ACTA ARITHMETICA
117.2 (2005)

The Piltz divisor problem in number fields: An improved lower bound by Soundararajan's method

by

K. Girstmair (Innsbruck), M. Kühleitner (Wien),
W. Müller (Graz) and W. G. Nowak (Wien)

Dedicated to Professor Franz Halter-Koch on his 60th birthday

1. Introduction. The branch of analytic number theory which relates the size of certain arithmetic functions to the number of lattice points in certain domains has a long and very prolific history. Enlightening expositions can be found in the monographs by F. Fricker [1], E. Krätzel [11], [12], and M. N. Huxley [7]. The most classical topics of this theory are the Dirichlet divisor problem and the Gaussian circle problem: they are concerned with the error terms $\Delta(x)$ and $P(x)$ in the identities

$$(1)\qquad \sum_{1\le n\le x} d(n) = x\log x + (2\gamma-1)x + \Delta(x), \qquad \sum_{1\le n\le x} r(n) = \pi x + P(x),$$

where $d(n)$ counts the number of divisors of $n\in\mathbb{N}$ and $r(n)$ the number of ways to write n as a sum of two squares. The sharpest upper bounds to date are due to M. N. Huxley [6], [8], the present "records" reading

$$\Delta(x) \ll x^{131/416}(\log x)^{26957/8320}, \qquad P(x) \ll x^{131/416}(\log x)^{18637/8320},$$

where $\frac{131}{416} = 0.3149\ldots$.

Concerning lower bounds, J. L. Hafner [2], [3] in 1981 improved upon G. H. Hardy's classical results [5], showing that ([1])

$$\Delta(x) = \Omega_+(x^{1/4}(\log x)^{1/4}(\log_2 x)^{(3+2\log 2)/4}\exp(-c_1(\log_3 x)^{1/2})),$$
$$P(x) = \Omega_-(x^{1/4}(\log x)^{1/4}(\log_2 x)^{(\log 2)/4}\exp(-c_2(\log_3 x)^{1/2})),$$

with certain $c_1, c_2 > 0$. Very recently, K. Soundararajan [19] developed an ingenious new method by which he sharpened these bounds (up to the am-

2000 *Mathematics Subject Classification*: 11N37, 11P21, 11R45.
Key words and phrases: the Piltz divisor problem, Chebotarev density theorem.

([1]) For the definitions of the different Ω-symbols, cf. E. Krätzel [11, p. 14]. Here and throughout, $\log_j$ stands for the j-fold iterated logarithm.

biguity of the sign) to

$$\begin{aligned}\Delta(x) &= \Omega(x^{1/4}(\log x)^{1/4}(\log_2 x)^{(3/4)(2^{4/3}-1)}(\log_3 x)^{-5/8}),\\ P(x) &= \Omega(x^{1/4}(\log x)^{1/4}(\log_2 x)^{(3/4)(2^{1/3}-1)}(\log_3 x)^{-5/8}).\end{aligned} \tag{2}$$

To visualize the refinement in the exponent of the $\log_2 x$-factor, note that $\frac{1}{4}(3+2\log 2) = 1.0965\ldots$, $\frac{1}{4}\log 2 = 0.1732\ldots$, while $\frac{3}{4}(2^{4/3}-1) = 1.1398\ldots$, $\frac{3}{4}(2^{1/3}-1) = 0.1949\ldots$.

The objective of the present article is to extend Soundararajan's approach to a much more general situation which includes the two classical problems as special cases. Let K be an arbitrary algebraic number field of degree k, and denote by $\mathfrak{o}_K$ its ring of algebraic integers. For a positive integer m we consider the arithmetic function $d_{K,m}(n)$ defined by the relation

$$\zeta_K^m(s) = \sum_{n=1}^{\infty} d_{K,m}(n)n^{-s} \qquad (\mathrm{Re}(s) > 1),$$

where ζ_K is the Dedekind zeta-function of K. In other words, $d_{K,m}(n)$ counts the number of m-tuples $(\mathfrak{n}_1,\ldots,\mathfrak{n}_m)$ of $\mathfrak{o}_K$-ideals with $N_K(\mathfrak{n}_1\cdots\mathfrak{n}_m) = n$. Here $N_K(\mathfrak{n})$ denotes the absolute norm of $\mathfrak{n}$. In analogy to (1), we are interested in the behavior of the error term in the identity

$$\sum_{1\le n\le x} d_{K,m}(n) = \operatorname*{res}_{s=1}\left(\zeta_K^m(s)\,\frac{x^s}{s}\right) + \Delta_{K,m}(x) \tag{3}$$

as $x\to\infty$. The investigation of $\Delta_{\mathbb{Q},m}$ is known as the *Piltz divisor problem.* Concerning upper bounds for $\Delta_{K,m}(x)$, see A. Ivić [9] (where the rational case is discussed in detail), W. G. Nowak [16], and also the references in W. Narkiewicz [15, Ch. 7].

In this article we apply Soundararajan's method to establish a sharp lower estimate for $\Delta_{K,m}(x)$. To describe the result we have to introduce some notations. For $0\le\nu\le k$ let P_ν denote the set of all rational primes which are unramified in K, and which are divisible by exactly ν $\mathfrak{o}_K$-prime ideals of degree 1. The P_ν are disjoint and together with the finitely many ramified primes exhaust the rational primes. Our result depends on the Dirichlet densities δ_ν of the sets P_ν. These densities can be calculated as follows. Let L be the minimal normal extension of K, $G = \mathrm{Gal}(L/\mathbb{Q})$ its Galois group, and $H = \mathrm{Gal}(L/K)$ the subgroup of G corresponding to K via Galois theory. Then

$$\delta_\nu = |G|^{-1}\big|\{\tau\in G : |\{\sigma\in G : \tau\in\sigma H\sigma^{-1}\}| = \nu|H|\}\big|. \tag{4}$$

The constants δ_ν satisfy $\sum_{\nu=1}^{k}\nu\delta_\nu = 1$. If K is normal, then P_ν is empty for $1\le\nu<k$. Hence $\delta_\nu = 0$ for $1\le\nu<k$ and $\delta_k = 1/k$ in this case. Additionally, denote by R the number of $1\le\nu\le k$ with $\delta_\nu > 0$.

THEOREM 1. *Let $km > 1$. Then for $x \to \infty$ the error term $\Delta_{K,m}$ in* (3) *satisfies*

$$\Delta_{K,m}(x) = \Omega((x\log x)^{\frac{km-1}{2km}}(\log_2 x)^{\kappa}(\log_3 x)^{-\lambda}), \tag{5}$$

where

$$\kappa = \frac{km+1}{2km}\Big(\sum_{\nu=1}^{k}\delta_\nu(\nu m)^{\frac{2km}{km+1}} - 1\Big), \qquad \lambda = \frac{km+1}{4km}R + \frac{km-1}{2km}.$$

Furthermore, let r_1 denote the number of real conjugates of K. If $mr_1 \equiv 3 \pmod 8$ then (5) *remains true if Ω is replaced by Ω_+. If $mr_1 \equiv 7 \pmod 8$ then* (5) *remains true if Ω is replaced by Ω_-.*

This theorem contains (2) as special cases. The Dirichlet divisor problem corresponds to $K = \mathbb{Q}$ and $m = 2$, and the circle problem corresponds to $K = \mathbb{Q}(i)$ and $m = 1$. In Section 4 we discuss further special examples. Theorem 1 should be compared with the classical estimate

$$\Delta_{K,m}(x) = \Omega_*((x\log x)^{\frac{km-1}{2km}}(\log_2 x)^{m-1})$$

due to P. Szegő and A. Walfisz [20], [21], and with Hafner's [4] refinement

$$\Delta_{K,m}(x) = \Omega_*((x\log x)^{\frac{km-1}{2km}}(\log_2 x)^{\kappa'}\exp(-c_3(\log_3 x)^{1/2})).$$

Here $c_3 > 0$ is a constant (depending on K and m) and

$$\kappa' = \frac{km-1}{2km}\Big(m\log m + \Big(\sum_{\nu=1}^{k}\delta_\nu \nu\log\nu\Big)m - m + 1\Big) + m - 1.$$

In both of these last Ω-statements,

$$\Omega_* = \begin{cases} \Omega_\pm & \text{if } km \ge 4 \text{ or } K \text{ is cubic and not totally real,}\\ \Omega_- & \text{if } m = 1 \text{ and } K \text{ is quadratic imaginary,}\\ \Omega_+ & \text{if } m = 2,3 \text{ and } K = \mathbb{Q},\\ & \text{or } m = 1,\ k = 2,3 \text{ and } K \text{ is totally real.}\end{cases}$$

Soundararajan's method in most cases fails to control the sign of the large values exhibited. But note that always $\kappa > \kappa'$. For k fixed and $m \to \infty$, we see that κ roughly grows like cm^2 with $c = \frac{1}{2}\sum_{\nu=1}^{k}\nu^2\delta_\nu$, while Hafner's κ' only behaves like $\frac{1}{2}m\log m$.

