Maass operators and van der Pol-type identities for Ramanujan’s tau function

by

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1. Introduction. For $z$ in the upper half plane $\mathcal{H}$ let $e(z) = e^{2\pi i z}$. Ramanujan’s tau function $\tau(n)$ is then defined by the expansion

\[
\Delta(z) = e(z) \prod_{n=1}^{\infty} (1 - e(nz))^{24} = \sum_{n=1}^{\infty} \tau(n) e(nz),
\]

where $\Delta(z)$ is the cuspform of weight 12 and level 1. Using differential equations satisfied by $\Delta(z)$, Eisenstein series, and certain other functions van der Pol [9] (and Resnikoff in [10]) established identities relating $\tau(n)$ to sum-of-divisors functions. For example, van der Pol showed

\[
\tau(n) = n^2 \sigma_7(n) - 540 \sum_{m=1}^{n-1} m(n-m) \sigma_3(m) \sigma_3(n-m)
\]

where $\sigma_k(n) = \sum_{d|n} d^k$.

In this paper we use Maass operators (see [7])

\[
\delta_\kappa = \frac{1}{2\pi i} \left( \kappa \frac{\kappa}{2iy} + \frac{\partial}{\partial z} \right)
\]

to prove a number of similar identities relating Ramanujan’s tau function to sum-of-divisors functions, and in particular we establish the van der Pol identities in a natural way. Our method is analogous to the classical method of establishing identities among Fourier coefficients of modular forms of low weight. That is, for $E_\kappa(z)$ the normalized Eisenstein series of weight $\kappa$ and level 1, we have relations like

\[
E_4(z) E_8(z) = E_{12}(z) + \frac{432000}{691} \Delta(z)
\]

and from (1) we can obtain identities for $\tau(n)$. Here we study the explicit structure of the non-holomorphic modular form $\delta_\kappa^{(q)} E_\kappa(z) \cdot \delta_\mu^{(r)} E_\mu(z)$ and

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obtain twelve identities for $\tau(n)$ (essentially) including van der Pol’s identities. These methods can of course be applied to the Fourier coefficients of other modular forms, but for simplicity we restrict our interest to $\tau(n)$ and holomorphic Eisenstein series. Some of these ideas were studied in [6] and applied to special values of $L$-functions.

We classify the identities into four theorems, based on the appearance of $\sigma_k$’s in the summations. Some identities in each theorem are equivalent to each other, while some follow from the others in the theorem using elementary methods and classical identities of sum-of-divisors functions.

**Theorem 1.**

(i) \[ \tau(n) = n^2\sigma_7(n) - 540 \sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m), \]

(ii) \[ \tau(n) = -\frac{5}{4} n^2\sigma_7(n) + \frac{9}{4} n^2\sigma_3(n) + 540 \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_3(n-m), \]

(iii) \[ \tau(n) = n^2\sigma_7(n) - \frac{1080}{n} \sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m), \]

(iv) \[ \tau(n) = -\frac{1}{2} n^2\sigma_7(n) + \frac{3}{2} n^2\sigma_3(n) + \frac{360}{n} \sum_{m=1}^{n-1} m^3\sigma_3(m)\sigma_3(n-m). \]

Note that (i) and (ii) are equivalent and (iii) and (iv) are equivalent. Identity (i) is equation (2), essentially proven in [9] but with an error, corrected in [10].

Theorem 1 yields the congruences

\[ \tau(n) \equiv n^2\sigma_7(n) \pmod{540}, \]

which is congruence (7.3) from [5], and

\[ \tau(n) \equiv n^2\sigma_3(n) \pmod{240}, \]

which improves a congruence from [9].

