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A generalization of Rademacher's reciprocity law

by

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1. Introduction. For a rational number h/k with $(h,k)=1$, $k>1$, and a complex number a, let

$$\mathrm{c}_a\left(\frac{h}{k}\right) := k^a \sum_{m=1}^{k-1} \cot\left(\frac{\pi h m}{k}\right)\zeta\left(-a, \frac{m}{k}\right),$$

where $\zeta(s, m/k)$ is the Hurwitz zeta-function. These cotangent sums arise in analytic number theory in the value at $s=0$,

$$D\left(0, a, \frac{h}{k}\right) = -\frac{1}{2}\zeta(-a) + \frac{i}{2}\,\mathrm{c}_a\left(\frac{h}{k}\right),$$

of the Estermann function

$$D\left(s, a, \frac{h}{k}\right) := \sum_{n=1}^{\infty} \frac{\sigma_a(n)\,\mathrm{e}(nh/k)}{n^s},$$

which is initially defined for $\Re(s) > 1+\max(0, \Re(a))$, but can be analytically continued to $\mathbb{C}\setminus\{1, 1+a\}$. The Estermann function satisfies a functional equation and it is useful in studying the asymptotic of the mean square of the Riemann zeta function $\zeta(s)$ multiplied by a Dirichlet polynomial (see, for example, [BCH]). Here and in the following we write, as usual, $\sigma_a(n) := \sum_{d|n} d^a$ and $\mathrm{e}(x) := e^{2\pi i x}$.

The cotangent sum $\mathrm{c}_a(h/k)$ is most interesting in the cases $a=-1$ (note that the poles in the sum defining c_a cancel) and $a=0$. In the former case, c_{-1} is, up to a constant, the Dedekind sum,

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$$s\left(\frac{h}{k}\right) := \frac{1}{4k}\sum_{m=1}^{k-1}\cot\left(\frac{\pi hm}{k}\right)\cot\left(\frac{\pi m}{k}\right) = \sum_{m=1}^{k-1}\left(\left(\frac{mh}{k}\right)\right)\left(\left(\frac{m}{k}\right)\right)$$
$$= \frac{1}{2\pi}\,\mathrm{c}_{-1}\left(\frac{h}{k}\right),$$

where $((\cdot))$ is the sawtooth function,

$$((x)) := \begin{cases} \{x\} - 1/2, & x \notin \mathbb{Z}, \\ 0, & x \in \mathbb{Z}, \end{cases}$$

and $\{x\}$ is the fractional part of x.

The Dedekind sum appears in the root number in the functional equation of the Dedekind eta-function and has been much studied in number theory and other branches of mathematics. The main property of the Dedekind sum is that it satisfies a reciprocity formula

$$s\left(\frac{h}{k}\right) + s\left(\frac{k}{h}\right) - \frac{1}{12hk} = \frac{1}{12}\left(\frac{h}{k} + \frac{k}{h} - 3\right) \tag{1.1}$$

for $(h,k) = 1$, $h,k \in \mathbb{N}_{>0}$. This formula, due to Dedekind, has been generalized by Rademacher, who proved that

$$s\left(\frac{a\overline{b}}{c}\right) + s\left(\frac{b\overline{c}}{a}\right) + s\left(\frac{c\overline{a}}{b}\right) = \frac{a^2+b^2+c^2}{12abc} - \frac{1}{4} \tag{1.2}$$

for $(a,b) = (b,c) = (a,c) = 1$, $a,b,c \in \mathbb{N}_{>0}$, and where $\overline{b}$ (respectively $\overline{c},\overline{a}$) denotes the inverse of b (resp. c,a) modulo c (resp. a,b).

For $a = 0$, one has the cotangent sum

$$\mathrm{c}_0\left(\frac{h}{k}\right) = \sum_{m=1}^{k-1}\left\{\frac{m}{k}\right\}\cot\left(\frac{\pi mh}{k}\right),$$

which is relevant to the Nyman–Beurling–Báez-Duarte approach to the Riemann hypothesis. This asserts that the Riemann hypothesis is true if and only if $\lim_{N\to\infty} d_N = 0$, where

$$d_N^2 = \inf_{A_N}\frac{1}{2\pi}\int_{-\infty}^{\infty}\left|1 - \zeta\left(\frac{1}{2}+it\right)A_N\left(\frac{1}{2}+it\right)\right|^2\frac{dt}{1/4+t^2}$$