Soundararajan gives a heuristic reason why the exponent of $\log_2 x$ in (2) may be best possible. His argument carries over to our more general situation. Arrange the sequence $d_{K,m}(n)n^{-\frac{km-1}{2km}}$ in descending order, and let $S(M)$ denote the sum of the first M largest values. Then an optimal Omega result is expected to be of the form $\Delta_{K,m}(n) = \Omega(x^{\frac{km-1}{2km}}S(\log x))$. It can be proved that $S(M) = M^{\frac{km-1}{2km}}(\log M)^{\kappa+o(1)}$; thus Theorem 1 yields the expected maximal order of $\Delta_{K,m}(x)$ up to a factor $(\log_2 x)^{o(1)}$.

The proof of Theorem 1 runs along the same lines as in Soundararajan's paper [19]. We start with the known asymptotic expansion of a Borel mean value of $\varDelta_{K,m}$. To this we apply Soundararajan's key lemma (Lemma 1 below) to deduce the Omega result. In order to do this we have to count the natural numbers $n \le x$ which have exactly r_ν distinct prime divisors in the set P_ν for $1 \le \nu \le k$ and no prime divisors which ramify in K. Moreover, we need a result which is uniform in $r = r_1 + \cdots + r_k \le B \log_2 x$. Here $B \ge 1$ is a given constant. What we need is a special case of the following theorem (it may be viewed as a generalization of the Chebotarev density theorem). Note that for a prime p, which is unramified in K, the type of the prime ideal decomposition of $p\mathfrak{o}_K$ in K is determined by $H = \mathrm{Gal}(L/K)$ and the Frobenius conjugacy class of p in $L/\mathbb{Q}$ (see Section 2).

Let Q be an algebraic number field, K an algebraic extension of Q, and L an algebraic extension of K which is normal over Q. Set $G = \mathrm{Gal}(L/Q)$ and $H = \mathrm{Gal}(L/K)$. For every prime ideal $\mathfrak{p}$ in $\mathfrak{o}_Q$ denote by $(\mathfrak{p}, L/Q)$ the Frobenius conjugacy class of $\mathfrak{p}$ in L/Q (see Section 3).

Theorem 2. *Let $G = \bigcup_{\nu=1}^d A_\nu$ be a partition of G into d sets $A_\nu \ne \emptyset$ which are unions of conjugacy classes of G. For $r_\nu \ge 0$, $1 \le \nu \le d$, denote by $\pi_{K/Q}(x, r_1, \ldots, r_d)$ the number of $\mathfrak{o}_Q$-ideals $\mathfrak{n}$ with $N_Q(\mathfrak{n}) \le x$, such that $\mathfrak{n}$ has no prime divisor which is ramified in L/Q, and r_ν distinct prime divisors $\mathfrak{p}$ with $(\mathfrak{p}, L/Q) \subseteq A_\nu$ for $1 \le \nu \le d$. For given $B \ge 1$ and $r = \sum_{\nu=1}^d r_\nu$, we have, uniformly in $1 \le r \le B \log_2 x$ and $x \ge 3$,*

$$\pi_{K/Q}(x, r_1, \ldots, r_d) = \left(\prod_{\nu=1}^d \frac{\delta_\nu^{r_\nu}}{r_\nu!}\right) \frac{rx}{\log x} (\log_2 x)^{r-1} \left(\mu(x, r_1, \ldots, r_d) + O\left(\frac{r}{(\log_2 x)^2}\right)\right).$$

Here $\delta_\nu = |A_\nu|/|G|$ denotes the density of A_ν, and

$$\mu(x, r_1, \ldots, r_d) = H\left(\frac{(r-1)r_1}{r\delta_1 \log_2 x}, \ldots, \frac{(r-1)r_d}{r\delta_d \log_2 x}\right),$$

where H is the function defined in (24). *Furthermore, uniformly for $1 \le r \le B \log_2 x$ and $x \ge 3$,*

$$\mu(x, r_1, \ldots, r_d) = 1 + O\left(\frac{r}{\log_2 x}\right), \qquad \mu(x, r_1, \ldots, r_d) \asymp 1.$$

Corollary. *Let $q \ge 1$, and let $a_1, \ldots, a_d$ be d integers which are incongruent modulo q and relatively prime to q. Denote by $\pi(x, r_1, \ldots, r_d)$ the number of integers $n \le x$ which have r_ν distinct prime divisors $p \equiv a_\nu \pmod q$ for $1 \le \nu \le d$, and no other ones. For given $B \ge 1$, and $r = \sum_{\nu=1}^d r_\nu$, we have, uniformly in $1 \le r \le B \log_2 x$ and $x \ge 3$,*

$$\pi(x, r_1, \ldots, r_d) = \frac{r}{r_1! \ldots r_d! \varphi(q)^r} \frac{x}{\log x} (\log_2 x)^{r-1} \left(1 + O\left(\frac{r}{\log_2 x}\right)\right),$$
$$\pi(x, r_1, \ldots, r_d) \asymp \frac{r}{r_1! \ldots r_d! \varphi(q)^r} \frac{x}{\log x} (\log_2 x)^{r-1}.$$

Proof. The Corollary is a special case of Theorem 2. Set $Q = \mathbb{Q}$ and let K be the cyclotomic field $\mathbb{Q}(\zeta_q)$. Here ζ_q denotes a primitive root of unity of order q. $K/\mathbb{Q}$ is normal and $|\mathrm{Gal}(K/\mathbb{Q})| = \varphi(q)$. A prime p is ramified in $K/\mathbb{Q}$ if and only if $p \mid q$. For an integer a with $(a, q) = 1$, let $\sigma_a \in \mathrm{Gal}(K/\mathbb{Q})$ be uniquely defined by $\sigma_a(\zeta_q) = \zeta_q^a$. For $p \nmid q$ it is well known that $(p, K/\mathbb{Q}) = \{\sigma_a\}$ if and only if $p \equiv a \pmod{q}$ (cf. G. J. Janusz [10, p. 104]). This proves the Corollary. ∎

2. Proof of Theorem 1. We start with the asymptotic expansion for the *Borel mean value* of the error term $\Delta_{K,m}(t)$. For real parameters $T \geq 40$ and $t \in [T/2, T^2]$, and arbitrary $\varepsilon > 0$,

$$B(t) := \frac{1}{\Gamma(h+1)} \int_0^\infty e^{-u} u^h \Delta_{K,m}(X u^{km/2})\, du = c_0 t^{(km-1)/2} S(t) + O(t^{km/2 - 3/4 + \varepsilon}) \tag{6}$$

where

$$S(t) = \sum_{n \leq T^{\varepsilon_0}} d_{K,m}(n)\, n^{\theta - 1} \exp(-c_2 (nX)^{2/(km)}) \cos(2\pi c_1 n^{1/(km)} t + \beta_0)$$

with

$$X := (\log T)^{-2}, \qquad h = h(t) := (t X^{-1/(km)})^2,$$
$$\theta = \frac{km - 1}{2km}, \qquad \beta_0 = -\frac{3\pi}{4} + \frac{\pi}{4} m r_1.$$

Here $\varepsilon_0 > 0$ is a sufficiently small constant, c_j are positive constants which depend only on the field K and on m. This asymptotic expansion goes back to G. Szegő and A. Walfisz [20], [21]. For a more recent treatment in the context of asymmetric divisor problems, see W. G. Nowak [17, p. 269].

The following lemma is due to K. Soundararajan [19]. It gives a general lower bound for trigonometric series. Let $(f(n))_{n \geq 1}$ be a sequence of non-negative real numbers and $(\lambda_n)_{n \geq 1}$ be a non-decreasing sequence of non-negative real numbers. Suppose that $\sum_{n \geq 1} f(n) < \infty$ and consider the trigonometric series

$$F(t) := \sum_{n \geq 1} f(n) \cos(2\pi \lambda_n t + \beta),$$

where $\beta \in \mathbb{R}$.

