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Maass operators and van der Pol-type identities for Ramanujan's tau function

by

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1. Introduction. For z in the upper half plane $\mathcal{H}$ let $e(z) = e^{2\pi i z}$. Ramanujan's tau function $\tau(n)$ is then defined by the expansion

$$\Delta(z) = e(z)\prod_{n=1}^{\infty}(1-e(nz))^{24} = \sum_{n=1}^{\infty}\tau(n)e(nz), \tag{1}$$

where $\Delta(z)$ is the cuspform of weight 12 and level 1. Using differential equations satisfied by $\Delta(z)$, Eisenstein series, and certain other functions van der Pol [9] (and Resnikoff in [10]) established identities relating $\tau(n)$ to sum-of-divisors functions. For example, van der Pol showed

$$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m) \tag{2}$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

In this paper we use Maass operators (see [7])

$$\delta_\kappa = \frac{1}{2\pi i}\left(\frac{\kappa}{2iy} + \frac{\partial}{\partial z}\right)$$

to prove a number of similar identities relating Ramanujan's tau function to sum-of-divisors functions, and in particular we establish the van der Pol identities in a natural way. Our method is analogous to the classical method of establishing identities among Fourier coefficients of modular forms of low weight. That is, for $E_\kappa(z)$ the normalized Eisenstein series of weight κ and level 1, we have relations like

$$E_4(z)E_8(z) = E_{12}(z) + \frac{432000}{691}\Delta(z)$$

and from (1) we can obtain identities for $\tau(n)$. Here we study the explicit structure of the non-holomorphic modular form $\delta_\kappa^{(q)} E_\kappa(z)\cdot\delta_\mu^{(r)} E_\mu(z)$ and

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obtain twelve identities for $\tau(n)$ (essentially) including van der Pol's identities. These methods can of course be applied to the Fourier coefficients of other modular forms, but for simplicity we restrict our interest to $\tau(n)$ and holomorphic Eisenstein series. Some of these ideas were studied in [6] and applied to special values of L-functions.

We classify the identities into four theorems, based on the appearance of σ_k's in the summations. Some identities in each theorem are equivalent to each other, while some follow from the others in the theorem using elementary methods and classical identities of sum-of-divisors functions.

THEOREM 1.

(i) $$\tau(n) = n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),$$

(ii) $$\tau(n) = -\frac{5}{4}n^2\sigma_7(n) + \frac{9}{4}n^2\sigma_3(n) + 540\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_3(n-m),$$

(iii) $$\tau(n) = n^2\sigma_7(n) - \frac{1080}{n}\sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m),$$

(iv) $$\tau(n) = -\frac{1}{2}n^2\sigma_7(n) + \frac{3}{2}n^2\sigma_3(n) + \frac{360}{n}\sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m).$$

Note that (i) and (ii) are equivalent and (iii) and (iv) are equivalent. Identity (i) is equation (2), essentially proven in [9] but with an error, corrected in [10].

Theorem 1 yields the congruences

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{540},$$

which is congruence (7.3) from [5], and

$$\tau(n) \equiv n^2\sigma_3(n) \pmod{240},$$

which improves a congruence from [9].

THEOREM 2.

(i) $$\tau(n) = -\frac{11}{24}n\sigma_9(n) + \frac{35}{24}n\sigma_5(n) + 350\sum_{m=1}^{n-1}(n-m)\sigma_3(m)\sigma_5(n-m),$$

(ii) $$\tau(n) = \frac{11}{36}n\sigma_9(n) + \frac{25}{36}n\sigma_3(n) - 350\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_5(n-m),$$

(iii) $$\tau(n) = \frac{1}{6}n\sigma_9(n) + \frac{5}{6}n\sigma_3(n) - \frac{420}{n}\sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_5(n-m),$$

(iv) $\tau(n) = n\sigma_9(n) - \frac{2100}{n}\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_5(n-m),$

(v) $\tau(n) = -\frac{1}{4}n\sigma_9(n) + \frac{5}{4}n\sigma_5(n) + \frac{300}{n}\sum_{m=1}^{n-1}(n-m)^2\sigma_3(m)\sigma_5(n-m).$

It is easy to see that (i) and (ii) are equivalent and (iii), (iv), and (v) are equivalent.