and the inf is over all the Dirichlet polynomials $A_N(s) = \sum_{n=1}^{N} a_n/n^s$ of length N. When computing this integral, one is led to consider integrals of the form

$$\nu\left(\frac{h}{k}\right) := \frac{1}{2\pi\sqrt{hk}}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+it\right)\right|^2\left(\frac{h}{k}\right)^{it}\frac{dt}{1/4+t^2}, \tag{1.3}$$

which can be re-expressed (see [Vas]) as

$$\nu\left(\frac{h}{k}\right) = \frac{\log 2\pi - \gamma}{2}\left(\frac{1}{h} + \frac{1}{k}\right) + \frac{k-h}{2hk}\log\frac{h}{k} - \frac{\pi}{2hk}\left(V\left(\frac{h}{k}\right) + V\left(\frac{k}{h}\right)\right),$$

where

$$V\left(\frac{h}{k}\right) = -\,\mathrm{c}_0\left(\frac{\overline{h}}{k}\right)$$

is the Vasyunin sum and $\overline{h}$ is the inverse of h modulo k. It should be remarked that the convexity bound, $\zeta(1/2+it) \ll |t|^{1/4+\varepsilon}$, implies that the integral in (1.3) extends to a continuous function of $h/k \in \mathbb{R}_{>0}$, though it follows from the work of Báez-Duarte, Balazard, Landreau and Saias [BBLS] that this function is not differentiable at any rational number.

In [**?**], Conrey and the author showed that a natural generalization of the Dedekind reciprocity formula to c_0 is

$$\mathrm{c}_0\left(\frac{h}{k}\right) + \frac{k}{h}\,\mathrm{c}_0\left(\frac{k}{h}\right) - \frac{1}{\pi h} = \omega\left(\frac{h}{k}\right), \tag{1.4}$$

where $\omega(x)$ is an explicit holomorphic function on $\mathbb{C}' := \mathbb{C}\setminus\mathbb{R}_{\le 0}$. (A generalization to all a was given by the same authors in [BC2]). This formula shows that c_0 can be interpreted as an "imperfect" quantum modular form of weight 1, in the sense of Zagier [Zag].

It is the purpose of the present paper to provide the analogue of (a generalization of) Rademacher's formula (1.2) for c_a for all $a \in \mathbb{C}$.

THEOREM 1.1. *Let $a \in \mathbb{C}$ and let M be any integer greater than or equal to $-\frac{1}{2}\min(0, \Re(a))$. Let $h,k,p,q \in \mathbb{N}_{>0}$, with $(h,k) = (p,q) = 1$, and let $d = (pk+h, q)$. Then*

(1.5)

$$\begin{aligned}\mathrm{c}_a\left(\frac{pk+h}{qk}\right) - \left(\frac{k}{h}\right)^{1+a}\mathrm{c}_a\left(\frac{-\overline{p}h-k}{qh}\right) - \mathrm{c}_a\left(\frac{p}{q}\right) + a\zeta(1-a)\frac{(kq)^a d^{1-a}}{\pi h}\\ = -2i\sum_{m=1}^{2M} D\left(-m, a, \frac{p}{q}\right)\frac{(2\pi i\frac{h}{kq})^m}{m!} + g_{a,M}\left(\frac{h}{k}, \frac{p}{q}\right)\\ + 2\left(2\pi\frac{h}{k}\right)^{-1} q^a\zeta(1-a) - \cot\frac{\pi a}{2}\zeta(-a)\left(\frac{k}{h}\right)^{1+a},\end{aligned}$$

where $\overline{p}$ is the inverse of p modulo q and

$$\begin{aligned} g_{a,M}\left(z, \frac{h}{k}\right) := \frac{1}{\pi i}\int_{(-1/2-2M)} \mathit{\Gamma}(s)\frac{\cos\frac{\pi a}{2}}{\sin\pi(s-a)}\\ \times\left(e^{-\pi i(s-a)/2}D\left(s,a,\frac{h}{k}\right) + D\left(s,a,-\frac{h}{k}\right)e^{\pi i(s-a)/2}\right)\left(\frac{2\pi z}{k}\right)^{-s} ds.\end{aligned} \tag{1.6}$$

In particular, for all $(p,q)=1$, the left hand side of (1.5) *can be continued to a function of h/k which is holomorphic on $\mathbb{C}' := \mathbb{C}\setminus\mathbb{R}_{\leq 0}$.*

