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Edmund Hlawka (1916–2009)

by

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I. A short biography ([1]). Professor Edmund Hlawka, a leading figure of Austrian mathematics since the second world war, passed away on February 19, 2009, in his ninety-third year. He was the author of numerous important and influential research papers, but also was an enthusiastic teacher. His mathematical work and his lively lectures inspired many disciples and admirers in Vienna, in Austria and beyond.

Born on November 5, 1916 in Bruck an der Mur (in Styria, Austria), he soon moved with his parents to Vienna where he attended primary and secondary school. In a short account of his youth (see the article in [145]) he writes about a large table in the living room of his parents where he often worked at that time, and remarks that later on, after his apartment had been bombed during the war, he for a while had to work at kitchen tables. Although a very good student of mathematics at the Realgymnasium (a type of school between the more technically oriented Realschule and the humanistic Gymnasium), where he sometimes prepared the homework for all the other students, he first did not plan to study mathematics. He was more interested in chemistry. But formulas involving derivatives in chemistry books intrigued him so much that he turned to mathematics after all.

In 1934 he began his studies in mathematics, physics and astronomy at the University of Vienna. His Ph.D. thesis in 1938 was on a topic in diophantine approximation. In 1944 he proved a long standing conjecture of Minkowski in the Geometry of Numbers. This result, now known as the Minkowski–Hlawka Theorem, established his fame in the international mathematics community. In the same year he married Ms. Rosa Reiter, who later taught mathematics at secondary schools. After some low level positions, Hlawka in 1948 was named ordentlicher Professor (full Professor) at the University of Vienna, a position he held until 1981. From 1981 until his retirement in 1987 he was Professor at the Technical University of Vienna.

([1]) I am indebted to the article by Dr. Christa Binder in [145, pp. 233–240] where further details may be found.

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For many years he was an editor of Acta Arithmetica and of Monatshefte für Mathematik. During his long career he had more than 130 Doktoranden (Ph.D. students) some of whom became prominent mathematicians, and he conducted final exams for more than 800 future mathematics teachers at secondary schools. He thus had a strong impact on mathematics education in Austria.

Hlawka's great reputation led to many honors, as well as invitations to the Institute for Advanced Study in Princeton, to Cal Tech in California, to the Sorbonne in Paris, as well as to the ETH in Zürich. He was a member of the Austrian Academy of Sciences, as well as of several German Academies, and the Academy in Bologna. He received the Dannie–Heinemann Prize of the Göttinger Academy of Sciences, the Gauss medal of the DDR Academy of Sciences and many honors in Austria, such as the Ehrenzeichen (loosely translated as medal of honor) for the sciences and art by the Republic of Austria, and the Golden Ehrenzeichen for Merits for the Republic of Austria. He held honorary doctor degrees from the University of Vienna, as well as the universities of Salzburg, Graz, Erlangen and the Technical University of Vienna. He was an honorary member of the Austrian Mathematical Society as well as of the Schrödinger Society.

The enclosed short description of some of Hlawka's works and the list of publications attests to his large range of interests and accomplishments.

II. Some personal memories. As a first year student in 1951/52 I was deeply impressed by Hlawka's Vorlesung (lecture course) on differential and integral calculus. With him, the epsilon and delta arguments were enjoyable to learn. He always lectured without notes. Except that sometimes he wrote specific numbers for examples on the back side of streetcar tickets. He once told us with a grin that he could not proceed because the conductor had punched a hole in his ticket just where he had written such a number. He had a great sense of humor, especially a Viennese type of humor. Using a commonly known phrase, when he made a mistake at the blackboard he once said he was a "Kipfel" (the Austrian version of croissant). He clearly enjoyed teaching and had the habit to continue lecturing after he had left the room. He was admired by all the students. He was rather thin, looked frail, and we worried about his health. But he fooled us all: He outlived most of his more robust colleagues.

At first I had intended to study mathematical physics. So when Hlawka gave a course on the Geometry of Numbers I enrolled in a physics course. But on the side I studied notes of Hlawka's lectures, taken by my fellow student Hans Hejtmanek. I liked these lectures on the Geometry of Numbers so much that I decided to get my degree in mathematics, and eventually I got my Ph.D. under the direction of Hlawka. While I was still a student, he and his wife accepted me in their very informal mathematical circle, which involved numerous seminars and Nachsitzungen (after-sessions) in coffee houses. Always a very colloquial Viennese language was spoken. I felt very honored to be allowed to address him by his first name Edmund, and by “du” instead of the more formal “Sie”. There is much that I owe to Hlawka. He introduced me to the Geometry of Numbers and to Uniform Distribution, areas which have been important in my own work.

