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Shimura's mass formula for an orthogonal group over real quadratic fields

by

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Introduction. Let V be an n-dimensional row vector space over a totally real algebraic number field F, and let φ be a totally positive definite symmetric matrix with entries in F. Put $\varphi[x] = x\varphi \cdot {}^t x$ for $x \in V$. By a $\mathfrak{g}$-*maximal lattice* L with respect to φ, we understand a $\mathfrak{g}$-lattice L in V which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. Here $\mathfrak{g}$ is the ring of integers of F. Put $G^{\varphi} = \{\alpha \in \mathrm{GL}_n(F) \mid \alpha\varphi \cdot {}^t\alpha = \varphi\}$ and let $\{L_i\}_{i=1}^{k}$ be representatives of classes in the genus of L with respect to G^{φ}. Then we define the *mass of the genus of a* $\mathfrak{g}$-*maximal lattice* L *with respect to* φ by

$$\mathfrak{m}(L) = \sum_{i=1}^{k} [\Gamma_i : 1]^{-1},$$

where $\Gamma_i = \{\gamma \in G^{\varphi} \mid L_i\gamma = L_i\}$. Shimura's mass formula for orthogonal groups determines $\mathfrak{m}(L)$ for an arbitrary totally real algebraic number field F and arbitrary φ ([6, Theorem 5.8]).

In this paper, we consider only the case where F is a real quadratic field $\mathbb{Q}(\sqrt{m})$, and φ is the unit matrix 1_n of size n $(n > 1)$. The purpose of this paper is to digest Shimura's mass formula, applying it to the case of real quadratic fields, and to state the formula in a simpler form, from which we can compute $\mathfrak{m}(L)$. Applying this formula to the case where $F = \mathbb{Q}(\sqrt{5})$ and $\varphi = 1_4$, we see that the genus belonging to a $\mathfrak{g}$-maximal lattice L with respect to φ consists of one class, and further give the number $N(L, h) = \#\{x \in L \mid \varphi[x] = h\}$ for a totally positive element h in $\mathfrak{g}$ by specializing the formula due to Shimura [7, Theorem 1.5] to the present situation (Section 4). At the end of this paper, we give a numerical table of $\mathfrak{m}(L)$ for several quadratic fields F (Section 5).

Applying the formula in [6, Theorem 5.8] to our case, we can reduce the calculation of the mass to the following two arguments. One is to compute the special values of the Dedekind zeta function of F and the L-function

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of F with the Hecke character of F corresponding to $F(\sqrt{-1})/F$ (Section 2). These values can be obtained by calculating the values of the Riemann zeta function and Dirichlet L-functions, since $F(\sqrt{-1})/\mathbb{Q}$ is an abelian extension. The other is to determine a Witt decomposition for 1_n over the local field F_v at a nonarchimedean prime v of F (Section 3). To do this, we first take an anisotropic part θ_p of a Witt decomposition for 1_n over $\mathbb{Q}_p$ for the rational prime $v \mid p$. Then we decompose θ_p on F_v. In particular, nontrivial cases are $v \mid 2$ and $n \equiv \pm 3,\ 4 \pmod 8$. In these cases, we can determine a Witt decomposition for θ_p over F_v by using the local theory of quaternion algebras. Summing up these results, we obtain the mass $\mathfrak{m}(L)$ of the orthogonal group G^φ of $\varphi = 1_n$ over real quadratic fields F (Theorem 3.6).

We end this introduction with the following remark: As was mentioned in [6], maximal lattices were introduced by Eichler. This *maximal* lattice differs from a *unimodular* lattice in general. Shimura gave the mass formula for the case of maximal lattices with respect to an arbitrary φ exactly in [6, Theorem 5.8]. We use this formula to compute the mass in our case. Clear-cut explanation why we work on maximal lattices is given in the introduction of [6] and also in that of [7].

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Notation. If R is an associative ring with identity element and if M is an R-module, then we write $R^\times$ for the group of all invertible elements of R, and M_n^m for the R-module of $m \times n$-matrices with entries in M. We write 1_n for the identity element of the matrix ring R_n^n. We put $\mathrm{GL}_n(R) = (R_n^n)^\times$. We denote by ${}^t x$ and $\det(x)$ the transpose and determinant of a matrix $x \in R_n^n$. If $x_1, \ldots, x_r$ are square matrices, $\mathrm{diag}[x_1, \ldots, x_r]$ denotes the matrix with $x_1, \ldots, x_r$ in the diagonal blocks and 0 in all other blocks. For a finite set X, we denote by $\#X$ the number of elements in X. For a symmetric matrix $A \in R_n^n$, we put

$$A(x, y) = xA \cdot {}^t y \quad \text{and} \quad A[x] = xA \cdot {}^t x \qquad (x, y \in R_n^1).$$

Let F be a number field. We denote by $\mathbf{a}$ and $\mathbf{h}$ the sets of archimedean primes and nonarchimedean primes of F. We identify v with the prime ideal of F corresponding to v. For $v \in \mathbf{h}$, π_v and q_v denote a prime element of F_v and the norm of the prime ideal at v. We denote by B_l and $B_{l,\psi}$ the lth Bernoulli number and lth generalized Bernoulli number associated with a Dirichlet character ψ. For a rational prime p, $\left(\frac{}{p}\right)$ denotes the Legendre symbol.

1. Preliminaries. Throughout the paper, F is a real quadratic field and $\varphi = 1_n$. Let $\mathfrak{g}$ be the ring of integers of F and put $V = F_n^1$. We put

$$G = G^\varphi = \{\gamma \in \mathrm{GL}_n(F) \mid \gamma\varphi \cdot {}^t\gamma = \varphi\} = \{\gamma \in \mathrm{GL}_n(F) \mid \gamma \cdot {}^t\gamma = 1_n\}.$$

For a $\mathfrak{g}$-*lattice* L in V, which is a finitely generated $\mathfrak{g}$-submodule in V containing a basis of V, and $h \in \mathfrak{g}$, we put

$$N(L,h) = \#\{x \in L \mid \varphi[x] = h\},$$
$$\widetilde{L} = \{y \in V \mid 2\varphi(x,y) \in \mathfrak{g} \text{ for every } x \in L\}.$$

Let $G_{\mathbf{A}}$ be the adelization of G.

For $\alpha \in G_{\mathbf{A}}$, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in V such that $(L\alpha)_v = L_v\alpha_v$ for any nonarchimedean prime v of F. Here L_v is the $\mathfrak{g}_v$-lattice in $(F_v)_n^1$ which is spanned by L over $\mathfrak{g}_v$. We call $\{L\alpha \mid \alpha \in G_{\mathbf{A}}\}$ (resp. $\{L\alpha \mid \alpha \in G\}$) the *genus* (resp. *class*) of L with respect to G. We put

$$C = \{\alpha \in G_{\mathbf{A}} \mid L\alpha = L\}, \qquad \Gamma^\alpha = G \cap \alpha C \alpha^{-1} \quad (\alpha \in G_{\mathbf{A}}).$$

The mapping $G\alpha C \mapsto L\alpha^{-1}$ gives a bijection of $G\backslash G_{\mathbf{A}}/C$ onto the set of classes in the genus of L. It is known that $G\backslash G_{\mathbf{A}}/C$ is a finite set ([5, Lemma 8.7(4)]). Thus the genus of L is decomposed into a disjoint union of finitely many classes. Let $\mathfrak{B}$ be a complete set of representatives for $G\backslash G_{\mathbf{A}}/C$. We then put

$$\mathfrak{m}(L) = \mathfrak{m}(G,C) = \sum_{\alpha \in \mathfrak{B}} [\Gamma^\alpha : 1]^{-1},$$
$$R(L,h) = \sum_{\alpha \in \mathfrak{B}} [\Gamma^\alpha : 1]^{-1} N(L\alpha^{-1}, h).$$

We call $\mathfrak{m}(G,C)$ the *mass of G relative to C*.

By a $\mathfrak{g}$-*maximal* lattice L with respect to φ, we understand a $\mathfrak{g}$-lattice L in V which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. The genus of a $\mathfrak{g}$-maximal lattice L consists of all $\mathfrak{g}$-maximal lattices (cf. [5, Lemma 5.9]).

Let $v \in \mathbf{a} \cup \mathbf{h}$. For symmetric matrices A and B in $\mathrm{GL}_n(F_v)$, we say that A is *equivalent* to B over F_v if there exists $X \in \mathrm{GL}_n(F_v)$ such that $XA \cdot {}^tX = B$. We call A *isotropic* over F_v if there exists $0 \neq x \in (F_v)_n^1$ such that $xA \cdot {}^tx = 0$. Otherwise we say that A is *anisotropic* over F_v. Put

$$\eta_r = \begin{bmatrix} 0 & 1_r \\ 1_r & 0 \end{bmatrix}.$$

Then η_r is a symmetric matrix of $\mathrm{GL}_{2r}(F_v)$. For every $x \in (F_v)_r^1$, we have $\eta_r\big[[x\ 0_r^1]\big] = 0$, and so η_r is isotropic over F_v.

For φ and $v \in \mathbf{a} \cup \mathbf{h}$ there exists $\alpha_v \in \mathrm{GL}_n(F_v)$ such that

$$\alpha_v \varphi \cdot {}^t\alpha_v = \begin{bmatrix} 0 & 0 & 1_{r_v} \\ 0 & \theta_v & 0 \\ 1_{r_v} & 0 & 0 \end{bmatrix} \tag{1.1}$$

with an anisotropic symmetric matrix $\theta_v \in \mathrm{GL}_{t_v}(F_v)$ over F_v. We call (1.1) a *Witt decomposition* for φ over F_v. Then $n = 2r_v + t_v$. It is known that $t_v \leq 4$ for $v \in \mathbf{h}$ (cf. [5, Proposition 5.2]). In Section 3, we determine a Witt decomposition for φ over F_v ($v \in \mathbf{h}$).

2. Mass formula for $\varphi = 1_n$ over real quadratic fields. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. The formula of $\mathfrak{m}(L)$ for G is given in [6, Theorem 5.8]. In fact, we can apply this formula to the case of the maximal lattices with respect to an arbitrary totally definite quadratic form over an arbitrary totally real algebraic number field.