COROLLARY 1.2. *Let $h,k,p,q\in\mathbb{N}_{>0}$, with $(h,k)=(p,q)=1$, and let $d=(pk+h,q)$. Let $\overline{p}$ be the inverse of p modulo q. Then*

$$\mathrm{c}_0\bigg(\frac{pk+h}{qk}\bigg)+\frac{k}{h}\,\mathrm{c}_0\bigg(\frac{\overline{p}h+k}{qh}\bigg)-\mathrm{c}_0\bigg(\frac{p}{q}\bigg)-\frac{d}{\pi h}=f\bigg(\frac{h}{k},\frac{p}{q}\bigg),$$

where

$$f\bigg(z,\frac{p}{q}\bigg) := -\frac{\log(2\pi qz)-\gamma}{\pi z} + \frac{1}{\pi i}\int_{(-1/2)}\frac{\Gamma(s)}{\sin\pi s}\bigg(e^{-\pi is/2}D\bigg(s,0,\frac{p}{q}\bigg)+e^{\pi is/2}D\bigg(s,0,-\frac{p}{q}\bigg)\bigg)\bigg(2\pi\frac{z}{q}\bigg)^{-s}ds$$

is a holomorphic function of z on $\mathbb{C}'$.

In the case of $a=-1$ Theorem 1.1 yields the following corollary.

COROLLARY 1.3. *Let $h,k,p,q\in\mathbb{N}_{>0}$, with $(h,k)=(p,q)=1$, and $d=(pk+h,q)$. Then*

$$s\bigg(\frac{pk+h}{qk}\bigg)+s\bigg(\frac{\overline{p}h+k}{qh}\bigg)-s\bigg(\frac{p}{q}\bigg)=\frac{k^2+d^2+h^2}{12hkq}-\frac{1}{4}. \tag{1.7}$$

This is an extension of Rademacher's formula and is equivalent to Lemma 7 of [CFKS] (which is itself equivalent to an analogous formula of Dieter [Die]), as we shall show at the end of Section 3. Finally, it should be noticed that for negative odd integer a, the identities we obtain involve, as in the case when $a=-1$, only cotangent sums and a rational function and are particular cases of the formulae obtained by Beck [Beck].

One of the main ingredients in the proof of Theorem 1.1 comes from providing the analytic continuation for the "period function" (in the sense of [LZ])

$$\psi\bigg(z,a,\frac{h}{k}\bigg) := \mathcal{S}\bigg(\frac{z}{k},a,\frac{h}{k}\bigg)-\frac{1}{z^{1+a}}\,\mathcal{S}\bigg(-\frac{1}{kz},a,\frac{-\overline{h}}{k}\bigg)$$

of

$$\mathcal{S}\bigg(z,a,\frac{h}{k}\bigg) := \sum_{n=1}^{\infty}\sigma_a(n)\,\mathrm{e}\bigg(n\frac{h}{k}\bigg)\,\mathrm{e}(nz).$$

The function $\mathcal{S}(z,a,h/k)$ is defined only for $\Im(z)>0$ (notice that $\Im(z)>0$ iff $\Im(-1/z)>0$), however, its period function $\psi(z,a,h/k)$ can be analytically continued to $\mathbb{C}'$, as shown by the following theorem which extends the work of Lewis and Zagier [LZ] (see also Theorem 1 in [BC2]).

THEOREM 1.4. *Let $h/k \in \mathbb{Q}$ with $(h,k)=1$, $k>0$. Let $a \in \mathbb{C}$ and let M be any integer greater than or equal to $-\frac{1}{2}\min(0,\Re(a))$. Then $\psi(z,a,h/k)$ extends to an analytic function of z on $\mathbb{C}'$ via the representation*

$$\psi\left(z,a,\frac{h}{k}\right) = r_{a,M}\left(z,\frac{h}{k}\right) + \frac{i}{2} g_{a,M}\left(z,\frac{h}{k}\right), \tag{1.8}$$

where $g_{a,M}(z,h/k)$ is as in (1.6) *and*

$$r_{a,M}\left(z,\frac{h}{k}\right) := ik^a\frac{\zeta(1-a)}{2\pi z} + e^{\pi i(1+a)/2}\Gamma(1+a)\frac{\zeta(1+a)}{(2\pi z)^{1+a}} + \sum_{m=1}^{2M} D\left(-m,a,\frac{h}{k}\right)\frac{i^m}{m!}(2\pi z/k)^m + D\left(0,a,\frac{h}{k}\right).$$

This result is of independent interest, as the (smoothed) second moment of $\zeta(s)$ times a Dirichlet polynomial can be expressed in terms of $\mathcal{S}(z,a,h/k)$. We remark that the asymptotics for these moments are needed, for example, for theorems which give a lower bound for the portion of zeros of $\zeta(s)$ on the critical line (see [Iwa] and [Con]).