From Theorem 2 we get the congruences

$$\tau(n) \equiv n\sigma_9(n) \pmod{2100} \quad \text{for } 2,3,5,7 \nmid n,$$
$$\tau(n) \equiv n\sigma_5(n) \pmod{240} \quad \text{for } 2,3,5,7 \nmid n,$$
$$\tau(n) \equiv n\sigma_3(n) \pmod{504} \quad \text{for } 2,3,5,7 \nmid n.$$

Combining these and a previous congruence gives

$$(n-1)\sigma_3(n) \equiv 0 \pmod{24} \quad \text{for } 2,3,5,7 \nmid n.$$

A catalog of similar congruences for σ_k's and $\tau(n)$ is given in [4] and [5].

Theorem 3.

$$\tau(n) = \frac{65}{756}\sigma_{11}(n) + \frac{691}{756}\sigma_5(n) - \frac{2\cdot 691}{3n}\sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n-m).$$

For the last theorem we have σ_3 and σ_7 in the summations.

Theorem 4.

(i) $\tau(n) = -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_3(n) + \frac{4\cdot 691}{5n}\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_7(n-m),$

(ii) $\tau(n) = -\frac{91}{600}\sigma_{11}(n) + \frac{691}{600}\sigma_7(n) + \frac{2\cdot 691}{5n}\sum_{m=1}^{n-1}(n-m)\sigma_3(m)\sigma_7(n-m).$

It is easy to see that (i) and (ii) here are equivalent.

As a consequence of these identities we get some other relations among the sum-of-divisors functions, such as

$$\sum_{m=1}^{n-1} m(n-m)(n-2m)\sigma_3(m)\sigma_3(n-m) = 0.$$

2. Eisenstein series and Maass operators. In this section we introduce our Eisenstein series, Maass operators, and prove the key proposition. For

$$\Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathbb{Z} \right\}$$

denote the Eisenstein series of weight κ and level 1 by

$$E_\kappa(z) = \sum_{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} (cz+d)^{-\kappa} = 1 + \frac{2}{\zeta(1-\kappa)} \sum_{n=1}^{\infty} \sigma_{\kappa-1}(n) e(nz)$$

where $\zeta(s)$ is the Riemann zeta function.

Let $\mathcal{M}_\kappa$ denote the space of modular forms of weight κ and level 1 and let $\mathcal{C}_\kappa$ denote the subspace of cuspforms. Consider the differential operator

$$\delta_\kappa^{(r)} = \left(\frac{1}{2\pi i}\right)^r \left(\frac{\kappa+2r-2}{2iy} + \frac{\partial}{\partial z}\right)\left(\frac{\kappa+2r-4}{2iy} + \frac{\partial}{\partial z}\right) \cdots \left(\frac{\kappa}{2iy} + \frac{\partial}{\partial z}\right)$$

where $\delta_\kappa^{(0)}$ is just the identity operator and $z = x + iy$. Note that from Maass [7] the operator $\delta_\kappa^{(r)}$ preserves automorphy of $g_\kappa(z) \in \mathcal{M}_\kappa$ but not holomorphy. Further, note that if $g_\kappa(z) \in \mathcal{M}_\kappa$ then $\delta_\kappa^{(r)} g_\kappa(z)$ is a non-holomorphic modular form of weight $\kappa + 2r$ (see [2] and [3]).

We now study the structure of $\delta_\kappa^{(q)} E_\kappa(z) \cdot \delta_\mu^{(r)} E_\mu(z)$.

From [1] and [8] the action of $\delta_\kappa^{(r)}$ on $g_\kappa(z) = \sum_{n=0}^{\infty} a_n e(nz)$ is given explicitly by

$$\delta_\kappa^{(r)} g_\kappa(z) = \sum_{n=0}^{\infty} a_n \Big(\sum_{j=0}^{r} P_{j,\kappa}^{(r)} (-4\pi y)^{-j} n^{r-j} \Big) e(nz) \tag{3}$$

where

$$P_{j,\kappa}^{(r)} = \binom{r}{j} \frac{\Gamma(\kappa + r)}{\Gamma(\kappa + r - j)}$$

and $\Gamma(s)$ is the usual gamma function.

The following is a lemma in [11], and see [12] for a more general discussion of nearly holomorphic modular forms and differential operators.