LEMMA 1. *Let $L \geq 2$ and $N \geq 1$ be integers. Let $\mathcal{M}$ be a set of integers such that $\lambda_m \in [\lambda_N/2, 3\lambda_N/2]$ for each $m \in \mathcal{M}$. For any $T \geq 2$ there exists a point $t \in [T/2, (6L)^{|\mathcal{M}|+1}T]$ such that*

$$|F(t)| \geq \frac{1}{8} \sum_{m\in\mathcal{M}} f(m) - \frac{1}{L-1} \sum_{\substack{n \\ \lambda_n \leq 2\lambda_N}} f(n) - \frac{4}{\pi^2 X\lambda_N} \sum_{n\geq 1} f(n). \tag{7}$$

If $\beta \equiv 0 \pmod{2\pi}$ then the conclusion (7) *holds with $F(t)$ in place of $|F(t)|$. If $\beta \equiv \pi \pmod{2\pi}$ then the conclusion* (7) *holds with $-F(t)$ in place of $|F(t)|$.*

To prove Theorem 1 we apply Lemma 1 with

$$f(n) = d_{K,m}(n)n^{\theta-1}\exp(-c_2(nX)^{2/(km)}) \tag{8}$$

if $n \leq T^{\varepsilon_0}$ and $f(n) = 0$ else, $\lambda_n = c_1 n^{1/(km)}$ and $\beta = \beta_0$. For $0 \leq \nu \leq k$ let P_ν be the set of all rational primes which are unramified in K and which are divisible by exactly ν $\mathfrak{o}_K$-prime ideals of degree 1. Let $I = \{1 \leq \nu \leq k : \delta_\nu > 0\}$, where δ_ν denotes the Dirichlet density of P_ν as in the introduction. Then $R = |I|$.

We choose $\mathcal{M}$ to be the set of all $n \in [2^{-km}N, (3/2)^{km}N]$ such that n has r_ν distinct prime factors from P_ν for all $\nu \in I$ and no others. Here $r_\nu = [\lambda_\nu \log_2 N]$ with some parameters $\lambda_\nu > 0$ (the optimal choice turns out to be $\lambda_\nu = \delta_\nu(\nu m)^{2km/(km+1)}$).

For $n \in \mathcal{M}$ the value of $d_{K,m}(n)$ is large. Indeed, if $p \in P_\nu$ is prime then the factorization of $p\mathfrak{o}_K$ contains exactly ν prime ideals $\mathfrak{p}$ with $N_K(\mathfrak{p}) = p$. Hence $d_{K,1}(p) = \nu$. Since for every prime ideal $\mathfrak{p}$ there are m tuples $(\mathfrak{n}_1, \ldots, \mathfrak{n}_m)$ with $N_K(\mathfrak{n}_1 \cdots \mathfrak{n}_m) = p$ (take one of the $\mathfrak{n}_i$ equal to $\mathfrak{p}$, all others equal to $\mathfrak{o}_K$) we find $d_{K,m}(p) = \nu m$. The same argument proves $d_{K,m}(p^e) \geq \nu m$ for $e \geq 1$ and $p \in P_\nu$. Hence

$$d_{K,m}(n) \geq \prod_{\nu\in I} (\nu m)^{r_\nu} \qquad (n \in \mathcal{M}). \tag{9}$$

Next we use Theorem 2 to estimate the cardinality of $\mathcal{M}$. Let L denote the minimal normal extension of K. A rational prime p is ramified in K if and only if p is ramified in L (cf. W. Narkiewicz [15, Cor. 2 to Prop. 4.12]). For an unramified p let $(p, L/\mathbb{Q})$ be the Frobenius symbol of p in $L/\mathbb{Q}$. This is a class of conjugate elements of $G = \mathrm{Gal}(L/\mathbb{Q})$. Together with $H = \mathrm{Gal}(L/K)$ it determines the type of factorization of p (unramified) in the following way (cf. G. J. Janusz [10, Prop. 2.8]). Take any $\varphi \in (p, L/\mathbb{Q})$. If the set of right cosets of H in G is partitioned into sets

$$\{H\sigma_1, H\sigma_1\varphi, \ldots, H\sigma_1\varphi^{f_1-1}\}, \ldots, \{H\sigma_j, H\sigma_j\varphi, \ldots, H\sigma_j\varphi^{f_j-1}\}$$

with $\sigma_i \in G$ and $f_i \geq 1$, then $p\mathfrak{o}_K = \mathfrak{p}_1 \cdots \mathfrak{p}_j$. Here $\mathfrak{p}_i$ are prime ideals in $\mathfrak{o}_K$ with initial degree $f(\mathfrak{p}_i|\mathbb{Q}) = f_i$, thus $N_K(\mathfrak{p}_i) = p^{f_i}$. Hence $p \in P_\nu$ if and only if p is unramified in K, and there are exactly ν right cosets $H\sigma$ with

$H\sigma\varphi = H\sigma$. In other words

$$P_\nu = \{p \text{ unramified in } K : |\{\sigma \in G : \varphi \in \sigma^{-1}H\sigma\}| = \nu|H|\}.$$

The set on the right hand side is independent of the choice of $\varphi \in (p, L/\mathbb{Q})$. Let

$$A_\nu = \{\tau \in G : |\{\sigma \in G : \tau \in \sigma^{-1}H\sigma\}| = \nu|H|\}.$$

Then $p \in P_\nu$ if and only if $(p, L/\mathbb{Q}) \subseteq A_\nu$. The density of A_ν is $\delta_\nu = |A_\nu|/|G|$. This explains (4). If we write $c_1 = (1/2)^{km}$, $c_2 = (3/2)^{km}$ for the moment, Theorem 2 implies (with the choice of the numbers r_ν made earlier)

$$\begin{aligned}|\mathcal{M}| \asymp r\biggl(\prod_{\nu\in I}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\biggr)\biggl(&H\biggl(\frac{(r-1)r_1}{r\delta_1\log_2(c_2N)}, \ldots, \frac{(r-1)r_d}{r\delta_d\log_2(c_2N)}\biggr)\\ &\times \frac{c_2N}{\log(c_2N)}\,(\log_2(c_2N))^{r-1}\\ &- H\biggl(\frac{(r-1)r_1}{r\delta_1\log_2(c_1N)}, \ldots, \frac{(r-1)r_d}{r\delta_d\log_2(c_1N)}\biggr)\frac{c_1N}{\log(c_1N)}\,(\log_2(c_1N))^{r-1}\biggr).\end{aligned}$$

For N large, the arguments of H here differ only by $o(1)$ in each component, hence the same is true for the two values of H involved. Observe that they are also $\asymp 1$. However,

$$\frac{c_2N}{\log(c_2N)}\,(\log_2(c_2N))^{r-1} - \frac{c_1N}{\log(c_1N)}\,(\log_2(c_1N))^{r-1} \asymp \frac{N}{\log N}\,(\log_2 N)^{r-1}.$$

Therefore,

$$|\mathcal{M}| \asymp r\biggl(\prod_{\nu\in I}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\biggr)\frac{N}{\log N}\,(\log_2 N)^{r-1} \asymp \frac{N}{\log N}\prod_{\nu\in I}\frac{(\delta_\nu\log_2 N)^{r_\nu}}{r_\nu!}.$$

Using $n! \sim \sqrt{2\pi}\,n^{n+1/2}e^{-n}$ we find

$$|\mathcal{M}| \asymp N(\log N)^{\alpha}(\log_2 N)^{\beta}, \tag{10}$$

where $\alpha = -1 + \sum_{\nu\in I}\lambda_\nu(1 + \log\delta_\nu - \log\lambda_\nu)$ and $\beta = -R/2$.

To complete the proof of Theorem 1, let $T \ge 40$ be a real parameter, and choose $L = \lceil(\log_2 T)^m\rceil$ and

$$N = \lceil c\log T(\log_2 T)^{-\alpha}(\log_3 T)^{-1-\beta}\rceil.$$

Here c denotes a positive constant, which we choose sufficiently small to ensure

$$(6L)^{|\mathcal{M}|+1} \le T. \tag{11}$$

Moreover, this choice implies $nX \ll 1$ for $n \ll N$ and $t \in \bigl[\frac{1}{2}T, T^2\bigr]$. By (8)–(10) we find

$$\sum_{n\in\mathcal{M}} f(n) \gg N^{\theta-1}|\mathcal{M}|\prod_{\nu\in I}(\nu m)^{r_\nu} \gg N^{\theta}(\log N)^{\alpha+\sum_{\nu\in I}\lambda_\nu \log(\nu m)}(\log_2 N)^{\beta}$$
$$\gg (\log T)^{\theta}(\log_2 T)^{(1-\theta)\alpha+\sum_{\nu\in I}\lambda_\nu\log(\nu m)}(\log_3 T)^{(1-\theta)\beta-\theta}.$$

The choice $\lambda_\nu = \delta_\nu(\nu m)^{2km/(km+1)}$, $\nu\in I$, maximizes the exponent of $\log_2 T$. This yields

(12) $$\sum_{m\in\mathcal{M}} f(n) \gg (\log T)^{\theta}(\log_2 T)^{\kappa}(\log_3 T)^{-\lambda},$$

with κ and λ as in Theorem 1.