We conclude the paper by showing that the coefficients of the Taylor series of $g_{a,M}(z,h/k)$ are exceptionally small when $\Re(z)>0$. In particular, the Taylor series converges (absolutely) on the boundary of the disk of convergence. This is particularly relevant, since it can be used to give an exact formula for the smoothed second moment of $\zeta(s)$ times a Dirichlet polynomial.

THEOREM 1.5. *Let $h,k \in \mathbb{N}_{>0}$ with $(h,k)=1$. Let $a\in\mathbb{C}$ be fixed and let M be any integer greater than or equal to $-\frac{1}{2}\min(0,\Re(a))$. Let τ be a complex number with positive real part and, for $|y|<1$, let*

$$g_{a,M}\left(\tau+\tau y,\frac{h}{k}\right) = \sum_{m=0}^{\infty}\rho_{a,m,M}\left(\tau,\frac{h}{k}\right)(-y)^m$$

be the Taylor series of $g_{a,M}(z,h/k)$ at $z=\tau$. Then

$$\rho_{a,m,M}\left(\tau,\frac{h}{k}\right) = \cos\frac{\pi a}{2}\,2^{\frac{7}{4}-\frac{a}{2}}\pi^{-\frac{3}{4}-\frac{a}{2}}\tau^{-\frac{3}{4}-\frac{a}{2}}k^{\frac{1}{4}+\frac{a}{2}}m^{-\frac{1}{4}+\frac{a}{2}}e^{-2\sqrt{\pi m/(\tau k)}} \times\left(\cos\left(\frac{\pi}{4}\left(a-\frac{1}{2}\right)+\frac{\pi}{\tau k}-2\sqrt{\frac{\pi m}{\tau k}}+2\pi\frac{\overline{h}}{k}\right)+O\left(\sqrt{\frac{|\tau|k}{m}}\right)\right)$$

uniformly in $h,k\ge 1$, $m\ge 2M+1$ and $|\tau|>K$ for any fixed $K>0$.

2. The period function. The next lemma gives the functional equation for $D(s,a,h/k)$ and can be proved easily by the following decomposition of D in terms of the Hurwitz zeta-function:

$$D\left(s,a,\frac{h}{k}\right)=\frac{1}{k^{2s-a}}\sum_{m,n=1}^{k}\mathrm{e}\left(\frac{mnh}{k}\right)\zeta\left(s-a,\frac{m}{k}\right)\zeta\left(s,\frac{n}{k}\right).$$

Lemma 2.1. *For $(h,k)=1$, $k>0$ and $a\in\mathbb{C}$,*

$$D\left(s,a,\frac{h}{k}\right)-k^{1+a-2s}\zeta(s-a)\zeta(s)$$

is an entire function of s. Moreover, $D(s,a,h/k)$ satisfies the functional equation

$$D\left(s,a,\frac{h}{k}\right)=-\frac{2}{k}\left(\frac{k}{2\pi}\right)^{2-2s+a}\Gamma(1-s+a)\Gamma(1-s)\times\left(\cos\left(\frac{\pi}{2}(2s-a)\right)D\left(1-s,-a,-\frac{\overline{h}}{k}\right)-\cos\frac{\pi a}{2}D\left(1-s,-a,\frac{\overline{h}}{k}\right)\right). \tag{2.1}$$

We can now prove Theorem 1.4.

Proof of Theorem 1.4. Firstly observe that we can assume $0\neq|\Re(a)|<1$, since the lemma will then follow by analytic continuation in a. Now, we notice that $\mathcal{S}(z/k,a,h/k)$ can be written as

$$\mathcal{S}\left(\frac{z}{k},a,\frac{h}{k}\right)=\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}D\left(s,a,\frac{h}{k}\right)e^{\pi is/2}\Gamma(s)(2\pi z/k)^{-s}\,ds$$

and, by contour integration, this is equal to

$$\mathcal{S}\left(\frac{z}{k},a,\frac{h}{k}\right)=\frac{1}{2\pi i}\int_{(-1/2-2M)}D\left(s,a,\frac{h}{k}\right)e^{\pi is/2}\Gamma(s)(2\pi z/k)^{-s}\,ds+r_{a,M}\left(\frac{z}{k},\frac{h}{k}\right). \tag{2.2}$$