**Theorem 2.**

(i) \[ \tau(n) = -\frac{11}{24} n\sigma_9(n) + \frac{35}{24} n\sigma_5(n) + 350 \sum_{m=1}^{n-1} (n-m)\sigma_3(m)\sigma_5(n-m), \]

(ii) \[ \tau(n) = \frac{11}{36} n\sigma_9(n) + \frac{25}{36} n\sigma_3(n) - 350 \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_5(n-m), \]

(iii) \[ \tau(n) = \frac{1}{6} n\sigma_9(n) + \frac{5}{6} n\sigma_3(n) - \frac{420}{n} \sum_{m=1}^{n-1} m^2\sigma_3(m)\sigma_5(n-m), \]
(iv) \( \tau(n) = n\sigma_9(n) - \frac{2100}{n} \sum_{m=1}^{n-1} m(n - m)\sigma_3(m)\sigma_5(n - m), \)

(v) \( \tau(n) = -\frac{1}{4} n\sigma_9(n) + \frac{5}{4} n\sigma_5(n) + \frac{300}{n} \sum_{m=1}^{n-1} (n - m)^2\sigma_3(m)\sigma_5(n - m). \)

It is easy to see that (i) and (ii) are equivalent and (iii), (iv), and (v) are equivalent.

From Theorem 2 we get the congruences

\( \tau(n) \equiv n\sigma_9(n) \pmod{2100} \quad \text{for} \ 2, 3, 5, 7 \mid n, \)

\( \tau(n) \equiv n\sigma_5(n) \pmod{240} \quad \text{for} \ 2, 3, 5, 7 \mid n, \)

\( \tau(n) \equiv n\sigma_3(n) \pmod{504} \quad \text{for} \ 2, 3, 5, 7 \mid n. \)

Combining these and a previous congruence gives

\( (n - 1)\sigma_3(n) \equiv 0 \pmod{24} \quad \text{for} \ 2, 3, 5, 7 \mid n. \)

A catalog of similar congruences for \( \sigma_k \)'s and \( \tau(n) \) is given in [4] and [5].

THEOREM 3.

\[ \tau(n) = \frac{65}{756} \sigma_{11}(n) + \frac{691}{756} \sigma_5(n) - \frac{2 \cdot 691}{3n} \sum_{m=1}^{n-1} m\sigma_5(m)\sigma_5(n - m). \]

For the last theorem we have \( \sigma_3 \) and \( \sigma_7 \) in the summations.

THEOREM 4.

(i) \( \tau(n) = -\frac{91}{600} \sigma_{11}(n) + \frac{691}{600} \sigma_3(n) + \frac{4 \cdot 691}{5n} \sum_{m=1}^{n-1} m\sigma_3(m)\sigma_7(n - m), \)

(ii) \( \tau(n) = -\frac{91}{600} \sigma_{11}(n) + \frac{691}{600} \sigma_7(n) + \frac{2 \cdot 691}{5n} \sum_{m=1}^{n-1} (n - m)\sigma_3(m)\sigma_7(n - m). \)

It is easy to see that (i) and (ii) here are equivalent.

As a consequence of these identities we get some other relations among the sum-of-divisors functions, such as

\[ \sum_{m=1}^{n-1} m(n - m)(n - 2m)\sigma_3(m)\sigma_3(n - m) = 0. \]

2. Eisenstein series and Maass operators. In this section we introduce our Eisenstein series, Maass operators, and prove the key proposition. For

\[ \Gamma_{\infty} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| n \in \mathbb{Z} \right\} \]
denote the Eisenstein series of weight $\kappa$ and level 1 by

$$E_\kappa(z) = \sum_{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_\infty \setminus \text{SL}_2(\mathbb{Z})} (cz + d)^{-\kappa} = 1 + \frac{2}{\zeta(1 - \kappa)} \sum_{n=1}^{\infty} \sigma_{\kappa-1}(n) e(nz)$$

where $\zeta(s)$ is the Riemann zeta function.

Let $M_\kappa$ denote the space of modular forms of weight $\kappa$ and level 1 and let $C_\kappa$ denote the subspace of cuspforms. Consider the differential operator

$$\delta^{(r)}_\kappa = \left( \frac{1}{2\pi i} \right)^r \left( \frac{\kappa + 2r - 2}{2iy} + \frac{\partial}{\partial z} \right) \left( \frac{\kappa + 2r - 4}{2iy} + \frac{\partial}{\partial z} \right) \cdots \left( \frac{\kappa}{2iy} + \frac{\partial}{\partial z} \right)$$

where $\delta^{(0)}_\kappa$ is just the identity operator and $z = x + iy$. Note that from Maass [7] the operator $\delta^{(r)}_\kappa$ preserves automorphy of $g_\kappa(z) \in M_\kappa$ but not holomorphy. Further, note that if $g_\kappa(z) \in M_\kappa$ then $\delta^{(r)}_\kappa g_\kappa(z)$ is a non-holomorphic modular form of weight $\kappa + 2r$ (see [2] and [3]).