We first define a constant λ_v in [6, Theorem 5.8]. Let $\mathfrak{e}$ be the product of all nonarchimedean primes v satisfying $\widetilde{L}_v \neq L_v$. For $v \mid \mathfrak{e}$, put

$$\lambda_v = \begin{cases} 1 & \text{if } t_v = 1, \\ 2^{-1}(1+q_v)^{-1}(1+q_v^{1-u})(1+q_v^{-u}) & \\ \qquad\qquad \text{if } t_v = 2,\ \mathfrak{d}_v = \mathfrak{r}_v, \text{ and } c_v \in \pi_v\mathfrak{g}_v^{\times}, & \\ 2^{-1} & \text{if } t_v = 2,\ \mathfrak{d}_v \neq \mathfrak{r}_v, \\ 2^{-1}(1+q_v)^{-1}(1-q_v^{1-n}) & \text{if } t_v = 3, \\ 2^{-1}(1+q_v)^{-1}(1-q_v^{1-u})(1-q_v^{-u}) & \text{if } t_v = 4, \end{cases} \tag{2.1}$$

where $u = 2^{-1}n$, and if $t_v = 2$, then $\mathfrak{r}_v$ is the maximal order of $K_v = F_v(\sqrt{-\det(\theta_v)})$, $\mathfrak{d}_v$ is the different of K_v relative to F_v, and c_v is a constant from [6, §3.2]. Here c_v can be taken as follows: Let θ_v be as in (1.1). This is equivalent to a matrix $\mathrm{diag}[a_v, b_v]$ over F_v with a suitable basis $\{e_i\}$ such that $a_v = \theta_v[e_1]$, $b_v = \theta_v[e_2] \in F_v^{\times}$ and $\theta_v(e_1, e_2) = 0$. We may assume that $b_v \in \mathfrak{g}_v^{\times} \cup \pi_v^{-1}\mathfrak{g}_v^{\times}$. The mapping $y_1e_1 + y_2e_2 \mapsto b_v y_2 + y_1\sqrt{-a_v b_v}$ yields an isomorphism $(F_v)_2^1 \cong K_v$ such that

$$\theta_v[y_1e_1 + y_2e_2] = b_v^{-1} \cdot N_{K_v/F_v}(b_v y_2 + y_1\sqrt{-a_v b_v}).$$

Here $N_{K_v/F_v}(x) = xx^{\varrho}$, where ϱ is the nontrivial automorphism of K_v over F_v. Then we can take $c_v = b_v^{-1}$. Notice that $v \nmid \mathfrak{e}$ if $t_v = 0$, or $t_v = 2$, $\mathfrak{d}_v = \mathfrak{r}_v$ and $c_v \in \mathfrak{g}_v^{\times}$ (see (3.6) below).

Applying [6, Theorem 5.8] to the case where F is a real quadratic field and $\varphi = 1_n$, we have the following

PROPOSITION 2.1. *Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer m, and let $n > 1$. Put $V = F_n^1$ and $\varphi = 1_n$. Let L be a $\mathfrak{g}$-maximal lattice in V*

with respect to φ, $\mathfrak{e}$ *the product of all nonarchimedean primes* v *satisfying* $\widetilde{L}_v \neq L_v$, *and* χ *the Dirichlet character corresponding to* $F/\mathbb{Q}$.

(1) *Suppose* n *is odd; let* $\mathfrak{f}$ *be the product of all nonarchimedean primes* v *satisfying* $t_v = 3$. *Then*

$$\mathfrak{m}(L) = 2^{1-n}\Big(\prod_{k=1}^{(n-1)/2}(4k)^{-2}B_{2k}B_{2k,\chi}\Big)[\widetilde{L}:L]^{(n-1)/2}\prod_{v|\mathfrak{f}}\frac{1-q_v^{1-n}}{2(1+q_v)}. \tag{2.2}$$

(2) *Suppose* n *is even; put* $F' = \mathbb{Q}(\sqrt{-1})$ *and* $F'' = \mathbb{Q}(\sqrt{-m})$, *and denote by* χ' *(resp.* χ''*) the Dirichlet character corresponding to* $F'/\mathbb{Q}$ *(resp.* $F''/\mathbb{Q}$*). Let* λ_v *be as in* (2.1).

(i) *If* $n \equiv 0 \pmod 4$, *then*

$$\begin{aligned}\mathfrak{m}(L) = n^{-2}B_{n/2}B_{n/2,\chi}&\Big(\prod_{k=1}^{[(n-1)/2]}(4k)^{-2}B_{2k}B_{2k,\chi}\Big)\\ &\times[\widetilde{L}:L]^{(n-1)/2}\prod_{v|\mathfrak{e}}\lambda_v.\end{aligned} \tag{2.3}$$

(ii) *If* $n \equiv 2 \pmod 4$, *then*

$$\begin{aligned}\mathfrak{m}(L) = n^{-2}&B_{n/2,\chi'}B_{n/2,\chi''}\\ &\times\Big(\prod_{k=1}^{[(n-1)/2]}(4k)^{-2}B_{2k}B_{2k,\chi}\Big)[\widetilde{L}:L]^{(n-1)/2}\\ &\times\prod_{v|\mathfrak{e}}\lambda_v\cdot\begin{cases}2^{2(1-n)} & \text{if } m\equiv 1 \pmod 4,\\ 2^{1-n} & \text{if } m \equiv 2 \pmod 4,\\ 1 & \text{if } m\equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{2.4}$$

Proof. By [6, Lemma 5.6(1)], we have

$$\mathfrak{m}(G,C) = \frac{1}{2}\,\mathfrak{m}(G_+,C^+),$$

where

$$G_+ = \{\gamma \in G \mid \det(\gamma) = 1\}$$

and $C^+ = C \cap (G_+)_{\mathbf{A}}$ with the adelization $(G_+)_{\mathbf{A}}$ of G_+. Taking $\mathfrak{g}$ as an ideal $\mathfrak{c}$ of F in [6, Theorem 5.8], we have $C^+ = D^+$, where D^+ is a subgroup of $(G_+)_{\mathbf{A}}$ in that theorem (cf. [6, (5.7.3), (5.7.4), and (5.7.5)]). Since φ is anisotropic over F and satisfies the condition [6, (5.7.1)], we can apply [6, Theorem 5.8] to $\mathfrak{m}(G_+,C^+)$. Hence we obtain

$$\mathfrak{m}(G_+,C^+) = \mathfrak{m}_n(\mathfrak{g})[\widetilde{L}:L]^{\mu}\prod_{v|\mathfrak{e}}\lambda_v,$$

where

$$\mathfrak{m}_n(\mathfrak{g}) = 2D_F^{[\mu^2]} \prod_{k=1}^{[\mu]} \{D_F^{1/2}((2k-1)!(2\pi)^{-2k})^2 \zeta_F(2k)\}$$
$$\times \begin{cases} 2^{-2\mu} & \text{if } n \text{ is odd,} \\ D_F^{1/2}((u-1)!(2\pi)^{-u})^2 L(u, \psi_{K/F}) & \text{if } n \text{ is even.} \end{cases}$$

Here $\mu = (n-1)/2$, and D_F is the discriminant of F; if n is even, put $u = n/2$, $K = F(\sqrt{(-1)^u})$, and denote by $\psi_{K/F}$ the Hecke character of F corresponding to K/F.

Suppose n is odd. Then we have $D_F^{[\mu^2]} = \prod_{k=1}^{[\mu]} D_F^{2k-1}$, and hence

$$\mathfrak{m}_n(\mathfrak{g}) = 2^{-(n-2)} \prod_{k=1}^{[\mu]} D_F^{2k-1/2}((2k-1)!(2\pi)^{-2k})^2 \zeta_F(2k). \tag{2.5}$$

Put $\widehat{\zeta}_F(s) = \pi^{-s}\Gamma(s/2)^2\zeta_F(s)$, where $\Gamma(s)$ is the gamma function ($s \in \mathbb{C}$). By the functional equation $\widehat{\zeta}_F(s) = |D_F|^{1/2-s}\widehat{\zeta}_F(1-s)$, we have

$$\zeta_F(2k) = |D_F|^{1/2-2k}\pi^{4k}\left(\frac{2^{2k-1}}{(2k-1)!}\right)^2 \zeta_F(1-2k).$$

Thus

$$D_F^{2k-1/2}((2k-1)!(2\pi)^{-2k})^2\zeta_F(2k) = 2^{-2}\zeta_F(1-2k).$$

Since

$$\begin{aligned} \zeta_F(s) &= \zeta_{\mathbb{Q}}(s)L(s,\chi), \\ \zeta_{\mathbb{Q}}(1-2k) &= -B_{2k}/2k, \\ L(1-2k,\chi) &= -B_{2k,\chi}/2k, \end{aligned}$$

we have

$$D_F^{2k-1/2}((2k-1)!(2\pi)^{-2k})^2\zeta_F(2k) = (4k)^{-2}B_{2k}B_{2k,\chi}.$$

Substituting this into (2.5) gives

$$\mathfrak{m}_n(\mathfrak{g}) = 2^{-(n-2)} \prod_{k=1}^{[\mu]} (4k)^{-2}B_{2k}B_{2k,\chi},$$

and hence we obtain (2.2).

Suppose n is even. Since $D_F^{[\mu^2]} = D_F^{u-1}\prod_{k=1}^{[\mu]} D_F^{2k-1}$, we have

$$\mathfrak{m}_n(\mathfrak{g}) = 2D_F^{\mu}((u-1)!(2\pi)^{-u})^2 L(u,\psi_{K/F}) \prod_{k=1}^{[\mu]} (4k)^{-2}B_{2k}B_{2k,\chi}.$$

Then

$$\mathfrak{m}(L) = D_F^{\mu}((u-1)!(2\pi)^{-u})^2 L(u,\psi_{K/F}) \times \Big(\prod_{k=1}^{[\mu]}(4k)^{-2}B_{2k}B_{2k,\chi}\Big)[\widetilde{L}:L]^{\mu}\prod_{v|\mathfrak{e}}\lambda_v. \tag{2.6}$$

Next, we compute the value $L(u,\psi_{K/F})$. If $n \equiv 0 \pmod 4$, then $K=F$, and so $L(u,\psi_{K/F}) = \zeta_F(u)$. Since this value can be obtained as above, we easily deduce (2.3) from (2.6). Let $n \equiv 2 \pmod 4$. Then K is a totally imaginary quadratic field $F(\sqrt{-1}) = \mathbb{Q}(\sqrt{m},\sqrt{-1})$ over F. In this case, the functional equation is $R(s,\psi_{K/F}) = R(1-s,\psi_{K/F})$ by [5, Theorems A6.2, A6.3], where

$$R(s,\psi_{K/F}) = |D_F N(D_{K/F})|^{s/2}(\pi^{-(s+1)/2}\Gamma((s+1)/2))^2 L(s,\psi_{K/F}).$$

From this,

$$L(s,\psi_{K/F}) = |D_F N(D_{K/F})|^{(1-2s)/2}\pi^{2s-1}\left(\frac{\Gamma((2-s)/2)}{\Gamma((s+1)/2)}\right)^2 L(1-s,\psi_{K/F}).$$

Now, since $K/\mathbb{Q}$, $F/\mathbb{Q}$, and K/F are abelian extensions, we have

$$\begin{aligned}\zeta_K(s) &= \zeta_{\mathbb{Q}}(s)L(s,\chi)L(s,\chi')L(s,\chi''),\\ \zeta_F(s) &= \zeta_{\mathbb{Q}}(s)L(s,\chi),\\ \zeta_K(s) &= \zeta_F(s)L(s,\psi_{K/F}).\end{aligned}$$

Hence we obtain

$$L(s,\psi_{K/F}) = \zeta_K(s)\zeta_F(s)^{-1} = L(s,\chi')L(s,\chi''). \tag{2.7}$$

It is known that $L(1-k,\omega) = -B_{k,\omega}/k$ for a positive odd integer k with $\omega = \chi'$ or χ''. Combining this with (2.7), we find that $L(1-u,\psi_{K/F}) = u^{-2}B_{u,\chi'}B_{u,\chi''}$. Observing

$$\frac{\Gamma((2-u)/2)}{\Gamma((u+1)/2)} = \frac{(-1)^{(u-1)/2}2^{u-1}\pi^{1/2}}{(u-1)!},$$

we see that

$$D_F^{\mu}((u-1)!(2\pi)^{-u})^2 L(u,\psi_{K/F}) = n^{-2}N(D_{K/F})^{-\mu}B_{u,\chi'}B_{u,\chi''}.$$