Consider

$$\begin{aligned}\frac{1}{(zk)^{1+a}}&\mathcal{S}\left(-\frac{1}{zk},a,-\frac{\overline{h}}{k}\right)\\&=\frac{1}{(zk)^{1+a}}\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}D\left(s,a,-\frac{\overline{h}}{k}\right)\Gamma(s)e^{\pi is/2}\left(2\pi\frac{-1}{zk}\right)^{-s}ds\\&=\frac{1}{2\pi i}\int_{(2+\max(0,\Re(a)))}D\left(s,a,-\frac{\overline{h}}{k}\right)\Gamma(s)e^{-\pi is/2}(2\pi)^{-s}(zk)^{s-1-a}\,ds,\end{aligned}$$

since in this context $0<\arg z<\pi$ and $0<\arg(-1/z)<\pi$, so the identity $\arg(-1/z)=\pi-\arg z$ holds. Applying the functional equation (2.1), we find that this is

$$-\frac{2}{k}\frac{1}{2\pi i}\int\limits_{(2+\max(0,\Re(a)))}\left(\frac{k}{2\pi}\right)^{2-2s+a}\Gamma(1-s+a)\Gamma(1-s)$$
$$\times\left(\cos\left(\frac{\pi}{2}(2s-a)\right)D\left(1-s,-a,\frac{h}{k}\right)-\cos\frac{\pi a}{2}D\left(1-s,-a,-\frac{h}{k}\right)\right)$$
$$\times\,\Gamma(s)e^{-\pi is/2}(2\pi)^{-s}(zk)^{s-1-a}\,ds.$$

Observing that $D(s,-a,-h/k)=D(s+a,a,-h/k)$ and using Euler's reflection formula, we see that this is equal to

$$-\frac{2\pi}{k}\frac{1}{2\pi i}\int\limits_{(2+\max(0,\Re(a)))}\left(\frac{k}{2\pi}\right)^{2-2s+a}\Gamma(1-s+a)$$
$$\times\left(\cos\left(\frac{\pi}{2}(2s-a)\right)D\left(1-s+a,a,\frac{h}{k}\right)\right.$$
$$\left.-\cos\frac{\pi a}{2}D\left(1-s+a,a,-\frac{h}{k}\right)\right)\frac{e^{-\pi is/2}}{\sin\pi s}(2\pi)^{-s}(zk)^{s-1-a}\,ds.$$

Now, we make the change of variable $s\mapsto 1-s+a$ and then move the line of integration to $-1/2-2M$ without crossing any pole. Thus, we get

(2.3)
$$\frac{1}{(zk)^{1+a}}\mathcal{S}\left(-\frac{1}{zk},a,\frac{\overline{h}}{k}\right)=-\frac{1}{2\pi k}\int\limits_{(-1/2-2M)}k^{-a}\Gamma(s)\frac{e^{\pi i(s-a)/2}}{\sin\pi(s-a)}\left(\frac{2\pi z}{k}\right)^{-s}$$
$$\times\left(\cos\left(\frac{\pi}{2}(2s-a)\right)D\left(s,a,\frac{h}{k}\right)+\cos\frac{\pi a}{2}D\left(s,a,-\frac{h}{k}\right)\right)ds.$$

The lemma then follows by taking the difference between (2.2) and (2.3), thanks to the identity

$$e^{\pi is/2}+i\frac{\cos(\frac{\pi}{2}(2s-a))}{\sin\pi(s-a)}e^{\pi i(s-a)/2}=i\frac{e^{-\pi i(s-a)/2}\cos\frac{\pi a}{2}}{\sin\pi(s-a)}.\ \blacksquare$$

3. A generalization of Rademacher's formula. We can now prove the extension (1.5) of Rademacher's reciprocity formula to the sum $\mathrm{c}_a(h/k)$. The proof follows the method used to prove Theorem 4 in [BC2].

Proof of Theorem 1.1. Firstly observe that we can assume $0\neq|\Re(a)|<1$, since the result will then follow by analytic continuation in a.