It remains to show that the other two terms on the right hand side of (7) are small. The bound $\sum_{n\le x} d_{K,m}(n) \ll x(\log x)^{m-1}$ together with partial summation yields $\sum_{\lambda_n\le\lambda_N} f(n) \ll N^{\theta}(\log N)^{m-1}$. After division by $L-1$ this is small compared to the right hand side of (12). Similarly, for $t\in\left[\frac{1}{2}T, T^2\right]$,

$$\frac{4}{\pi^2 T\lambda_N}\sum_{n=1}^{\infty} f(n) \ll \frac{1}{TN}\sum_{1\le n\le T^{\varepsilon_0}} d_{K,m}(n)\, n^{\theta-1} \ll T^{3\varepsilon_0-1}.$$

This is again small compared with (12). Using (7) and (11) we conclude that for arbitrary $T\ge 40$, there exists a value $t\in\left[\frac{1}{2}T, T^2\right]$ with

(13) $$|B(t)| \gg t^{(km-1)/2}(\log t)^{\theta}(\log_2 t)^{\kappa}(\log_3 t)^{-\lambda}.$$

Now let us assume that (5) is false. Then for every $\varepsilon_1 > 0$ there is a constant c such that, for all $u>0$,

(14) $$|\Delta_{K,m}(u)| \le c + \varepsilon_1 u^{\theta}\mathcal{L}(u),$$

where

$$\mathcal{L}(u) := (\log u)^{\theta}(\log_2 u)^{\kappa}(\log_3 u)^{-\lambda}$$

for $u\ge 20$, and $\mathcal{L}(u) = \mathcal{L}(20)$ else. By the definition (6) of $B(t)$, this implies that

$$|B(t)| \le c + \frac{\varepsilon_1}{\Gamma(h+1)}\int_0^{\infty} e^{-u}u^h (Xu^{km/2})^{\theta}\mathcal{L}(Xu^{km/2})\,du$$

for all $t>0$. Estimating this integral by Hafner's Lemma 2.3.6 in [3], we obtain

$$|B(t)| \le c + c^*\varepsilon_1(Xh^{km/2})^{\theta}\mathcal{L}(Xh^{km/2}) = c + c^*\varepsilon_1 t^{(km-1)/2}\mathcal{L}(t^{km/2}).$$

Together with (13), this yields a positive lower bound for ε_1. This proves (5). If β_0 is an integer multiple of π, Lemma 1 yields (13) with $|B(t)|$ replaced by $B(t)$ or $-B(t)$. Completing the argument as before, we obtain, for this case, the more precise information stated in Theorem 1. ∎

3. Proof of Theorem 2. We use a variant of the method of Selberg–Delange to prove Theorem 2. For a description of this method see G. Tenenbaum [22]. Let $\mathfrak{d}$ be the relative discriminant of L/Q. An $\mathfrak{o}_Q$-prime ideal $\mathfrak{p}$ is ramified in L/Q if and only if $\mathfrak{p} \mid \mathfrak{d}$. For an $\mathfrak{o}_Q$-ideal $\mathfrak{n}$ denote by $\omega_\nu(\mathfrak{n})$ the number of distinct $\mathfrak{o}_Q$-prime ideal divisors $\mathfrak{p}$ of $\mathfrak{n}$ which are unramified in L/Q and which satisfy $(\mathfrak{p}, L/Q) \subseteq A_\nu$. For $s \in \mathbb{C}$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ consider the Dirichlet series

$$F(s,z) = \sum_{(\mathfrak{n},\mathfrak{d})=1} \Big(\prod_{\nu=1}^{d} z_\nu^{\omega_\nu(\mathfrak{n})} \Big) N_Q(\mathfrak{n})^{-s} \qquad (\mathrm{Re}(s) > 1).$$

The sum runs over all $\mathfrak{o}_Q$-ideals $\mathfrak{n}$ relatively prime to $\mathfrak{d}$. It is absolutely convergent in the half-plane $\mathrm{Re}(s) > 1$. Since $z_\nu^{\omega_\nu(\mathfrak{n})}$ is multiplicative in $\mathfrak{n}$ we find

$$F(s,z) = \prod_{\nu=1}^{d} \prod_{\substack{\mathfrak{p} \nmid \mathfrak{d} \\ (\mathfrak{p},L/Q) \subseteq A_\nu}} \Big(1 + \frac{z_\nu}{N_Q(\mathfrak{p})^s - 1} \Big).$$

The plan of the proof is as follows. After the analytic continuation of F, standard methods give the asymptotic behavior of

$$S(x,z) = \sum_{\substack{N_Q(\mathfrak{n}) \le x \\ (\mathfrak{n},\mathfrak{d})=1}} \prod_{\nu=1}^{d} z_\nu^{\omega_\nu(\mathfrak{n})}. \tag{15}$$

Then Theorem 2 follows by a d-fold application of Cauchy's theorem.

To study the analytic properties of F we use Artin L-series (see E. de Shalit [18] for a recent survey of Artin L-series). They are defined as follows. Let L/Q be a finite normal extension of Q with Galois group G. For a prime ideal $\mathfrak{P}$ of $\mathfrak{o}_L$ lying over the prime ideal $\mathfrak{p}$ of $\mathfrak{o}_Q$, one defines the *decomposition group* of $\mathfrak{P}$ by

$$D_\mathfrak{P} = \{\tau \in G : \tau(\mathfrak{P}) = \mathfrak{P}\}$$

and the *inertia subgroup* by

$$I_\mathfrak{P} = \{\tau \in G : \tau(x) \equiv x \ (\mathrm{mod}\, \mathfrak{P}) \text{ for all } x \in \mathfrak{o}_L\}.$$

The map $D_\mathfrak{P} \to \mathrm{Gal}(L_\mathfrak{P}/Q_\mathfrak{p})$, induced by the restriction of $\tau \in D_\mathfrak{P}$ to $\mathfrak{o}_L$, is surjective with kernel $I_\mathfrak{P}$. Here $L_\mathfrak{P}$ and $Q_\mathfrak{p}$ denote the residue class fields. The Galois group $\mathrm{Gal}(L_\mathfrak{P}/Q_\mathfrak{p})$ is cyclic with generator $x \mapsto x^{N_Q(\mathfrak{p})}$. Every element $\sigma_\mathfrak{P} \in D_\mathfrak{P}$ which maps to this generator is called a *Frobenius element* of $\mathfrak{P}$ in L/Q. It is only unique modulo $I_\mathfrak{P}$. The ideal $\mathfrak{p}$ is unramified if and only if $I_\mathfrak{P} = 1$. In this case the Frobenius element is unique. For unramified $\mathfrak{p}$ the Frobenius elements $\sigma_\mathfrak{P}$ of all $\mathfrak{P} \mid \mathfrak{p}$ are conjugate. This conjugacy class is called the *Frobenius conjugacy class* of $\mathfrak{p}$ in L/Q and denoted by $(\mathfrak{p}, L/Q)$.