It is known that $D_K = D_F D_{F'} D_{F''}$. Since $D_K = N(D_{K/F})D_F^2$, we have

$$\begin{aligned} N(D_{K/F}) &= D_F^{-1}D_{F'}D_{F''} = 4D_F^{-1}D_{F''}\\ &= \begin{cases} 2^4 & \text{if } m \equiv 1 \pmod 4,\\ 2^2 & \text{if } m \equiv 2 \pmod 4,\\ 1 & \text{if } m \equiv 3 \pmod 4.\end{cases}\end{aligned} \tag{2.8}$$

Substituting these into (2.6), we obtain (2.4). ∎

3. Computation of the mass

3.1. *Local Witt decompositions for $\varphi = 1_n$.* Throughout this section, L is a $\mathfrak{g}$-maximal lattice in V with respect to φ. To compute $\mathfrak{m}(L)$, we need to obtain $\mathfrak{e}$, $\mathfrak{f}$, $[\widetilde{L} : L]$, and λ_v of Proposition 2.1. First of all, we determine a Witt decomposition for $\varphi = 1_n$ at each nonarchimedean prime v of F (Lemma 3.3). For this purpose, we use a quaternion algebra for $v \mid 2$ and $n \equiv \pm 3, 4 \pmod 8$. By Wedderburn's theorem, every quaternion algebra over F_v is isomorphic to the matrix algebra $(F_v)_2^2$ or a division algebra. For a quaternion algebra B over F and $v \in \mathbf{a} \cup \mathbf{h}$, we say that B is *ramified* at v if $B \otimes_F F_v$ is a division algebra, and that B is *unramified* at v if $B \otimes_F F_v$ is isomorphic to $(F_v)_2^2$. It is well known that B is ramified at v if and only if the reduced norm of $B \otimes_F F_v$ is anisotropic over F_v, and that the reduced norm of the matrix algebra $(F_v)_2^2$ is equivalent to the isotropic symmetric matrix η_2.

LEMMA 3.1. *Let p be a rational prime. Then we can take an anisotropic symmetric matrix θ_p of a Witt decomposition for φ over $\mathbb{Q}_p$ of the following form:*

(1) *If* $p \equiv 1 \pmod 4$, *then*

$$\theta_p = \begin{cases} \emptyset & \textit{if } n \textit{ is even}, \\ 1 & \textit{if } n \textit{ is odd}. \end{cases}$$

(2) *If* $p \equiv 3 \pmod 4$, *then*

$$\theta_p = \begin{cases} \emptyset & \textit{if } n \equiv 0 \pmod 4, \\ \pm 1 & \textit{if } n \equiv \pm 1 \pmod 4, \\ 1_2 & \textit{if } n \equiv 2 \pmod 4. \end{cases}$$

(3) *If* $p = 2$, *then*

$$\theta_p = \begin{cases} \emptyset & \textit{if } n \equiv 0 \pmod 8, \\ \pm 1 & \textit{if } n \equiv \pm 1 \pmod 8, \\ \pm 1_2 & \textit{if } n \equiv \pm 2 \pmod 8, \\ \pm 1_3 & \textit{if } n \equiv \pm 3 \pmod 8, \\ 1_4 & \textit{if } n \equiv 4 \pmod 8. \end{cases}$$

Here $\theta_p = \emptyset$ means that $t_p = 0$, i.e. φ is equivalent to $\eta_{n/2}$ over $\mathbb{Q}_p$, and "$\theta_p = \pm 1$ if $n \equiv \pm 1 \pmod 4$" means that $\theta_p = 1$ (resp. $\theta_p = -1$) if $n \equiv 1 \pmod 4$ (resp. $n \equiv -1 \pmod 4$). The other double signs should be read similarly.

This fact can be seen by [6, Examples 5.16] or [8, (1.2)]. We remark that the constant c_p in [8, (1.2)] is determined by the determinant of the decomposition.

We next give a Witt decomposition for φ over F_v for a totally real number field F, though we are interested in real quadratic ones. To do this, we define the Hilbert symbol $(a,b)_{F_v}$ over F_v by

$$(a,b)_{F_v} = \begin{cases} 1 & \text{if } \mathrm{diag}[a,b,-1] \text{ is isotropic over } F_v, \\ -1 & \text{otherwise,} \end{cases} \tag{3.1}$$

for $a,b \in F_v^\times$. We can verify that $(a,b)_{F_v} = 1$ if and only if there exists $0 \neq [x\ y] \in (F_v)^1_2$ such that $ax^2 + by^2 = 1$.

LEMMA 3.2. *Let F be a totally real number field. Let $v \in \mathbf{h}$, and let p be the rational prime which lies below v. Then we can take an anisotropic symmetric matrix θ_v of a Witt decomposition for φ over F_v of the following form:*

(1) *Suppose $v \nmid 2$.*

(i) *If $p \equiv 1 \pmod 4$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases}$$

(ii) *If $p \equiv 3 \pmod 4$ and $\sqrt{-1} \notin F_v$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod 4, \\ \pm 1 & \text{if } n \equiv \pm 1 \pmod 4, \\ 1_2 & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

(iii) *If $p \equiv 3 \pmod 4$ and $\sqrt{-1} \in F_v$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases}$$

(2) *Suppose $v \mid 2$.*

(i) *If $n \equiv 0 \pmod 8$, then $\theta_v = \emptyset$.*

(ii) *If $n \equiv \pm 1 \pmod 8$, then $\theta_v = \pm 1$.*

(iii) *If $n \equiv \pm 2 \pmod 8$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } \sqrt{-1} \in F_v, \\ \pm 1_2 & \text{if } \sqrt{-1} \notin F_v. \end{cases}$$

(iv) *If $n \equiv \pm 3 \pmod 8$, then*

$$\theta_v = \begin{cases} \pm 1_3 & \text{if } (-1,-1)_{F_v} = -1, \\ \mp 1 & \text{if } (-1,-1)_{F_v} = 1. \end{cases}$$

(v) *If $n \equiv 4 \pmod 8$, then*

$$\theta_v = \begin{cases} 1_4 & \text{if } (-1,-1)_{F_v} = -1, \\ \emptyset & \text{if } (-1,-1)_{F_v} = 1. \end{cases}$$

Here $\theta_v = \emptyset$ means that $t_v = 0$.

Proof. (1) Suppose $v \nmid 2$. Let θ_p be as in Lemma 3.1. If $p \equiv 1 \pmod 4$, then we can take $\theta_v = \theta_p$, and thus we have (i).

To prove (ii) and (iii), let $p \equiv 3 \pmod 4$. We show only the case of $n \equiv 2 \pmod 4$, since the other cases are trivial. In this case,

$$1_2 \text{ is isotropic over } F_v \ \Leftrightarrow\ \sqrt{-1} \in F_v, \tag{3.2}$$

which proves (ii) and (iii).

(2) Suppose $v \mid 2$. If $n \equiv 0$, ± 1, or $\pm 2 \pmod 8$, then we can determine a decomposition for φ over F_v in the same way as in (1), which proves (i)–(iii).

To prove (iv) and (v), we take the quaternion algebra B_0 of [6, Examples 5.16] which is only ramified at 2 and infinity: $B_0 = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$, where $a^2 = b^2 = -1$ and $ba = -ab$. Then the reduced norm of B_0 is equivalent to 1_4 over $\mathbb{Q}$. Put $B = B_0 \otimes_{\mathbb{Q}} F$. Then it can be verified that

$$\begin{aligned}(-1,-1)_{F_v} = 1 \ &\Leftrightarrow\ -1 \in N_{F_v(\sqrt{-1})/F_v}(F_v(\sqrt{-1})^{\times})\\ &\Leftrightarrow\ \text{the reduced norm } N_{B/F} \text{ of } B \text{ is isotropic over } F_v,\end{aligned}$$

which leads to (v). Furthermore, set $W = \{x \in B \mid \mathrm{Tr}_{B/F}(x) = 0\}$ and let ψ be the restriction of $N_{B/F}$ to W. Then ψ is equivalent to 1_3 over F, and it can be verified that $N_{B/F}$ is isotropic over F_v if and only if ψ is isotropic over F_v, which proves (iv). ■

Returning to real quadratic fields, we have

LEMMA 3.3. *Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer m. Let $v \in \mathbf{h}$, and let p be the rational prime which lies below v. Then we can take an anisotropic symmetric matrix θ_v of a Witt decomposition for φ over F_v of the following form*:

(1) *Suppose $v \nmid 2$.*

(i) *If $p \equiv 1 \pmod 4$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even},\\ 1 & \text{if } n \text{ is odd}.\end{cases}$$

(ii) *If $p \equiv 3 \pmod 4$ and $\left(\frac{D_F}{p}\right) \neq -1$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod 4,\\ \pm 1 & \text{if } n \equiv \pm 1 \pmod 4,\\ 1_2 & \text{if } n \equiv 2 \pmod 4.\end{cases}$$

(iii) *If $p \equiv 3 \pmod 4$ and $\left(\frac{D_F}{p}\right) = -1$, then*

$$\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even},\\ 1 & \text{if } n \text{ is odd}.\end{cases}$$

(2) *Suppose $v \mid 2$.*

(i) *If* $n \equiv 0 \pmod 8$, *then* $\theta_v = \emptyset$.

(ii) *If* $n \equiv \pm 1 \pmod 8$, *then* $\theta_v = \pm 1$.

(iii) *If* $n \equiv \pm 2 \pmod 8$, *then*

$$\theta_v = \begin{cases} \emptyset & \textit{if } m \equiv -1 \pmod 8, \\ \pm 1_2 & \textit{if } m \not\equiv -1 \pmod 8. \end{cases}$$

(iv) *If* $n \equiv \pm 3 \pmod 8$, *then*

$$\theta_v = \begin{cases} \pm 1_3 & \textit{if } m \equiv 1 \pmod 8, \\ \mp 1 & \textit{if } m \not\equiv 1 \pmod 8. \end{cases}$$

(v) *If* $n \equiv 4 \pmod 8$, *then*

$$\theta_v = \begin{cases} 1_4 & \textit{if } m \equiv 1 \pmod 8, \\ \emptyset & \textit{if } m \not\equiv 1 \pmod 8. \end{cases}$$

Here $\theta_v = \emptyset$ *means that* $t_v = 0$.

Proof. First, we note that $\sqrt{d} \in \mathbb{Q}_p$ if and only if p splits in the quadratic field $\mathbb{Q}(\sqrt{d})$. We see that if $p \not\equiv 1 \pmod 4$, that is, $\sqrt{-1} \notin \mathbb{Q}_p$, then

$$\sqrt{-1} \in F_v \ \Leftrightarrow\ \sqrt{-m} \in \mathbb{Q}_p. \tag{3.3}$$

To show this equivalence, we may assume that $[F_v : \mathbb{Q}_p] = 2$. Let $\sqrt{-1} \in F_v$. As $\sqrt{m} \in F_v$, there exist $x, y \in \mathbb{Q}_p$ such that $\sqrt{-m} = x + y\sqrt{m}$. Thus $-m = x^2 + my^2 + 2xy\sqrt{m}$. If $x = 0$, then $\sqrt{-1} \in \mathbb{Q}_p$. This contradicts $\sqrt{-1} \notin \mathbb{Q}_p$. Hence $y = 0$ and $x^2 = -m$. The converse is trivial.