Lemma 1 (Shimura). *Let $G(z)$ be a function on $\mathcal{H}$ so that for $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ we have*

$$G(\gamma(z)) = (cz+d)^\kappa G(z) \quad \text{and} \quad G(z) = \sum_{j=0}^{r} a_j y^{-j} g_{\kappa-2j}(z)$$

where $g_{\kappa-2j}(z)$ is holomorphic on $\mathcal{H}$ and has a Fourier expansion. Then

$$G(z) = \sum_{j=0}^{r} \widetilde{a}_j \, \delta_{\kappa-2j}^{(j)} \, \widetilde{g}_{\kappa-2j}(z)$$

where $\widetilde{g}_{\kappa-2j}(z) \in \mathcal{M}_{\kappa-2j}$.

As a consequence of Lemma 1, for $g_\kappa(z) \in \mathcal{M}_\kappa$ and $g_\mu(z) \in \mathcal{M}_\mu$ we have the decomposition

$$\delta_\kappa^{(q)} g_\kappa(z) \cdot \delta_\mu^{(r)} g_\mu(z) = \sum_{l=0}^{q+r} \delta_{\kappa+\mu+2q+2r-2l}^{(l)} (\alpha_l E_{\kappa+\mu+2q+2r-2l}(z) + \beta_l F(l,z))$$

where $F(l,z) \in \mathcal{C}_{\kappa+\mu+2q+2r-2l}$ and $\alpha_l, \beta_l \in \mathbb{C}$.

PROPOSITION 1. *For $q \le r$,*

$$\delta_\kappa^{(q)} E_\kappa(z) \cdot \delta_\mu^{(r)} E_\mu(z) = \frac{\Gamma(\kappa+q)\Gamma(\mu+r)\Gamma(\kappa+\mu)}{\Gamma(\kappa)\Gamma(\mu)\Gamma(\kappa+\mu+q+r)}\, \delta_{\kappa+\mu}^{(q+r)} E_{\kappa+\mu}(z) + \sum_{l=0}^{q+r} \beta_l\, \delta_{\kappa+\mu+2q+2r-2l}^{(l)} F(l,z)$$

where the $F(l,z) \in \mathcal{C}_{\kappa+\mu+2r+2r-2l}$ are normalized so their first non-zero Fourier coefficients are 1 and the $\beta_l \in \mathbb{C}$ consist of integer values of Γ-functions and $\zeta(s)$.

Proof. For simplicity write $E_\kappa(z) = \sum_{m=0}^\infty c_\kappa(m)e(mz)$; then from equation (3) we get

$$\delta_\kappa^{(q)} E_\kappa(z) = \sum_{m=0}^\infty c_\kappa(m)\Big(\sum_{k=0}^q P_{k,\kappa}^{(q)}(-4\pi y)^{-k} m^{q-k}\Big)e(mz).$$

For $E_\mu(x) = \sum_{n=0}^\infty c_\mu(n)e(nz)$ we then have

$$\begin{aligned}
(4)\qquad & \delta_\kappa^{(q)} E_\kappa(z) \cdot \delta_\mu^{(r)} E_\mu(z) \\
&= \Big(\sum_{m=0}^\infty c_\kappa(m)\Big(\sum_{k=0}^q P_{k,\kappa}^{(q)}(-4\pi y)^{-k} m^{q-k}\Big)e(mz)\Big) \\
&\quad \times \Big(\sum_{n=0}^\infty c_\mu(n)\Big(\sum_{j=0}^r P_{j,\mu}^{(r)}(-4\pi y)^{-j} n^{r-j}\Big)e(nz)\Big) \\
&= \sum_{t=0}^\infty \sum_{s=0}^{q+r} \Big(\sum_{m+n=t} \Big(\sum_{j+k=s} c_\kappa(m)c_\mu(n)P_{k,\kappa}^{(q)} m^{q-k} P_{j,\mu}^{(r)} n^{r-j}\Big)(-4\pi y)^{-s}\Big)e(tz).
\end{aligned}$$

For simplicity again, put

$$E_{\kappa+\mu+2q+2r-2l}(z) = \sum_{n=0}^\infty c(l,n)e(nz), \qquad F(l,z) = \sum_{n=1}^\infty d(l,n)e(nz)$$

where we normalize $F(l,z)$ so its first non-zero Fourier coefficient is 1. Then

$$\delta_{\kappa+\mu+2q+2r-2l}^{(l)} E_{\kappa+\mu+2q+2r-2l}(z) = \sum_{t=0}^\infty c(l,t)\Big(\sum_{j=0}^l p_j^l(t) y^{-j}\Big)e(tz)$$

where $p_j^l(t) = P_{j,\kappa+\mu+2q+2r-2l}^{(l)}(-4\pi)^{-j} t^{l-j}$ and

$$\delta_{\kappa+\mu+2q+2r-2l}^{(l)} F(l,z) = \sum_{t=1}^\infty d(l,t)\Big(\sum_{j=0}^l p_j^l(t) y^{-j}\Big)e(tz).$$