Hlawka’s life took a downturn after the death of his wife, and he was confined to his apartment during his last years. But I hardly ever heard him complain. So despite his old age, the news of his death came as a shock to me. With his decease, an era of Austrian mathematics has come to an end.

III. Mathematical works. Hlawka wrote more than 150 articles, as well as half a dozen books. It is impossible to do justice to all this work in a short article. Moreover, the choice of results highlighted here is no doubt subjective, colored by my interests and lack of expertise in certain areas. On the other hand, Hlawka himself in [114] gives a good survey of some of his work and its context.

1. Hlawka’s expertise in classical analysis was already demonstrated in his first publication [1] on Laguerre polynomials, and his paper [2] on linear differential equations of second order.

2. But he soon turned to number theory, in particular to **Diophantine Approximation**. His work [3] deals with approximation properties of two inhomogeneous linear forms in two variables. Suppose

$$F(x,y) = (\alpha x + \beta y - \xi)(\gamma x + \delta y - \eta) \tag{1}$$

is a product of two such forms with $\alpha\delta - \beta\gamma = 1$. Minkowski had shown that when all the coefficients $\alpha, \beta, \gamma, \delta, \xi, \eta$ are real, then there are rational integers x, y with $|F(x,y)| \le 1/4$. Hlawka now takes up the case when the coefficients are complex, and shows that there are Gaussian integers x, y with $|F(x,y)| \le 1/2$. Minkowski’s estimate is best possible for $F(x,y) = (x-1/2)(y-1/2)$, and Hlawka’s is best possible for $F(x,y) = (x-(i+1)/2)(y-(i+1)/2)$. His proof was rather difficult; further proofs and results for other number fields were given by Mahler, Chalk, Perron, Mordell and Niven. In the subsequent paper [4], Hlawka dealt with the inequality

$$|\beta y - x - \xi| < c/|y|. \tag{2}$$

When β, ξ are real, with β irrational and ξ not of the type $m\beta + n$ with integers m, n, then according to Minkowski there are infinitely many solutions of (2) with x, y in $\mathbb{Z}$, $y > 0$, provided $c = 1/4$. Hlawka now established an analogous result with β, ξ complex, with x, y Gaussian integers and $c = 1/2$. The proof that $c = 1/2$ is best possible requires new methods different from the real case. Hlawka also shows that when $\alpha, \beta, \gamma, \delta, \xi, \eta$ are complex and as in (1), with $\beta\gamma$ real and positive, then there are Gaussian integers x, y with

$$|\alpha x + \beta y - \xi| \leq |\alpha|/\sqrt{2}, \qquad |\gamma x + \delta y - \eta| \leq |\delta|/\sqrt{2}.$$

In the third paper [5] of the series, he turns to homogeneous linear forms and gives a new proof of Minkowski's results that for complex $\alpha, \beta, \gamma, \delta$ with $\alpha\delta - \beta\gamma = 1$ there are Gaussian integers x, y with $(x, y) \neq (0, 0)$ having

$$\max(|\alpha x + \beta y|^2, |\gamma x + \delta y|^2) \leq (1 + 1/\sqrt{3})/\sqrt{2}.$$

It is not known what corresponding results for three complex linear forms in three variables should be like, more precisely, what the best constants should be.

Let us now turn to approximation results [143], [148], [152] obtained much later in Hlawka's career. Triples (x, y, z) of rational integers are called *Pythagorean triples* if $x^2 + y^2 = z^2$. Triples without common divisor and $z > 0$ may be written as $(x, y, z) = (v^2 - u^2, 2uv, v^2 + u^2)$ with coprime nonzero integers u, v. Given such a triple, the pairs

$$\left(\frac{x}{z}, \frac{y}{z}\right) = \left(\frac{v^2 - u^2}{v^2 + u^2}, \frac{2uv}{v^2 + u^2}\right)$$

lie on the unit circle and will now be called *Pythagorean points*. Every rational point on the unit circle is such a point. Now Hlawka shows that when $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ are points on the unit circle, then for sufficiently large N there are integers $v, u_1, \ldots, u_n$ with $1 \leq v \leq N^n$ and $1 \leq u_j \leq v$ $(j = 1, \ldots, n)$, such that the Pythagorean points

$$(r_j, s_j) = \left(\frac{x_j}{z_j}, \frac{y_j}{z_j}\right) = \left(\frac{v^2 - u_j^2}{v^2 + u_j^2}, \frac{2u_j v}{v^2 + u_j^2}\right) \tag{3}$$

have

$$\max(|\alpha_j - r_j|, |\beta_j - s_j|) < 2/Nv \qquad (j = 1, \ldots, n).$$

Observe that this statement has some similarity to Dirichlet's Theorem on simultaneous approximation. The exponent n in N^n is best possible. The denominators of the points (3) are at most $2v^2 \leq 2N^{2n}$.