(1) Suppose $v \nmid 2$. Let θ_p be as in Lemma 3.1. If $p \equiv 1 \pmod 4$, then we can take $\theta_v = \theta_p$, and thus we have (i).

To prove (ii) and (iii), let $p \equiv 3 \pmod 4$. We show only the case of $n \equiv 2 \pmod 4$, since the other cases are trivial. In this case, $\theta_p = 1_2$. If p splits in F, then $\sqrt{-1} \notin \mathbb{Q}_p = F_v$. If p does not split in F, then

$$p \text{ remains prime in } F \ \Leftrightarrow\ \left(\frac{-m}{p}\right) = -\left(\frac{D_F}{p}\right) = 1 \ \Leftrightarrow\ \sqrt{-m} \in \mathbb{Q}_p,$$

since $D_F = m$ or $4m$, $\left(\frac{2}{p}\right) \neq 0$, and $\left(\frac{-1}{p}\right) = -1$. Thus, by (3.3) and (3.2), if p splits or ramifies in F, then 1_2 is anisotropic over F_v. If p remains prime in F, then 1_2 is isotropic, and equivalent to η_1 over F_v by

$$\alpha 1_2 \cdot {}^t\alpha = \eta_1 \quad \text{with} \quad \alpha = \begin{bmatrix} 1 & \sqrt{-1} \\ 1/2 & -\sqrt{-1}/2 \end{bmatrix}.$$

(2) Suppose $v \mid 2$. We note that $\sqrt{-1} \notin \mathbb{Q}_2$. By (3.3) and (3.2), we know that $m \equiv -1 \pmod 8$ if and only if $\sqrt{-1} \in F_v$. In this case, using the same method as in case (1)(iii), we can find a decomposition for φ over F_v, since 1 is equivalent to -1 over F_v.

We assume $m \not\equiv -1 \pmod 8$ until the end of this proof. If $n \equiv 0,\ \pm 1$, or $\pm 2 \pmod 8$, then we can take the decomposition for φ over F_v as in Lemma 3.2 because of (3.3), which proves (i)–(iii) for $m \not\equiv -1 \pmod 8$.

To prove (v), though we have given a decomposition for 1_4 by using the Hilbert symbol in Lemma 3.2, we now determine it in another way. We again take B_0 and B as in Lemma 3.2. Since $B_0 \otimes_{\mathbb{Q}} \mathbb{Q}_2$ is a division algebra, 1_4 is anisotropic over $\mathbb{Q}_2$. By the local theory of quaternion algebras over algebraic number fields, a quaternion algebra over a nonarchimedean local field splits over an arbitrary quadratic field over the local field (cf. [1, VII, §2, Satz 4]). From this fact, we obtain

$$\begin{aligned} B \otimes_F F_v &\cong (B_0 \otimes_{\mathbb{Q}} \mathbb{Q}_2) \otimes_{\mathbb{Q}_2} F_v \\ &\cong \begin{cases} B_0 \otimes_{\mathbb{Q}} \mathbb{Q}_2 & \text{if 2 splits in } F, \\ (F_v)_2^2 & \text{if 2 ramifies or remains prime in } F. \end{cases} \end{aligned}$$

Therefore we have

$$\begin{cases} 1_4 \text{ is anisotropic over } F_v & \text{if } m \equiv 1 \pmod 8, \\ 1_4 \text{ is equivalent to } \eta_2 \text{ over } F_v & \text{if } m \not\equiv 1 \pmod 8). \end{cases}$$

Next, we prove the case $n \equiv 3 \pmod 8$; the proof of the case $n \equiv -3 \pmod 8$ can be obtained in the same way. Then we have

$$\begin{cases} 1_3 \text{ is anisotropic over } F_v & \text{if } m \equiv 1 \pmod 8, \\ 1_3 \text{ is equivalent to } \mathrm{diag}[\eta_1, -1] \text{ over } F_v & \text{if } m \not\equiv 1 \pmod 8). \end{cases}$$

To show this, we first assume that $m \equiv 1 \pmod 8$. Since 1_4 is anisotropic over F_v, so is 1_3. Next assume $m \not\equiv 1 \pmod 8$. Then there exists an element β_v in $\mathrm{GL}_4(F_v)$ such that $\beta_v 1_4 \cdot {}^t\beta_v = \eta_2$. We identify $(F_v)_3^1$ with the subspace of $(F_v)_4^1$ by the mapping $[x_1\, x_2\, x_3] \mapsto [x_1\, x_2\, x_3\, 0]$. Put $H = (F_v)_4^1 \beta_v^{-1}$ and $W = (F_v)_3^1 \beta_v^{-1}$. Now $W \cap (F_v)_2^1 \neq \{0\}$ in H. Hence there exists $0 \neq y \in (F_v)_3^1$ such that $y\beta_v^{-1} \in W \cap (F_v)_2^1$. Then $1_3[y] = 1_4[y] = \eta_2[y\beta_v^{-1}] = 0$, and so 1_3 is isotropic over F_v. Put $y = [y_1\, y_2\, y_3]$. Then $y_1^2 + y_2^2 + y_3^2 = 0$. We may assume that $y_2 \neq 0$. Then we have $N_{F_v(\sqrt{-1})/F_v}(z) = -1$ with $z = y_1\sqrt{-1}(y_2 + y_3\sqrt{-1})^{-1} \in F_v(\sqrt{-1})$. From this, 1_2 is equivalent to -1_2 over F_v. Thus $1_3 = \mathrm{diag}[1, 1_2]$ is equivalent to $\mathrm{diag}[\eta_1, -1]$, which is the desired decomposition. ∎

3.2. *The relative discriminant $D_{K/F}$.* Put $K = F(\sqrt{-1})$. We give the relative discriminant $D_{K/F}$ and the relative different $\mathfrak{d}_{K/F}$ of K/F, which is needed in the proof of Lemma 3.4:

$$D_{K/F} = \begin{cases} 4\mathfrak{g} & \text{if } m \equiv 1 \pmod 4, \\ 2\mathfrak{g} & \text{if } m \equiv 2 \pmod 4, \\ \mathfrak{g} & \text{if } m \equiv 3 \pmod 4, \end{cases} \tag{3.4}$$

$$\mathfrak{d}_{K/F} = \begin{cases} \prod_v v\mathfrak{r} & \text{if } m \equiv 1, 2 \pmod 4, \\ \mathfrak{r} & \text{if } m \equiv 3 \pmod 4, \end{cases} \tag{3.5}$$

where $\mathfrak{r}$ is the ring of integers of K, and the product runs through primes v in F dividing 2. To show these, we first determine $D_{K/F}$. We see that $v \mid D_{K/F}$ only if $v \mid 2$ by (2.8). If $m \equiv 5 \pmod 8$, then $D_{K/F} = 2^2\mathfrak{g}$, since 2 remains prime in F and (2.8) holds. If $m \equiv 2 \pmod 4$, then $D_{K/F} = 2\mathfrak{g}$, since 2 ramifies in F and we have (2.8). If $m \equiv 3 \pmod 4$, then $D_{K/F} = \mathfrak{g}$ by (2.8). Let $m \equiv 1 \pmod 8$. Then 2 splits in F. For every $\sigma \in \mathrm{Gal}(K/\mathbb{Q})$, we have $\mathfrak{d}_{K/F}^\sigma = \mathfrak{d}_{K/\mathbb{Q}}^\sigma/(\mathfrak{d}_{F/\mathbb{Q}}\mathfrak{r})^\sigma = \mathfrak{d}_{K/\mathbb{Q}}/\mathfrak{d}_{F/\mathbb{Q}}\mathfrak{r} = \mathfrak{d}_{K/F}$, since $K/\mathbb{Q}$ is an abelian extension. Thus $D_{K/F}^\sigma = D_{K/F}$, which implies $D_{K/F} = 2^2\mathfrak{g}$ by (2.8).

Next we prove (3.5). The case $m \equiv 3 \pmod 4$ is clear. If $m \not\equiv 3 \pmod 4$, then v is ramified in K for any $v \mid 2$ from (3.4) and Dedekind's discriminant theorem, that is, there exists a prime w in K such that $v\mathfrak{r} = w^2$. If $m \equiv 1 \pmod 8$, then $D_{K/F} = N_{K/F}(\mathfrak{d}_{K/F}) = 2^2\mathfrak{g} = \prod_v v^2$ by (3.4), and hence we have $\mathfrak{d}_{K/F} = \prod_{v|2}\prod_{w|v} w^2 = \prod_v v\mathfrak{r}$. The other cases can be shown in the same way.

3.3. *The index* $[\widetilde{L}_v : L_v]$. The index $[\widetilde{L}_v : L_v]$ for $v \in \mathbf{h}$ is given by [6, (3.2.1)]:

$$[\widetilde{L}_v : L_v] = \begin{cases} 1 & \text{if } t_v = 0, \\ [\mathfrak{g}_v : 2\mathfrak{g}_v] & \text{if } t_v = 1 \text{ and } v \mid 2, \\ 1 & \text{if } t_v = 1 \text{ and } v \nmid 2, \\ q_v^2 & \text{if } t_v = 2,\ \mathfrak{d}_v = \mathfrak{r}_v, \text{ and } c_v \in \pi_v\mathfrak{g}_v^\times, \\ 1 & \text{if } t_v = 2,\ \mathfrak{d}_v = \mathfrak{r}_v, \text{ and } c_v \in \mathfrak{g}_v^\times, \\ [\mathfrak{r}_v : \mathfrak{d}_v] & \text{if } t_v = 2 \text{ and } \mathfrak{d}_v \neq \mathfrak{r}_v, \\ [\mathfrak{g}_v : 2\mathfrak{g}_v]q_v^2 & \text{if } t_v = 3, \\ q_v^2 & \text{if } t_v = 4. \end{cases} \tag{3.6}$$

Here $\mathfrak{r}_v$, $\mathfrak{d}_v$, and c_v are as in (2.1), and $\widetilde{L}_v = \{y \in (F_v)_n^1 \mid 2\varphi(x, y) \in \mathfrak{g}_v$ for every $x \in L_v\}$. Using this result, we have the following

LEMMA 3.4. *Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Let $v \in \mathbf{h}$.*

(1) *Suppose $v \nmid 2$. Then $[\widetilde{L}_v : L_v] = 1$.*
(2) *Suppose $v \mid 2$.*

 (i) *If $n \equiv 0 \pmod 8$, then $[\widetilde{L}_v : L_v] = 1$.*
 (ii) *If $n \equiv \pm 1 \pmod 8$, then*

$$[\widetilde{L}_v : L_v] = \begin{cases} 2 & \text{if } m \equiv 1 \pmod 8, \\ 2^2 & \text{otherwise.} \end{cases}$$

(iii) *If* $n \equiv \pm 2 \pmod 8$, *then*

$$[\widetilde{L}_v : L_v] = \begin{cases} 1 & \textit{if } m \equiv 3 \pmod 4, \\ 2^4 & \textit{if } m \equiv -3 \pmod 8, \\ 2^2 & \textit{otherwise.} \end{cases}$$

(iv) *If* $n \equiv \pm 3 \pmod 8$, *then*

$$[\widetilde{L}_v : L_v] = \begin{cases} 2^3 & \textit{if } m \equiv 1 \pmod 8, \\ 2^2 & \textit{otherwise.} \end{cases}$$

(v) *If* $n \equiv 4 \pmod 8$, *then*

$$[\widetilde{L}_v : L_v] = \begin{cases} 2^2 & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. (1) There exists a $\mathfrak{g}$-maximal lattice M in V with respect to φ including $\mathfrak{g}_n^1$. Then $\mathfrak{g}_n^1 \subset M \subset \widetilde{M} \subset \widetilde{\mathfrak{g}_n^1}$. Since $\widetilde{\mathfrak{g}_n^1} = 2^{-1}\mathfrak{g}_n^1$, we have $\widetilde{(\mathfrak{g}_v)_n^1} = (\mathfrak{g}_v)_n^1$ for every $v \nmid 2$, so that $\widetilde{M_v} = M_v$. Since M is in the genus of L, there exists $\alpha \in G_{\mathbf{A}}$ such that $L\alpha = M$. Thus we have $L_v\alpha_v = M_v$. Hence $[\widetilde{L}_v : L_v] = [\widetilde{M}_v : M_v] = 1$ if $v \nmid 2$.

