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On the symmetry of the divisor function in almost all short intervals

by

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1. Introduction and statement of the results. In this paper we study the symmetry of the divisor function in almost all short intervals. As in the previous paper [2] on the symmetry of the function $\omega^*(n)$ (i.e. the number of prime divisors $p \mid n$ with $p < \sqrt{n}$), we use the Large Sieve to derive a similar bound for the divisor function, $d(n)$ (number of divisors of n). This function has the useful property (lacked by both $\omega(n)$ and $\omega^*(n)$) of "flipping", i.e. $d(n)$ can be written as

$$d(n) = \sum_{\substack{d|n \\ d<\sqrt{n}}} 1 + \sum_{\substack{d|n \\ d>\sqrt{n}}} 1 = 2 \sum_{\substack{d|n \\ d<\sqrt{n}}} 1$$

(if n is not a square). This time the technical difficulties arising from the "large" prime divisors (encountered, for example, in [2] with $\omega(n)$) disappear because of the flipping property.

This property can be applied, more generally, when dealing with the symmetry of an arithmetical function $f(n)$ which is the Dirichlet convolution of a (fixed) arithmetical function $g(n)$ with itself: $f = g * g$, i.e. (if n is not a square)

$$f(n) = 2 \sum_{\substack{d|n \\ d<\sqrt{n}}} g(d) g\left(\frac{n}{d}\right).$$

However, to apply our method, we also need the hypothesis that g is "smooth ($g \in C^1([1, \infty[)$ suffices) and with small derivative".

A more general study of such classes of arithmetical functions f (which we call "quasi-symmetric") will be the object of a future paper.

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Here we confine ourselves to the case $f(n) = d(n)$ ($g(n) = 1$, so it is the smoothest, and with lowest derivative, i.e. 0), also because of the links of this case to the Dirichlet problem for the divisor function in almost all short intervals.

First of all, we consider n to be in a short interval of the kind $[x-h, x+h]$ (we call it *short*, as usual, if $h = o(x)$ and $h(x) \to \infty$ as $x \to \infty$) and we form the *symmetry sum* (here and in what follows $x \in \mathbb{N}$ and $x \to \infty$)

$$S^{\pm}(x) := \sum_{x-h \le n \le x+h} d(n)\mathrm{sgn}(n-x)$$

(here $\mathrm{sgn}(t) := t/|t|$ for $t \in \mathbb{R} - \{0\}$, $\mathrm{sgn}(0) = 0$); then we consider its *mean square* (with $h = h(N) \in \mathbb{N}$ independent of x, where N is also an integer)

$$I(N,h) := \sum_{N < x \le 2N} |S^{\pm}(x)|^2,$$

which has *diagonal* ($\| \, \|$ is the distance from $\mathbb{Z}$)

$$D(N,h) := 8N \sum_{t \le \sqrt{N}} \frac{\mu(t)}{t^2} \sum_{k \le \sqrt{N}/t} \left\| \frac{h}{k} \right\| \log^2 \frac{\sqrt{N}}{tk}.$$

Our main result is the following (in what follows $L := \log N$):

THEOREM 1. *Let N and $h = h(N) < \sqrt{N}/2$ be large enough natural numbers (with $h \to \infty$ as $N \to \infty$) and $I(N,h)$, $D(N,h)$ be defined as above. Then*

$$I(N,h) = D(N,h) + \mathcal{O}(NhL^{5/2}\sqrt{\log L}).$$

Then, by simple calculations (see Section 3), we get from it the following asymptotic estimate:

COROLLARY 1. *Under the hypothesis of Theorem 1 we have*

$$I(N,h) = \frac{16}{\pi^2} Nh \log^3 \frac{\sqrt{N}}{h} + \mathcal{O}(NhL^{5/2}\sqrt{\log L}).$$

Our method also allows us to prove estimates for the mean square of the remainder of the Dirichlet divisor problem in a short interval $[x, x+h]$.

We remark that these results, in particular our Corollary 2, have been obtained (but without the explicit constant and under slightly less general hypotheses) by Kiuchi and Tanigawa in [6, Corollary]; however, they use the Voronoï formula in their proofs (see also [5]), while our approach is elementary (using only the Large Sieve, see Lemmas 1 and 3). Nonetheless, we will not give the proofs of our Theorem 2 and Corollary 2 as they are a straightforward adaptation of the proofs of Theorem 1 and Corollary 1.