Let $z=\frac{h}{k}(1+i\xi)$ for a small $\xi>0$ and let $\alpha=pk+h$, $\beta=qk$. We have

$$\begin{aligned}\mathcal{S}\Big(\frac{z}{q},a,\frac{p}{q}\Big)&=\sum_{n=1}^{\infty}\sigma_a(n)\,\mathrm{e}\Big(n\frac{\alpha}{\beta}\Big)\,\mathrm{e}\Big(in\frac{h}{\beta}\xi\Big)\\&=\frac{1}{2\pi i}\int\limits_{(2+\max(\Re(a)),0)}\Gamma(s)D\Big(s,a,\frac{\alpha}{\beta}\Big)\Big(2\pi\frac{h}{\beta}\xi\Big)^{-s}\,ds.\end{aligned}$$

Moving the line of integration to $\Re(s)=-1/2$ and picking up the residue encountered, by Lemma 2.1 we deduce that this is equal to

$$\begin{aligned}\mathcal{S}\Big(\frac{z}{q},a,\frac{p}{q}\Big)&=(\beta/d)^{a-1}\zeta(1-a)\Big(2\pi\frac{h}{\beta}\xi\Big)^{-1}+\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha}{\beta}\Big)-\frac{1}{2}\zeta(-a)\\&\quad+\Gamma(1+a)(\beta/d)^{-1-a}\zeta(1+a)\Big(2\pi\frac{h}{\beta}\xi\Big)^{-1-a}+O(\xi^{1/2}).\end{aligned}\tag{3.1}$$

In the same way, writing

$$-\frac{1}{z}=-\frac{k}{h}(1-i\xi'),\qquad \xi'=\frac{\xi}{1+i\xi}=\xi-i\xi^2+O(\xi^3)$$

and $\alpha'=-\overline{p}h-k$, $\beta'=qh$ (note that $(p,q)=(h,k)=(\alpha,q)=1$ implies $(\alpha',q)=1$), we have

$$\begin{aligned}\mathcal{S}\Big(-\frac{1}{qz},a,-\frac{\overline{p}}{q}\Big)&=\sum_{n=1}^{\infty}\sigma_a(n)\,\mathrm{e}\Big(n\frac{\alpha'}{\beta'}\Big)\,\mathrm{e}\Big(in\frac{k}{\beta'}\xi'\Big)\\&=(\beta'/d)^{a-1}\zeta(1-a)\Big(2\pi\frac{k}{\beta'}\xi'\Big)^{-1}+\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha'}{\beta'}\Big)-\frac{1}{2}\zeta(-a)\\&\quad+\Gamma(1+a)(\beta'/d)^{-1-a}\zeta(1+a)\Big(2\pi\frac{k}{\beta'}\xi'\Big)^{-1-a}+O((\xi')^{1/2})\end{aligned}$$

and thus

$$\begin{aligned}\mathcal{S}\Big(\frac{-1}{qz},a,-\frac{\overline{p}}{q}\Big)&=\Big(\frac{\beta'}{d}\Big)^{a-1}\zeta(1-a)\Big(2\pi\frac{k}{\beta'}\xi\Big)^{-1}(1+i\xi)+\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha'}{\beta'}\Big)-\frac{\zeta(-a)}{2}\\&\quad+\Gamma(1+a)(\beta'/d)^{-1-a}\zeta(1+a)\Big(2\pi\frac{k}{\beta'}\xi\Big)^{-1-a}(1+i\xi)^{1+a}+O(\xi^{1/2}).\end{aligned}\tag{3.2}$$

Therefore, from (3.1) and (3.2) it follows that

$$\begin{aligned}\mathcal{S}\Big(\frac{z}{q},a,\frac{p}{q}\Big)-\frac{1}{z^{1+a}}\,\mathcal{S}\Big(-\frac{1}{qz},a,-\frac{\overline{p}}{q}\Big)&=\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha}{\beta}\Big)-\frac{1}{z^{1+a}}\,\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha'}{\beta'}\Big)\\&\quad-\frac{1}{2}\zeta(-a)+ia\zeta(1-a)\frac{(kq)^a d^{1-a}}{2\pi h}+\frac{1}{z^{1+a}}\,\frac{1}{2}\zeta(-a)+O(\xi^{1/2})\end{aligned}$$

and thus

$$\lim_{\xi\to 0^+} \mathcal{S}\Big(\frac{z}{q}, a, \frac{p}{q}\Big) - \frac{1}{z^{1+a}}\,\mathcal{S}\Big(-\frac{1}{qz}, a, -\frac{\overline{p}}{q}\Big) = \frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha}{\beta}\Big) - \Big(\frac{k}{h}\Big)^{1+a}\frac{i}{2}\,\mathrm{c}_a\Big(\frac{\alpha'}{\beta'}\Big)$$
$$- \frac{1}{2}\zeta(-a) + ia\zeta(1-a)\frac{(kq)^a d^{1-a}}{2\pi h} + \Big(\frac{k}{h}\Big)^{1+a}\frac{1}{2}\zeta(-a).$$