(2) Suppose $v \,|\, 2$. (i) If $n \equiv 0 \pmod 8$, then 1_n is equivalent to $\eta_{n/2}$ over F_v, that is, $t_v = 0$ for every $v \in \mathbf{h}$ by Lemma 3.3. Thus the assertion (i) can be easily obtained by (3.6).

(ii) If $n \equiv \pm 1 \pmod 8$, then $t_v = 1$ for every $v \in \mathbf{h}$ by Lemma 3.3. Thus, by (3.6), we have

$$\begin{aligned} [\widetilde{L}_v : L_v] &= [\mathfrak{g}_v : 2\mathfrak{g}_v] \\ &= \begin{cases} [\mathfrak{g} : v] & \text{if 2 splits in } F, \\ [\mathfrak{g} : 2\mathfrak{g}] & \text{if 2 remains prime in } F, \\ [\mathfrak{g} : v]^2 & \text{if 2 ramifies in } F \end{cases} \\ &= \begin{cases} 2 & \text{if } m \equiv 1 \pmod 8, \\ 2^2 & \text{otherwise.} \end{cases} \end{aligned}$$

Thus we have (ii).

(iii) If $m \equiv -1 \pmod 8$, then $t_v = 0$ by Lemma 3.3, and so $[\widetilde{L}_v : L_v] = 1$. If $m \not\equiv -1 \pmod 8$, then $t_v = 2$ by Lemma 3.3. In this case, by Dedekind's discriminant theorem and (3.4), $F_v(\sqrt{-1})/F_v$ is a ramified (resp. an unramified) quadratic extension if $m \equiv 1, 2 \pmod 4$ (resp. $m \equiv 3 \pmod 8$). Let $m \equiv 3 \pmod 8$. Then $t_v = 2$ and $\mathfrak{d}_v = \mathfrak{r}_v$. Since we can take $\theta_v = \pm 1_2$ in Lemma 3.3, we have $c_v = b_v^{-1} = \pm 1 \in \mathfrak{g}_v^\times$ as shown below (2.1). Thus $[\widetilde{L}_v : L_v] = 1$ by (3.6). If $m \equiv 1, 2 \pmod 4$, then $\mathfrak{d}_v = \pi_{K_v}^2 \mathfrak{r}_v$ by (3.5). Here π_{K_v} is a prime element of $K_v = F_v(\sqrt{-1})$. Thus by (3.6), we have $[\widetilde{L}_v : L_v] = [\mathfrak{r}_v : \mathfrak{d}_v] = [\mathfrak{r}_v : \pi_{K_v}^2 \mathfrak{r}_v]$. Since v ramifies in K, we have

$\mathfrak{r}_v/\pi_{K_v}^2\mathfrak{r}_v \cong \mathfrak{g}_v/\pi_v\mathfrak{g}_v$, and hence $[\mathfrak{r}_v : \pi_{K_v}^2\mathfrak{r}_v] = q_v^2$. Therefore

$$[\widetilde{L}_v : L_v] = q_v^2 = \begin{cases} 2^2 & \text{if } m \equiv 1, \pm 2 \pmod 8, \\ 2^4 & \text{if } m \equiv -3 \pmod 8). \end{cases}$$

(iv) Let $n \equiv \pm 3 \pmod 8$. Combining Lemma 3.3 with (3.6), we have

$$[\widetilde{L}_v : L_v] = \begin{cases} [\mathfrak{g}_v : 2\mathfrak{g}_v]q_v^2 & \text{if } m \equiv 1 \pmod 8, \\ [\mathfrak{g}_v : 2\mathfrak{g}_v] & \text{if } m \not\equiv 1 \pmod 8, \end{cases}$$

and hence we get (iv).

(v) Let $n \equiv 4 \pmod 8$. Then

$$[\widetilde{L}_v : L_v] = \begin{cases} q_v^2 & \text{if } m \equiv 1 \pmod 8, \\ 1 & \text{if } m \not\equiv 1 \pmod 8) \end{cases}$$

by Lemma 3.3 and (3.6), which proves (v). ∎

We note that $[\widetilde{L}_v : L_v] \neq 1$ if and only if $v \mid D_{K/F}$ when $n \equiv \pm 2 \pmod 8$.

It is well known that $[\widetilde{L} : L] = \prod_{v \in \mathbf{h}}[\widetilde{L}_v : L_v]$. Combining this with Lemma 3.4, we have the following

LEMMA 3.5. *Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ.*

(1) *If* $n \equiv 0 \pmod 8$, *then* $[\widetilde{L} : L] = 1$.
(2) *If* $n \equiv \pm 1 \pmod 8$, *then* $[\widetilde{L} : L] = 2^2$.
(3) *If* $n \equiv \pm 2 \pmod 8$, *then*

$$[\widetilde{L} : L] = \begin{cases} 1 & \textit{if } m \equiv 3 \pmod 4, \\ 2^4 & \textit{if } m \equiv -3 \pmod 8, \\ 2^2 & \textit{otherwise.} \end{cases}$$

(4) *If* $n \equiv \pm 3 \pmod 8$, *then*

$$[\widetilde{L} : L] = \begin{cases} 2^6 & \textit{if } m \equiv 1 \pmod 8, \\ 2^2 & \textit{otherwise.} \end{cases}$$

(5) *If* $n \equiv 4 \pmod 8$, *then*

$$[\widetilde{L} : L] = \begin{cases} 2^4 & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{otherwise.} \end{cases}$$

3.4. *Formula for computation.* Summing up Lemmas 3.3–3.5 and Proposition 2.1, we obtain the following

THEOREM 3.6. *Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer m, and let $n > 1$. Let L be a $\mathfrak{g}$-maximal lattice in V with respect to φ. Let χ, χ', and χ'' be the Dirichlet characters corresponding to $F/\mathbb{Q}$, $\mathbb{Q}(\sqrt{-1})/\mathbb{Q}$, and $\mathbb{Q}(\sqrt{-m})/\mathbb{Q}$, respectively.*

(1) *If* $n \equiv 0 \pmod 8$, *then*

$$\mathfrak{m}(L) = n^{-2} B_{n/2} B_{n/2,\chi} \Big(\prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k,\chi} \Big).$$

(2) *If* $n \equiv \pm 1 \pmod 8$, *then*

$$\mathfrak{m}(L) = \prod_{k=1}^{(n-1)/2} (4k)^{-2} B_{2k} B_{2k,\chi}.$$

(3) *If* $n \equiv \pm 2 \pmod 8$, *then*

$$\mathfrak{m}(L) = n^{-2} B_{n/2,\chi'} B_{n/2,\chi''} \Big(\prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k,\chi} \Big) \times \begin{cases} 2^{-2} & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{if } m \equiv 3 \pmod 4, \\ 2^{-1} & \textit{otherwise.} \end{cases}$$

(4) *If* $n \equiv \pm 3 \pmod 8$, *then*

$$\mathfrak{m}(L) = \Big(\prod_{k=1}^{(n-1)/2} (4k)^{-2} B_{2k} B_{2k,\chi} \Big) \times \begin{cases} 2^{-2} \cdot 3^{-2} (2^{n-1} - 1)^2 & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{otherwise.} \end{cases}$$

(5) *If* $n \equiv 4 \pmod 8$, *then*

$$\mathfrak{m}(L) = n^{-2} B_{n/2} B_{n/2,\chi} \Big(\prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k,\chi} \Big) \times \begin{cases} 2^{-2} \cdot 3^{-2} (2^{n/2-1} - 1)^2 (2^{n/2} - 1)^2 & \textit{if } m \equiv 1 \pmod 8, \\ 1 & \textit{otherwise.} \end{cases}$$

Proof. (1) If $n \equiv 0 \pmod 8$, then substituting $\mathfrak{e} = \mathfrak{g}$ and $[\widetilde{L} : L] = 1$ into (2.3), we obtain (1).

(2) If $n \equiv \pm 1 \pmod 8$, then $t_v = 1$ for every $v \in \mathbf{h}$ and $\mathfrak{f} = \mathfrak{g}$ by Lemma 3.3. By Lemma 3.4, we have $\mathfrak{e} = \prod_{v|2} v$. Hence by Lemma 3.5 and (2.1),

$$[\widetilde{L} : L]^{(n-1)/2} \prod_{v|\mathfrak{e}} \lambda_v = 2^{n-1}.$$

Substituting this into (2.2), we obtain (2).

(3) Suppose $n \equiv \pm 2 \pmod 8$. If $m \equiv 1, 2 \pmod 4$ and $v \,|\, 2$, then $t_v = 2$ by Lemma 3.3. By (2.1) and Lemma 3.4, we have $\lambda_v = 2^{-1}$ and $\mathfrak{e} = \prod_{v|2} v$.

If $m \equiv 3 \pmod 4$, then $\mathfrak{e} = \mathfrak{g}$. Thus by Lemma 3.5,

$$[\widetilde{L} : L]^{(n-1)/2} \prod_{v|\mathfrak{e}} \lambda_v = \begin{cases} 2^{2(n-2)} & \text{if } m \equiv 1 \pmod 8, \\ 2^{2n-3} & \text{if } m \not\equiv -3 \pmod 8, \\ 2^{n-2} & \text{if } m \equiv 2 \pmod 4, \\ 1 & \text{if } m \equiv 3 \pmod 4. \end{cases}$$

Combining this with (2.4), we obtain (3).