Define (as in [6]) the remainder in the Dirichlet problem:

$$\Delta(x) := \sum_{n\le x} d(n) - \left(x\log x + (2\gamma-1)x + \frac{1}{4}\right)$$

(γ is the Euler constant). Then we get

$$\begin{aligned} J(N,h) &:= \sum_{x=N}^{2N} |\Delta(x+h)-\Delta(x)|^2 \\ &= \sum_{x=N}^{2N} \Big| \sum_{x<n\le x+h} d(n) - h\log x - 2\gamma h + \mathcal{O}\Big(\frac{h^2}{x}+1\Big)\Big|^2 \\ &= \sum_{x=N}^{2N} \Big| \sum_{x<n\le x+h} d(n) - h\log x - 2\gamma h\Big|^2 + \mathcal{O}(Nh^{1/2}L^{3/2}), \end{aligned}$$

by Cauchy's inequality, if $h<\sqrt{N}/2$ (see the proof of Theorem 1); since

$$\sum_{x<n\le x+h} d(n) = 2\sum_{d\le\sqrt{x+h}}\ \sum_{x/d<m\le (x+h)/d} 1$$

and (see [13, p. 6])

$$2h\sum_{d\le\sqrt{x}}\frac{1}{d} = h\log x + 2\gamma h + \mathcal{O}\Big(\frac{h}{\sqrt{x}}\Big),$$

we get, under the hypothesis $h<\sqrt{N}/2$ (here $\{\}$ is the fractional part)

$$J(N,h) = 4\sum_{x=N}^{2N}\Big|\sum_{d\le\sqrt{x}}\Big(\Big\{\frac{x+h}{d}\Big\} - \Big\{\frac{x}{d}\Big\}\Big)\Big|^2 + \mathcal{O}(Nh^{1/2}L^{3/2}).$$

Then we have the following two results.

THEOREM 2. *Under the hypotheses of Theorem* 1 *we have*

$$J(N,h) = 4N\sum_{t\le\sqrt{N}}\frac{\mu(t)}{t^2}\sum_{k\le\sqrt{N}/t}\Big(\Big\{\frac{h}{k}\Big\} - \Big\{\frac{h}{k}\Big\}^2\Big)\log^2\frac{\sqrt{N}}{tk} + \mathcal{O}(NhL^{5/2}\sqrt{\log L}).$$

COROLLARY 2. *Under the hypothesis of Theorem* 2 *we have*

$$J(N,h) = \frac{8}{\pi^2}Nh\log^3\frac{\sqrt{N}}{h} + \mathcal{O}(NhL^{5/2}\sqrt{\log L}).$$

The paper is organized as follows: in Section 2 we give the lemmas necessary to prove Theorem 1, and in Section 3 we prove Theorem 1 and Corollary 1.

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2. Lemmas

LEMMA 1. *Let Q and N be natural numbers, M be an integer and $\lambda_{a,q}$ be complex numbers* ($\forall a, q \in \mathbb{N}$); *then*

$$\sum_{n=M+1}^{M+N} \Big| \sum_{q \le 2Q} \sum_{a \le q}{}^{*} \lambda_{a,q} e_q(an) \Big|^2 = (N + \mathcal{O}(Q^2)) \sum_{q \le 2Q} \sum_{a \le q}{}^{*} |\lambda_{a,q}|^2.$$

(Here, as usual, the $*$ in the sum means that $(a, q) = 1$.)

This is a version of the well known Large Sieve inequality, usually obtained by applying the Duality Principle (e.g. [10, p. 134]) to the "main" version (see, for example, [1, p. 13]); here we have a more precise statement, obtained from the proof of the next Lemma 3 (using Hilbert's inequality instead of Lemma 2: see the remarks after the next lemma). In fact, also Lemma 3 gives an estimate of the off-diagonal terms. On the other hand, on the diagonal "well spaced" property of the arguments a/q of the exponentials does not hold and the sum over n simply counts the number of terms, giving the "main term".