3. Soon after his initial work on approximation, Hlawka turned to the **Geometry of Numbers**. His proof in [6] of a long-standing conjecture of Minkowski, now called the Minkowski–Hlawka Theorem, made his fame. The theorem is as follows.

Suppose $\mathcal{S} \subset \mathbb{R}^n$ *is a symmetric star body, i.e., a set with nonempty interior such that with* $\mathbf{x} \in \mathcal{S}$ *also* $\lambda\mathbf{x} \in \mathcal{S}$ *for every* λ *with* $|\lambda| \leq 1$. *Then given* $\epsilon > 0$, *there is a lattice of determinant less than* $(2\zeta(n))^{-1}\,\mathrm{Vol}(\mathcal{S}) + \epsilon$ *having no point in* $\mathcal{S}$ *besides the origin.*

An essential tool in the proof is Hlawka's "Deformationssatz": *Let* $\chi(\mathbf{x})$ *be a bounded Riemann-integrable function with support on a bounded region of* $\mathbb{R}^n$. *Then given* $\epsilon > 0$ *there is a lattice* Λ *of determinant* 1 *such that*

$$\sum_{\mathbf{x}\in\Lambda\setminus\{\mathbf{0}\}} \chi(\mathbf{x}) < \int_{\mathbb{R}^n} \chi(\mathbf{x})\,d\mathbf{x} + \epsilon.$$

Like later proofs, Hlawka's argument involves taking a mean value over lattices. Siegel in fact proved the result without the ϵ and with $\leq$ in place of $<$. It is an immediate consequence of the Deformationssatz that for a bounded Jordan-measurable set $\mathcal{S} \subset \mathbb{R}^n$ there is a lattice Λ of determinant less than $\mathrm{Vol}(\mathcal{S}) + \epsilon$ with $\Lambda \cap \mathcal{S} = \{\mathbf{0}\}$ or $\emptyset$. The Minkowski–Hlawka Theorem follows by some inversion argument involving the Möbius μ-function. The factor $(2\zeta(n))^{-1}$ in the theorem has been somewhat improved by Rogers and Schmidt; see [a], Ch. VI.

The paper [6], which led to Hlawka's "Habilitation" (venia legendi) contains a further interesting theorem, concerning two alternatives:

Suppose $k_0 \in \mathbb{N}$ (*where* $\mathbb{N} = \mathbb{Z}_{>0}$) *and* $0 \leq r \leq k_0 - \lceil k/2 \rceil - 1$. *Let* $\mathcal{K} \subset \mathbb{R}^n$ *be a symmetric* (*about* $\mathbf{0}$) *convex body of volume at least* $2^{n-1}k_0$. *Then either there are* $k_0 - r$ *pairs* $\mathbf{g}, -\mathbf{g}$ *of nonzero integer points in* $\mathcal{K}$, *or for every* $\mathbf{x}_0 \in \mathbb{R}^n$ *there are at least* $r+1$ *integer points* $\mathbf{h}$ *with* $\mathbf{h} + \mathbf{x}_0 \in \mathcal{K}$.

Let us also mention [17], which contains the following. If $\mathbf{a}_1, \dots, \mathbf{a}_n$ are linearly independent in $\mathbb{R}^n$, and $V > 0$, then there is a parallelepiped $\mathcal{P} \subset \mathbb{R}^n$ of volume V, centered at $\mathbf{0}$ and with faces respectively perpendicular to $\mathbf{a}_1, \dots, \mathbf{a}_n$, which contains at most $A(n)$ pairs $\mathbf{g}, -\mathbf{g}$ of nonzero integer points. One may take $A(n) = \frac{1}{n}(n!)^2 2^{n(n-1)/2}$.

