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Arithmetic Clifford's theorem for Hermitian vector bundles

by

TOHRU NAKASHIMA (Tokyo)

1. Introduction. Let K be a number field and let $\mathcal{O}_K$ denote its ring of integers. A Hermitian vector bundle $\overline{E}$ on an arithmetic curve $\operatorname{Spec}\mathcal{O}_K$ is a projective $\mathcal{O}_K$-module E equipped with a Hermitian metric $E \otimes_v \mathbb{C}$ for each infinite place $v : K \hookrightarrow \mathbb{C}$. We expect that Hermitian bundles have properties similar to those of vector bundles on an algebraic curve defined over a field.

Recently the notion of *size* $h^0(\overline{L})$ has been introduced for a Hermitian line bundle $\overline{L}$ on an arithmetic curve. This invariant may be considered as an arithmetic analogue of the dimension of the space of global sections. For example, a Riemann–Roch type theorem holds for them ([GS]).

In [G] Groenewegen has proved an arithmetic analogue of Clifford's theorem by means of the size function. The purpose of this note is to generalize the result to *semistable* Hermitian vector bundles. We notice that this is also an arithmetic analogue of a theorem obtained in [BGN] in the geometric case. We also consider an example of a semistable Hermitian bundle which comes from arithmetic abelian schemes ([B]).

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2. Statement of the result. Let K be a number field and let S^∞ denote the set of infinite places of K. The scheme $S = \operatorname{Spec}\mathcal{O}_K$ is said to be an *arithmetic curve*. If we denote by r_1 and r_2 the number of real resp. complex embeddings of K, then $n = [K : \mathbb{Q}] = r_1 + 2r_2$.

Let $\overline{E}$ be a Hermitian vector bundle of rank r on S. In other words, E is a projective $\mathcal{O}_K$-module of rank r equipped with Hermitian metrics $\| \, \|_v$ on the complex vector spaces $E_v = E \otimes_v \mathbb{C}$ associated with the embeddings

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$v \in S^\infty$. Let

$$E_{\mathbb{C}} \cong \bigoplus_{v \in S^\infty} E_v.$$

Let $E \otimes_{\mathbb{Z}} \mathbb{R}$ denote the real subspace of $E_{\mathbb{C}}$ fixed under complex conjugation. We equip $E_{\mathbb{R}}$ with the norm which is the restriction of the norm on $E_{\mathbb{C}}$ which is defined for $x \in E_{\mathbb{C}}$ as follows:

$$\|x\|^2 = \sum_{v \in S^\infty} \|x\|_v^2.$$

We consider E as a lattice of rank rn in $E_{\mathbb{R}}$.

The *arithmetic degree* of a Hermitian line bundle $\overline{L}$ is defined by

$$\widehat{\deg}\,\overline{L} = \log \sharp(L/\mathcal{O}_K s) - \sum_{v \in S^\infty} \log \|s\|_v$$

where s is a nonzero element of L. For a general Hermitian bundle $\overline{E}$, we define $\widehat{\deg}\,\overline{E}$ by

$$\widehat{\deg}\,\overline{E} = \widehat{\deg} \det \overline{E}.$$

We have the equality

$$\widehat{\deg}\,\overline{E} = -\log \operatorname{covol} \overline{E} + \frac{r \log |\Delta_K|}{2}$$

where we denote by $\operatorname{covol} \overline{E}$ the covolume of $\overline{E}$ and by Δ_K the discriminant of K. We define the *norm* of $\overline{E}$ by

$$N(\overline{E}) = e^{\widehat{\deg}\,\overline{E}}.$$

The $\mathcal{O}_K$-module

$$\omega_S = \operatorname{Hom}_{\mathbb{Z}}(\mathcal{O}_K, \mathbb{Z})$$

is locally free of rank one. We equip ω_S with the Hermitian metric which is defined, for each $v \in S^\infty$, by $\|\mathrm{Tr}\|_v = 1$ for the trace map $\mathrm{Tr} \in \omega_S$. Let $\overline{\omega}_S$ denote the resulting Hermitian line bundle. Then we have

$$\widehat{\deg}\,\overline{\omega}_S = \log |\Delta_K|.$$

For a Hermitian vector bundle $\overline{E}$, we denote by $\overline{E}^\vee = \operatorname{Hom}_{\mathcal{O}_K}(E, \mathcal{O}_K)$ the dual vector bundle equipped with the dual metric. We let

$$\overline{E}^* = \overline{E}^\vee \otimes \overline{\omega}_S$$

denote the Hermitian bundle equipped with the tensor product of the dual metric and the metric on ω_S defined above. Its degree is given by

$$\widehat{\deg}\,\overline{E}^* = -\widehat{\deg}\,\overline{E} + r\widehat{\deg}\,\overline{\omega}_S.$$