Let V be a finite-dimensional $\mathbb{C}$-vector space and $\varrho : G \to \mathrm{GL}(V)$ a representation of G with character χ. Furthermore, denote by $V_{\mathfrak{P}} = \{x \in V : \varrho(\sigma)(x) = x \text{ for all } \sigma \in I_{\mathfrak{P}}\}$ the vector space of all $I_{\mathfrak{P}}$-fixed points. Note that $V_{\mathfrak{P}} = V$ if $\mathfrak{p}$ is unramified in L/Q. The *Artin L-series* is defined as

$$L(s,\chi) = \prod_{\mathfrak{p}} \det((\mathrm{Id}_V - \varrho(\sigma_{\mathfrak{P}})N_Q(\mathfrak{p})^{-s})|_{V_{\mathfrak{P}}})^{-1} \quad (\mathrm{Re}(s) > 1).$$

The product runs over all prime ideals $\mathfrak{p}$ in $\mathfrak{o}_Q$. It is absolutely and uniformly convergent in every compact subset of the half-plane $\mathrm{Re}(s) > 1$. Hence $L(s,\chi)$ is holomorphic in this domain. Moreover,

$$\log L(s,\chi) = \sum_{\mathfrak{p}} \sum_{k \geq 1} \frac{\widetilde{\chi}(\sigma_{\mathfrak{P}}^k)}{kN_Q(\mathfrak{p})^s} \quad (\mathrm{Re}(s) > 1), \tag{16}$$

where

$$\widetilde{\chi}(\sigma_{\mathfrak{P}}^k) := \frac{1}{|I_{\mathfrak{P}}|} \sum_{\alpha \in I_{\mathfrak{P}}} \chi(\sigma_{\mathfrak{P}}^k \alpha).$$

Here and in the following, log always denotes that branch of the logarithm which is real on the positive real axis. If $\mathfrak{p}$ is unramified, e.g. $\mathfrak{p} \nmid \mathfrak{d}$, then $\widetilde{\chi}(\sigma_{\mathfrak{P}}^k) = \chi(\sigma_{\mathfrak{P}}^k)$. The Artin L-series can be continued meromorphically to the entire complex plane. There is a real constant $c = c(L) > 0$ such that, for all irreducible characters χ,

$$L(s,\chi) \neq 0 \quad \text{for } s \in \mathcal{D} := \{\sigma + it : \sigma \geq \psi(t)\}, \tag{17}$$

where $\psi(t) := 1 - c/\log(|t| + 3)$. Moreover, $L(s,\chi)$ is holomorphic in $\mathcal{D}$ (up to a simple pole at $s = 1$ if χ is the trivial character) and, for $s \in \mathcal{D}$ with $|t| \geq \delta > 0$,

$$\log L(s,\chi) \ll \log_2(|t| + 3). \tag{18}$$

This is well known for Hecke L-series: For a proof of (17) see J. C. Lagarias and A. M. Odlyzko [13, Chapter 8]. Using (5.9) of [13], the bound (18) is readily verified on classical lines (cf. G. Tenenbaum [22, II, Theorem 16]). If one knows (17) and (18) for Hecke L-series, both assertions follow for general Artin L-series, since every Artin L-series can be expressed as a quotient of products of Hecke L-series (this is a consequence of the Brauer induction theorem). Note that we do not try to give results which are uniform in the field L.

For $\mathrm{Re}(s) > 1$, define

$$Z_A(s) = \prod_{\substack{\mathfrak{p} \nmid \mathfrak{d} \\ (\mathfrak{p}, L/Q) \subseteq A}} (1 - N_Q(\mathfrak{p})^{-s})^{-1}.$$

Here $A = \bigcup_{j\in J} C_j$ is a non-empty disjoint union of conjugacy classes C_j of G. From the representation theory of finite groups we use the following facts. If G splits into h conjugacy classes $C_1,\ldots,C_h$, then there are exactly h irreducible representations, whose characters we denote by $\chi_1,\ldots,\chi_h$. The χ_i are constant on every conjugacy class. They satisfy the orthogonality relations

$$|G|^{-1}\sum_{s\in G}\chi_i(s)\chi_j(s^{-1}) = \begin{cases} 1 & \text{if } i=j,\\ 0 & \text{else.}\end{cases}$$

This implies that the indicator function I_A of A can be written as $I_A = \sum_{i=1}^h \alpha_i(A)\chi_i$, where $\alpha_i(A) = |G|^{-1}\sum_{j\in J}\chi_i(C_j^{-1})|C_j|$. Together with (16) this gives

$$\begin{aligned}\sum_{i=1}^h &\alpha_i(A)\log L(s,\chi_i)\\ &= \sum_{\substack{\mathfrak{p}\nmid\mathfrak{d}\\ (\mathfrak{p},L/Q)\subseteq A}} N_Q(\mathfrak{p})^{-s} + \sum_{i=1}^h \alpha_i(A)\Big\{\sum_{\mathfrak{p}\mid\mathfrak{d}}\frac{\widetilde{\chi}_i(\sigma_{\mathfrak{P}})}{N_Q(\mathfrak{p})^s} + \sum_{\mathfrak{p}}\sum_{k\ge 2}\frac{\widetilde{\chi}_i(\sigma_{\mathfrak{P}}^k)}{kN_Q(\mathfrak{p})^{ks}}\Big\}.\end{aligned}$$

On the other hand, for $\mathrm{Re}(s) > 1$,

$$\log Z_A(s) = \sum_{\substack{\mathfrak{p}\nmid\mathfrak{d}\\ (\mathfrak{p},L/Q)\subseteq A}} N_Q(\mathfrak{p})^{-s} + \sum_{\substack{\mathfrak{p}\nmid\mathfrak{d}\\ (\mathfrak{p},L/Q)\subseteq A}}\sum_{k\ge 2}\frac{1}{kN_Q(\mathfrak{p})^{ks}}.$$

If χ_1 denotes the trivial character, then $\alpha_1(A) = |A|/|G| = \delta(A)$. Altogether, we find

$$\log Z_A(s) = \delta(A)\log\zeta_L(s) + \xi_A(s) + \eta_A(s),$$

where

$$\xi_A(s) = \sum_{i=2}^h \alpha_i(A)\log L(s,\chi_i)$$

and η_A denotes a function which is holomorphic and bounded in every half-plane $\mathrm{Re}(s) \ge \frac{1}{2}+\varepsilon$, $\varepsilon > 0$. Note that $(s-1)\zeta_L(s)$ and $L(s,\chi_i)$, $i > 1$, are holomorphic and non-zero in $\mathcal{D}$. Thus, for $v\in\mathbb{C}$,

$$\begin{aligned}(19)\qquad Z_A^v(s) := \exp(&-\delta(A)v\log(s-1)\\ &+ \delta(A)v\log((s-1)\zeta_L(s)) + v\xi_A(s) + v\eta_A(s))\end{aligned}$$

gives the analytic continuation of $Z_A^v(s)$ to $\overline{\mathcal{D}} := \mathcal{D}\setminus[\psi(0),1]$. From (18) it follows that $\log Z_A(s) \ll \log_2(|t|+3)$ for $s\in\mathcal{D}$, $|t|\ge\delta$. Hence, there is a constant $c(B)$ such that

$$(20)\qquad Z_A^v(s) \ll (\log|t|)^{c(B)}$$

uniformly for $|v|\le B$, $s\in\mathcal{D}$, $|t|\ge\delta$.

For $\mathrm{Re}(s) > 1$ and $z = (z_1, \ldots, z_d) \in \mathbb{C}^d$ we write

$$F(s,z) = G(s,z) \prod_{\nu=1}^{d} Z_{A_\nu}^{z_\nu}(s), \tag{21}$$

where

$$G(s,z) = \prod_{\nu=1}^{d} \prod_{\substack{\mathfrak{p} \nmid \mathfrak{d} \\ (\mathfrak{p}, L/Q) \subseteq A_\nu}} \left(1 + \frac{z_\nu}{N_Q(\mathfrak{p})^s - 1}\right)\left(1 - \frac{1}{N_Q(\mathfrak{p})^s}\right)^{z_\nu}.$$

As a function of s, G is analytic in $\mathrm{Re}(s) > 1/2$, and uniformly bounded in $\mathrm{Re}(s) \geq 1/2 + \varepsilon > 1/2$, $|z_\nu| \leq B$, $1 \leq \nu \leq d$ (cf. G. Tenenbaum [22, p. 201] for a more detailed argument). Thus (21) gives the analytic continuation of F to $\overline{\mathcal{D}}$.

Let $x \geq 4$, $a = 1 + 1/\log x$ and set $T = \exp(\sqrt{\log x}) - 3$. Using Perron's formula and (20) we find, for an arbitrary $\varepsilon > 0$,

$$\begin{aligned}\int_0^x S(u,z)\,du &= \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} F(s,z) x^{s+1} \frac{ds}{s(s+1)} \\ &= \frac{1}{2\pi i} \int_{a-iT}^{a+iT} F(s,z) x^{s+1} \frac{ds}{s(s+1)} + O(x^2 T^{-1+\varepsilon}).\end{aligned}$$

Here and in the following, all estimates are uniform in $|z_1| \leq B, \ldots, |z_d| \leq B$. The constants implied in the O- and $\ll$-notation may depend on B and the field L.