4. In the paper [16] on **packing and covering** we are given two convex sets $\mathcal{B}$, $\mathcal{K}$ in $\mathbb{R}^n$ with nonempty interiors and finite volumes. When $A \subset \mathbb{R}^n$ is of finite cardinality $|A|$, set

$$S(\mathcal{B}, \mathcal{K}, A) = |A| \,\mathrm{Vol}(\mathcal{K}) / \mathrm{Vol}(\mathcal{B}).$$

We call A a *packing set of* $\mathcal{K}$ *in* $\mathcal{B}$ if the translated sets

$$\mathcal{K} + \mathbf{a} \quad (\mathbf{a} \in A) \tag{4}$$

are mutually disjoint and contained in $\mathcal{B}$. We call it a packing set involving a lattice Γ if moreover $\mathbf{a} - \mathbf{a}' \in \Gamma$ for any $\mathbf{a}, \mathbf{a}'$ in A. The quantity

$$S^*(\mathcal{B}, \mathcal{K}) = \sup S(\mathcal{B}, \mathcal{K}, A),$$

with the supremum taken over all packing sets A of $\mathcal{K}$ in $\mathcal{B}$, is the *maximum packing density* of $\mathcal{B}$ in $\mathcal{K}$. One defines $S^*(\mathcal{B}, \mathcal{K}, \Gamma)$ in an analogous way, but with A restricted to packing sets involving Γ. We call A a *covering set* of $\mathcal{B}$ by $\mathcal{K}$ if the union of the sets (4) contains $\mathcal{B}$, and we put

$$\Sigma^*(\mathcal{B}, \mathcal{K}) = \inf S(\mathcal{B}, \mathcal{K}, A)$$

where the infimum is over covering sets A. Finally, $\Sigma^*(\mathcal{B},\mathcal{K},\Gamma)$ is defined in the obvious way. Clearly

$$(5)\qquad S^*(\mathcal{B},\mathcal{K},\Gamma)\le S^*(\mathcal{B},\mathcal{K})\le 1\le \Sigma^*(\mathcal{B},\mathcal{K})\le \Sigma^*(\mathcal{B},\mathcal{K},\Gamma).$$

Hlawka gives upper and lower bounds for these quantities. For instance, suppose $\rho<\beta$, and that there is a ball of radius ρ containing $\mathcal{K}$, and a ball of radius β contained in $\mathcal{B}$. Then

$$S^*(\mathcal{B},\mathcal{K},\Gamma)\ge (1-\rho/\beta)^n(\mu_1/2)^n$$

where μ_1 is the first minimum of $\mathcal{K}$ with respect to Γ (in the sense of Minkowski). A more complicated *upper* bound for $S^*(\mathcal{B},\mathcal{K},\Gamma)$, involving the mixed volumes of $\mathcal{B}$, is also given. Hlawka further studies the behavior of the quantities in (5) when $\mathcal{K}$ is replaced by $r\mathcal{K}$ (consisting of points $r\mathbf{x}$ with $\mathbf{x}\in\mathcal{K}$) and $r\to 0$.

Among further works on the Geometry of Numbers let me just mention [13] on lattice points in cylinders, [7], [12] on power sums of linear forms, [24] on the "Figurengitter", as well as [27], [28].

5. The long and difficult papers [18], [19] [23], [25] about **integrals on convex sets** are motivated by questions in the Geometry of Numbers. They display Hlawka's expertise on, and fondness for classical analysis. The main interest here is on

$$G_1(\boldsymbol{\ell})=\int_{\mathcal{B}}(n+i\boldsymbol{\ell}\mathbf{x})e^{i\boldsymbol{\ell}\mathbf{x}}\,d\mathbf{x},\qquad G(\boldsymbol{\ell})=\int_{\mathcal{B}}e^{i\boldsymbol{\ell}\mathbf{x}}\,d\mathbf{x}$$

where $\mathcal{B}$ is a symmetric convex body in $\mathbb{R}^n$, where $\boldsymbol{\ell}\in\mathbb{R}^n$ and $\boldsymbol{\ell}\mathbf{x}$ is the inner product of $\boldsymbol{\ell}$ and $\mathbf{x}$. When $\mathcal{B}$ is a ball, then there are known explicit formulas for $G_1(\boldsymbol{\ell})$ and $G(\boldsymbol{\ell})$ in terms of Bessel functions.