Taking the Fourier expansions and switching the order of summations yields

$$\sum_{l=0}^{q+r} \delta^{(l)}_{\kappa+\mu+2q+2r-2l}(\alpha_l E_{\kappa+\mu+2q+2r-2l}(z) + \beta_l F(l,z))$$

$$= \sum_{l=0}^{q+r} \Big[\alpha_l \sum_{t=0}^{\infty} c(l,t)\Big(\sum_{s=0}^{l} p^l_s(t) y^{-s}\Big) e(tz)$$

$$+ \beta_l \sum_{t=1}^{\infty} d(l,t)\Big(\sum_{s=0}^{l} p^l_s(t) y^{-s}\Big) e(tz)\Big]$$

$$= \sum_{s=0}^{q+r} \Big(\sum_{l=s}^{q+r} \alpha_l c(l,0) p^l_s(0)\Big) y^{-s}$$

$$+ \sum_{t=1}^{\infty} \Big[\sum_{s=0}^{q+r}\Big(\sum_{l=s}^{q+r} (\alpha_l c(l,t) p^l_s(t) + \beta_l d(l,t) p^l_s(t))\Big) y^{-s}\Big] e(tz).$$

From Lemma 1 we can set the above result equal to (4). Then for $t = 0$ we set the terms indexed by $(-4\pi y)^{-s}$ equal and get the equation

$$(5) \qquad \sum_{j+k=s} (P^{(q)}_{k,\kappa} m^{q-k} P^{(r)}_{j,\mu} n^{r-j})|_{\substack{m=0\\n=0}} = \sum_{l=j+k}^{q+r} \alpha_l p^l_{j+k}(0).$$

Now,

$$p^{j+k}_{j+k}(0) = (-4\pi)^{-j-k} \frac{\Gamma(\kappa+\mu+2q+2r-j-k)}{\Gamma(\kappa+\mu+2q+2r-2j-2k)} \neq 0$$

and $p^l_{j+k}(0) = 0$ for $l \neq j+k$. Therefore equation (5) becomes

$$(P^{(q)}_{k,\kappa} m^{q-k})|_{m=0} (P^{(r)}_{j,\mu} n^{r-j})|_{n=0} = \alpha_{j+k} p^{j+k}_{j+k}(0).$$

But $(P^{(q)}_{k,\kappa} m^{q-k})|_{m=0} = 0$ for $k \neq q$ and $P^{(q)}_{q,\kappa} = \Gamma(\kappa+q)/\Gamma(\kappa) \neq 0$, and similarly for $(P^{(r)}_{j,\mu} n^{r-j})|_{n=0}$. This implies $\alpha_s = 0$ for $s < q+r$ and also

$$\alpha_{q+r} = \frac{(P^{(q)}_{k,\kappa} m^{q-k})|_{\substack{k=q\\m=0}} (P^{(r)}_{j,\mu} n^{r-j})|_{\substack{j=r\\n=0}}}{p^{q+r}_{q+r}(0)} = \frac{\Gamma(\kappa+q)\Gamma(\mu+r)\Gamma(\kappa+\mu)}{\Gamma(\kappa)\Gamma(\mu)\Gamma(\kappa+\mu+q+r)}.$$

For $t = 1$ we set the $(-4\pi y)^{-s} e(z)$ terms equal. As $c_\kappa(1) = 2/\zeta(1-\kappa)$ and $c_\mu(1) = 2/\zeta(1-\mu)$, we get the equation

$$(6) \qquad \sum_{j+k=s} \frac{2}{\zeta(1-\mu)} (P^{(q)}_{k,\kappa} m^{q-k})|_{m=0} P^{(r)}_{j,\mu} + \frac{2}{\zeta(1-\kappa)} P^{(q)}_{k,\kappa} (P^{(r)}_{j,\mu} n^{r-j})|_{n=0}$$

$$= \sum_{l=j+k}^{q+r} \alpha_l c(l,1) p^l_{j+k}(1) + \beta_l d(l,1) p^l_{j+k}(1).$$