Following [GS], we define the *effectivity* of $\overline{E}$ to be the number

$$k^0(\overline{E}) = \sum_{x \in E} e^{-\pi \|x\|^2}.$$

Then the *size* of $\overline{E}$ is defined as follows:

$$h^0(\overline{E}) = \log k^0(\overline{E}).$$

$h^0(\overline{E})$ may be considered as an arithmetic analogue of the dimension of the space of global sections in the geometric case. For example, as a consequence of the Poisson summation formula we have the following result, which is stated in [GS, 9] in a different form.

PROPOSITION 2.1 (Riemann–Roch). *Let $\overline{E}$ be a Hermitian vector bundle of rank r on S. Then*

$$h^0(\overline{E}) - \widehat{\deg}\,\overline{E}/2 = h^0(\overline{E}^*) - \widehat{\deg}\,\overline{E}^*/2.$$

For a Hermitian bundle $\overline{E}$, we define its *slope* $\widehat{\mu}(\overline{E})$ by

$$\widehat{\mu}(\overline{E}) = \widehat{\deg}\,\overline{E}/\mathrm{rk}\,E.$$

$\overline{E}$ is said to be *semistable* if $\widehat{\mu}(\overline{F}) \le \widehat{\mu}(\overline{E})$ for every subbundle $\overline{F}$ of $\overline{E}$ with the induced metric.

The main result of this note is the following

THEOREM 2.2. *Let $\overline{E}$ be a semistable Hermitian vector bundle of rank r on $S = \mathrm{Spec}\,\mathcal{O}_K$.*

(1) *If* $\widehat{\deg}\,\overline{E} \le 0$, *then*

$$h^0(\overline{E}) < \frac{3^{rn}\pi}{\pi - r\log 3}\, e^{-\pi e^{-2(\widehat{\deg}\,\overline{E})/(rn)}}.$$

(2) *If* $\widehat{\deg}\,\overline{E} \ge 0$, *then*

$$h^0(\overline{E}) \le rn(\log\omega + \log r + 2^{-1}\log n) + \widehat{\deg}\,\overline{E}.$$

(3) *If* $0 \le \widehat{\mu}(\overline{E}) \le \widehat{\deg}\,\overline{\omega}_S$, *then*

$$h^0(\overline{E}) \le rn(\log\omega + \log r + \log n) + \widehat{\deg}\,\overline{E}/2.$$

We notice that the above theorem has a geometric counterpart. Let C be a smooth projective curve over $\mathbb{C}$ and let E be a semistable vector bundle of rank r on C. Let $\mu(E) = \deg E/r$ denote its slope. Then $h^0(E) = \dim H^0(E)$ has the following properties.

PROPOSITION 2.3. (1) *If* $\deg E < 0$, *then* $h^0(E) = 0$.
(2) *If* $\deg E \ge 0$, *then* $h^0(E) \le r + \deg E$.
(3) *If* $0 \le \mu(E) \le 2g - 2$, *then* $h^0(E) \le r + \deg E/2$.

Proof. (1) is clear from the definition and (3) has been proved in [BGN, Theorem 2.1]. We prove (2) by induction on r, following the argument in *loc.cit.* The case $r = 1$ is easy. Assume that the inequality holds for semistable bundles of rank less than r. Let E_1 denote a subbundle of E which has the maximal slope among all proper subbundles. Let $E_2 = E/E_1$. It is clear that E_1, E_2 are both semistable. Since we may assume that $h^0(E) > 0$, we obtain $\mu(E_1) \geq 0$. We also have $\mu(E_1) \geq \mu(E) \geq 0$ by the semistability of E. Hence by the induction assumption we obtain

$$h^0(E) \leq h^0(E_1) + h^0(E_2) \leq \operatorname{rk} E_1 + \deg E_1 + \operatorname{rk} E_2 + \deg E_2 = r + \deg E. \ \blacksquare$$

3. Proof of Theorem 2.2. For a lattice Λ, its *minimum* is defined to be the minimum norm of nonzero elements in Λ. The following result has been proved by Groenewegen ([G, Proposition 5.4, Corollary 5.7]).

