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Addendum to "On the p^λ problem"

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by

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In [Bai] we proved the following mean value estimate for products of shifted and ordinary Dirichlet polynomials.

THEOREM 1 ([Bai, Theorem 4]). *Suppose that* $\alpha \neq 0$, $0 \leq \theta < 1$, $T > 0$, $K \geq 1$, $L \geq 1$. *If* $\theta \neq 0$, *then additionally suppose that* $L \leq T^{1/2}$. *Let* (a_k) *and* (b_l) *be arbitrary sequences of complex numbers. Suppose that* $|a_k| \leq A$ *for all* $k \sim K$ *and* $|b_l| \leq B$ *for all* $l \sim L$. *Then*

$$(1)\qquad \int\limits_0^T \Big| \sum_{k\sim K} a_k k^{it} \Big|^2 \Big| \sum_{l\sim L} b_l (l+\theta)^{i\alpha t} \Big|^2 \, dt \ll A^2B^2(T+KL)KL\log^3(2KLT),$$

the implied $\ll$*-constant depending only on* α. *If* $\theta = 0$, *then* $\log^3(2KLT)$ *on the right side of* (1) *may be replaced by* $\log^2(2KLT)$.

We then used this mean value estimate to prove the following result on the p^λ problem.

THEOREM 2 ([Bai, Theorem 3]). *Suppose that* $\varepsilon > 0$, $B > 0$, $\lambda \in (0, 1/2]$ *and a real* θ *are given. If* θ *is irrational*, *then suppose that* $\lambda < 5/19$. *Let* $N \geq 3$. *Let* $\mathbb{A}$ *be an arbitrarily given subset of the set of positive integers. Define*

$$F_\theta(\lambda) := \begin{cases} \max\limits_{k\in\mathbb{N}} \min\left\{ \dfrac{5}{12} - \dfrac{(k+6)\lambda}{6(k+1)}, \dfrac{5}{11} - \dfrac{(5k+1)\lambda}{11} \right\} & \textit{if } \theta \textit{ is rational}, \\ \dfrac{5}{12} - \dfrac{7\lambda}{12} & \textit{otherwise}. \end{cases}$$

Suppose that

$$N^{-F_\theta(\lambda)+\varepsilon\lambda} \leq \delta \leq 1.$$

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Then

$$\sum_{\substack{N<n\le 2N\\ \{n^\lambda-\theta\}<\delta\\ [n^\lambda]\in\mathbb{A}}} \Lambda(n) = \frac{\delta}{\lambda}\cdot \sum_{\substack{N^\lambda<n\le(2N)^\lambda\\ n\in\mathbb{A}}} n^{1/\lambda-1} + O\left(\frac{\delta N}{(\log N)^B}\right).$$

At the end of the last section in [Bai] we pointed out that if the condition $L \le T^{1/2}$ in the above Theorem 1 could be omitted, then the condition $\lambda < 5/19$ in Theorem 2 could be omitted as well. In the following we will see that the condition $L \le T^{1/2}$ in Theorem 1 is actually superfluous if we allow ourselves to weaken the mean value estimate (1) slightly. We establish the following

THEOREM 3. *Let $\theta, \xi, \alpha, \beta$ be real numbers with $0 \le \theta, \xi < 1$ and $\alpha\beta \ne 0$. Suppose that $T, K, L \ge 1$, $|a_k| \le 1$ and $|b_l| \le 1$. Then*

$$\int_0^T \Big|\sum_{k\sim K} a_k(k+\theta)^{i\alpha t}\Big|^2 \Big|\sum_{l\sim L} b_l(l+\xi)^{i\beta t}\Big|^2 dt \ll (T+KL)KL(\log T)^{15}. \tag{2}$$

In accordance with the proof of [Bai, Theorem 3], from the above Theorem 3 with $\xi = 0$ it can be deduced that Theorem 2 holds true with the condition $\lambda < 5/19$ omitted.