Next, Cauchy's theorem is used to replace the path of integration by a path on the boundary of $\overline{\mathcal{D}}$. The new integration path is $\mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{H} \cup \mathcal{G}_3 \cup \mathcal{G}_4$, where $\mathcal{G}_1$ is the line connecting $a - iT$ with $\psi(T) - iT$, $\mathcal{G}_2$ is the curve $\psi(t) - it$, t running from $-T$ to 0, $\mathcal{G}_3 = -\overline{\mathcal{G}}_2$, $\mathcal{G}_4 = -\overline{\mathcal{G}}_1$ and $\mathcal{H}$ is the Hankel type integration path connecting $\psi(0)$ with $1 - \varrho$ (on this part of the path $\arg(s) = -\pi$), turning around $s = 1$ at a circle with radius $\varrho := (2\log x)^{-1}$ and connecting $1 - \varrho$ with $\psi(0)$ (here $\arg(s) = \pi$). Using (20) we find that the contribution of the integral over $\mathcal{G}_1$ and $\mathcal{G}_4$ is $O(x^2 T^{-2+\varepsilon})$. Similarly, the contribution of the integrals over $\mathcal{G}_2$ and $\mathcal{G}_3$ is $O(x^{1+\sigma(T)}) = O(x^2 \exp(-c\sqrt{\log x}))$. Hence, by (15),

$$\int_0^x S(u,z)\,du = \Phi(x,z) + O(x^2 \exp(-c\sqrt{\log x})), \tag{22}$$

where $0 < c < 1$ and

$$\Phi(x,z) = \frac{1}{2\pi i} \int_{\mathcal{H}} F(s,z) x^{s+1} \frac{ds}{s(s+1)}.$$

Taylor series expansion in (19) yields

$$Z_A^v(s) = (s-1)^{-\delta(A)v} e^{\beta(A)v}(1+O(|s-1|)) \quad (s \in \mathcal{H},\ v \in \mathbb{C},\ |v| \le B)$$

with $\beta(A) = \delta(A)\operatorname{res}_{s=1}\zeta_L(s) + \xi_A(1) + \eta_A(1)$. It follows that, for $s \in \mathcal{H}$,

$$\frac{1}{s} F(s,z) = H(z)\Gamma(w+1)(s-1)^{-w} + O(|s-1|^{1-\mathrm{Re}(w)}), \tag{23}$$

where $w = \sum_{\nu=1}^{d} \delta_\nu z_\nu$ and

$$H(z) := \frac{1}{\Gamma(w+1)} G(1,z) \prod_{\nu=1}^{d} e^{\beta(A_\nu) z_\nu}. \tag{24}$$

Note that $H((0,\ldots,0)) = 1$. Using (23) we find

$$\begin{aligned} \frac{\partial^2 \Phi}{\partial x^2}(x,z) &= \frac{1}{2\pi i}\int_{\mathcal{H}} F(s,z) x^{s-1}\,ds \\ &\ll \int_{\mathcal{H}} |s-1|^{-\mathrm{Re}(w)} |x^{s-1}|\,|ds| \ll (\log x)^{\mathrm{Re}(w)} \end{aligned} \tag{25}$$

and

$$\begin{aligned} \frac{\partial \Phi}{\partial x}(x,z) &= \frac{1}{2\pi i}\int_{\mathcal{H}} F(s,z) x^{s}\,\frac{ds}{s} \\ &= H(z) w\Gamma(w)\,\frac{1}{2\pi i}\int_{\mathcal{H}} (s-1)^{-w} x^s\,ds + O(R(x)). \end{aligned}$$

Here

$$\begin{aligned} R(x) &= \int_{\mathcal{H}} |s-1|^{1-\mathrm{Re}(w)} |x^s|\,|ds| \ll \int_{\psi(0)}^{1-\varrho} (1-\sigma)^{1-\mathrm{Re}(w)} x^\sigma\,d\sigma + x^{1+\varrho}\varrho^{2-\mathrm{Re}(w)} \\ &\ll x(\log x)^{\mathrm{Re}(w)-2}\Big(\int_{1/2}^{\infty} u^{1-\mathrm{Re}(w)} e^{-u}\,du + 1\Big) \ll x(\log x)^{\mathrm{Re}(w)-2}. \end{aligned}$$

By G. Tenenbaum [22, II.5, Corollary 2.1],

$$\frac{1}{2\pi i}\int_{\mathcal{H}} (s-1)^{-w} x^s\,ds = x(\log x)^{w-1}(\Gamma(w)^{-1} + O(x^{-\psi(0)/2}))$$

uniformly for $|w| \le B'$, $B' > 0$. Hence

$$\frac{\partial \Phi}{\partial x}(x,z) = wH(z)x(\log x)^{w-1} + O(x(\log x)^{\mathrm{Re}(w)-2}). \tag{26}$$

Using (22) and (25) we infer, with $h := \exp(-(c/2)\sqrt{\log x})$,

$$h^{-1}\int\limits_x^{x+h} S(u,z)\,du = h^{-1}(\Phi(x+h,z)-\Phi(x,z)) + O(x^2h)$$
$$= \frac{\partial\Phi}{\partial x}(x,z) + h\int\limits_0^1 (1-t)\,\frac{\partial^2\Phi}{\partial x^2}(x+th,z)\,dt + O(x^2h)$$
$$= \frac{\partial\Phi}{\partial x}(x,z) + O(x^2h).$$

Together with (26) this implies

$$h^{-1}\int\limits_x^{x+h} S(u,z)\,du = wH(z)x(\log x)^{w-1} + O(x(\log x)^{\mathrm{Re}(w)-2}). \tag{27}$$

Furthermore,

$$S(x,z) = h^{-1}\int\limits_x^{x+h} S(u,z)\,du + O(D(x)), \tag{28}$$

where, with $\mathbf{B} := (B,\ldots,B) \in \mathbb{R}_+^s$,

$$D(x) = h^{-1}\int\limits_x^{x+h} |S(u,z)-S(x,z)|\,du \tag{29}$$
$$\le h^{-1}\int\limits_x^{x+h} (S(u,\mathbf{B})-S(x,\mathbf{B}))\,du$$
$$\le h^{-1}\int\limits_x^{x+h} S(u,\mathbf{B})\,du - h^{-1}\int\limits_{x-h}^{x} S(u,\mathbf{B})\,du.$$

Here we used the fact that $S(u,\mathbf{B}) \ge 0$ is non-decreasing in u. Applying (27) in (28) and (29) three times, we obtain

$$S(x,z) = wH(z)x(\log x)^{w-1} + O(x(\log x)^{\mathrm{Re}(w)-2}) \tag{30}$$

uniformly for $|z_1| \le B,\ldots,|z_d| \le B$. Remember that $w = \sum_{\nu=1}^d \delta_\nu z_\nu$.

Now a d-fold application of Cauchy's theorem yields

$$\pi(x) = \pi_{K/Q}(x,r_1,\ldots,r_d)$$
$$= \frac{1}{(2\pi i)^d}\int\limits_{C(\varrho_1)}\cdots\int\limits_{C(\varrho_d)} S(x,z)\Big(\prod_{\nu=1}^d z_\nu^{-r_\nu-1}\Big)dz_1\ldots dz_d.$$

Here $C(\varrho_\nu)$ denotes the positively oriented circle with radius $\varrho_\nu := \max(1,r_\nu)/X_\nu$, $X_\nu := \delta_\nu\log_2 x$, and center in the origin. Using (30) we find

$$\pi(x)=\frac{x}{\log x}\,\frac{1}{(2\pi i)^d}$$
$$\times\int\limits_{C(\varrho_1)}\ldots\int\limits_{C(\varrho_d)}\sum_{\mu=1}^{d}\delta_\mu z_\mu H(z)\prod_{\nu=1}^{d}e^{X_\nu z_\nu}z_\nu^{-r_\nu-1}\,dz_\nu+O(R_1(x))$$

with

$$R_1(x)\ll\frac{x}{(\log x)^2}\prod_{\nu=1}^{d}\Big(\varrho_\nu^{-r_\nu}\int\limits_0^{2\pi}e^{\max(1,r_\nu)\cos\varphi}\,d\varphi\Big).$$

The first of the two elementary bounds (cf. Tenenbaum [22, p. 204])

$$\begin{gathered}\int\limits_0^{2\pi}e^{x\cos\varphi}\,d\varphi\ll x^{-1/2}e^x,\\ \int\limits_0^{2\pi}e^{x\cos\varphi}(1-\cos\varphi)\,d\varphi\ll x^{-3/2}e^x\end{gathered}\qquad(x>0)\tag{31}$$

and $n!\sim\sqrt{2\pi}\,n^{n+1/2}e^{-n}$ yield (recall $r=\sum_{\nu=1}^{d}r_\nu$)

$$\begin{aligned}R_1(x)&\ll\frac{x}{(\log x)^2}(\log_2 x)^r\prod_{\nu=1}^{d}(\delta_\nu^{r_\nu}\max(1,r_\nu)^{-r_\nu-1/2}e^{r_\nu})\\&\ll\frac{x}{(\log x)^2}(\log_2 x)^r\prod_{\nu=1}^{d}\frac{\delta_\nu^{r_\nu}}{r_\nu!}.\end{aligned}$$

