We have $f(X) = a_1^4(X) + \cdots + a_r^4(X)$, and if $\gamma \in \mathbb{R}$ is such that $f(\gamma) = 0$, then $a_i(\gamma) = 0$ for $i = 1,\ldots,n$ and this is true for all the conjugates of γ. Hence the irreducible polynomial $p(X) \in \mathbb{Q}[X]$ defining γ must divide each $a_i(X)$, which implies that $p^4(X)$ divides $f(X)$. Repeating the process we have $f(X) = P^4(X)F_1(X)$, where $F_1(x) \geq \mu > 0$ for all $x \in \mathbf{K}$.

If $f(X) = b_1^2(X) + \cdots + b_n^2(X)$, then $P^2(X)$ divides each $b_i(X)$ and we have a representation of $F_1(X)$ as a sum of n squares. Since $4 \mid \partial F_1(X)$, by (2), we have $F_1(X) = g_1^2(X)+\cdots+b_n^2(X)$ where $g_i(x) \geq \mu_i > 0$ for all $x \in \mathbf{K}$. We then have the representation $f(X) = P^4(X)F_1(X) = (P^2(X)g_1(X))^2 + \cdots + (P^2(X)g_n(X))^2$, where, for $1 \leq i \leq n$, $P^2(x)g_i(x) \geq 0$ for all $x \in \mathbf{K}$.

We shall prove (2) by induction on n. If $n = 1$, then $f(X) = a_1^2(X)$ and the result is immediate.

The above remarks imply the following induction hypothesis: (2) *holds for all* $f(X) \in P(\mathbf{K}(X),4)$ *that can be written as a sum of at most* $n-1$ *squares in* $\mathbf{K}(X)$. This will be used in the proof of Lemma 5.4.

In order to prove the case n of the induction step, we need the following four lemmas, which show that starting from any representation of $f(X)$ as a sum of squares we can construct representations with increasing positivity conditions on one or more of the summands.

Lemma 5.2. *If* $\mathbf{K}$ *is a field, not of characteristic* 2, *and*

$$f = a_1^2 + a_2^2 + \cdots + a_n^2, \qquad f, a_i \in \mathbf{K} \text{ for } 1 \leq i \leq n,$$

then the general solution $(u_0,\ldots,u_n)$ *of* $fU_0^2 = U_1^2 + \cdots + U_n^2$ *is*

$$u_0 = \sum_{j=1}^n T_j^2, \qquad u_i = 2T_i\Big(\sum_{j=1}^n a_jT_j\Big) - a_i\sum_{j=1}^n T_j^2$$

where $T_j \in \mathbf{K}$ *for* $1 \leq i \leq n$.

Proof. If $\mathbf{a} = (a_1,\ldots,a_n)$, where $f = a_1^2+\cdots+a_n^2$, $\mathbf{t} = (t_1,\ldots,t_n) \in \mathbf{K}^n$, then the line joining $\mathbf{a}$ and $\mathbf{t}$ intersects the quadric $fU_0^2 = U_1^2 + \cdots + U_n^2$ in a second $\mathbf{K}$-rational point. If we write $T_i = t_i - a_i$ for $i = 1,\ldots,n$, then the coordinates of this $\mathbf{K}$-rational point are given by the above formulae. ∎

Lemma 5.3. *If* $f(X) \in \mathbf{K}[X]$ *is strictly positive definite,* $4 \mid \partial f$ *and* $f = a_1^2 + \cdots + a_n^2$, $a_i \in \mathbf{K}(X)$, *then there exist* $g_0(X), g_1(X),\ldots,g_n(X) \in \mathbf{K}[X]$ *such that* $f\cdot g_0^2 = g_1^2+\cdots+g_n^2$, $\partial g_1(X) = \cdots = \partial g_n(X)$, *and* $g_0(X)$ *is strictly positive definite.*