The main idea of our proof of Theorem 3 is to relate the shifted Dirichlet polynomials on the left-hand side of (2) to the corresponding Hurwitz zeta functions. For technical reasons we here define the Hurwitz zeta function $\zeta(s, y)$ in a slightly different manner to normal usage. For $0 \le y < 1$ and $\operatorname{Re} s > 1$ we write

$$\zeta(s, y) := \sum_{n=1}^{\infty} (n+y)^{-s}.$$

In the usual definition the series on the right-hand side starts with $n = 0$, and the case $y = 0$ is excluded, which we seek to avoid here.

As a function of s, the Hurwitz zeta function has a meromorphic continuation to the entire complex plane, with a simple pole at $s = 1$ (see [Ivi]). At first, we establish the following fourth power moment estimate for the Hurwitz zeta function on the critical line.

THEOREM 4. *Suppose that $V > 2\pi$ and $0 \le y < 1$. Then*

$$\int_{-V}^{V} |\zeta(1/2+it, y)|^4 \, dt \ll V(\log V)^{10}.$$

Proof. By $\zeta(\bar{s},y)=\overline{\zeta(s,y)}$, it suffices to show that

$$\int_{2\pi}^{V} |\zeta(1/2+it,y)|^4\,dt \ll V(\log V)^{10}. \tag{3}$$

By [Tch, Lemma 1], the Hurwitz zeta function satisfies an approximate functional equation of the form

$$\zeta(1/2+it,y)=\sum_{1\le m\le M}(m+y)^{-1/2-it}+\chi(1/2+it)\sum_{1\le n\le N}e(-ny)n^{-1/2+it} + O(1+M^{-3/2}|t|^{1/2})$$

if $|t|\ge 2\pi$, $1\le M\le |t|$, $N\ge 1$ and $2\pi MN=|t|$, where $|\chi(1/2+it)|=1$. Hence, we have

$$\int_{2\pi}^{V} |\zeta(1/2+it,y)|^4\,dt \ll V+\int_{2\pi}^{V}\Big|\sum_{1\le m\le\sqrt{t/(2\pi)}}(m+y)^{-1/2-it}\Big|^4\,dt + \int_{2\pi}^{V}\Big|\sum_{1\le n\le\sqrt{t/(2\pi)}}e(-ny)n^{-1/2+it}\Big|^4\,dt. \tag{4}$$

By the orthogonality relation

$$\int_0^1 e(zu)\,du=\begin{cases}1 & \text{if } z=0,\\ 0 & \text{if } z\in\mathbb{Z}\setminus\{0\},\end{cases}$$

we get

$$\sum_{1\le m\le\sqrt{t/(2\pi)}}(m+y)^{-1/2-it}=\int_0^1\sum_{1\le m\le\sqrt{V}}(m+y)^{-1/2-it}e(mu)K(t,u)\,du \tag{5}$$

for $2\pi\le t\le V$, where

$$K(t,u):=\sum_{1\le n\le\sqrt{t/(2\pi)}}e(-nu).$$

If $2\pi\le t\le V$, then the geometric sum $K(t,u)$ can be estimated by

$$K(t,u)\ll\min\{\sqrt{V},\|u\|^{-1}\}. \tag{6}$$

This yields

$$\int_0^1 |K(t,u)|\,du\ll\log V. \tag{7}$$

Using Hölder's inequality, from (5) and (7), we obtain

$$(8)\qquad \Big|\sum_{1\le m\le\sqrt{t/(2\pi)}}(m+y)^{-1/2-it}\Big|^4 \ll (\log V)^3\int_0^1\Big|\sum_{1\le m\le\sqrt{V}}(m+y)^{-1/2-it}e(mu)\Big|^4|K(t,u)|\,du.$$

Employing Hölder's inequality and [Har, Lemma 3] after dividing the sum on the right-hand side of (8) into $O(\log V)$ sums of the form

$$\sum_{M<m\le 2M}(m+y)^{-1/2-it}e(mu),$$

we obtain

$$(9)\qquad \int_{2\pi}^{V}\Big|\sum_{1\le m\le\sqrt{V}}(m+y)^{-1/2-it}e(mu)\Big|^4 dt \ll V(\log V)^6,$$

where the implied $\ll$-constant does not depend on u. Combining (6), (8) and (9), we get

$$(10)\qquad \int_{2\pi}^{V}\Big|\sum_{1\le m\le\sqrt{t/(2\pi)}}(m+y)^{-1/2-it}\Big|^4 dt \ll V(\log V)^{10}.$$

In a similar manner, we can prove

$$(11)\qquad \int_{2\pi}^{V}\Big|\sum_{1\le n\le\sqrt{t/(2\pi)}}e(-ny)n^{-1/2+it}\Big|^4 dt \ll V(\log V)^{10}.$$

Combining (4), (10) and (11), we obtain (3). This completes the proof. ■

To all appearances, there is no result like Theorem 4 in the literature.