With these ideas in mind, we will prove Lemma 3 by means of the following:

LEMMA 2. *Let R, S be natural numbers, λ_j be arbitrary* (*distinct*) *real numbers and $u_1, \ldots, u_R$, $v_1, \ldots, v_S$ be arbitrary complex numbers with ℓ^2-norms*

$$\|u\|_2 := \Big(\sum_{r \le R} |u_r|^2 \Big)^{1/2}, \quad \|v\|_2 := \Big(\sum_{s \le S} |v_s|^2 \Big)^{1/2};$$

then, writing $\|\beta\|$ for the distance of $\beta \in \mathbb{R}$ from the nearest integer and letting $\delta > 0$ we have

$$\|\lambda_r - \lambda_s\| \ge \delta \ \forall r \ne s \ \Rightarrow \Big| \sum_{r \le R} \sum_{\substack{s \le S \\ s \ne r}} \frac{u_r \overline{v}_s}{\|\lambda_r - \lambda_s\|} \Big| \le \frac{2 + \log(RS)}{\delta} \|u\|_2 \|v\|_2.$$

We remark that the idea to use a lemma of this kind (similar to Hilbert's inequality) to prove an estimate like Lemma 3 (similar to the Large Sieve) is not new. See, for example, the work of Elliott [4] and also [7]–[9], [11], [12].

However, we will prove both our lemmas in full detail, due to the dependence of the inner sums of Lemma 3 on x (i.e., due to the complicating function $\alpha_{j,d}(x)$).

In fact, we cannot use Hilbert's inequality as it stands (i.e., with $\lambda_r - \lambda_s$ instead of $\|\lambda_r - \lambda_s\|$), because we need to estimate the exponential sum over x in absolute value (to bound the terms $\alpha_{j,d}(x)$; see the proof of Lemma 3); this justifies the additional logarithm we get.

In order to make the arguments and the presence of each term in the bounds clear, we present a complete proof of Lemma 2.

Proof of Lemma 2. Since $\|\lambda_r - \lambda_s\| \ge \delta > 0$ for $r \ne s$ we may suppose (on reordering r, s if necessary) that $\|\lambda_r - \lambda_s\| \ge \delta|r - s|$, whence

$$\frac{1}{\|\lambda_r - \lambda_s\|} \le \frac{1}{\delta}\,\frac{1}{|r-s|} \qquad \forall r \ne s.$$

By Cauchy's inequality,

$$\Bigg|\sum_{r\le R}\sum_{\substack{s\le S\\ s\ne r}} \frac{u_r\overline{v}_s}{\|\lambda_r-\lambda_s\|}\Bigg|^2 \le \Bigg(\sum_{r\le R}\sum_{\substack{s\le S\\ s\ne r}}\frac{|u_r|^2}{\delta|r-s|}\Bigg)\Bigg(\sum_{s\le S}\sum_{\substack{r\le R\\ r\ne s}}\frac{|v_s|^2}{\delta|s-r|}\Bigg),$$

whence it suffices to prove that (due to the symmetry in r, s)

$$\sum_{\substack{s\le S\\ s\ne r}} \frac{1}{|s-r|} \le 2 + \log(RS).$$

Here the left-hand side is (the sums, now, could be empty)

$$\sum_{s=1}^{r-1}\frac{1}{r-s} + \sum_{s=r+1}^{S}\frac{1}{s-r} = \sum_{j\le r-1}\frac{1}{j} + \sum_{j\le S-r}\frac{1}{j} \le \log R + \log S + 2,$$

by the elementary estimate

$$\sum_{j\le n}\frac{1}{j} \le 1 + \int_1^n \frac{dt}{t} = 1 + \log n \qquad \forall n \ge 1;$$

this gives the stated inequality and hence the lemma.

REMARK. The constant $2 + \log(RS)$ is not optimal, but it is sufficient for our purposes.