We now study the structure of $\delta^{(q)}_\kappa E_\kappa(z) \cdot \delta^{(r)}_\mu E_\mu(z)$.

From [1] and [8] the action of $\delta^{(r)}_\kappa$ on $g_\kappa(z) = \sum_{n=0}^{\infty} a_n e(nz)$ is given explicitly by

$$\delta^{(r)}_\kappa g_\kappa(z) = \sum_{n=0}^{\infty} a_n \left( \sum_{j=0}^{r} P^{(r)}_{j,\kappa} (-4\pi y)^{-j} n^{r-j} \right) e(nz) \tag{3}$$

where

$$P^{(r)}_{j,\kappa} = \binom{r}{j} \frac{\Gamma(\kappa + r)}{\Gamma(\kappa + r - j)}$$

and $\Gamma(s)$ is the usual gamma function.

The following is a lemma in [11], and see [12] for a more general discussion of nearly holomorphic modular forms and differential operators.

**Lemma 1 (Shimura).** Let $G(z)$ be a function on $\mathcal{H}$ so that for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$ we have

$$G(\gamma(z)) = (cz + d)^{\kappa} G(z) \quad \text{and} \quad G(z) = \sum_{j=0}^{r} a_j y^{-j} g_{\kappa-2j}(z)$$

where $g_{\kappa-2j}(z)$ is holomorphic on $\mathcal{H}$ and has a Fourier expansion. Then

$$G(z) = \sum_{j=0}^{r} \tilde{a}_j \delta^{(j)}_{\kappa-2j} \tilde{g}_{\kappa-2j}(z)$$

where $\tilde{g}_{\kappa-2j}(z) \in M_{\kappa-2j}$.

As a consequence of Lemma 1, for $g_\kappa(z) \in M_\kappa$ and $g_\mu(z) \in M_\mu$ we have the decomposition

$$\delta^{(q)}_\kappa g_\kappa(z) \cdot \delta^{(r)}_\mu g_\mu(z) = \sum_{l=0}^{q+r} \delta^{(l)}_{\kappa+\mu+2q+2r-2l}(\alpha_l E_{\kappa+\mu+2q+2r-2l} + \beta_l F(l, z))$$

where $F(l, z) \in C_{\kappa+\mu+2q+2r-2l}$ and $\alpha_l, \beta_l \in \mathbb{C}$.
Proposition 1. For \( q \leq r \),

\[
\delta_{\kappa}^{(q)} E_{\kappa}(z) \cdot \delta_{\mu}^{(r)} E_{\mu}(z) = \frac{\Gamma(\kappa + q)\Gamma(\mu + r)\Gamma(\kappa + \mu)}{\Gamma(\kappa)\Gamma(\mu)\Gamma(\kappa + \mu + q + r)} \delta_{\kappa+\mu}^{(q+r)} E_{\kappa+\mu}(z)
\]

\[+ \sum_{l=0}^{q+r} \beta_l \delta_{\kappa+\mu+2q+2r-2l}^{(l)} F(l, z)\]

where the \( F(l, z) \in C_{\kappa+\mu+2q+2r-2l} \) are normalized so their first non-zero Fourier coefficients are 1 and the \( \beta_l \in \mathbb{C} \) consist of integer values of \( \Gamma \)-functions and \( \zeta(s) \).