Hlawka now makes the assumption on $\mathcal{B}$ that its boundary $\partial(\mathcal{B})$ is very smooth. More precisely the distance function of $\mathcal{B}$, as well as the support function $H(\mathbf{u})$ (defined for $\mathbf{u}$ with Euclidean norm $|\mathbf{u}|=1$, and such that $\mathcal{B}$ consists of the points $\mathbf{x}$ with $\mathbf{xu}\le H(\mathbf{u})$ for each $\mathbf{u}$) have continuous partial derivatives up to order $6n$. Furthermore, if $K(\mathbf{x})$ for $\mathbf{x}\in\partial(\mathcal{B})$ is the product of the $n-1$ principal radii of curvature, it is assumed that $K(\mathbf{x})\ge\rho>0$ for every $\mathbf{x}$ in question. In the case when $\boldsymbol{\ell}=(\omega,0,\dots,0)$, Hlawka derives a main term as well as an error term for G_1 and G, with these terms depending on ω as well as on $H(\pm1,0,\dots,0),K(\pm1,0,\dots,0)$, and the main terms being of order $\omega^{-(m-1)/2},\omega^{-(m+1)/2}$, respectively.

When $\mathcal{B}$ is as above, Hlawka derives a number of applications. For instance, when $\mathcal{B}'$ is a translate of $\mathcal{B}$ and $\Phi(\mathcal{B}',u)$ the number of integer points in $u\mathcal{B}'$, then

$$\Phi(\mathcal{B}',u)=u^n\operatorname{Vol}(\mathcal{B})+O(u^{n(n-1)/(n+1)}).$$

For the ellipsoid this had been shown by Landau, and has since been improved.

It easily follows from standard arguments in the Geometry of Numbers that if $\mathcal{B}$ contains exactly S pairs $\mathbf{g}, -\mathbf{g}$ of integer points, then $\mathrm{Vol}(\mathcal{B}) \le 2^n(S+1)$. Under the condition on $\mathcal{B}$ stated above, Hlawka strengthens this by asserting that if S_t is the number of pairs $\mathbf{g}, -\mathbf{g}$ in the expanded region $t\mathcal{B}$, then for large t,

$$V(t\mathcal{B}) \le 2^{n-1}(S_t+1)(1+(1-\rho C t^{n-1}(S_t+1)^{-2})^{1/2})$$

where $C = C(n) > 0$.

Many more results are established.

6. In [30] Hlawka turns to **uniform distribution**, an area that held his interest for much of his later career. In fact he considers uniform distribution in a compact group G, with Haar measure normalized so that $\int_G d\mathbf{x} = 1$. Let $D_0(\mathbf{x}), D_1(\mathbf{x}), \dots$ be the unitary matrices corresponding to the classes of irreducible representations of G, with D_0 the trivial representation. A sequence $\omega = (\mathbf{x}_i)_{i\in\mathbb{N}}$ of elements of G is uniformly distributed if for every continuous function $f : G \to \mathbb{C}$ we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} f(\mathbf{x}_i) = \int_G f(\mathbf{x})\, d\mathbf{x}. \tag{6}$$

Weyl's Theorem holds in this general context, i.e., *the sequence is uniformly distributed precisely if for every* $j \ne 0$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} G_j(\mathbf{x}_i) = 0.$$

On the other hand, *the sequence is uniformly distributed if and only if* (6) *holds for the indicator function* $f_{\mathcal{B}}$ *of every closed set* $\mathcal{B} \subset G$.

A set S of sequences is called *uniformly uniformly distributed* (u.u.d.) if the limit relation (6) for continuous f holds uniformly for all the sequences in S. *When* $\omega = (\mathbf{x}_i)$ *is uniformly distributed, then the set* S *of sequences* $\omega_c = (c\mathbf{x}_i)$ *with* $c \in G$ *is u.u.d.* A single sequence $\omega = (\mathbf{x}_i)$ is called u.u.d. if the set S of sequences $\omega_h = (\mathbf{x}_{i+h-1})_{i\in\mathbb{N}}$ with $h \in \mathbb{N}$ is u.u.d. *If* $(\mathbf{x}_i)$ *is u.u.d. and* $\mathcal{B} \subset G$ *is measurable with* $\mu(\mathcal{B}) = \int_{\mathcal{B}} d\mathbf{x} > 0$, *then there is an* $N \subset \mathbb{N}$ *such that for every* h, *at least one of the elements* $\mathbf{x}_h, \mathbf{x}_{h+1}, \dots, \mathbf{x}_{h+N}$ *is in* $\mathcal{B}$.