LEMMA 3. *Let A, B and N be natural numbers, M be an integer and $c_{j,d}$ be complex numbers* ($\forall j, d \in \mathbb{N}$); *for a sequence $a_n > 0$ $\forall n \in \mathbb{N}$, define*

$$\alpha_{j,d}(x) := \sum_{n\in\mathbb{N}} a_n \chi_{\mathcal{I}(j,d,n)}(x)$$

where $\mathcal{I}(j,d,n)$ is an interval whose endpoints depend on these three (inte-

ger) variables and $\chi_{\mathcal{I}(j,d,n)}(x)$ *indicates its characteristic function. Then*

$$\sum_{x=M+1}^{M+N}\Big|\sum_{d=A}^{B}\sum_{j\le d}{}^{*}\alpha_{j,d}(x)c_{j,d}e_d(jx)\Big|^2=\sum_{d=A}^{B}\sum_{j\le d}{}^{*}|c_{j,d}|^2\sum_{x=M+1}^{M+N}|\alpha_{j,d}(x)|^2$$
$$+\mathcal{O}\Big(\alpha^2B^2\log B\sum_{d=A}^{B}\sum_{j\le d}|c_{j,d}|^2\Big),$$

with $(\alpha>0)$

$$\alpha:=\max_{\substack{M<x\le M+N\\ j,d}}|\alpha_{j,d}(x)|\ll 1.$$

(Here the implied constant depends at most on A,B,M,N.)

Proof. We first expand the square on the left-hand side and we isolate the main term (explicitly given), so that we have only to estimate the off-diagonal terms. All of these have well-spaced j/d:

$$\|j/d-j'/d'\|\ge\frac{1}{dd'}\ge 1/B^2\qquad\forall j/d\ne j'/d',$$

provided $d,d'\le B$ (which we suppose; here $\|\ \|$ is the distance from an integer).

Thus the off-diagonal terms are (here the dash indicates $j/d\ne j'/d'$)

$$\sum_{d,d'=A}^{B}\sum_{j,j'}{}'c_{j,d}\overline{c_{j',d'}}\sum_{x=M+1}^{M+N}\alpha_{j,d}(x)\overline{\alpha_{j',d'}(x)}e((j/d-j'/d')x);$$

the sum over x can be estimated (in absolute value) as (see the lemma for the definition of α)

$$\ll\alpha^2\Big|\sum_x e((j/d-j'/d')x)\Big|;$$

now the range of the sum over x also depends on the variables j,d,j',d'.

However, by the definition of the functions $\alpha_{j,d}(x)$, x is an integer in an interval of (consecutive) integers and by a well known estimate ([3, p. 143])

$$\sum_{x=M+1}^{M+N}\alpha_{j,d}(x)\overline{\alpha_{j',d'}(x)}e((j/d-j'/d')x)\ll\frac{\alpha^2}{\|j/d-j'/d'\|}.$$

In fact, from the definition of $\alpha_{j,d}(x)$, we can write

$$\alpha(x):=\alpha_{j,d}(x)\alpha_{j',d'}(x)=\sum_n b(n)\chi_{I(n)}(x),$$

where $I(n)=[u_n,v_n]$ is an interval depending on n and $b(n)$ is positive.

Then we have

$$\sum_x \alpha(x)e(x\xi) = \sum_n b(n)\Big(\sum_{x\in I(n)} e(x\xi)\Big) = \sum_n b(n)\,\frac{e(\xi u_n)-e(\xi v_n)}{1-e(\xi)} \ll \frac{\alpha^2}{\|\xi\|}.$$

We then transform the double summation over j,d into summation over $k := j/d$ and let $c_{j,d} := C_k$ (note that $(j,d)=1$ allows us to do so); in this way the off-diagonal terms are bounded by

$$\alpha^2 \sum_{\substack{k_r,k_s\in\mathcal{K}\\ k_r\neq k_s}} |C_{k_r}|\,|C_{k_s}|\,\frac{1}{\|k_r-k_s\|},$$

where $\mathcal{K}$ is a set of well-spaced numbers (i.e. $\|k_r-k_s\|\geq 1/B^2$ by the above), whose cardinality is $\mathcal{O}(B^2)$.

Since we can apply Lemma 2 with $R=S=\mathcal{O}(B^2)$, $k_r=\lambda_r$ and $k_s=\lambda_s$, setting $\delta=1/B^2$ we get the $\mathcal{O}$-estimate of Lemma 3 (the diagonal being easy to evaluate).

Remark. We explicitly point out that the additional logarithm $(\log B)$, due to the estimates of Lemma 2, is not a problem, since it will give a negligible contribution to the terms with $\sqrt{N}<d\leq\sqrt{x}$ (see the proof of Theorem 1 in the next section).