(4) Let $m \equiv 1 \pmod 8$. By Lemma 3.3, we have $t_v = 3$ if $v \,|\, 2$. Thus, by Lemma 3.4, $\mathfrak{e} = \prod_{v|2} v = \mathfrak{f}$. If $m \not\equiv 1 \pmod 8$, then $\mathfrak{f} = \mathfrak{g}$ and $\mathfrak{e} = \prod_{v|2} v$. Hence by Lemma 3.5,

$$2^{1-n}[\widetilde{L} : L]^{(n-1)/2} \prod_{v|\mathfrak{e}} \lambda_v = \begin{cases} 2^{-2} \cdot 3^{-2}(2^{n-1} - 1)^2 & \text{if } m \equiv 1 \pmod 8, \\ 1 & \text{otherwise.} \end{cases}$$

(5) Suppose $n \equiv 4 \pmod 8$. If $m \equiv 1 \pmod 8$, then $t_v = 4$ for $v \,|\, 2$ and $\mathfrak{e} = \prod_{v|2} v$ by Lemmas 3.3 and 3.4. If $m \not\equiv 1 \pmod 8$, then $\mathfrak{e} = \mathfrak{g}$. By Lemma 3.5,

$$[\widetilde{L} : L]^{(n-1)/2} \prod_{v|\mathfrak{e}} \lambda_v = \begin{cases} 2^{-2} \cdot 3^{-2}(2^{n/2-1} - 1)^2(2^{n/2} - 1)^2 & \text{if } m \equiv 1 \pmod 8, \\ 1 & \text{otherwise,} \end{cases}$$

which proves (5). ∎

4. Numerical example. In this section, we take $F = \mathbb{Q}(\sqrt{5})$ and $\varphi = 1_4$, and as an application of Theorem 3.6 we show that the genus of a $\mathfrak{g}$-maximal lattice in $V = F_4^1$ with respect to φ consists of one class; moreover, we give a formula for $N(L, h)$ by using this fact and the formula due to Shimura in [7, Theorem 1.5].

Let $\mathfrak{g} = \mathbb{Z} + \mathbb{Z}\omega$ with $\omega = (1 + \sqrt{5})/2$. We note that ω is a fundamental unit of F. We consider a $\mathfrak{g}$-maximal lattice L with respect to φ. Set

$$L = \sum_{i=1}^{4} \mathfrak{g}\alpha_i = \mathfrak{g}_4^1\alpha, \quad \alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_4 \end{bmatrix} \in \mathrm{GL}_4(F).$$

A $\mathfrak{g}$-lattice L in V is $\mathfrak{g}$-maximal with respect to φ if and only if $\varphi[x] \in \mathfrak{g}$ for every $x \in L$ and $[\widetilde{L}_v : L_v] = 1$ for every $v \in \mathbf{h}$. These conditions are equivalent to

$$\begin{cases} \varphi[\alpha_i] \in \mathfrak{g} \quad (1 \le i \le 4), \quad 2\varphi(\alpha_i, \alpha_j) \in \mathfrak{g} \quad (i \ne j), \\ \det(\alpha) \in 2^{-2}\mathfrak{g}^\times. \end{cases} \tag{4.1}$$

The last condition can be seen by using elementary divisors. Put

$$\alpha = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1/2 & (1+\omega)/2 & \omega/2 & 0 \\ 1/2 & 1/2 & 1/2 & 1/2 \end{bmatrix}; \tag{4.2}$$

then α satisfies (4.1). Hence $L = \mathfrak{g}_4^1\alpha$ is a $\mathfrak{g}$-maximal lattice with respect to φ. We remark that L contains $\mathfrak{g}_4^1$.

We shall calculate the order of $\Gamma = \{\gamma \in G \mid L\gamma = L\}$ for this L. We take the subgroup

$$\Gamma_1 = \{\gamma \in \Gamma \mid e_1\gamma = e_1\},$$

where $\{e_i\}_{i=1}^4$ is the standard basis of V. We will calculate $[\Gamma : \Gamma_1]$ and $[\Gamma_1 : 1]$. For $\gamma = {}^t[{}^t\gamma_1 \ \cdots \ {}^t\gamma_4] \in F_4^4$, $\gamma \in \Gamma_1$ if and only if

$$\begin{cases} \gamma_i \in n(L,1) \quad (1 \le i \le 4), \quad \gamma_i \cdot {}^t\gamma_j = 0 \quad (i \ne j), \\ 2^{-1}(\gamma_1 + (1+\omega)\gamma_2 + \omega\gamma_3) \in n(L, 1+\omega), \\ 2^{-1}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \in n(L,1). \end{cases} \tag{4.3}$$

Here $n(M,h) = \{x \in M \mid \varphi[x] = h\}$ for a $\mathfrak{g}$-lattice M in V and $h \in F$. This can be verified as in [2].

We need $n(L,1)$ to use condition (4.3). As $\mathfrak{g}_4^1 \subset L$, we see that $L \subset \widetilde{L} \subset \widetilde{\mathfrak{g}_4^1} = (2^{-1}\mathfrak{g})_4^1$. So we only have to determine $n((2^{-1}\mathfrak{g})_4^1, 1)$, and find the elements in the set satisfying the following condition: For $y = [x_1/2 \ \cdots \ x_4/2] \in (2^{-1}\mathfrak{g})_4^1$ and $x_i = a_i + b_i\omega$, the condition $y \in L$ is equivalent to the conditions

$$\begin{cases} a_2 + b_3 + a_4 + b_4 \equiv b_2 + a_3 + b_3 + a_4 \equiv 0 \pmod 2, \\ a_1 + a_3 + b_3 + b_4 \equiv b_1 + a_3 + a_4 + b_4 \equiv 0 \pmod 2). \end{cases} \tag{4.4}$$

The set $n((2^{-1}\mathfrak{g})_4^1, 1)$ can be determined in the following way: Let $x/2$ be an element of $(2^{-1}\mathfrak{g})_4^1$, and solve the equation $\varphi[x] = 4$ instead of $\varphi[x/2] = 1$. Furthermore, $\varphi[x]$ is of the form

$$\varphi[x] = \begin{bmatrix} 1_4 & 0 \\ 0 & 1_4 \end{bmatrix}[\widehat{x}] + \begin{bmatrix} 0 & 1_4 \\ 1_4 & 1_4 \end{bmatrix}[\widehat{x}] \cdot \omega.$$

Here we set $\widehat{x} = [a_1 \ \cdots \ a_4 \ b_1 \ \cdots \ b_4] \in \mathbb{Z}_8^1$ for $x = [x_1 \ \cdots \ x_4] \in \mathfrak{g}_4^1$ and $x_i = a_i + b_i\omega$. Calculating $\widehat{x}$ in $\mathbb{Z}_8^1$ satisfying the conditions

$$\begin{cases} \begin{bmatrix} 1_4 & 0 \\ 0 & 1_4 \end{bmatrix}[\widehat{x}] = 4, \\ \begin{bmatrix} 0 & 1_4 \\ 1_4 & 1_4 \end{bmatrix}[\widehat{x}] = 0, \end{cases}$$

we get $n((2^{-1}\mathfrak{g})_4^1, 1)$. Employing condition (4.4) for this set, we find

$$\begin{aligned} n(L,1) &= n((2^{-1}\mathfrak{g})_4^1, 1) \cap L \\ &= \{\pm e_i \mid 1 \le i \le 4\} \cup \{(\delta_1/2, \ldots, \delta_4/2)\} \\ &\quad \cup \{(0, \delta_2/2, \delta_3\omega/2, \delta_4\omega^\varrho/2)\} \cup \{(0, \delta_2\omega/2, \delta_3\omega^\varrho/2, \delta_4/2)\} \\ &\quad \cup \{(0, \delta_2\omega^\varrho/2, \delta_3/2, \delta_4\omega/2)\} \cup \{(\delta_1/2, 0, \delta_3\omega^\varrho/2, \delta_4\omega/2)\} \\ &\quad \cup \{(\delta_1\omega/2, 0, \delta_3/2, \delta_4\omega^\varrho/2)\} \cup \{(\delta_1\omega^\varrho/2, 0, \delta_3\omega/2, \delta_4/2)\} \\ &\quad \cup \{(\delta_1/2, \delta_2\omega/2, 0, \delta_4\omega^\varrho/2)\} \cup \{(\delta_1\omega/2, \delta_2\omega^\varrho/2, 0, \delta_4/2)\} \\ &\quad \cup \{(\delta_1\omega^\varrho/2, \delta_2/2, 0, \delta_4\omega/2)\} \cup \{(\delta_1/2, \delta_2\omega^\varrho/2, \delta_3\omega/2, 0)\} \\ &\quad \cup \{(\delta_1\omega/2, \delta_2/2, \delta_3\omega^\varrho/2, 0)\} \cup \{(\delta_1\omega^\varrho/2, \delta_2\omega/2, \delta_3/2, 0)\}. \end{aligned}$$

Here we allow δ_i to take the values ± 1 at random in each set, and $\omega^\varrho = 1 - \omega = -\omega^{-1}$ with the nontrivial automorphism ϱ of F over $\mathbb{Q}$.

We compute $[\varGamma : \varGamma_1]$. For $\gamma, \delta \in \varGamma$, we denote by γ_1 the first row vector of $\gamma \in \varGamma$. Since $\varGamma_1\gamma = \varGamma_1\delta$ if and only if $\gamma_1 = \delta_1$, we see that $[\varGamma : \varGamma_1] \le N(L,1)$; moreover, for any $\gamma_1 \in n(L,1)$ there exists an element of $\varGamma$ which has γ_1 as the first row vector. This can be verified by using $n(L,1)$ and taking an element of $\varGamma$ satisfying (4.3). Thus

$$[\varGamma : \varGamma_1] = N(L,1) = 120.$$

To compute $[\varGamma_1 : 1]$, we use the condition

$$2^{-1}(\omega^{-1}\gamma_1 + \omega^{-1}(1+\omega)\gamma_2 + \gamma_3) \in n(L,1)$$

instead of

$$2^{-1}(\gamma_1 + (1+\omega)\gamma_2 + \omega\gamma_3) \in n(L, 1+\omega)$$

in (4.3), because $n(L, 1+\omega) = \omega \cdot n(L,1)$. From this, $\varGamma_1$ consists of all elements $\gamma = {}^t[{}^te_1\ {}^t\gamma_2\ {}^t\gamma_3\ {}^t\gamma_4] \in F_4^4$ such that

$$(4.5)\qquad \begin{cases} \gamma_i \in n(L,1) \quad (2 \le i \le 4), \quad \gamma_i \cdot {}^t\gamma_j = 0 \quad (i \ne j), \\ 2^{-1}(\omega^{-1}\gamma_1 + \omega^{-1}(1+\omega)\gamma_2 + \gamma_3) \in n(L,1), \\ 2^{-1}(\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4) \in n(L,1). \end{cases}$$

Since $n(L,1)$ has been determined, we can choose the pair $(e_1, \gamma_2, \gamma_3, \gamma_4)$ satisfying (4.5). After some calculations, we obtain $[\varGamma_1 : 1] = 120$. Thus $[\varGamma : 1] = 120 \cdot 120 = 2^6 \cdot 3^2 \cdot 5^2$.

On the other hand, we see that $\mathfrak{m}(L) = 2^{-6} \cdot 3^{-2} \cdot 5^{-2}$ by Theorem 3.6. Therefore the genus of L consists of one class.