Proof. For simplicity write \( E_{\kappa}(z) = \sum_{m=0}^{\infty} c_{\kappa}(m)e(mz) \); then from equation (3) we get

\[
\delta_{\kappa}^{(q)} E_{\kappa}(z) = \sum_{m=0}^{\infty} c_{\kappa}(m) \left( \sum_{k=0}^{q} P_{k,\kappa}^{(q)}(-4\pi y)^{-k} m^q m^{-k} \right) e(mz).
\]

For \( E_{\mu}(x) = \sum_{n=0}^{\infty} c_{\mu}(n)e(nx) \) we then have

\[
(4) \quad \delta_{\kappa}^{(q)} E_{\kappa}(z) \cdot \delta_{\mu}^{(r)} E_{\mu}(z)
= \left( \sum_{m=0}^{\infty} c_{\kappa}(m) \left( \sum_{k=0}^{q} P_{k,\kappa}^{(q)}(-4\pi y)^{-k} m^q m^{-k} \right) e(mz) \right) \times \left( \sum_{n=0}^{\infty} c_{\mu}(n) \left( \sum_{j=0}^{r} P_{j,\mu}^{(r)}(-4\pi y)^{-j} n^r n^{-j} \right) e(nz) \right)
= \sum_{t=0}^{\infty} \sum_{s=0}^{\infty} \left( \sum_{m+n=t} \left( \sum_{j+k=s} c_{\kappa}(m)c_{\mu}(n)P_{k,\kappa}^{(q)}m^q m^{-k} P_{j,\mu}^{(r)}n^r n^{-j} \right) (-4\pi y)^{-s} \right) e(tz).
\]

For simplicity again, put

\[
E_{\kappa+\mu+2q+2r-2l}(z) = \sum_{n=0}^{\infty} c(l, n)e(nz), \quad F(l, z) = \sum_{n=1}^{\infty} d(l, n)e(nz)
\]

where we normalize \( F(l, z) \) so its first non-zero Fourier coefficient is 1. Then

\[
\delta_{\kappa+\mu+2q+2r-2l}^{(l)} E_{\kappa+\mu+2q+2r-2l}(z) = \sum_{l=0}^{\infty} c(l, t) \left( \sum_{j=0}^{l} p_{j}^{l}(t)y^{-j} \right) e(tz)
\]

where \( p_{j}^{l}(t) = P_{j,\kappa+\mu+2q+2r-2l}^{(l)}(-4\pi)^{-j} t^{l-j} \) and

\[
\delta_{\kappa+\mu+2q+2r-2l}^{(l)} F(l, z) = \sum_{l=0}^{\infty} d(l, t) \left( \sum_{j=0}^{l} p_{j}^{l}(t)y^{-j} \right) e(tz).
\]
Taking the Fourier expansions and switching the order of summations yields
\[
\sum_{l=0}^{q+r} \delta_{\kappa+\mu+2q+2r-2l}(\alpha_l E_{\kappa+\mu+2q+2r-2l}(z) + \beta_l F(l, z))
\]
\[
= \sum_{l=0}^{q+r} \left[ \alpha_l \sum_{t=0}^{\infty} c(l, t) \left( \sum_{s=0}^{l} p_s^l(t) y^{-s} \right) e(tz) \right.
\]
\[
+ \beta_l \sum_{t=1}^{\infty} d(l, t) \left( \sum_{s=0}^{l} p_s^l(t) y^{-s} \right) e(tz) \left]\right]
\]
\[
= \sum_{s=0}^{q+r} \left( \sum_{l=s}^{q+r} \alpha_l c(l, 0) p_s^l(0) \right) y^{-s}
\]
\[
+ \sum_{l=1}^{\infty} \left[ \sum_{s=0}^{q+r} \left( \sum_{l=s}^{q+r} (\alpha_l c(l, t) p_s^l(t) + \beta_l d(l, t) p_s^l(t)) \right) y^{-s} \right] e(tz).
\]

From Lemma 1 we can set the above result equal to (4). Then for \( t = 0 \) we set the terms indexed by \((-4\pi y)^{-s}\) equal and get the equation
\[
\sum_{j+k=s}^{q+r} (P_{k,\kappa}^{(q)} m^{q-k} P_{j,\mu}^{(r)} n^{r-j})_{m=0}^{n=0} = \sum_{l=0}^{q+r} \alpha_l p_{j+k}^l(0).
\]