LEMMA 3.1. *Let Λ be a lattice of rank n with minimum λ and let Λ^* denote the dual lattice with minimum λ^*. Let $\omega = k^0(\mathbb{Z})$ where $\mathbb{Z}$ is equipped with the trivial metric. For $1 \leq i \leq n$, let γ_i denote the ith Hermite constant. Then*

$$k^0(\Lambda) \leq \omega^n \prod_{i=1}^{n} \max\{1, \gamma_i/\lambda\}.$$

Furthermore, we have either

$$k^0(\Lambda) \leq \omega^n \max\{1, 1/\lambda\}^{n/2} n^n \quad \text{or} \quad k^0(\Lambda^*) \leq \omega^n \max\{1, 1/\lambda^*\}^{n/2} n^n.$$

LEMMA 3.2. *If $\overline{E}$ is a semistable Hermitian bundle, then $\overline{E}^*$ is also semistable.*

Proof. Assume that $\overline{E}$ is semistable and let $\overline{F} \subset \overline{E}^*$ be a sub-$\mathcal{O}_K$-module with the induced metric. Considering the saturation of $\overline{F}$, we may assume that $\overline{E}^*/\overline{F}$ is projective. Then we obtain an injection $(\overline{E}^*/\overline{F})^* \hookrightarrow \overline{E}$. Since $\overline{E}$ is assumed to be semistable, we have $\widehat{\mu}((\overline{E}^*/\overline{F})^*) \leq \widehat{\mu}(\overline{E})$, which implies $\widehat{\mu}(\overline{F}) \leq \widehat{\mu}(\overline{E}^*)$ as desired. $\blacksquare$

Let λ (resp. λ^*) denote the minimum of $\overline{E}$ (resp. $\overline{E}^*$). Since $\overline{E}$ is semistable, so is $\overline{E}^*$ by Lemma 3.2. For any nonzero $s \in E$, let $\overline{L}$ denote the Hermitian line bundle which is generated by s in E, with the induced metric. Then

$$\widehat{\mu}(\overline{E}) \geq \widehat{\deg}\, \overline{L} \geq -\sum_{v \in S^\infty} \log \|s\|_v.$$

Hence, by the geometric-arithmetic mean inequality, we have

$$\|s\|^2 = \sum_{v \in S^\infty} \|s_v\|_v^2 \geq n\Big(\prod_{v \in S^\infty} \|s\|_v^2\Big)^{1/n} \geq ne^{-2\widehat{\deg}\, \overline{E}/(rn)} = nN(\overline{E})^{-2/(rn)}.$$

This yields

$$\lambda \geq \sqrt{n}\, N(\overline{E})^{-1/(rn)}.$$

Similarly, by Lemma 3.2,

$$\lambda^* \geq \sqrt{n}\, N(\overline{E}^*)^{-1/(rn)}.$$

If $\widehat{\deg}\,\overline{E} \le 0$, then $N(\overline{E}) \le 1$ and hence $\lambda \ge \sqrt{n}$. By [G, Prop. 4.4], we obtain

$$k^0(\overline{E}) \le 1 + \frac{3^{rn}\pi}{\pi - \log 3}\, e^{-\pi\lambda^2}.$$

This implies (1).

Since $\gamma_i \le rn$ for all i, Lemma 3.1 yields

$$k^0(\overline{E}) \le \omega^{rn} \max\{1, (rn/\lambda)^{rn}\}.$$

If $\widehat{\deg}\,\overline{E} \ge 0$, then $\max\{1, (rn/\lambda)^{rn}\} \le (r\sqrt{n})^{rn} e^{\widehat{\deg}\,\overline{E}}$. Thus (2) follows.