By Theorem 1.4, this is also equal to $r_{a,M}(h/k, p/q) + (i/2)g_{a,M}(h/k, p/q)$ and thus Theorem 1.1 follows after using the functional equation for the Riemann zeta-function. ∎

Corollary 1.2 follows immediately by applying Theorem 1.1 to the case $a = 0$. We remark that replacing k with qk in Corollary 1.2, we obtain, for all $M \in \mathbb{Z}_{\geq 0}$,

$$(3.3)\qquad c_0\Big(\frac{pqk+h}{q^2k}\Big) + \frac{qk}{h}c_0\Big(\frac{ph+kq}{qh}\Big) - c_0\Big(\frac{\overline{p}}{q}\Big) - \frac{1}{\pi h}$$
$$= \frac{q}{\pi^2}\sum_{m=1}^{M}(-1)^m(2m)!D_{\sin}\Big(1+2m, \frac{p}{q}\Big)\Big(\frac{h}{2\pi k}\Big)^{2m} + q\mu_M\Big(\frac{h}{k}, \frac{p}{q}\Big),$$

where $\mu_M(x, y)$ is holomorphic in x for $x \in \mathbb{C}'$ and $C^{2M+1}(\mathbb{R})$ in y, and where

$$D_{\sin}(s, x) := \frac{D(s, 0, x) - D(s, 0, -x)}{2i}$$

for $\Re(s) > 1 + \max(0, \Re(a))$.

Applying Theorem 1.1 to $a = -1$, one obtains immediately the generalization (1.7) of Rademacher's reciprocity formula, since for $a = -1$ one sees that $g_{1,M}$ is identically zero.

We conclude the section by showing how to obtain Lemma 7 of [CFKS] from (1.7). This lemma states that, if $a, c, \ell, m \in \mathbb{N}_{>0}$ with $(a, c) = (\ell, m) = 1$, and b, d are such $ad - bc = 1$, then

$$(3.4)\qquad s\Big(\frac{a}{c}\Big) + s\Big(\frac{\ell}{m}\Big) - s\Big(\frac{x}{y}\Big) = \frac{c^2+m^2+y^2}{12cmy} - \frac{1}{4},$$

where $x = a\ell + bm$ and $y = c\ell + dm$.

To prove this result we apply Corollary 1.3 to $p = x$, $q = y$, $k = c/u$ and $h = m/u$, where $u = (c, m)$. We have

$$(3.5)\qquad u(pk+h) = xc + m = ac\ell + m(bc+1) = ac\ell + ad = ay = aq$$

and

$$\ell(\overline{p}h + k)u = \ell(\overline{p}m + c) = \ell\overline{p}m + \ell c = \ell\overline{p}m + q - dm = (\ell\overline{p} - d)m + q$$
$$= ((\overline{p}p - 1)d - \overline{p}bq)m + q,$$

where we used $\ell = dp - bq$, which can be obtained from the definition of x and y and the condition $ad - bc = 1$. Therefore

$$\ell(\overline{p}h + k)u/q \equiv 1 \pmod{m}. \tag{3.6}$$

Thus (3.4) follows from (3.5) and (3.6) by observing that

$$s\left(\frac{\overline{\ell}}{m}\right) = s\left(\frac{\ell}{m}\right).$$

4. The Taylor coefficients. First, we need the following lemma from [BC2].

LEMMA 4.1. *Let $a \in \mathbb{C}$ be fixed and let M be any integer greater than or equal to $-\frac{1}{2}\min(0, \Re(a))$. Let z be a complex number with positive real part and let*

$$I^{\pm}_{m,a}(z) := -\frac{1}{4\pi} \int_{(-1/2-2M)} \Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)(\pm 2\pi i z)^s \, ds.$$

Then, for $m \geq 2M + 1$ and $|z| \geq K$ for some fixed $K > 0$, we have

$$I^{\pm}_{m,a}(z) = \pm 2^{\frac{1}{4}+\frac{a}{2}} \pi^{\frac{7}{4}+\frac{a}{2}} e^{\frac{\pm\pi i(a-1/2)}{4}} z^{\frac{3}{4}+\frac{a}{2}} e^{\pm i\pi z} e^{-2(1\pm i)\sqrt{\pi m z}} e^{-m} m^{m+\frac{1}{4}+\frac{a}{2}} \times \left(1 + O\left(\frac{1}{\sqrt{m|z|}}\right)\right)$$

uniformly in m and z.