To analyze the main term we remark that Taylor series expansion of H at $z=0$ is sufficient to prove an asymptotic expansion of $\pi(x)$ with an error term of order $O(r(\log_2 x)^{-1})$. This is too weak to deduce a lower bound for $\pi(x)$ if $r\gg\log_2 x$. Alternatively, we expand H at $a=(a_1,\ldots,a_d)$ and choose a in such a way that the contribution of the linear terms vanishes. Applying $F(1)=F(0)+F'(0)+\int_0^1(1-t)F''(t)\,dt$ to $F(t)=H(a+t(z-a))$ we find

$$H(z)=H(a)+\sum_{\kappa=1}^{d}\frac{\partial H}{\partial z_\kappa}(a)(z_\kappa-a_\kappa)+\sum_{\kappa,\iota=1}^{d}(z_\kappa-a_\kappa)(z_\iota-a_\iota)H_{\kappa,\iota}(z)$$

with

$$H_{\kappa,\iota}(z)=\int\limits_0^1(1-t)\,\frac{\partial^2 H}{\partial z_\kappa\partial z_\iota}(a+t(z-a))\,dt\ll 1.$$

The contribution of the linear terms to $\pi(x)$ vanishes if we choose $a_\nu:=$

$(1-1/r)r_\nu/X_\nu$. Indeed,

$$\frac{1}{(2\pi i)^d}\int\limits_{C(\varrho_1)}\cdots\int\limits_{C(\varrho_d)}\sum_{\mu=1}^{d}\delta_\mu z_\mu(z_\kappa-a_\kappa)\prod_{\nu=1}^{d}e^{X_\nu z_\nu}z_\nu^{-r_\nu-1}\,dz_\nu$$

$$=\Big(\prod_{\nu=1}^{d}\frac{X_\nu^{r_\nu}}{r_\nu!}\Big)\Big(\sum_{\mu\neq\kappa}\delta_\mu\,\frac{r_\mu}{X_\mu}\Big(\frac{r_\kappa}{X_\kappa}-a_\kappa\Big)+\delta_\kappa\,\frac{r_\kappa}{X_\kappa}\Big(\frac{r_\kappa-1}{X_\kappa}-a_\kappa\Big)\Big)$$

$$=(\log_2 x)^{r-1}\Big(\prod_{\nu=1}^{d}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\Big)\Big((r-1)\,\frac{r_\kappa}{X_\kappa}-ra_\kappa\Big)=0.$$

The contribution of the second derivatives of H to $\pi(x)$ is

$$R_2(x):=\frac{x}{\log x}\sum_{\mu,\kappa,\iota=1}^{d}I_{\mu,\kappa,\iota},$$

where

$$I_{\mu,\kappa,\iota}=\frac{1}{(2\pi i)^d}$$
$$\times\int\limits_{C(\varrho_1)}\cdots\int\limits_{C(\varrho_d)}\delta_\mu z_\mu(z_\kappa-a_\kappa)(z_\iota-a_\iota)H_{\kappa,\iota}(z)\prod_{\nu=1}^{d}e^{X_\nu z_\nu}z_\nu^{-r_\nu-1}\,dz_\nu.$$

If $a_\nu>0$ the integration path $C(\varrho_\nu)$ can be replaced by $C(a_\nu)$. Set $b_\nu=a_\nu$ if $a_\nu>0$ and $b_\nu=\varrho_\nu$ else. Since $H_{\kappa,\iota}\ll 1$ it follows that

$$I_{\mu,\kappa,\iota}\ll\Big(\prod_{\nu=1}^{d}b_\nu^{-r_\nu}\Big)b_\mu b_\kappa b_\iota$$
$$\times\int\limits_{0}^{2\pi}\cdots\int\limits_{0}^{2\pi}|1-I_{\{a_\kappa>0\}}e^{i\varphi_\kappa}|\,|1-I_{\{a_\iota>0\}}e^{i\varphi_\iota}|\prod_{\nu=1}^{d}e^{X_\nu b_\nu\cos\varphi_\nu}\,d\varphi_\nu.$$

Using Cauchy's inequality and the bounds (31) we find

$$I_{\mu,\kappa,\iota}\ll\Big(\prod_{\nu=1}^{d}b_\nu^{-r_\nu}\Big)b_\mu b_\kappa b_\iota(X_\kappa b_\kappa)^{-1/2}(X_\iota b_\iota)^{-1/2}\prod_{\nu=1}^{d}(X_\nu b_\nu)^{-1/2}e^{X_\nu b_\nu}.$$

Note that $X_\kappa b_\kappa=1$ if $a_\kappa=0$. It follows that

$$R_2(x)\ll\frac{x}{\log x}\Big(\prod_{\nu=1}^{d}b_\nu^{-r_\nu-1/2}X_\nu^{-1/2}e^{X_\nu b_\nu}\Big)\sum_{\mu=1}^{d}b_\mu\Big(\sum_{\kappa=1}^{d}b_\kappa^{1/2}X_\kappa^{-1/2}\Big)^2$$

$$\begin{aligned}&\ll \frac{x}{\log x}(\log_2 x)^{r-3}\Big(\prod_{\nu=1}^{d}\delta_\nu^{r_\nu}(X_\nu b_\nu)^{-r_\nu-1/2}e^{X_\nu b_\nu}\Big)\\ &\quad\times\sum_{\mu=1}^{d}\max(1,r_\mu)\Big(\sum_{\kappa=1}^{d}\max(1,r_\kappa)^{1/2}\Big)^2\\ &\ll \frac{x}{\log x}(\log_2 x)^{r-3}\Big(\prod_{\nu=1}^{d}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\Big)r^2.\end{aligned}$$

Finally, the contribution of the term $H(a)$ to $\pi(x)$ is

$$\frac{rx}{\log x}(\log_2 x)^{r-1}\Big(\prod_{\nu=1}^{d}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\Big)\mu(x,r_1,\ldots,r_d),$$

where

$$\mu(x,r_1,\ldots,r_d):=H\Big(\frac{(r-1)r_1}{r\delta_1\log_2 x},\ldots,\frac{(r-1)r_d}{r\delta_d\log_2 x}\Big).$$

Altogether, we have proved

$$\pi(x)=\Big(\prod_{\nu=1}^{d}\frac{\delta_\nu^{r_\nu}}{r_\nu!}\Big)\frac{rx}{\log x}(\log_2 x)^{r-1}\Big(\mu(x,r_1,\ldots,r_d)+O\Big(\frac{r}{(\log_2 x)^2}\Big)\Big).$$

Since $H(z)=1+O(|z_1|)+\cdots+O(|z_d|)$ and $H(z)>0$ for real $z_1\ge 0,\ldots,z_d\ge 0$, we find $\mu(x,r_1,\ldots,r_d)=1+O(r(\log_2 x)^{-1})$ and $\mu(x,r_1,\ldots,r_d)\asymp 1$. This completes the proof of Theorem 2. ■

4. Some special examples

4.1. For $K=\mathbb{Q}$ and $m\ge 2$ arbitrary, Theorem 1 contains Soundararajan's result on the Piltz divisor problem established already in [19]:

$$\begin{aligned}&\Delta_{\mathbb{Q},m}(x)\\ &\quad=\Omega((x\log x)^{1/2-1/2m}(\log_2 x)^{(1/2+1/2m)(m^{2m/(m+1)}-1)}(\log_3 x)^{-3/4+1/4m}),\end{aligned}$$

where Ω can be replaced by Ω_+ if $m\equiv 3 \pmod 8$, and by Ω_- if $m\equiv 7 \pmod 8$.