3. Proof of Theorem 1 and of Corollary 1. We start by writing the symmetry sum in order to apply the Large Sieve (for both Lemmas 1 and 3):

$$S^{\pm}(x) = 2\sum_{d<\sqrt{x+h}}\ \sideset{}{'}\sum_{(x-h)/d\leq m\leq(x+h)/d} \operatorname{sgn}(md-x) + \mathcal{O}\Big(\sum_{\sqrt{x-h}\leq d\leq\sqrt{x+h}} 1\Big);$$

here the remainder is clearly $\mathcal{O}(h/\sqrt{x}+1)$ and the dash means that the second sum has the further limitation on m given by $m>d$; this has no influence when $d\leq\sqrt{x-h}$, while the range $\sqrt{x-h}<d<\sqrt{x+h}$ contributes

$$S^{\pm}(x) = 2\sum_{d\leq\sqrt{x}}\ \sum_{(x-h)/d\leq m\leq(x+h)/d} \operatorname{sgn}(md-x) + \mathcal{O}\Big(\frac{h^2}{x}+\frac{h}{\sqrt{x}}+1\Big).$$

Here the remainders are $\mathcal{O}(1)$, which gives rise to $\mathcal{O}(hN\log N)$ in the mean square $I(N,h)$ after using the trivial estimate

$$\mathcal{O}\Big(\sum_{x\sim N}\sum_{d<\sqrt{x}}\ \sum_{(x-h)/d\leq m\leq(x+h)/d} 1\Big).$$

We remark that the inner sum on m:

$$\chi_q(x) := \sum_{|m-x/q|\leq h/q} \operatorname{sgn}(mq-x)$$

is treated in [2] in a slightly different manner (which, in this case, would entail small technical problems), while here we use directly the orthogonality of additive characters (as usual $e_q(k) := e^{2\pi ik/q}$)

$$\frac{1}{q}\sum_{j\le q} e_q(j(n-a)) = \begin{cases} 1 & \text{if } n\equiv a \pmod q,\\ 0 & \text{otherwise}\end{cases}$$

(see [14, Chapter 1, Lemma 5]) to write it as an exponential sum; in fact

$$\chi_q(x) = \sum_{\substack{|r|\le h\\ r\equiv -x \pmod q}} \operatorname{sgn}(r) = \sum_{j<q} c_{j,q} e_q(jx),$$

whereas this function is defined by fractional parts in [2], while now

$$c_{j,q} := \frac{1}{q}\sum_{|r|\le h} \operatorname{sgn}(r) e_q(rj) \qquad (\Rightarrow c_{q,q} = 0).$$

When x is an integer mod q the functions $\chi_q(x)$ depend on the sign of x in the residue classes, and they have absolute value 1, otherwise they vanish; this implies, by the already quoted orthogonality,

$$\sum_{j<q} |c_{j,q}|^2 = 2\left\|\frac{h}{q}\right\|.$$

In order to eliminate the dependence of $S^{\pm}(x)$ on x in its range of summation, we split it as (the remainder being negligible)

$$S^{\pm}(x) = S_0^{\pm}(x) + S_1^{\pm}(x) + \mathcal{O}(1),$$

where

$$S_0^{\pm}(x) := 2\sum_{q\le\sqrt{N}} \chi_q(x), \qquad S_1^{\pm}(x) := 2\sum_{\sqrt{N}<q\le\sqrt{x}} \chi_q(x).$$

We will apply the Large Sieve to both their mean squares: Lemma 1 to $S_0^{\pm}(x)$ and Lemma 3 to $S_1^{\pm}(x)$; first, we introduce the (reduced) fractions j/d and then we isolate the main term.

We are not yet ready to apply the Large Sieve, because $j<q$ does not mean $(j,q)=1$ (as is the case when q is prime); we need the following property of $c_{j,q}$:

$$c_{dj',dq'} = \frac{1}{d} c_{j',q'} \qquad \forall d, j', q' \in \mathbb{N},$$

to get

$$S_0^{\pm}(x) = 2\sum_{q\le\sqrt{N}}\sum_{j<q} c_{j,q}e_q(jx) = 2\sum_{q\le\sqrt{N}}\sum_{d|q}\sum_{\substack{j<q\\ (j,q)=d}} c_{j,q}e_q(jx)$$