It is further shown that *if* $\omega' = (\mathbf{y}_i)$ *is u.u.d. and* $\omega = (\mathbf{x}_i)$ *has*

$$\lim_{i\to\infty} \mathbf{y}_{i+1}^{-1}\mathbf{x}_{i+1}\mathbf{x}_i^{-1}\mathbf{y}_i = 1$$

(*the identity of* G), *then also* ω *is u.u.d.* In the special case of the additive group $G = \mathbb{R}/\mathbb{Z}$, suppose α is irrational and set $y_i = i\alpha$. Then (y_i) is u.u.d., and if $\omega = (x_i)$ has $x_{i+1} - x_i \to \alpha$ as $i \to \infty$, then $-y_{i+1} + x_{i+1} - x_i + y_i \to 0$, and therefore (x_i) is u.u.d., which had been shown by van der Corput.

Hlawka also points out that the “fundamental inequality” of van der Corput remains true for compact groups. As a consequence, he shows that if for a sequence $\omega = (\mathbf{x}_i)$, every sequence $\omega^h = (\mathbf{x}_i^{-1}\mathbf{x}_{i+h})_{i\in\mathbb{N}}$ with $h \in \mathbb{N}$ is uniformly distributed, then so is ω. Thus the property of a sequence to be uniformly distributed is “erblich” (inheritable): a property P is erblich if ω has P when ω^h has P for every $h \in \mathbb{N}$. This property is studied in some detail in [35].

Finally, it is shown that any compact group G has uniformly distributed sequences. In fact, in some sense “almost every” sequence is uniformly distributed.

The papers [31] about sequences in compact spaces, and [42], [49] on “rhythmic sequences” are loosely related to uniform distribution. In [31] we are given a compact space X with countable basis, and the object of study are limit measures (“Häufungsmaße”) of sequences ω on X. Let $C(X)$ be the space of continuous functions $f : X \to \mathbb{C}$, with norm $\|f\| = \sup_{\mathbf{x}\in X} |f(\mathbf{x})|$. Further, let $M(X)$ be the space of all Radon measures μ on X with the weak topology, so that $\mu_n \to \mu$ if $\mu_n(f) \to \mu(f)$ for every $f \in C(X)$.

Now if $\omega = (\mathbf{x}_i)$ is a sequence of elements of X, set

$$\lambda_N(\omega, f) = \frac{1}{N}\sum_{i=1}^{N} f(\mathbf{x}_i) \tag{7}$$

for $f \in C(X)$. A *limit measure* μ of ω is a measure $\mu \in M(X)$ such that there is a sequence $N_1 < N_2 < \cdots$ of natural numbers with

$$\lim_{t\to\infty} \lambda_{N_t}(\omega, f) = \mu(f) = \int_X f(\mathbf{x})\, d\mu(\mathbf{x})$$

for every $f \in C(X)$. It turns out that *every sequence ω has at least one limit measure.* Also, *the set of limit measures of a given sequence ω is closed.*

In [42], [49] the intriguing notion of rhythmic sequences, first considered by van der Corput for the circle group $\mathbb{R}/\mathbb{Z}$, is generalized to an arbitrary compact group G. The letter $\mathcal{U}$ will denote an open neighborhood of the identity of G. When $\omega = (\mathbf{x}_i)_{i\in\mathbb{Z}}$ (note that i runs through all of $\mathbb{Z}$ rather than $\mathbb{N}$), and when $\ell \in \mathbb{Z}$, $N \in \mathbb{N}$, let $\omega(\ell, N)$ be the finite sequence $\mathbf{x}_\ell, \mathbf{x}_{\ell+1}, \ldots, \mathbf{x}_{\ell+N-1}$. A sequence $\omega(m, N)$ will be said to be a *repetition of* $\omega(\ell, N)$ *up to* $\mathcal{U}$ if

$$\mathbf{x}_{m+k}\mathbf{x}_{\ell+k}^{-1} \in \mathcal{U} \quad \text{for } 0 \le k < N.$$

A sequence ω is called *rhythmic* if for every ℓ, N and every $\mathcal{U}$ there is an L such that in every interval of length L there is an m such that $\omega(m, N)$ repeats $\omega(\ell, N)$ up to $\mathcal{U}$. Among the results let us just mention that *if $f : G \to G$ is continuous and $\omega = (\mathbf{x}_i)$ is rhythmic, then so is $\omega^* = (f(\mathbf{x}_i))$.*

7. We now return to uniform distribution, in fact **uniform distribution on the torus group** $T^n = (\mathbb{R}/\mathbb{Z})^n$. We will be interested in quantitative results.