Proof. By hypothesis, $f = a_1^2(X) + \cdots + a_n^2(X)$, $a_i(X) \in \mathbf{K}(X)$. By Cassels' theorem [3], there are $b_1,\ldots,b_n \in \mathbf{K}[X]$ such that $f = b_1^2+\cdots+b_n^2$. Since $\partial f = 4m$, at least one of the b_i, say b_1, has $\partial b_1 = 2m$ and $\partial b_i \leq 2m$ for $i = 2,\ldots,n$. In Lemma 5.2, choose T_1 to be an irreducible polynomial of degree $d > 1$ and, for $j > 1$, T_j to be polynomials of degree $< d$. We then see that $u_0(X)$ is a strictly positive definite polynomial of degree $2d$ and $\partial u_i(X) = 2m + d$ for $i = 1, \ldots, n$. ∎

LEMMA 5.4. *If $f(X) \in \mathbf{K}[X]$ is strictly positive definite, $4 \mid \partial f$ and $f \cdot a_0^2 = a_1^2 + \cdots + a_n^2$, $a_i(X) \in \mathbf{K}[X]$, $a_0(X)$ strictly positive definite and $\partial a_1 = \cdots = \partial a_n$, then there exist $g_0(X), g_1(X), \ldots, g_n(X) \in \mathbf{K}[X]$ such that $f \cdot g_0^2 = g_1^2 + \cdots + g_n^2$, $\partial g_1 = \cdots = \partial g_n$, $g_0(X)$ is strictly positive definite and $g_1(X)$ is positive definite.*

Proof. From the representation $f \cdot a_0^2 = a_1^2 + \cdots + a_n^2$ we have $F(X) = a_0^2 f(X) a_n^2 = a_1^2 + \cdots + a_{n-1}^2$, and since $4 \mid \partial F(X)$, by the induction hypothesis we have $c_0^2 F = c_1^2 + \cdots + c_{n-1}^2$, where $c_0(x) \geq \mu > 0$ and, for $i = 1, \ldots, n$, $c_i(x) \geq 0$ for all $x \in \mathbf{K}$. Hence $(a_0 c_0)^2 f = c_1^2 + \cdots + c_{n-1}^2 + (c_0 a_n)^2$ and we can take $g_0 = a_0 c_0$ and $g_1 = c_1$. ∎

LEMMA 5.5. *If $f(X) \in P(\mathbf{K}[X], 4)$ is strictly positive definite and $f \cdot a_0^2 = a_1^2 + \cdots + a_n^2$, $a_i \in \mathbf{K}[X]$, $a_0(X)$ strictly positive definite, $a_1(X)$ positive definite and $\partial a_1 = \cdots = \partial a_n$, then there exist $g_0(X), g_1(X), \ldots, g_n(X) \in \mathbf{K}[X]$ such that $f \cdot g_0^2 = g_1^2 + \cdots + g_n^2$, $\partial g_1 = \cdots = \partial g_n$ and $g_0(X)$, $g_1(X)$ are both strictly positive definite.*

Proof. By hypothesis $f = a_1^2 + \cdots + a_n^2$, where $a_1(x) \geq 0$ for all $x \in \mathbf{K}$ and $\partial a_1 = \cdots = \partial a_n = 2m$. The $a_i(X)$ cannot have any common real zeros since $f(x) \geq \mu > 0$ for all $x \in \mathbf{K}$.

In the general solution of $fU_0^2 = U_1^2 + \cdots + U_n^2$ given by Lemma 5.2 we choose $T_1 = 1$ and, for $j = 2, \ldots, n$, we choose $T_j = a_j/(AX^{2m} + C)$, where A, C are positive rational numbers chosen so that

$$\frac{a_2^2 + \cdots + a_n^1}{(Ax^{2m} + C)^2} \leq \frac{1}{2}$$

for all $x \in \mathbf{K}$, and we write $b(X) = 1 - (a_2^2 + \cdots + a_n^2)/(Ax^{2m} + C)^2 \geq 1/2$.