$$= 2\sum_{q\le\sqrt{N}}\sum_{\substack{d|q\\ q'=q/d}} \frac{1}{d}\sum_{j'<q'}^{*} c_{j',q'}e_{q'}(j'x)$$

$$= 2 \sum_{q \le \sqrt{N}} \sum_{\substack{d|q \\ q'=q/d}} \frac{1}{q'} \sum_{j<d}{}^{*} c_{j,d} e_d(jx)$$

$$= 2 \sum_{d \le \sqrt{N}} \sum_{j<d}{}^{*} \Bigg(\sum_{q' \le \sqrt{N}/d} \frac{1}{q'} \Bigg) c_{j,d} e_d(jx) = \sum_{d \le \sqrt{N}} \alpha_d \sum_{j<d}{}^{*} c_{j,d} e_d(jx),$$

where $\alpha_d := \sum_{n \le \sqrt{N}/d} 2/n$, and in the same way

$$S_1^{\pm}(x) = \sum_{\sqrt{N} < q \le \sqrt{x}} \sum_{j<q} c_{j,q} e_q(jx) = \sum_{d \le \sqrt{2N}} \sum_{j<d}{}^{*} \alpha_{j,d}(x) c_{j,d} e_d(jx),$$

where

$$\alpha_{j,d}(x) := \sum_{\sqrt{N}/d < n \le \sqrt{x}/d} \frac{2}{n}.$$

Before applying the lemmas, we need to calculate (in the diagonals)

$$f^*(d) := \sum_{j<d}{}^{*} |c_{j,d}|^2 = \sum_{\substack{j<d \\ (j,d)=1}} |c_{j,d}|^2,$$

for which we have the trivial bound $\mathcal{O}(\min(1, h/d))$; we will accomplish this by Möbius inversion, since (by the already quoted properties of $c_{j,q}$)

$$2 \left\| \frac{h}{q} \right\| = \sum_{j<q} |c_{j,q}|^2 = \sum_{d|q} \frac{1}{d^2} \sum_{j'<q/d}{}^{*} |c_{j',q/d}|^2 = \left(\frac{1}{N^2} * f^* \right)(q),$$

where $\frac{1}{N^2}(n) := \frac{1}{n^2}$ is completely multiplicative, whence its (Dirichlet) inverse is $\frac{\mu}{N^2}(n) := \frac{\mu(n)}{n^2}$. This gives

$$f^*(d) = \sum_{j<d}{}^{*} |c_{j,d}|^2 = 2 \sum_{t|d} \frac{\mu(t)}{t^2} \left\| \frac{ht}{d} \right\|.$$

We will use this to calculate the diagonal of the mean square of first $S_0^{\pm}(x)$ and then $S_1^{\pm}(x)$.

The first terms on the right-hand side of both lemmas, 1 and 3, come from the diagonal parts, and the second terms from the off-diagonal parts. In fact, by Lemma 1, the mean-square diagonal of $S_0^{\pm}(x)$ is

$$N \sum_{d \le \sqrt{N}} \Bigg(\sum_{q' \le \sqrt{N}/d} \frac{2}{q'} \Bigg)^2 \sum_{j<d}{}^{*} |c_{j,d}|^2 = 2N \sum_{d \le \sqrt{N}} \alpha_d^2 \sum_{t|d} \frac{\mu(t)}{t^2} \left\| \frac{ht}{d} \right\|,$$

where by the elementary estimate (see [13, p. 6])

$$\sum_{n \le B} \frac{1}{n} = \log B + \mathcal{O}(1),$$

we have

$$\alpha_d := \sum_{n\le\sqrt{N}/d} \frac{2}{n} = 2\log\frac{\sqrt{N}}{d} + \mathcal{O}(1);$$

hence the diagonal is

$$8N\sum_{d\le\sqrt{N}} \log^2\frac{\sqrt{N}}{d}\sum_{t|d}\frac{\mu(t)}{t^2}\left\|\frac{ht}{d}\right\| + \mathcal{O}\Big(NL\sum_{d\le\sqrt{N}}\min\Big(1,\frac{h}{d}\Big)\Big),$$

by the above estimates for α_d and $f^*(d) \ll \min(1,h/d)$; if we exchange the sums, this is

$$8N\sum_{t\le\sqrt{N}}\frac{\mu(t)}{t^2}\sum_{k\le\sqrt{N}/t}\left\|\frac{h}{k}\right\|\log^2\frac{\sqrt{N}}{tk} + \mathcal{O}(NhL^2) = D(N,h) + \mathcal{O}(NhL^2).$$

Since this is the main term in Theorem 1 (and $\mathcal{O}(NhL^2)$ is a negligible remainder), we still have to show that all the other terms belong to the remainder $\mathcal{O}(NhL^{5/2}(\log L)^{1/2})$.