In a similar way to what happened for (5), equation (6) gives us

$$\sum_{l=s}^{q+r}\beta_l d(l,1)p_s^l(1) = -\alpha_{q+r}c(q+r,1)p_s^{q+r}(1) + \begin{cases} 0, & q>s,\\ \dfrac{2}{\zeta(1-\mu)}\,(P_{k,\kappa}^{(q)}m^{q-k})\Big|_{\substack{k=q\\ m=0}} P_{s-q,\mu}^{(r)}, & r>s\ge q,\\ \dfrac{2}{\zeta(1-\mu)}\,(P_{k,\kappa}^{(q)}m^{q-k})\Big|_{\substack{k=q\\ m=0}} P_{s-q,\mu}^{(r)} & \\ \quad +\dfrac{2}{\zeta(1-\kappa)}\,P_{s-r,\kappa}^{(q)}(P_{j,\mu}^{(r)}n^{r-j})\Big|_{\substack{j=r\\ n=0}}, & s\ge r,\end{cases}$$

and evaluating these, we get

$$\sum_{l=s}^{q+r}\beta_l d(l,1)p_s^l(1) = -\alpha_{q+r}c(q+r,1)p_s^{q+r}(1) + \begin{cases} 0, & q>s,\\ \dfrac{2}{\zeta(1-\mu)}\dbinom{r}{s-q}\dfrac{\Gamma(\kappa+q)\Gamma(\mu+r)}{\Gamma(\kappa)\Gamma(\kappa+q+r-s)}, & r>s\ge q,\\ \left[\dfrac{2}{\zeta(1-\mu)}\dbinom{r}{s-q}+\dfrac{2}{\zeta(1-\kappa)}\dbinom{q}{s-r}\right] & \\ \quad\times\dfrac{\Gamma(\kappa+q)\Gamma(\mu+r)}{\Gamma(\kappa)\Gamma(\kappa+q+r-s)}, & s\ge r.\end{cases}$$

Substituting α_{q+r} and the equation above into equation (6) we can solve for the β_l. ∎

In order to illustrate an application of this result note that from Proposition 1 we have

$$E_4(z)\cdot\delta_4\,E_4(z) = \frac{1}{2}\,\delta_8\,E_8(z).$$

Setting the Fourier coefficients of the $(-4\pi y)^{-1}e(nz)$ terms from both sides of this equation equal we get

$$n\sigma_7(n) = n\sigma_3(n) + 240\sum_{m=1}^{n-1} m\sigma_3(m)\sigma_3(n-m).$$

This is formula (7.5) from [4].

3. Proofs of the identities for Ramanujan's tau function. We give a proof for Theorem 1. The proofs for the other theorems are similar and we give appropriate indications for those. From Proposition 1 we have

$$(7)\quad \delta_4^{(q)} E_4(z)\cdot\delta_4^{(r)} E_4(z)$$
$$= \frac{\Gamma(4+q)\Gamma(4+r)\Gamma(8)}{\Gamma(4)^2\Gamma(8+q+r)}\,\delta_8^{(q+r)} E_8(z)$$
$$+\beta_0\Phi_{8+2q+2r}(z)+\beta_1\delta^{(1)}_{6+2q+2r}\Phi_{6+2q+2r}(z)+\ldots+\beta_{q+r-2}\delta_{12}^{(q+r-2)}\Delta(z).$$

Lemma 2.

$$\text{(i)}\qquad \delta_4\, E_4(z)\cdot\delta_4\, E_4(z) = \frac{2}{9}\,\delta_8^{(2)}\, E_8(z) - \frac{320}{3}\,\Delta(z),$$

$$\text{(ii)}\qquad E_4(z)\cdot\delta_4^{(2)}\, E_4(z) = \frac{5}{18}\,\delta_8^{(2)}\, E_8(z) + \frac{320}{3}\,\Delta(z),$$

$$\text{(iii)}\qquad \delta_4^{(2)}\, E_4(z)\cdot\delta_4\, E_4(z) = \frac{1}{9}\,\delta_8^{(3)}\, E_8(z) - \frac{160}{3}\,\delta_{12}\,\Delta(z),$$

$$\text{(iv)}\qquad E_4(z)\cdot\delta_4^{(3)}\, E_4(z) = \frac{1}{6}\,\delta_8^{(3)}\, E_8(z) + 160\,\delta_{12}\,\Delta(z).$$