From Lemma 5.2 we have $g_0(x) = 1 + T_2^2 + \cdots + T_n^2 \geq 1$ for all $x \in \mathbf{K}$ and

$$g_1(x) = a_1(x) b(x) + \frac{2}{Ax^{2m} + C} \sum_{j=2}^{n} a_j^2(x).$$

Thus, $g_1(x)$ is the sum of n positive terms, hence $g_1(x) \geq 0$ for all $x \in \mathbf{K}$. If $\gamma \in \mathbb{R}$ is such that $g_1(\gamma) = 0$, then $a_1(\gamma) = \cdots = a_n(\gamma) = 0$. This is impossible, as $f(X)$ is strictly positive, hence $g_1(x)$ is also strictly positive. ∎

We are now in a position to prove the induction step from $n - 1$ to n. Suppose that $f(X) \in P(\mathbf{K}, 4)$, $f(x) \geq \mu > 0$ for all $x \in \mathbf{K}$ and that $f(X) = a_1^2(X) + \cdots + a_n^2(X)$. By renumbering the functions $a_i(X)$ and using Lemmas 5.3–5.5 if necessary, we can assume that we have a representation $f(X) b_0^2(X) = b_1^2(X) + \cdots + b_n^2(X)$, where $b_0(x) \geq \mu_0 > 0$ and

$b_1(x) \geq \mu_1 > 0$ for all $x \in \mathbf{K}$ and $\partial b_1 = \cdots = \partial b_n$. We will now use the induction hypothesis and Lemma 4.1 to deduce that there is a representation $f(X)g_0^2(X) = g_1^2(X) + \cdots + g_n^2(X)$, where, for $1 \leq i \leq n$, we have $g_i(X) \geq \mu_i > 0$ for all $x \in \mathbf{K}$.

Let $\lambda(X) = b_1^2(X)/f(X)$. Then since $2\partial b_1 = \partial f$ and b_1, f are strictly positive definite, there exists $\lambda_0 > 0$ such that $\lambda(x) \geq \lambda_0 > 0$ for all $x \in \mathbf{K}$. From Lemma 4.1, for any $\lambda_0 > 0$, there exists $\phi(X) \in \mathbb{Q}[X]$ such that $f(x) < \phi^4(x) < (1+\lambda_0)f(x)$ for all $x \in \mathbf{K}$.

We now use the formulae of Lemma 5.2 to construct a new representation by taking $T_1 = 1$, $T_2 = (b_2(X) + \phi^2(X))/b_1(X)$ and $T_j = b_j(X)/b_1(X)$ for $j = 3, \ldots, n$. This gives:

$$h_0(x) = 1 + T_1^2 + \cdots + T_n^2 \geq 1 \quad \text{for all } x \in \mathbf{K},$$

$$h_1(x) = 2b_2T_2 + 2\sum_{j=3}^{n} b_jT_j - b_1(1 - T_2^2 - \cdots - T_n^2) = \frac{\phi^4 - f}{b_1} \geq \mu_1 > 0$$

and

$$\frac{h_1^2}{h_0^2} \leq \frac{(\phi^4 - f)^2}{h_0^2 b_1^2} \leq \frac{\lambda_0^2 f^2}{h_0^2 b_1^2} \leq \frac{b_1^4 f^2}{4f^2 b h_0^2 b_1^2} = \frac{b_1^2}{4h_0^2} \leq \frac{b_1^2}{4} \leq \frac{f}{4}.$$

Thus

$$f_1 = f - \frac{h_1^2}{h_0^2} = \left(\frac{h_2}{h_0}\right)^2 + \cdots + \left(\frac{h_n}{h_0}\right)^2 \geq \frac{3f}{4} \geq \frac{3\mu}{4} > 0.$$