Now,
\[
p_{j+k}^l(0) = (-4\pi)^{-j-k} \frac{\Gamma(\kappa + \mu + 2q + 2r - j - k)}{\Gamma(\kappa + \mu + 2q + 2r - 2j - 2k)} \neq 0
\]
and \( p_{j+k}^l(0) = 0 \) for \( l \neq j + k \). Therefore equation (5) becomes
\[
(P_{k,\kappa}^{(q)} m^{q-k})_{m=0}^{n=0} (P_{j,\mu}^{(r)} n^{r-j})_{n=0} = \alpha_{j+k} p_{j+k}^l(0).
\]

But \( (P_{k,\kappa}^{(q)} m^{q-k})_{m=0}^{n=0} = 0 \) for \( k \neq q \) and \( P_{q,\kappa}^{(q)} = \Gamma(\kappa + q) / \Gamma(\kappa) \neq 0 \), and similarly for \( (P_{j,\mu}^{(r)} n^{r-j})_{n=0} \). This implies \( \alpha_s = 0 \) for \( s < q + r \) and also
\[
\alpha_{q+r} = \frac{(P_{k,\kappa}^{(q)} m^{q-k})_{m=0}^{n=0} (P_{j,\mu}^{(r)} n^{r-j})_{n=0}}{P_{q+r}^{q+r}(0)} = \frac{\Gamma(\kappa + q) \Gamma(\mu + r) \Gamma(\kappa + \mu)}{\Gamma(\kappa) \Gamma(\mu) \Gamma(\kappa + \mu + q + r)}.
\]

For \( t = 1 \) we set the \((-4\pi y)^{-s} e(z)\) terms equal. As \( c_{\kappa}(1) = 2 / \zeta(1 - \kappa) \) and \( c_{\mu}(1) = 2 / \zeta(1 - \mu) \), we get the equation
\[
\sum_{j+k=s}^{q+r} \frac{2}{\zeta(1 - \mu)} (P_{k,\kappa}^{(q)} m^{q-k})_{m=0}^{n=0} P_{j,\mu}^{(r)} + \frac{2}{\zeta(1 - \kappa)} P_{k,\kappa}^{(q)} P_{j,\mu}^{(r)} n^{r-j})_{n=0} = \sum_{l=j+k}^{q+r} \alpha_l c(l, 1) p_{j+k}^l(1) + \beta_l d(l, 1) p_{j+k}^l(1).
In a similar way to what happened for (5), equation (6) gives us
\[ q + r X_l = s \]  
and evaluating these, we get
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\( \delta_4^{(q)} E_4(z) \cdot \delta_4^{(r)} E_4(z) \)
\[ = \frac{\Gamma(4 + q)\Gamma(4 + r)\Gamma(8)}{\Gamma(4)^2\Gamma(8 + q + r)} \delta_8^{(q+r)} E_8(z) \]
\[ + \beta_0 \Phi_{8+2q+2r}(z) + \beta_1 \delta_{6+2q+2r}^{(1)} \Phi_{6+2q+2r}(z) + \ldots + \beta_{q+r-2} \delta_{12}^{(q+r-2)} \Delta(z). \]

**Lemma 2.**

(i) \[ \delta_4 E_4(z) \cdot \delta_4 E_4(z) = \frac{2}{9} \delta_8^{(2)} E_8(z) - \frac{320}{3} \Delta(z), \]

(ii) \[ E_4(z) \cdot \delta_4^{(2)} E_4(z) = \frac{5}{18} \delta_8^{(2)} E_8(z) + \frac{320}{3} \Delta(z), \]

(iii) \[ \delta_4^{(2)} E_4(z) \cdot \delta_4 E_4(z) = \frac{1}{9} \delta_8^{(3)} E_8(z) - \frac{160}{3} \delta_{12} \Delta(z), \]

(iv) \[ E_4(z) \cdot \delta_4^{(3)} E_4(z) = \frac{1}{6} \delta_8^{(3)} E_8(z) + 160 \delta_{12} \Delta(z). \]

**Proof.** From (7) we have

\[ \delta_4 E_4(z) \cdot \delta_4 E_4(z) = \frac{2}{9} \delta_8^{(2)} E_8(z) + \beta_0 \Delta(z), \]

\[ E_4(z) \cdot \delta_4^{(2)} E_4(z) = \frac{5}{18} \delta_8^{(2)} E_8(z) + \beta_0 \Delta(z), \]

\[ \delta_4^{(2)} E_4(z) \cdot \delta_4 E_4(z) = \frac{1}{9} \delta_8^{(3)} E_8(z) + \beta_1 \delta_{12} \Delta(z), \]

\[ E_4(z) \cdot \delta_4^{(3)} E_4(z) = \frac{1}{6} \delta_8^{(3)} E_8(z) + \beta_1 \delta_{12} \Delta(z). \]