Next, to introduce the formula in [7, Theorem 1.5(II)], we define $\gamma_v(s)$ by the following formulas (cf. [7, §1.6]):

$$\gamma_v(s) = \begin{cases} \dfrac{1-q_v^{(\nu+1)(1-s)}}{1-q_v^{1-s}} & \text{if } t_v = 0, \\ \dfrac{1-(-q_v^{1-s})^{\nu+1}}{1+q_v^{1-s}} & \text{if } t_v = 2,\ \mathfrak{d}_v = \mathfrak{r}_v, \text{ and } c_v \in \mathfrak{g}_v^{\times}, \\ \dfrac{1-(-q_v^{1-s})^{\nu+2}+q_v^{2-s}(1-(-q_v^{1-s})^{\nu})}{(1+q_v^{-s})(1+q_v^{1-s})} & \\ & \text{if } t_v = 2,\ \mathfrak{d}_v = \mathfrak{r}_v, \text{ and } c_v \in \pi_v\mathfrak{g}_v^{\times}, \\ 1+\psi_v(c_v h)(|h|_v^{-1}N(\mathfrak{d}_v))^{1-s} & \text{if } t_v = 2 \text{ and } \mathfrak{d}_v \neq \mathfrak{r}_v, \\ \dfrac{1-q_v^{(\nu+2)(1-s)}-q_v^{2-s}(1-q_v^{\nu(1-s)})}{(1-q_v^{-s})(1-q_v^{1-s})} & \text{if } t_v = 4, \end{cases}$$

where $\nu \in \mathbb{Z}$ is determined by $|h|_v^{-1} = q_v^{\nu}$ with the normalized valuation $|\ |_v$ at v of F_v.

THEOREM 4.1 (Shimura). *Let F be a totally real algebraic number field, and put $n = 2u$ and $V = F_n^1$. Let φ be a totally positive symmetric matrix of $\mathrm{GL}_n(F)$, L a $\mathfrak{g}$-maximal lattice in V with respect to φ, and h a totally positive element of $\mathfrak{g}$. Put $K = F(\sqrt{(-1)^{n/2}\det(\varphi)})$, and denote by $\mathfrak{d}$ the different of K/F and by $\psi_{K/F}$ the Hecke character of F corresponding to the extension K/F. Then*

$$(4.6)\qquad \frac{R(L,h)}{\mathfrak{m}(L)} = \frac{N_{F/\mathbb{Q}}(h)^{u-1}}{D_F^{(n-1)/2}N(\mathfrak{d})^{1/2}N(\mathfrak{f})\{(u-1)!(2\pi)^{-u}\}^2 L(u,\psi_{K/F})} \times \prod_{v|h\mathfrak{e}} \gamma_v(u).$$

Here $\mathfrak{e}$ is the product of all primes v satisfying $\widetilde{L}_v \neq L_v$, $\mathfrak{f}$ is the product of prime factors of $\mathfrak{e}$ unramified in K, and $\gamma_v(s)$ is given as above.

We specialize this formula to the case where $F = \mathbb{Q}(\sqrt{5})$, $V = F_4^1$, and $\varphi = 1_4$. We easily see that $K = F$. By Lemma 3.4, we have $\mathfrak{e} = \mathfrak{g}$. Since $t_v = 0$ for every $v \in \mathbf{h}$ by Lemma 3.3, $\gamma_v(2)$ is given by

$$\gamma_v(2) = \frac{1-q_v^{-(e_v+1)}}{1-q_v^{-1}}$$

for a totally positive element h of $\mathfrak{g}$, where the principal ideal of h is

$(h)_F = \prod_v v^{e_v}$. Now, the genus of a $\mathfrak{g}$-maximal lattice with respect to φ consists of one class. Thus $N(L,h) = R(L,h)\mathfrak{m}(L)^{-1}$. Hence by Theorem 4.1, we obtain

$$\begin{aligned} N(L,h) &= \frac{N_{F/\mathbb{Q}}(h)}{5^{3/2}(2\pi)^{-4}\zeta_F(2)} \prod_{v|h} \frac{1-q_v^{-(e_v+1)}}{1-q_v^{-1}} \\ &= 120 \prod_{v|h} \Big(\sum_{i=0}^{e_v} q_v^i \Big). \end{aligned} \tag{4.7}$$

Thus $N(L,h)/120$ coincides with the Fourier coefficient of the Eisenstein series E_2 in the space $\mathcal{M}_{(2,2)}(\mathfrak{g}, \mathrm{id})$ of all Hilbert modular forms for GL(2) over $F = \mathbb{Q}(\sqrt{5})$ of weight $(2,2)$, level $\mathfrak{g}$, and with the identity character. It is known that the dimension of this space is 1 (see [3, §3.4], for example), that is, $\mathcal{M}_{(2,2)}(\mathfrak{g}, \mathrm{id}) = \mathbb{C} \cdot E_2$ in this case.

5. Numerical table of $\mathfrak{m}(L)$. Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer m, and let $\varphi = 1_n$. Using Theorem 3.6, we give a table of $\mathfrak{m}(L)$ for several quadratic fields F.

We note that the lth generalized Bernoulli number $B_{l,\psi}$ associated with a Dirichlet character ψ can be computed by the following recursion formula (cf. [4]):

$$\begin{aligned} B_{l,\psi} &= \frac{1}{f}\sum_{a=1}^{f} \psi(a)a^l - \sum_{k=1}^{l-1} \frac{f^{l-k}}{k}\binom{l}{k-1} B_{k,\psi} \quad (0 < l \in \mathbb{Z}), \\ B_{0,\psi} &= 0. \end{aligned}$$

Here f is the conductor of ψ and we put $\binom{l}{r} = l!/(l-r)!r!$.

$m \backslash n$	2	3	4	5	6
2	$\frac{1}{2^4}$	$\frac{1}{2^4 \cdot 3}$	$\frac{1}{2^8 \cdot 3^2}$	$\frac{11}{2^9 \cdot 3^2 \cdot 5}$	$\frac{11}{2^{13} \cdot 3 \cdot 5}$
3	$\frac{1}{2^3 \cdot 3}$	$\frac{1}{2^3 \cdot 3}$	$\frac{1}{2^6 \cdot 3^2}$	$\frac{23}{2^7 \cdot 3^2 \cdot 5}$	$\frac{23}{2^9 \cdot 3^4 \cdot 5}$
5	$\frac{1}{2^3}$	$\frac{1}{2^3 \cdot 3 \cdot 5}$	$\frac{1}{2^6 \cdot 3^2 \cdot 5^2}$	$\frac{1}{2^7 \cdot 3^2 \cdot 5^2}$	$\frac{1}{2^{10} \cdot 3 \cdot 5}$
6	$\frac{1}{2^3}$	$\frac{1}{2^3}$	$\frac{1}{2^6}$	$\frac{3 \cdot 29}{2^7 \cdot 5}$	$\frac{3 \cdot 23 \cdot 29}{2^{10} \cdot 5}$
7	$\frac{1}{2^3}$	$\frac{1}{2 \cdot 3}$	$\frac{1}{2^2 \cdot 3^2}$	$\frac{113}{2^3 \cdot 3^2 \cdot 5}$	$\frac{113}{2^2 \cdot 3^2 \cdot 5 \cdot 7}$
10	$\frac{1}{2^3}$	$\frac{7}{2^3 \cdot 3}$	$\frac{7^2}{2^6 \cdot 3^2}$	$\frac{7 \cdot 19 \cdot 83}{2^7 \cdot 3^2 \cdot 5}$	$\frac{7 \cdot 19 \cdot 79 \cdot 83}{2^{10} \cdot 3^2 \cdot 5}$
11	$\frac{1}{2^3}$	$\frac{7}{2^3 \cdot 3}$	$\frac{7^2}{2^6 \cdot 3^2}$	$\frac{7 \cdot 2153}{2^7 \cdot 3^2 \cdot 5}$	$\frac{7 \cdot 2153}{2^9 \cdot 3 \cdot 5}$
13	$\frac{1}{2^3}$	$\frac{1}{2^3 \cdot 3}$	$\frac{1}{2^6 \cdot 3^2}$	$\frac{29}{2^7 \cdot 3^2 \cdot 5}$	$\frac{29 \cdot 151}{2^{10} \cdot 3^2 \cdot 5}$
14	$\frac{1}{2^2}$	$\frac{5}{2^2 \cdot 3}$	$\frac{5^2}{2^4 \cdot 3^2}$	$\frac{2503}{2^5 \cdot 3^2}$	$\frac{11 \cdot 2503}{2^7}$