To prove (3), we note that, by Lemma 3.1 we have either

$$(*) \qquad k^0(\overline{E}) \le \omega^{rn} \max\{1, 1/\lambda\}^{rn/2} (rn)^{rn}$$

or

$$(**) \qquad k^0(\overline{E}^*) \le \omega^{rn} \max\{1, 1/\lambda^*\}^{rn/2} (rn)^{rn}.$$

By the assumption $0 \le \widehat{\mu}(\overline{E}) \le \widehat{\deg}\,\overline{\omega}_S$, we have $\widehat{\deg}\,\overline{E} \ge 0$ and $\widehat{\deg}\,\overline{E}^* \ge 0$. Assume that $(*)$ holds. Since $\max\{1, 1/\lambda\} = 1/\lambda$, we have

$$h^0(\overline{E}) \le rn(\log\omega + \log rn) + \widehat{\deg}\,\overline{E}/2.$$

Hence we are done in this case. Similarly, if $(**)$ holds, we obtain

$$h^0(\overline{E}^*) \le rn(\log\omega + \log rn) + \widehat{\deg}\,\overline{E}^*/2,$$

which yields, by Riemann–Roch,

$$h^0(\overline{E}) \le rn(\log\omega + \log rn) + \widehat{\deg}\,\overline{E}/2.$$

This completes the proof.

4. An example. In this section we shall give an example of semistable Hermitian bundle due to J.-B. Bost. For higher-dimensional Arakelov geometry, we refer to [SABK].

Let A be an abelian variety of dimension g over a number field K with $n = [K : \mathbb{Q}]$. Let L be an ample symmetric line bundle on A and let $\chi(A, L)$ denote its Euler characteristic. Then, by Riemann–Roch,

$$\chi(A, L) = L^g/g!.$$

Assume that A has good reduction and let the abelian scheme

$$\pi : \mathcal{A} \to S = \operatorname{Spec} \mathcal{O}_K$$

denote a model of A over S and let $\mathcal{L}$ be a line bundle $\mathcal{A}$ extending L. Let $\varepsilon : S \to \mathcal{A}$ be a zero section. For each $v \in S^\infty$ there exists an F_∞-invariant Hermitian metric $\|\ \|_v$ on L_v such that its curvature form is translation invariant. This metric is unique up to multiplication by positive constants. Let $\overline{\mathcal{L}}$ denote the Hermitian line bundle equipped with this metric, normalized so that $\varepsilon^*\overline{\mathcal{L}}$ is isometric to the trivial line bundle $\overline{\mathcal{O}}_S$ with the trivial metric. Then $\pi_*\mathcal{L}$ is a vector bundle of rank $\chi(A,L)$ on S. We equip it with the metric which is defined as follows. For each $v \in S^\infty$ and $s \in (\pi_*\mathcal{L})_v \cong H^0(A_v(\mathbb{C}), \mathcal{L}_v)$, one sets

$$\|s\|_v = \int_{\mathcal{A}_v(\mathbb{C})} \|s\|_{\mathcal{L}}^2 \, d\mu,$$

where $d\mu$ denotes the Haar measure of total volume one on $\mathcal{A}_v(\mathbb{C})$. J.-B. Bost proved that the resulting Hermitian bundle $\pi_*\overline{\mathcal{L}}$ is semistable and its slope is given by

$$\widehat{\mu}(\pi_*\overline{\mathcal{L}}) = -\frac{1}{2}\, h(A) + \frac{1}{4}\log\left(\frac{\chi(A,L)}{(2\pi)^g}\right),$$

where $h(A)$ denotes the Faltings height of A ([B,Théorème 4.2]). Thus, we may apply Theorem 2.2 to $\pi_*\overline{\mathcal{L}}$ to see that if

$$\log\left(\frac{\chi(A,L)}{(2\pi)^g}\right) \geq 2h(A),$$

then

$$h^0(\pi_*\overline{\mathcal{L}}) \leq \chi(A,L)\Bigg\{ n\left(\log\omega + \frac{1}{2}\log n + \log\chi(A,L)\right) - \frac{1}{2}\, h(A) + \frac{1}{4}\log\left(\frac{\chi(A,L)}{(2\pi)^g}\right)\Bigg\}.$$

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Department of Mathematics
Tokyo Metropolitan University
Minami-Ohsawa 1-1, Hachioji-shi
Tokyo 192-0397, Japan
E-mail: nakasima@comp.metro-u.ac.jp

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