We only need to look at the holomorphic part of each equation in order to solve for the \( \beta_j \)'s. As

\[ \delta_4 E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n + 4(-4\pi y)^{-1})e(nz), \]

\[ \delta_4^{(2)} E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n^2 + 10(-4\pi y)^{-1}n + 20(-4\pi y)^{-2})e(nz), \]

\[ \delta_4^{(3)} E_4(z) = \sum_{n=0}^{\infty} c_4(n)(n^3 + 18(-4\pi y)^{-1}n^2 \]
\[ + 90(-4\pi y)^{-2}n + 120(-4\pi y)^{-3})e(nz), \]

\[ \delta_8^{(2)} E_8(z) = \sum_{n=0}^{\infty} c_8(n)(n^2 + 18(-4\pi y)^{-1}n + 72(-4\pi y)^{-2})e(nz), \]

\[ \delta_8^{(3)} E_8(z) = \sum_{n=0}^{\infty} c_8(n)(n^3 + 30(-4\pi y)^{-1}n^2 \]
\[ + 270(-4\pi y)^{-2}n + 720(-4\pi y)^{-3})e(nz) \]
we can find explicitly the holomorphic part of $\delta_4^{(q)} E_4(z) \cdot \delta_4^{(r)} E_4(z)$. The $e(tz)$ term of the holomorphic part of $\delta_4 E_4(z) \cdot \delta_4 E_4(z)$ is
\[
\sum_{m+n=t} mc_4(m)nc_4(n)
\]
and for $t = 1$ this is 0. The $e(tz)$ term for the holomorphic part of $\frac{2}{9} \delta_8^{(2)} E_8(z) + \beta_0 \Delta(z)$ is $\frac{2}{9} t^2 c_8(t) + \beta_0 \tau(t)$. Setting $t = 1$ we get
\[
\beta_0 = -\frac{2}{9} c_8(1) = -\frac{2}{9} \frac{2}{\zeta(-7)} = -\frac{320}{3}.
\]
This gives (i).

The $e(tz)$ term for the holomorphic part of $E_4(z) \cdot \delta_4^{(2)} E_4(z)$ is
\[
\sum_{m+n=t} c_4(m)n^2c_4(n).
\]
The $e(tz)$ term for the holomorphic part of $\frac{5}{18} \delta_8^{(2)} E_8(z) + \beta_0 \Delta(z)$ is $\frac{5}{18} t^2 c_8(t) + \beta_0 \Delta(z)$. Setting $t = 1$ we get
\[
\beta_0 = c_4(1) - \frac{5}{18} c_8(1) = \frac{2}{\zeta(-3)} - \frac{5}{18} \frac{2}{\zeta(-7)} = \frac{320}{3},
\]
which gives (ii).

The $e(tz)$ term for the holomorphic part of $\delta_4^{(2)} E_4(z) \cdot \delta_4 E_4(z)$ is
\[
\sum_{m+n=t} m^2c_4(m)nc_4(n).
\]
This is 0 for $t = 1$. The $e(tz)$ term for the holomorphic part of $\frac{1}{9} \delta_8^{(3)} E_8(z) + \beta_1 \delta_{12} \Delta(z)$ is $\frac{1}{9} t^3 c_8(t) + \beta_0 t \tau(t)$. Setting $t = 1$ we get
\[
\beta_1 = -\frac{1}{9} \frac{2}{\zeta(-7)} = -\frac{160}{3},
\]
which gives (iii).