4.2. If K is a normal extension of $\mathbb{Q}$ of degree $k\ge 2$, then $\delta_\nu=0$ for $1\le\nu<k$ and $\delta_k=\frac{1}{k}$ (cf. W. Narkiewicz [15, p. 357, Cor. 4]). For arbitrary $m\ge 1$, Theorem 1 applies with

$$\kappa=\frac{1}{2}m(km+1)(km)^{-2/(km+1)}-\frac{1}{2}-\frac{1}{2m}.$$

The last exponent should be compared with Hafner's (see [4])

$$\kappa'=\frac{km-1}{2k}\log(km)+\frac{1}{2}(m-1)\Big(1+\frac{1}{km}\Big).$$

Note that, for km large, $\kappa \sim \frac{1}{2}km^2$, while $\kappa' \sim \frac{1}{2}m\log(km)$. This case clearly contains that of a quadratic field K: the result then reads

$$\Delta_{K,m}(x) = \Omega((x\log x)^{1/2-1/4m}(\log_2 x)^{\frac{1}{2}m(2m+1)(2m)^{-2/(2m+1)}-1/2-1/4m} \times (\log_3 x)^{-3/4+1/8m}).$$

Since $P(x) = 4\,\Delta_{\mathbb{Q}(i),1}(x)$, this in turn contains the second line of (2).

4.3. As the simplest example of a field K which is not a normal extension of the rationals, let us consider a cubic field whose discriminant D is not a perfect square (if the discriminant is a full square, the field K is normal). In this case $L = K(\sqrt{D})$ is the minimal normal extension. Its Galois group is isomorphic to S_3, the symmetric group of three elements, and $H = \mathrm{Gal}(L/K)$ is a cyclic subgroup of order 2. Using (4) we find $\delta_1 = \frac{1}{2}$, $\delta_2 = 0$ and $\delta_3 = \frac{1}{6}$. It follows that, for any $m \geq 1$,

$$\Delta_{K,m}(x) = \Omega((x\log x)^{1/2-1/6m}(\log_2 x)^{\kappa}(\log_3 x)^{-1}),$$

with

$$\kappa = \frac{1}{12}m^{(3m-1)/(3m+1)}(3m+1)(3^{(3m-1)/(3m+1)}+1) - \frac{1}{2} - \frac{1}{6m}.$$

For $m = 1$, this gives $\kappa = \frac{1}{3}(\sqrt{3}-1) \approx 0.2440$, while Hafner had only $\kappa' = \frac{1}{6}\log 3 \approx 0.1831$. Similarly, for $m = 2$, $\kappa = \frac{7}{12}(2^{5/7}+6^{5/7}-1) \approx 2.4714$, whereas $\kappa' = \frac{5}{12}\log 12 + \frac{7}{12} \approx 1.6187$.

4.4. As a last, more intrinsic example, let us consider $K = \mathbb{Q}(\alpha)$, where $\alpha \in \mathbb{C}$ is a zero of an (irreducible) polynomial f over $\mathbb{Q}$ of degree seven with Galois group $\mathrm{PSL}(3,2)$ (which is a simple group of order 168). In particular, $k = [K:\mathbb{Q}] = 7$. Infinitely many number fields K of this kind are known (see B. H. Matzat [14]), two of them being given by $f = X^7 - 7X + 3$ and $f = X^7 - 7X^3 + 14X^2 - 7X + 1$. There are exactly three non-empty sets P_ν belonging to K, namely P_1, P_3, and P_7, with corresponding Dirichlet densities $\delta_1 = 7/12$, $\delta_3 = 1/8$, and $\delta_7 = 1/168$. The set P_7 consists of those primes which split completely in K. The members of P_3 split into prime ideals whose degrees are given by the quintuple $(1,1,1,2,2)$. Finally, P_1 consists of two different types of primes whose decomposition is described by $(1,2,4)$ and $(1,3,3)$. It follows that, for any $m \geq 1$,

$$\Delta_{K,m}(x) = \Omega((x\log x)^{1/2-1/14m}(\log_2 x)^{\kappa}(\log_3 x)^{-5/4-1/28m}),$$

with

$$\kappa = m^{(7m-1)/(7m+1)}(7m+1)\left(\frac{7^{-2/(7m+1)}}{48} + \frac{3^{14m/(7m+1)}}{112} + \frac{1}{24}\right) - \frac{1}{2} - \frac{1}{14m}.$$

For $m = 1$, this gives $\kappa = \frac{1}{42}(3^{11/4} + 7^{3/4} - 10) \approx 0.3528$, while Hafner had only

$$\kappa' = \frac{1}{56}(\log 7 + 9\,\log 3) \approx 0.2113.$$

Similarly, for $m = 2$,

$$\kappa = \frac{1}{112}(45 \cdot 6^{13/15} + 70 \cdot 2^{13/15} + 5 \cdot 14^{13/15} - 60) \approx 2.94196,$$

whereas Hafner's κ' equals

$$\frac{1}{336}(13\,\log 7 + 117\,\log 3 + 312\,\log 2 + 180) \approx 1.63719.$$

References

[1] F. Fricker, *Einführung in die Gitterpunktlehre*, Birkhäuser, Basel, 1982.

[2] J. L. Hafner, *New Omega theorems for two classical lattice point problems*, Invent. Math. 63 (1981), 181–186.

[3] —, *On the average order of a class of arithmetical functions*, J. Number Theory 15 (1982), 36–76.

[4] —, *The distribution and average order of the coefficients of Dedekind ζ functions*, ibid. 17 (1983), 183–190.

[5] G. H. Hardy, *On Dirichlet's divisor problem*, Proc. London Math. Soc. (2) 15 (1916), 1–25.

[6] M. N. Huxley, *Exponential sums and lattice points II*, Proc. London Math. Soc. (3) 66 (1993), 279–301.

[7] —, *Area, Lattice Points, and Exponential Sums*, London Math. Soc. Monogr. (N.S.) 13, Oxford Univ. Press, Oxford, 1996.

[8] —, *Exponential sums and lattice points III*, Proc. London Math. Soc. (3) 87 (2003), 591–609.

[9] A. Ivić, *The Riemann Zeta-Function*, Wiley, New York, 1985.

[10] G. J. Janusz, *Algebraic Number Fields*, Academic Press, New York, 1973.

[11] E. Krätzel, *Lattice Points*, Deutscher Verlag Wiss., Berlin, 1988.

[12] —, *Analytische Funktionen in der Zahlentheorie*, Teubner, Stuttgart, 2000.

[13] J. C. Lagarias and A. M. Odlyzko, *Effective versions of the Chebotarev density theorem*, in: A. Fröhlich (ed.), Algebraic Number Fields, Academic Press, London, 1977, 409–464.

[14] B. H. Matzat, *Konstruktion von Zahl- und Funktionenkörpern mit vorgegebener Galoisgruppe*, J. Reine Angew. Math. 349 (1984), 179–220.

[15] W. Narkiewicz, *Elementary and Analytic Theory of Algebraic Numbers*, 2nd ed., Springer, Berlin, and PWN, Warszawa, 1990.

[16] W. G. Nowak, *On the distribution of integer ideals in algebraic number fields*, Math. Nachr. 161 (1993), 59–74.

[17] —, *On the general asymmetric divisor problem*, Abh. Math. Sem. Univ. Hamburg 65 (1995), 259–276.

[18] E. de Shalit, *Artin L functions*, in: P. Bernstein and S. Gelbart (eds.), An Introduction to the Langlands Program, Birkhäuser, Boston, 2003, 73–87.

[19] K. Soundararajan, *Omega results for the divisor and circle problems*, Internat. Math. Res. Notices 2003, no. 36, 1987–1998.

[20] G. Szegő und A. Walfisz, *Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern, erste Abhandlung*, Math. Z. 26 (1927), 138–156.
[21] —, —, *Über das Piltzsche Teilerproblem in algebraischen Zahlkörpern, zweite Abhandlung*, ibid., 467–486.
[22] G. Tenenbaum, *Introduction to Analytic and Probabilistic Number Theory*, Cambridge Stud. Adv. Math. 46, Cambridge Univ. Press, Cambridge, 1995.

Kurt Girstmair
Institut für Mathematik
Universität Innsbruck
Technikerstraße 25
6020 Innsbruck, Austria
E-mail: Kurt.Girstmair@uibk.ac.at
Web: http://mathematik.uibk.ac.at/users/girstmai/

Wolfgang Müller
Institut für Statistik
Technische Universität Graz
8010 Graz, Austria
E-mail: mueller@stat.tu-graz.ac.at

Manfred Kühleitner & Werner Georg Nowak
Institut für Mathematik
Department für Integrative Biologie
Universität für Bodenkultur Wien
Gregor Mendel-Straße 33
1180 Wien, Austria
E-mail: kleitner@edv1.boku.ac.at
nowak@mail.boku.ac.at
Web: http://www.boku.ac.at/math/nth.html

Received on 31.8.2004
and in revised form on 25.10.2004 (4842)