Since $4 \mid \partial f_1$ and f_1 is strictly positive definite we have $f_1 \in P(\mathbf{K}, 4)$. We also know that f_1 can be represented as the sum of $n-1$ squares, so, by our induction hypothesis, there exist $g_0, g_1, \ldots, g_{n-1}$ such that $f(X)g_0(X)^2 = g_1^2(X) + \cdots + g_{n-1}^2(X)$ where for $i = 0, 1, \ldots, n-1$, $g_i(x) \geq \mu_i > 0$ for all $x \in \mathbf{K}$. Hence

$$f(g_0h_0)^2 = (g_0h_1)^2 + g_1^2 + \cdots + g_{n-1}^2$$

and we have the desired representation for $f(X)$. This completes the induction step and the proof of Theorem 5.1. ∎

For $k = 3$, if $\mathbf{F}$ is any field not of characteristic 3, then $w(\mathbf{F}, 3) \leq 3$ and $g(\mathbf{K}(X), 3) \leq 3$. These follow from a classic identity due to Richmond (see [12, Notes to Chapter 13] and [16]): If $r, s \in \mathbf{F}$ and $s \neq 0$ and $t = 3r/s^3$, then

$$r = \left(\frac{s(1+t^3)}{3(1-t+t^2)}\right)^3 + \left(\frac{s(3t-1-t^3)}{3(1-t+t^2}\right)^3 + \left(\frac{s(3t-t^3)}{3(1-t+t^2}\right)^3.$$

This gives $w(\mathbf{F}, 3) \leq 3$. To deduce that $g(\mathbf{K}(X), 3) \leq 3$, we note that $1-t+t^2 = (t-1/2)^2 + 3/4$ is totally positive, so we must show that if $6 \mid \partial r$ and $r(x) > 0$ for all $x \in \mathbf{K}$, then $s(X) \in \mathbf{K}(X)$ can be chosen so that $s(x) > 0$, $(t(x) - t^2(x)) > 0$ and $3t(x) - 1 - t^3(x) > 0$ for all $x \in \mathbf{K}$.

A sufficient condition is that $1/2 < t(x) < 1$ for all $x \in \mathbf{K}$, namely $3r(x) < s^3(x) < 6r(x)$ for all $x \in \mathbf{K}$. If $r(X) \in \mathbf{K}[X]$ is strictly positive definite, of degree $6m$ and with leading coefficient a, then if $h(X) = bX^{2m} + 1$, where b is such that $3a < b^3 < 6a$, we have $1/6 < r(x)/h(x)^3 < 1/3$ for all $x \geq C(f)$. We now use Lemma 4.1 to construct $s(X)$ to conclude that $g(\mathbf{K}(X), 3) \leq 3$.

For $k = 4$, Choi et al. [5] use the fact that $g(\mathbb{R}(X), 2) = 2$ to show that $w(\mathbb{R}(X), 4) \leq 6$. A consequence of Theorem 5.1 and a theorem of Cassels [4] is: *If* $w(\mathbf{K}(X), 2) = 2$, *then* $w(\mathbf{K}(X), 4) \leq 6$. The Cassels theorem asserts that $w(\mathbf{K}(X), 2) \geq s(\mathbf{K}) + 1$. So $w(\mathbf{K}(X), 2) = 2$ implies that $s(\mathbf{K}) = 1$, i.e. every positive element of $\mathbf{K}$ is a square. In particular $\sqrt{3} \in \mathbf{K}$. If $f(X) \in \mathbf{K}[X]$ is strictly positive definite and $4 \mid \partial f$, then $f(X)/18$ can be represented as $f(X)/18 = g_1^2(X) + g_2^2(X)$ with $g_1(X)$ and $g_2(X)$ strictly positive definite. We then have $g_1(X) = a^2(X) + b^2(X)$ and $g_2(X) = c^2(X) + d^2(X)$. The specific Hilbert identity $18(u^2 + v^2)^2 = (u + \sqrt{3}b)^4 + (a - \sqrt{3}v)^4 + (2v)^4$ applied twice gives the result.

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