The $e(tz)$ term for the holomorphic part of $E_4(z) \cdot \delta_4^{(3)} E_4(z)$ is
\[
\sum_{m+n=t} c_4(m)n^3c_4(n).
\]
The $e(tz)$ term for the holomorphic part of $\frac{1}{6} \delta_8^{(3)} E_8(z) + \beta_1 \delta_{12} \Delta(z)$ is $\frac{1}{6} t^3 c_8(t) + \beta_1 t \tau(t)$. Setting $t = 1$ we get
\[
\beta_1 = c_4(1) - \frac{1}{6} c_8(1) = \frac{2}{\zeta(-3)} - \frac{1}{6} \frac{2}{\zeta(-7)} = 160,
\]
which gives (iv).
Theorem 1 follows from Lemma 2 by uniqueness of Fourier coefficients. Looking at the $e(nz)$ term of the holomorphic part of (i) gives us

$$
\sum_{m=0}^{n} mc_4(m)(n-m)c_4(n-m) = \frac{2}{9} n^2 c_8(n) - \frac{320}{3} \tau(n)
$$

so

$$
\tau(n) = \frac{3}{320} \frac{2}{9} \frac{2}{\zeta(-7)} n^2 \sigma_7(n) - \frac{3}{320} \frac{4}{\zeta(-3)^2} \sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m)
$$

$$
= n^2 \sigma_7(n) - 540 \sum_{m=1}^{n-1} m(n-m)\sigma_3(m)\sigma_3(n-m),
$$

which is (i) of Theorem 1. In a similar way we get (ii).

Looking at the $e(nz)$ term of the holomorphic part of (iii) gives us

$$
\sum_{m=0}^{n} m^2 c_4(m)(n-m)c_4(n-m) = \frac{1}{9} n^3 c_8(n) - \frac{160}{3} n\tau(n)
$$

so

$$
\tau(n) = \frac{3}{160} \frac{1}{9} \frac{2}{\zeta(-7)} n^2 \sigma_7(n)
$$

$$
- \frac{3}{160} \frac{4}{\zeta(-3)^2} \frac{1}{n} \sum_{m=1}^{n-1} m^2(n-m)\sigma_3(m)\sigma_3(n-m)
$$

$$
= n^2 \sigma_7(n) - \frac{1080}{n} \sum_{m=1}^{n-1} m^2(n-m)\sigma_3(n)\sigma_3(n-m),
$$

which is (iii) from Theorem 1. In a similar way we get (iv).

In the same way we can establish the following lemmas.

**Lemma 3.**

(i) \( \delta_4 E_4(z) \cdot E_6(z) = \frac{2}{5} \delta_{10} E_{10}(z) + \frac{1728}{5} \Delta(z) \),

(ii) \( E_4(z) \cdot \delta_6 E_6(z) = \frac{3}{5} \delta_{10} E_{10}(z) - \frac{1728}{5} \Delta(z) \),

(iii) \( \delta_4^{(2)} E_4(z) \cdot E_6(z) = \frac{2}{11} \delta^{(2)}_{10} E_{10}(z) + 288 \delta_{12} \Delta(z) \),

(iv) \( \delta_4 E_4(z) \cdot \delta_6 E_6(z) = \frac{12}{55} \delta^{(2)}_{10} E_{10}(z) + \frac{288}{5} \delta_{12} \Delta(z) \),

(v) \( E_4(z) \cdot \delta_6^{(2)} E_6(z) = \frac{21}{55} \delta^{(2)}_{10} E_{10}(z) - \frac{2016}{5} \delta_{12} \Delta(z) \).

**Lemma 4.**

\( \delta_6 E_6(z) \cdot E_6(z) = \frac{1}{2} \delta_{12} E_{12}(z) - \frac{381024}{691} \delta_{12} \Delta(z) \).
Lemma 5.

(i) \[ \delta_4 E_4(z) \cdot E_8(z) = \frac{1}{3} \delta_12 E_{12}(z) + \frac{144000}{691} \delta_{12} \Delta(z), \]

(ii) \[ E_4(z) \cdot \delta_8 E_8(z) = \frac{2}{3} \delta_12 E_{12}(z) + \frac{288000}{691} \delta_{12} \Delta(z). \]

From these lemmas we get Theorems 2, 3, and 4 just as was done for Theorem 1.

References

[5] —, On Ramanujan’s function \( \tau(n) \) and the divisor function \( \sigma_k(n) \). II, ibid. 39 (1947), 33–52.

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