We now prove Theorem 3 along the lines of the proof of [BaH, Theorem 3]. First we write

$$F(t):=\sum_{k\sim K}a_k(k+\theta)^{it},\qquad G(t):=\sum_{l\sim L}b_l(l+\xi)^{it},\qquad D(t):=\sum_{k\sim K}(k+\theta)^{it},$$
$$E(t):=\sum_{l\sim L}(l+\xi)^{it}.$$

Similarly to the proof of [BaH, Theorem 3], we can suppose that $K\le L\le T$, for otherwise the desired estimate follows from a classical mean value estimate for $G(t)$.

Analogously to [BaH, (17)], we have

$$\int_0^T |F(\alpha t)G(\beta t)|^2\,dt \ll (KL)^2 + \log T \max_{1\le V\le T} \int_V^{2V} |D(\alpha t)E(\beta t)|^2\,dt. \tag{12}$$

We fix V in the interval $1 \le V \le T$ for which the maximum is attained.

In the same manner like [BaH, (19)] one can prove

$$|D(\alpha t)| \ll K^{1/2} \int_{-V}^{V} |\zeta(1/2 + i\sigma - i\alpha t, \theta)|\varrho(\sigma)\,d\sigma + \frac{K\log T}{V} \tag{13}$$

as well as

$$|E(\beta t)| \ll L^{1/2} \int_{-V}^{V} |\zeta(1/2 + i\tau - i\beta t, \xi)|\varrho(\tau)\,d\tau + \frac{L\log T}{V}, \tag{14}$$

where $\varrho(x) := \min(1, 1/|x|)$. Using (13), (14) and the inequality of Cauchy–Schwarz, we deduce

$$\begin{aligned}
&\int_V^{2V} |D(\alpha t)E(\beta t)|^2\,dt \qquad (15)\\
&\ll \frac{(KL)^2\log^4 T}{V^3} + \frac{K^2L\log^3 T}{V^2} \int_{-V}^{V} \varrho(\tau) \int_V^{2V} |\zeta(1/2 + i\tau - i\beta t, \xi)|^2\,dt\,d\tau\\
&\quad + \frac{KL^2\log^3 T}{V^2} \int_{-V}^{V} \varrho(\sigma) \int_V^{2V} |\zeta(1/2 + i\sigma - i\alpha t, \theta)|^2\,dt\,d\sigma + KL\log^2 T\\
&\quad \times \int_{-V}^{V}\int_{-V}^{V} \varrho(\sigma)\varrho(\tau) \int_V^{2V} |\zeta(1/2 + i\sigma - i\alpha t, \theta)\zeta(1/2 + i\tau - i\beta t, \xi)|^2\,dt\,d\sigma\,d\tau\\
&\ll \frac{(KL)^2\log^4 T}{V^3} + \frac{K^2L\log^4 T}{V^2} \int_{-CV}^{CV} |\zeta(1/2 + it, \xi)|^2\,dt\\
&\quad + \frac{KL^2\log^4 T}{V^2} \int_{-CV}^{CV} |\zeta(1/2 + it, \theta)|^2\,dt\\
&\quad + KL\log^4 T\Big(\int_{-CV}^{CV} |\zeta(1/2 + it, \theta)|^4\,dt\Big)^{1/2}\Big(\int_{-CV}^{CV} |\zeta(1/2 + it, \xi)|^4\,dt\Big)^{1/2},
\end{aligned}$$

where C is a certain constant which depends only on α and β. From (12), (15), Theorem 4 and a similar *second* power moment estimate for the Hurwitz zeta function (which can be derived directly from Theorem 4 using the inequality of Cauchy–Schwarz), we obtain (2). This completes the proof of Theorem 3. ∎

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