When $\mathbf{x}_1, \ldots, \mathbf{x}_N$ is a finite sequence in T^n and $\mathcal{B} \subset T^n$ a *box*, i.e., the image in T^n of a box in $\mathbb{R}^n$ with sides parallel to the coordinate axes, set

$$\varDelta_{\mathcal{B}}(N) = \left| \frac{1}{N} z(\mathcal{B}) - \mu(\mathcal{B}) \right|$$

where $z(\mathcal{B})$ is the number of subscripts i with $\mathbf{x}_i \in \mathcal{B}$ and $\mu(\mathcal{B})$ the measure of $\mathcal{B}$. The *discrepancy* $\varDelta(N)$ is defined to be the supremum of $\varDelta_{\mathcal{B}}(N)$ over all boxes $\mathcal{B}$. When $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is an infinite sequence, define $\varDelta(N)$ in terms of the finite sequence $x_1, \ldots, x_N$. It is well known that $\mathbf{x}_1, \mathbf{x}_2, \ldots$ is uniformly distributed precisely if $\varDelta(N) \to 0$ as $N \to \infty$.

For a function $f : T^n \to \mathbb{C}$, Hlawka now employs a concept of total variation $V(f)$ in the sense of Hardy and Krause, which generalizes the total variation of functions $T \to \mathbb{C}$. In [37] it is shown that when f is of finite total variation and $\mathbf{x}_1, \ldots, \mathbf{x}_N$ are in T^n, then

$$\left| \lambda_N(f, \omega) - \int f(\mathbf{x})\, d\mathbf{x} \right| \le V(f) \varDelta(N), \tag{8}$$

which generalizes the one-dimensional case due to Koksma. For a somewhat related estimate, see [56].

It is well known that a sequence $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in T^n is uniformly distributed precisely if for each $\mathbf{t} \in \mathbb{Z}^n$, $\mathbf{t} \neq \mathbf{0}$, the sequence $\mathbf{t}\mathbf{x}_1, \mathbf{t}\mathbf{x}_2, \ldots$ is uniformly distributed in T. Defining $\varDelta_{\mathbf{t}}(N)$ in terms of the latter sequence, Hlawka estimates $\varDelta(N)$ in terms of the $\varDelta_{\mathbf{t}}(N)$ with $\mathbf{t}$ primitive (i.e., with coprime coordinates), thus giving a quantitative version of the above statement.

It had been shown by Rademacher that when $(s_n) = (\sigma_n + it_n)$ is the sequence of zeros of the Riemann zeta function in the critical strip with $0 < t_1 < t_2 < \cdots$, then assuming the Riemann hypothesis, the sequence $\alpha t_1, \alpha t_2, \ldots$ is uniformly distributed modulo 1 (i.e., its image in $\mathbb{R}/\mathbb{Z}$ is uniformly distributed) for every $\alpha \neq 0$. Hlawka in [67], using a result of Selberg, shows that this holds without the above assumption, and gives a generalization.

8. Let us turn to **uniform distribution on the sphere** and on products of spheres. Suppose $m > 1$, $n = m + 1$ and $S^m \subset \mathbb{R}^n$ is the sphere given by $x_1^2 + \cdots + x_n^2 = 1$. A basic goal of the long and difficult papers [86], [111] is to estimate

$$\epsilon_N(\omega, f) = |\lambda_N(\omega, f) - \mu(f)|$$

for sequences ω on S^m and functions f defined on S^m. Here λ_N is defined as in (7), and $\mu(f) = \int f(\mathbf{x})\, d\mu(\mathbf{x})$ where μ is the surface measure on S^m normalized so that $\mu(S^m) = 1$. Variations on the Koksma–Hlawka inequality (8) are given, depending on conditions on ω and f. For instance one may restrict f to satisfy a Lipschitz condition, to have suitably defined total variation or to be the indicator function of a closed set, e.g., of a spherical cap. The arguments depend on skillful handling of analysis. Uniform distribution on products of spheres, and the orthogonal and unitary groups are also considered.

The paper [134] is easier and describes a simple and explicit map $\Phi : U^m \to S^m$ where $U^m \subset \mathbb{R}^m$ is the unit cube. It has the property that if a finite sequence $\omega = (\mathbf{x}_1, \ldots, \mathbf{x}_N)$ in $\mathbb{R}^m$ has discrepancy $\Delta(N)$, and $\Phi(\omega) = (\Phi(\mathbf{x}_1), \ldots, \Phi(\mathbf{x}_N))$ with elements in S^m, then

$$\epsilon_N(\phi(\omega), f) \le c(f)\Delta(N)^{1/n}$$

when $f : S^m \to \mathbb{C}$ satisfies a Lipschitz condition.