$m \backslash n$	2	3	4	5	6
15	$\frac{1}{2^2}$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{3\cdot179}{2^3\cdot5}$	$\frac{3\cdot179}{2^2\cdot5}$
17	$\frac{1}{2^3}$	$\frac{1}{2^4\cdot3}$	$\frac{1}{2^6\cdot3^2}$	$\frac{5\cdot41}{2^7\cdot3^2}$	$\frac{41\cdot53}{2^8\cdot3\cdot5}$
19	$\frac{1}{2^3}$	$\frac{19}{2^3\cdot3}$	$\frac{19^2}{2^6\cdot3^2}$	$\frac{19\cdot109\cdot137}{2^7\cdot3^2\cdot5}$	$\frac{11\cdot19\cdot109\cdot137}{2^9\cdot3^2\cdot5}$
21	$\frac{1}{2^2}$	$\frac{1}{2^2\cdot3}$	$\frac{1}{2^4\cdot3^2}$	$\frac{7\cdot11}{2^5\cdot3^2\cdot5}$	$\frac{7\cdot11\cdot263}{2^7\cdot3^2\cdot5}$
22	$\frac{1}{2^3}$	$\frac{23}{2^3\cdot3}$	$\frac{23^2}{2^6\cdot3^2}$	$\frac{23\cdot24889}{2^7\cdot3^2\cdot5}$	$\frac{13\cdot23\cdot43\cdot24889}{2^{10}\cdot3^2\cdot5}$
23	$\frac{3}{2^3}$	$\frac{5}{2\cdot3}$	$\frac{5^2}{2^2\cdot3^2}$	$\frac{41\cdot173}{2^3\cdot3^2}$	$\frac{41\cdot173}{2^2\cdot3}$
26	$\frac{3}{2^3}$	$\frac{5^2}{2^3\cdot3}$	$\frac{5^4}{2^6\cdot3^2}$	$\frac{5\cdot31\cdot1409}{2^7\cdot3^2}$	$\frac{5\cdot31\cdot311\cdot1409}{2^{10}\cdot3}$
29	$\frac{3}{2^3}$	$\frac{1}{2^3}$	$\frac{1}{2^6}$	$\frac{157}{2^7\cdot5}$	$\frac{3\cdot11\cdot37\cdot157}{2^{10}\cdot5}$
30	$\frac{1}{2^2}$	$\frac{17}{2^2\cdot3}$	$\frac{17^2}{2^4\cdot3^2}$	$\frac{17\cdot36451}{2^5\cdot3^2\cdot5}$	$\frac{17\cdot127\cdot36451}{2^7\cdot3^2}$
31	$\frac{3}{2^3}$	$\frac{5}{3}$	$\frac{5^2}{3^2}$	$\frac{10357}{2\cdot3^2}$	$\frac{2\cdot10357}{3}$
33	$\frac{1}{2^3}$	$\frac{1}{2^4}$	$\frac{1}{2^6}$	$\frac{3\cdot5\cdot47}{2^7}$	$\frac{3\cdot11\cdot47\cdot73}{2^8\cdot5}$
34	$\frac{1}{2^2}$	$\frac{23}{2^2\cdot3}$	$\frac{23^2}{2^4\cdot3^2}$	$\frac{23\cdot57241}{2^5\cdot3^2\cdot5}$	$\frac{7\cdot11^2\cdot23\cdot57241}{2^7\cdot3^2\cdot5}$
35	$\frac{1}{2^2}$	$\frac{19}{2^2\cdot3}$	$\frac{19^2}{2^4\cdot3^2}$	$\frac{7\cdot19\cdot8819}{2^5\cdot3^2\cdot5}$	$\frac{3\cdot7\cdot19\cdot8819}{2^6\cdot5}$
37	$\frac{1}{2^3}$	$\frac{5}{2^3\cdot3}$	$\frac{5^2}{2^6\cdot3^2}$	$\frac{1129}{2^7\cdot3^2}$	$\frac{23\cdot89\cdot1129}{2^{10}\cdot3^2}$
38	$\frac{3}{2^3}$	$\frac{41}{2^3\cdot3}$	$\frac{41^2}{2^6\cdot3^2}$	$\frac{23\cdot41\cdot1429}{2^7\cdot3^2}$	$\frac{7\cdot23\cdot41\cdot113\cdot1429}{2^{10}\cdot3}$
39	$\frac{1}{2}$	$\frac{13}{2\cdot3}$	$\frac{13^2}{2^2\cdot3^2}$	$\frac{5\cdot13\cdot31\cdot59}{2^3\cdot3^2}$	$\frac{5\cdot11\cdot13\cdot31\cdot59}{2^2\cdot3^2}$
41	$\frac{1}{2^2}$	$\frac{1}{2^2\cdot3}$	$\frac{1}{2^2\cdot3^2}$	$\frac{2^2\cdot5\cdot7}{3^2}$	$\frac{2^2\cdot3^4\cdot7}{5}$
42	$\frac{1}{2^2}$	$\frac{3^2}{2^2}$	$\frac{3^4}{2^4}$	$\frac{3^4\cdot5^3\cdot7}{2^5}$	$\frac{3^4\cdot5^3\cdot7^2\cdot11\cdot19}{2^7}$
43	$\frac{1}{2^3}$	$\frac{3\cdot7}{2^3}$	$\frac{3^2\cdot7^2}{2^6}$	$\frac{3\cdot7\cdot11\cdot7873}{2^7\cdot5}$	$\frac{3\cdot7\cdot11\cdot83\cdot7873}{2^9\cdot5}$
46	$\frac{1}{2^2}$	$\frac{37}{2^2\cdot3}$	$\frac{37^2}{2^4\cdot3^2}$	$\frac{37\cdot164999}{2^5\cdot3^2\cdot5}$	$\frac{37\cdot359\cdot164999}{2^7\cdot3^2}$
47	$\frac{5}{2^3}$	$\frac{7}{3}$	$\frac{7^2}{3^2}$	$\frac{7\cdot43223}{2\cdot3^2\cdot5}$	$\frac{2\cdot7\cdot43223}{5}$
51	$\frac{1}{2^2}$	$\frac{13}{2^2}$	$\frac{13^2}{2^4}$	$\frac{3\cdot13\cdot5197}{2^5}$	$\frac{3\cdot13\cdot67\cdot5197}{2^6}$
53	$\frac{3}{2^3}$	$\frac{7}{2^3\cdot3}$	$\frac{7^2}{2^6\cdot3^2}$	$\frac{5^2\cdot7\cdot31}{2^7\cdot3^2}$	$\frac{5^2\cdot7\cdot13\cdot31\cdot139}{2^{10}\cdot3}$
55	$\frac{1}{2}$	$\frac{23}{2\cdot3}$	$\frac{23^2}{2^2\cdot3^2}$	$\frac{23^2\cdot6689}{2^3\cdot3^2\cdot5}$	$\frac{5\cdot23^2\cdot6689}{2^2\cdot3^2}$
57	$\frac{1}{2^3}$	$\frac{7}{2^4\cdot3}$	$\frac{7^2}{2^6\cdot3^2}$	$\frac{5\cdot7\cdot47\cdot61}{2^7\cdot3^2}$	$\frac{7\cdot31\cdot47\cdot61\cdot101}{2^8\cdot3^2\cdot5}$
58	$\frac{1}{2^3}$	$\frac{3\cdot11}{2^3}$	$\frac{3^2\cdot11^2}{2^6}$	$\frac{3\cdot11\cdot246839}{2^7\cdot5}$	$\frac{3\cdot11\cdot1259\cdot246839}{2^{10}}$
59	$\frac{3}{2^3}$	$\frac{5\cdot17}{2^3\cdot3}$	$\frac{5^2\cdot17^2}{2^6\cdot3^2}$	$\frac{17\cdot271\cdot2837}{2^7\cdot3^2}$	$\frac{17\cdot67\cdot271\cdot2837}{2^9\cdot3}$
61	$\frac{3}{2^3}$	$\frac{11}{2^3\cdot3}$	$\frac{11^2}{2^6\cdot3^2}$	$\frac{11\cdot17\cdot383}{2^7\cdot3^2\cdot5}$	$\frac{3\cdot11\cdot17\cdot271\cdot383}{2^{10}\cdot5}$
62	$\frac{1}{2}$	$\frac{7}{2}$	$\frac{7^2}{2^2}$	$\frac{7\cdot15193}{2^3}$	$\frac{3\cdot7\cdot673\cdot15193}{2^4}$
65	$\frac{1}{2^2}$	$\frac{1}{2\cdot3}$	$\frac{1}{3^2}$	$\frac{5\cdot1123}{2^2\cdot3^2}$	$\frac{11\cdot23\cdot1123}{2^2\cdot5}$
66	$\frac{1}{2}$	$\frac{2\cdot7}{3}$	$\frac{2^2\cdot7^2}{3^2}$	$\frac{7\cdot29\cdot4967}{3^2\cdot5}$	$\frac{7\cdot29\cdot2309\cdot4967}{2\cdot3^2\cdot5}$

$m \backslash n$	2	3	4	5	6
70	$\frac{1}{2^2}$	$\frac{67}{2^2\cdot 3}$	$\frac{67^2}{2^4\cdot 3^2}$	$\frac{5\cdot 7\cdot 67\cdot 4091}{2^5\cdot 3^2}$	$\frac{5^2\cdot 7\cdot 67\cdot 1019\cdot 4091}{2^7\cdot 3^2}$
71	$\frac{7}{2^3}$	$\frac{29}{2\cdot 3}$	$\frac{29^2}{2^2\cdot 3^2}$	$\frac{11\cdot 29\cdot 101\cdot 331}{2^3\cdot 3^2\cdot 5}$	$\frac{11\cdot 17\cdot 29\cdot 101\cdot 331}{2^2\cdot 3\cdot 5}$
73	$\frac{1}{2^3}$	$\frac{11}{2^4\cdot 3}$	$\frac{11^2}{2^6\cdot 3^2}$	$\frac{5^2\cdot 7\cdot 11\cdot 197}{2^7\cdot 3^2}$	$\frac{5\cdot 7^3\cdot 11\cdot 23\cdot 197}{2^8\cdot 3^2}$
74	$\frac{5}{2^3}$	$\frac{41}{2^3}$	$\frac{41^2}{2^6}$	$\frac{31\cdot 41\cdot 101\cdot 181}{2^7\cdot 5}$	$\frac{3^2\cdot 17\cdot 31\cdot 41\cdot 83\cdot 101\cdot 181}{2^{10}\cdot 5}$
77	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1193}{2^3\cdot 5}$	$\frac{3\cdot 1153\cdot 1193}{2^4\cdot 5}$
78	$\frac{1}{2^2}$	$\frac{23}{2^2}$	$\frac{23^2}{2^4}$	$\frac{3\cdot 23\cdot 31\cdot 3697}{2^5\cdot 5}$	$\frac{3\cdot 13\cdot 17\cdot 23\cdot 31^2\cdot 3697}{2^7\cdot 5}$
79	$\frac{5}{2^3}$	7	7^2	$\frac{7\cdot 37\cdot 2467}{2\cdot 5}$	$\frac{2\cdot 7\cdot 31\cdot 37\cdot 2467}{5}$
82	$\frac{1}{2^2}$	$\frac{3^3}{2^2}$	$\frac{3^6}{2^4}$	$\frac{3^3\cdot 179\cdot 463}{2^5}$	$\frac{3^3\cdot 17\cdot 179\cdot 443\cdot 463}{2^7}$
83	$\frac{3}{2^3}$	$\frac{43}{2^3}$	$\frac{43^2}{2^6}$	$\frac{7^3\cdot 43\cdot 2459}{2^7\cdot 5}$	$\frac{3\cdot 7^3\cdot 31\cdot 43\cdot 2459}{2^9}$
85	$\frac{1}{2^2}$	$\frac{3}{2^2}$	$\frac{3^2}{2^4}$	$\frac{3\cdot 3463}{2^5\cdot 5}$	$\frac{3\cdot 3463\cdot 8263}{2^7\cdot 5}$
86	$\frac{5}{2^3}$	$\frac{5\cdot 31}{2^3\cdot 3}$	$\frac{5^2\cdot 31^2}{2^6\cdot 3^2}$	$\frac{11\cdot 29^2\cdot 31\cdot 311}{2^7\cdot 3^2}$	$\frac{7\cdot 11\cdot 29^2\cdot 31\cdot 293\cdot 311}{2^{10}}$
87	$\frac{3}{2^2}$	$\frac{13}{2}$	$\frac{13^2}{2^2}$	$\frac{3\cdot 13\cdot 83903}{2^3\cdot 5}$	$\frac{3^5\cdot 13\cdot 83903}{2^2\cdot 5}$
89	$\frac{3}{2^3}$	$\frac{13}{2^4\cdot 3}$	$\frac{13^2}{2^6\cdot 3^2}$	$\frac{5^2\cdot 13\cdot 37\cdot 73}{2^7\cdot 3^2}$	$\frac{13\cdot 37\cdot 73\cdot 3371}{2^8\cdot 3}$
91	$\frac{1}{2^2}$	$\frac{103}{2^2\cdot 3}$	$\frac{103^2}{2^4\cdot 3^2}$	$\frac{7\cdot 13\cdot 103\cdot 19739}{2^5\cdot 3^2\cdot 5}$	$\frac{5\cdot 7\cdot 11\cdot 13\cdot 103\cdot 19739}{2^6\cdot 3^2}$
93	$\frac{1}{2^2}$	$\frac{3}{2^2}$	$\frac{3^2}{2^4}$	$\frac{3\cdot 4679}{2^5\cdot 5}$	$\frac{3\cdot 4679\cdot 10631}{2^7\cdot 5}$
94	$\frac{1}{2}$	$\frac{53}{2\cdot 3}$	$\frac{53^2}{2^2\cdot 3^2}$	$\frac{41\cdot 53\cdot 24527}{2^3\cdot 3^2\cdot 5}$	$\frac{13\cdot 41\cdot 53\cdot 83\cdot 24527}{2^4\cdot 3^2}$
95	1	$\frac{43}{2\cdot 3}$	$\frac{43^2}{2^2\cdot 3^2}$	$\frac{43\cdot 1017299}{2^3\cdot 3^2\cdot 5}$	$\frac{7\cdot 43\cdot 1017299}{2^2\cdot 3}$
97	$\frac{1}{2^3}$	$\frac{17}{2^4\cdot 3}$	$\frac{17^2}{2^6\cdot 3^2}$	$\frac{5\cdot 17^2\cdot 1097}{2^7\cdot 3^2}$	$\frac{17^2\cdot 29\cdot 79\cdot 1097}{2^8\cdot 3^2}$

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