Proof. This formula appears in the proof of Theorem 2 in [BC2]. ∎

We can now prove Theorem 1.5.

Proof of Theorem 1.5. Let $(h,k) = 1$, $k > 0$ and $a \in \mathbb{C}$. From Lemma 2.1 a simple computation shows that

$$C\left(s, a, \frac{h}{k}\right) := \Gamma(s)\left(\frac{k}{2\pi}\right)^s \left(e^{-\pi i(s-a)/2} D\left(s, a, \frac{h}{k}\right) + e^{\pi i(s-a)/2} D\left(s, a, -\frac{h}{k}\right)\right)$$

is a meromorphic function of s (with a simple pole at $s = 1$ only) and satisfies the functional equation

$$\sin(\pi s) C\left(s, a, \frac{h}{k}\right) = \sin(\pi(1-s+a)) C\left(1-s+a, a, \frac{\overline{h}}{k}\right).$$

Thus, we have

$$g_{a,M}\left(z,\frac{h}{k}\right)=\frac{1}{\pi i}\int_{(-1/2-2M)}C\left(s,a,\frac{h}{k}\right)\frac{\cos\frac{\pi a}{2}}{\sin(\pi(s-a))}z^{-s}\,ds$$
$$=\frac{1}{\pi i}\int_{(-1/2-2M)}C\left(1-s+a,a,\frac{\overline{h}}{k}\right)\frac{\cos\frac{\pi a}{2}}{\sin(\pi s)}z^{-s}\,ds.$$

Now,

$$\frac{d^m}{dz^m}z^{-s}=(-1)^m\frac{\Gamma(s+m)}{\Gamma(s)}z^{-s-m},$$

therefore, by the reflection formula for the Gamma function, one has

$$\begin{aligned}
&g_{a,M}^{(m)}\left(\tau,\frac{h}{k}\right)\\
&=\frac{(-1)^m}{\pi i}\int_{(-1/2-2M)}C\left(1-s+a,a,\frac{\overline{h}}{k}\right)\frac{\cos\frac{\pi a}{2}\Gamma(s+m)}{\sin(\pi s)\,\Gamma(s)}\tau^{-s-m}\,ds\\
&=\frac{(-1)^m}{\pi^2 i}\int_{(-1/2-2M)}\Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)\\
&\qquad\times C\left(1-s+a,a,\frac{\overline{h}}{k}\right)\cos\frac{\pi a}{2}\,\tau^{-s-m}\,ds\\
&=\frac{(-1)^m}{(\pi i)^2\tau^m}\left(\frac{k}{2\pi}\right)^{1+a}\int_{(-1/2-2M)}\Gamma(1-s)\Gamma(1-s+a)\Gamma(s+m)\\
&\quad\times\left(i^sD\left(s,a,\frac{h}{k}\right)-(-i)^sD\left(s,a,-\frac{h}{k}\right)\right)\cos\frac{\pi a}{2}\left(\frac{2\pi}{k\tau}\right)^s ds.
\end{aligned}$$

Expanding the D functions into their Dirichlet series, one gets

$$\begin{aligned}
g_{a,M}^{(m)}\left(\tau,\frac{h}{k}\right)&=4\frac{(-1)^m}{\pi\tau^m}\left(\frac{k}{2\pi}\right)^{1+a}\cos\frac{\pi a}{2}\sum_{\ell\ge 1}\frac{\sigma_a(\ell)}{\ell^{1+a}}\\
&\qquad\times\left(I_{m,a}^{+}\left(\frac{\ell}{\tau k}\right)\mathrm{e}\left(\frac{\overline{h}\ell}{k}\right)-I_{m,a}^{-}\left(\frac{\ell}{\tau k}\right)\mathrm{e}\left(-\frac{\overline{h}\ell}{k}\right)\right)
\end{aligned}$$

and Theorem 1.5 follows by Lemma 4.1 and Stirling's formula. ∎

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