Proof. From (7) we have

$$\delta_4\, E_4(z)\cdot\delta_4\, E_4(z) = \frac{2}{9}\,\delta_8^{(2)}\, E_8(z) + \beta_0\Delta(z),$$
$$E_4(z)\cdot\delta_4^{(2)}\, E_4(z) = \frac{5}{18}\,\delta_8^{(2)}\, E_8(z) + \beta_0\Delta(z),$$
$$\delta_4^{(2)}\, E_4(z)\cdot\delta_4\, E_4(z) = \frac{1}{9}\,\delta_8^{(3)}\, E_8(z) + \beta_1\,\delta_{12}\,\Delta(z),$$
$$E_4(z)\cdot\delta_4^{(3)}\, E_4(z) = \frac{1}{6}\,\delta_8^{(3)}\, E_8(z) + \beta_1\,\delta_{12}\,\Delta(z).$$

We only need to look at the holomorphic part of each equation in order to solve for the β_j's. As

$$\delta_4\, E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n+4(-4\pi y)^{-1})e(nz),$$

$$\delta_4^{(2)}\, E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n^2+10(-4\pi y)^{-1}n+20(-4\pi y)^{-2})e(nz),$$

$$\delta_4^{(3)}\, E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n^3+18(-4\pi y)^{-1}n^2$$
$$+90(-4\pi y)^{-2}n+120(-4\pi y)^{-3})e(nz),$$

$$\delta_8^{(2)}\, E_8(z) = \sum_{n=0}^{\infty} c_8(n)(n^2+18(-4\pi y)^{-1}n+72(-4\pi y)^{-2})e(nz),$$

$$\delta_8^{(3)}\, E_8(z) = \sum_{n=0}^{\infty} c_8(n)(n^3+30(-4\pi y)^{-1}n^2$$
$$+270(-4\pi y)^{-2}n+720(-4\pi y)^{-3})e(nz)$$

we can find explicitly the holomorphic part of $\delta_4^{(q)} E_4(z) \cdot \delta_4^{(r)} E_4(z)$. The $e(tz)$ term of the holomorphic part of $\delta_4 E_4(z) \cdot \delta_4 E_4(z)$ is

$$\sum_{m+n=t} mc_4(m)nc_4(n)$$

and for $t = 1$ this is 0. The $e(tz)$ term for the holomorphic part of $\frac{2}{9}\,\delta_8^{(2)} E_8(z) + \beta_0 \varDelta(z)$ is $\frac{2}{9}t^2 c_8(t) + \beta_0\tau(n)$. Setting $t = 1$ we get

$$\beta_0 = -\frac{2}{9}\,c_8(1) = -\frac{2}{9}\,\frac{2}{\zeta(-7)} = -\frac{320}{3}.$$

This gives (i).

The $e(tz)$ term for the holomorphic part of $E_4(z) \cdot \delta_4^{(2)} E_4(z)$ is

$$\sum_{m+n=t} c_4(m)n^2 c_4(n).$$

The $e(tz)$ term for the holomorphic part of $\frac{5}{18}\,\delta_8^{(2)} E_8(z) + \beta_0\varDelta(z)$ is $\frac{5}{18}t^2 c_8(t) + \beta_0\varDelta(z)$. Setting $t = 1$ we get

$$\beta_0 = c_4(1) - \frac{5}{18}\,c_8(1) = \frac{2}{\zeta(-3)} - \frac{5}{18}\,\frac{2}{\zeta(-7)} = \frac{320}{3},$$

which gives (ii).

The $e(tz)$ term for the holomorphic part of $\delta_4^{(2)} E_4(z) \cdot \delta_4 E_4(z)$ is

$$\sum_{m+n=t} m^2 c_4(m)nc_4(n).$$

This is 0 for $t = 1$. The $e(tz)$ term for the holomorphic part of $\frac{1}{9}\,\delta_8^{(3)} E_8(z) + \beta_1\,\delta_{12}\,\varDelta(z)$ is $\frac{1}{9}t^3 c_8(t) + \beta_0 t\tau(t)$. Setting $t = 1$ we get

$$\beta_1 = -\frac{1}{9}\,\frac{2}{\zeta(-7)} = -\frac{160}{3},$$

which gives (iii).