Let us evaluate the mean-square diagonal of $S_1^{\pm}(x)$; by Lemma 3 this is (using the above bounds and writing, as usual, $x\sim N$ for $N < x \le 2N$)

$$\begin{aligned}4\sum_{d\le\sqrt{2N}}\sum_{j<d}{}^{*}|c_{j,d}|^2\sum_{x\sim N}\Big|\sum_{\sqrt{N}/d<q'\le\sqrt{x}/d}\frac{1}{q'}\Big|^2 &\ll N\sum_{d\le\sqrt{2N}}\min\Big(1,\frac{h}{d}\Big)\\ &\ll NhL;\end{aligned}$$

this is acceptable, as also are the off-diagonal terms of $S_1^{\pm}(x)$:

$$\sum_{x\sim N}|S_1^{\pm}(x)|^2 \ll NhL + NL\sum_{d\le\sqrt{2N}}\min(1,h/d) \ll NhL^2.$$

(Here it is clear that loosing one logarithm off the diagonal, in Lemma 3, has no effect.)

Let us estimate the off-diagonal terms in the mean square of $S_0^{\pm}(x)$.

In this case, we need to distinguish “small” and “large” moduli:

$$\begin{aligned}S_0^{\pm}(x) &= \sum_{d\le\sqrt{N}/L}\alpha_d\sum_{j<d}{}^{*}c_{j,d}e_d(jx) + \sum_{\sqrt{N}/L<d\le\sqrt{N}}\alpha_d\sum_{j<d}{}^{*}c_{j,d}e_d(jx)\\ &= S_{0,1}^{\pm}(x) + S_{0,2}^{\pm}(x),\end{aligned}$$

whence, each sum being real, we get

$$|S_0^{\pm}(x)|^2 = (S_{0,1}^{\pm}(x))^2 + 2S_{0,1}^{\pm}(x)S_{0,2}^{\pm}(x) + (S_{0,2}^{\pm}(x))^2;$$

by Lemma 1 the off-diagonal terms of $S_{0,1}^{\pm}(x)$ give

$$\ll \frac{N}{L^2} L^2 \sum_{d \le \sqrt{N}/L} \min\left(1, \frac{h}{d}\right) \ll NhL,$$

while the contribution of those of $S_{0,2}^{\pm}(x)$ to $I(N,h)$ is

$$\ll NL^2 \sum_{\sqrt{N}/L < d \le \sqrt{N}} \frac{h}{d} \ll NhL^2 \log L;$$

by Cauchy's inequality the off-diagonal terms of $S_0^{\pm}(x)$ give

$$\ll NhL + \sqrt{NhL^3}\sqrt{NhL^2 \log L} + NhL^2 \log L \ll NhL^{5/2}\sqrt{\log L},$$

since we have the trivial estimate

$$D(N,h) \ll NL^2 \sum_{t \le \sqrt{N}} \frac{1}{t^2} \sum_{k \le 2h} 1 + NhL^2 \sum_{t \le \sqrt{N}} \frac{1}{t^2} \sum_{2h < k \le \sqrt{N}/t} \frac{1}{k} \ll NhL^3.$$

Finally, by Cauchy's inequality and since each sum is real (we ignore the previous remainders $\mathcal{O}(NhL)$):

$$|S^{\pm}(x)|^2 = (S_0^{\pm}(x))^2 + 2S_0^{\pm}(x)S_1^{\pm}(x) + (S_1^{\pm}(x))^2,$$

we get

$$I(N,h) = D(N,h) + \mathcal{O}(NhL^{5/2}(\log L)^{1/2}).$$

This concludes the proof of Theorem 1.