9. Now let us consider **applications of uniform distribution** to numerical analysis and physics. In view of what was said above, e.g. concerning (8), uniformly distributed sequences can be used to approximate integrals. With functions defined on $\mathbb{R}^n$ where n is large, there are problems in practice with this approach. Hlawka's method via good lattice points in [40], [47] is helpful here. Given $n \in \mathbb{N}$ and a prime p, he gives a definition of a *good lattice point* $\mathbf{g} \in \mathbb{Z}^n$ with respect to p, and shows such points always exist. *If* $\mathbf{g}$ *is such a point and* $f : \mathbb{R}^n \to \mathbb{R}$ *is periodic, more precisely if* $f(\mathbf{x} + \mathbf{m}) = f(\mathbf{x})$ *for* $\mathbf{m} \in \mathbb{Z}^n$, *and of total variation* $V(f)$, *then*

$$\left| \frac{1}{p} \sum_{k=0}^{p-1} f((k/p)\mathbf{g}) - \int_{U^n} f(\mathbf{x})\, d\mathbf{x} \right| < V(f)(10^3 \log p)^{2n}/p,$$

where U^n is the unit cube $0 \le x_i < 1$ $(i = 1, \ldots, n)$. Better estimates can be given if f possesses partial derivatives to a high order.

For other numerical applications see [66], [68], [85], also [44], [47] on integral equations, [91] on linear difference equations, [105] on the calculus of variation, [107] on differential equations, etc.

Let us now turn to the impressive paper [62] on models for the kinetic theory of gases. Hlawka begins with a historical and philosophical overview of the subject. Then the specific question of the behavior of N particles in a cube with reflecting walls is taken up. There is no loss of generality in assuming that the sides of the cube are parallel to the coordinate axes, in fact that the cube is the set $U^n_{1/2}$ of points $\mathbf{x}$ with $0 \le x_i < 1/2$ $(i = 1, \ldots, n)$. Suppose the particles have initial positions $\mathbf{u}_1, \ldots, \mathbf{u}_N$, and velocities $\mathbf{v}_1, \ldots, \mathbf{v}_N$, and are reflected at the walls of the cube. Setting $\widetilde{\mathbf{x}}_i(t) = \mathbf{u}_i + \mathbf{t}\mathbf{v}_i$, it is easily seen that the position of the ith particle at time t is

$$\mathbf{x}_i(t) = \min(\{\widetilde{\mathbf{x}}_i(t)\}, \{-\widetilde{\mathbf{x}}_i(t)\}) \quad (i = 1, \ldots, N)$$

where $\{(\xi_1, \ldots, \xi_n)\} = (\{\xi_1\}, \ldots, \{\xi_n\})$ and $\{\cdots\}$ denotes the fractional part. Thus the distribution in $U^n_{1/2}$ of the N points at time t is determined by the distribution of the N points $\widetilde{\mathbf{x}}_i(t)$ modulo 1. The discrepancy $\Delta(N)$ of N points in $U^n_{1/2}$ is now defined in an obvious way, and the goal is to show that the discrepancy $\Delta(N,t)$ of $\mathbf{x}_1(t),\dots,\mathbf{x}_N(t)$ is small for most of the time t.

Let us assume that the convex hull Ω of $\mathbf{v}_1,\dots,\mathbf{v}_N$ contains inner points. Hlawka now defines a certain discrepancy $\widehat{\Delta}$ of these velocities. It is always ≤ 1, and often very small. For $\alpha > 0$ and $T > 0$, let $\mathcal{W}_\alpha(T)$ be the set of t with $|t| \le T$ having $\Delta(N,t) \ge \alpha$. Then the measure $\mu(\mathcal{W}_\alpha(T))$ of $\mathcal{W}_\alpha(T)$ has

$$\frac{\mu(\mathcal{W}_\alpha(T))}{T} \le \frac{c(n)}{\alpha^2}\left(\frac{\phi(\Omega)}{T} + \widehat{\Delta}((\log \widehat{\Delta}^{-1})^{2n} + 1)\right),$$

where $\phi(\Omega)$ depends on Ω only. Thus when $\widehat{\Delta}$ is small, then $\Delta(N,t)$ will be small for most of the time t. A variation, with a different definition of $\widehat{\Delta}$, is given in [141].

10. There are also a number of papers of a more **general nature**, such as [81] on linguistics, [87] on the notion of number, [123] on entropy, [137] on statistics, as well as the papers [62], [149], and Hlawka's article 1.1 in [145], on physics, as well as quite a few obituaries.

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