The $e(tz)$ term for the holomorphic part of $E_4(z) \cdot \delta_4^{(3)} E_4(z)$ is

$$\sum_{m+n=t} c_4(m)n^3 c_4(n).$$

The $e(tz)$ term for the holomorphic part of $\frac{1}{6}\,\delta_8^{(3)} E_8(z) + \beta_1\,\delta_{12}\,\varDelta(z)$ is $\frac{1}{6}t^3 c_8(t) + \beta_1 t\tau(t)$. Setting $t = 1$ we get

$$\beta_1 = c_4(1) - \frac{1}{6}\,c_8(1) = \frac{2}{\zeta(-3)} - \frac{1}{6}\,\frac{2}{\zeta(-7)} = 160,$$

which gives (iv). ■

Theorem 1 follows from Lemma 2 by uniqueness of Fourier coefficients. Looking at the $e(nz)$ term of the holomorphic part of (i) gives us

$$\sum_{m=0}^{n} mc_4(m)(n-m)c_4(n-m) = \frac{2}{9}\,n^2c_8(n) - \frac{320}{3}\,\tau(n)$$

so

$$\tau(n) = \frac{3}{320}\,\frac{2}{9}\,\frac{2}{\zeta(-7)}\,n^2\sigma_7(n) - \frac{3}{320}\,\frac{4}{\zeta(-3)^2}\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m)$$
$$= n^2\sigma_7(n) - 540\sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),$$

which is (i) of Theorem 1. In a similar way we get (ii).

Looking at the $e(nz)$ term of the holomorphic part of (iii) gives us

$$\sum_{m=0}^{n} m^2c_4(m)(n-m)c_4(n-m) = \frac{1}{9}\,n^3c_8(n) - \frac{160}{3}\,n\tau(n)$$

so

$$\tau(n) = \frac{3}{160}\,\frac{1}{9}\,\frac{2}{\zeta(-7)}\,n^2\sigma_7(n)$$
$$- \frac{3}{160}\,\frac{4}{\zeta(-3)^2}\,\frac{1}{n}\sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m)$$
$$= n^2\sigma_7(n) - \frac{1080}{n}\sum_{m=1}^{n-1} m^2(n-m)\sigma_3(n)\sigma_3(n-m),$$

which is (iii) from Theorem 1. In a similar way we get (iv).

In the same way we can establish the following lemmas.

LEMMA 3.

(i) $$\delta_4\,E_4(z)\cdot E_6(z) = \frac{2}{5}\;\delta_{10}\,E_{10}(z) + \frac{1728}{5}\,\Delta(z),$$

(ii) $$E_4(z)\cdot \delta_6\,E_6(z) = \frac{3}{5}\;\delta_{10}\,E_{10}(z) - \frac{1728}{5}\,\Delta(z),$$

(iii) $$\delta_4^{(2)}\,E_4(z)\cdot E_6(z) = \frac{2}{11}\;\delta_{10}^{(2)}\,E_{10}(z) + 288\,\delta_{12}\,\Delta(z),$$

(iv) $$\delta_4\,E_4(z)\cdot \delta_6\,E_6(z) = \frac{12}{55}\;\delta_{10}^{(2)}\,E_{10}(z) + \frac{288}{5}\;\delta_{12}\,\Delta(z),$$

(v) $$E_4(z)\cdot \delta_6^{(2)}\,E_6(z) = \frac{21}{55}\;\delta_{10}^{(2)}\,E_{10}(z) - \frac{2016}{5}\;\delta_{12}\,\Delta(z).$$

LEMMA 4.

$$\delta_6\,E_6(z)\cdot E_6(z) = \frac{1}{2}\;\delta_{12}\,E_{12}(z) - \frac{381024}{691}\;\delta_{12}\,\Delta(z).$$

Lemma 5.

(i) $$\delta_4 E_4(z) \cdot E_8(z) = \frac{1}{3}\,\delta_{12} E_{12}(z) + \frac{144000}{691}\,\delta_{12} \Delta(z),$$

(ii) $$E_4(z) \cdot \delta_8 E_8(z) = \frac{2}{3}\,\delta_{12} E_{12}(z) + \frac{288000}{691}\,\delta_{12} \Delta(z).$$

From these lemmas we get Theorems 2, 3, and 4 just as was done for Theorem 1.

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