We now prove Corollary 1. We will show that

$$D(N,h) = \frac{16}{\pi^2} Nh \log^3 \frac{\sqrt{N}}{h} + \mathcal{O}(NhL^2).$$

We start by splitting the sum over k:

$$\begin{aligned} D(N,h) = 8N &\sum_{t \le \sqrt{N}} \frac{\mu(t)}{t^2} \sum_{k \le 2h} \left\| \frac{h}{k} \right\| \log^2 \frac{\sqrt{N}}{tk} \\ &+ 8Nh \sum_{t \le \sqrt{N}} \frac{\mu(t)}{t^2} \sum_{2h < k \le \sqrt{N}/t} \frac{1}{k} \log^2 \frac{\sqrt{N}}{tk} \\ = 8Nh &\sum_{t < \sqrt{N}/2h} \frac{\mu(t)}{t^2} \sum_{2h < k \le \sqrt{N}/t} \frac{1}{k} \left(\log^2 \frac{\sqrt{N}}{k} + \mathcal{O}(L \log t + (\log t)^2) \right) \\ &+ \mathcal{O}(NhL^2) \\ = 8Nh &\sum_{t < \sqrt{N}/2h} \frac{\mu(t)}{t^2} \sum_{2h < k \le \sqrt{N}/t} \frac{1}{k} \log^2 \frac{\sqrt{N}}{k} + \mathcal{O}(NhL^2), \end{aligned}$$

where the sum over t is non-empty (even if it can have only the term with $t=1$), by our hypothesis $h<\sqrt{N}/2$.

Thus it will suffice to prove that

$$\sum_{t<\sqrt{N}/2h}\frac{\mu(t)}{t^2}\sum_{2h<k\le\sqrt{N}/t}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}=\frac{2}{\pi^2}\log^3\frac{\sqrt{N}}{h}+\mathcal{O}(L^2).$$

By exchanging the sums, the left-hand side is

$$\sum_{2h<k\le\sqrt{N}}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}\sum_{t<\sqrt{N}/k}\frac{\mu(t)}{t^2}.$$

Using

$$\sum_{t\le T}\frac{\mu(t)}{t^2}=\frac{1}{\zeta(2)}+\mathcal{O}\left(\frac{1}{T}\right),$$

we get

$$\sum_{2h<k\le\sqrt{N}}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}\sum_{t<\sqrt{N}/k}\frac{\mu(t)}{t^2}=\frac{1}{\zeta(2)}\sum_{2h<k\le\sqrt{N}}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}+\mathcal{O}(L^2).$$

By partial summation, we have ($A<B$ are natural numbers)

$$\begin{aligned}\sum_{A<k\le B}\frac{1}{k}\log^2\frac{B}{k}&=\int_A^B\frac{[t]-A}{t^2}\log\frac{B}{t}\left(\log\frac{B}{t}+2\right)dt\\&=\int_A^B\frac{1}{t}\log^2\frac{B}{t}\,dt+\mathcal{O}(\log^2 B)\\&=\int_1^{B/A}\frac{\log^2 u}{u}\,du+\mathcal{O}(\log^2 B)\\&=\frac{\log^3(B/A)}{3}+\mathcal{O}(\log^2 B);\end{aligned}$$

letting $A=2h$ and $B=[\sqrt{N}]$, we get

$$\frac{1}{\zeta(2)}\sum_{2h<k\le\sqrt{N}}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}+\mathcal{O}(L^2)=\frac{1}{3\zeta(2)}\log^3\frac{[\sqrt{N}]}{2h}+\mathcal{O}(L^2);$$

since

$$\log^3\frac{[\sqrt{N}]}{2h}=\log^3\frac{\sqrt{N}}{2h}+\mathcal{O}(L^2)=\log^3\frac{\sqrt{N}}{h}+\mathcal{O}(L^2)$$

we finally obtain the required estimate:

$$\sum_{t<\sqrt{N}/2h}\frac{\mu(t)}{t^2}\sum_{2h<k\le\sqrt{N}/t}\frac{1}{k}\log^2\frac{\sqrt{N}}{k}=\frac{1}{3\zeta(2)}\log^3\frac{\sqrt{N}}{h}+\mathcal{O}(L^2)$$
$$=\frac{2}{\pi^2}\log^3\frac{\sqrt{N}}{h}+\mathcal{O}(L^2).$$

This proves Corollary 1.

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