Shimura’s mass formula for an orthogonal group over real quadratic fields

by

MANABU MURATA (Shiga)

Introduction. Let $V$ be an $n$-dimensional row vector space over a totally real algebraic number field $F$, and let $\varphi$ be a totally positive definite symmetric matrix with entries in $F$. Put $\varphi[x] = x\varphi \cdot ^tx$ for $x \in V$. By a $g$-maximal lattice $L$ with respect to $\varphi$, we understand a $g$-lattice $L$ in $V$ which is maximal among $g$-lattices on which the values $\varphi[x]$ are contained in $g$. Here $g$ is the ring of integers of $F$. Put $G^\varphi = \{ \alpha \in \text{GL}_n(F) \mid \alpha \varphi \cdot ^t\alpha = \varphi \}$ and let $\{L_i\}_{i=1}^k$ be representatives of classes in the genus of $L$ with respect to $G^\varphi$. Then we define the mass of the genus of a $g$-maximal lattice $L$ with respect to $\varphi$ by

$$m(L) = \sum_{i=1}^k [I_i : 1]^{-1},$$

where $I_i = \{ \gamma \in G^\varphi \mid L_i \gamma = L_i \}$. Shimura’s mass formula for orthogonal groups determines $m(L)$ for an arbitrary totally real algebraic number field $F$ and arbitrary $\varphi$ ([6, Theorem 5.8]).

In this paper, we consider only the case where $F$ is a real quadratic field $\mathbb{Q}(\sqrt{m})$, and $\varphi$ is the unit matrix $1_n$ of size $n$ ($n > 1$). The purpose of this paper is to digest Shimura’s mass formula, applying it to the case of real quadratic fields, and to state the formula in a simpler form, from which we can compute $m(L)$. Applying this formula to the case where $F = \mathbb{Q}(\sqrt{5})$ and $\varphi = 1_4$, we see that the genus belonging to a $g$-maximal lattice $L$ with respect to $\varphi$ consists of one class, and further give the number $N(L, h) = \#\{ x \in L \mid \varphi[x] = h \}$ for a totally positive element $h$ in $g$ by specializing the formula due to Shimura [7, Theorem 1.5] to the present situation (Section 4). At the end of this paper, we give a numerical table of $m(L)$ for several quadratic fields $F$ (Section 5).

Applying the formula in [6, Theorem 5.8] to our case, we can reduce the calculation of the mass to the following two arguments. One is to compute the special values of the Dedekind zeta function of $F$ and the $L$-function.
of $F$ with the Hecke character of $F$ corresponding to $F(\sqrt{-1})/F$ (Section 2). These values can be obtained by calculating the values of the Riemann zeta function and Dirichlet $L$-functions, since $F(\sqrt{-1})/\mathbb{Q}$ is an abelian extension. The other is to determine a Witt decomposition for $1_n$ over the local field $F_v$ at a nonarchimedean prime $v$ of $F$ (Section 3). To do this, we first take an anisotropic part $\theta_p$ of a Witt decomposition for $1_n$ over $\mathbb{Q}_p$ for the rational prime $v|p$. Then we decompose $\theta_p$ on $F_v$. In particular, nontrivial cases are $v|2$ and $n \equiv \pm 3, 4 \pmod{8}$. In these cases, we can determine a Witt decomposition for $\theta_p$ over $F_v$ by using the local theory of quaternion algebras. Summing up these results, we obtain the mass $m(L)$ of the orthogonal group $G^\varphi$ of $\varphi = 1_n$ over real quadratic fields $F$ (Theorem 3.6).

We end this introduction with the following remark: As was mentioned in [6], maximal lattices were introduced by Eichler. This maximal lattice differs from a unimodular lattice in general. Shimura gave the mass formula for the case of maximal lattices with respect to an arbitrary $\varphi$ exactly in [6, Theorem 5.8]. We use this formula to compute the mass in our case. Clear-cut explanation why we work on maximal lattices is given in the introduction of [6] and also in that of [7].

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**Notation.** If $R$ is an associative ring with identity element and if $M$ is an $R$-module, then we write $R^\times$ for the group of all invertible elements of $R$, and $M_{n}^{m}$ for the $R$-module of $m \times n$-matrices with entries in $M$. We write $1_n$ for the identity element of the matrix ring $R_n^n$. We put $\text{GL}_n(R) = (R_n^n)^\times$. We denote by $^tx$ and det$(x)$ the transpose and determinant of a matrix $x \in R_n^n$. If $x_1, \ldots, x_r$ are square matrices, diag$[x_1, \ldots, x_r]$ denotes the matrix with $x_1, \ldots, x_r$ in the diagonal blocks and 0 in all other blocks. For a finite set $X$, we denote by $\#X$ the number of elements in $X$. For a symmetric matrix $A \in R_n^n$, we put

$$A(x, y) = xA \cdot ^tx \quad \text{and} \quad A[x] = xA \cdot ^tx \quad (x, y \in R_1^n).$$

Let $F$ be a number field. We denote by $a$ and $h$ the sets of archimedean primes and nonarchimedean primes of $F$. We identify $v$ with the prime ideal of $F$ corresponding to $v$. For $v \in h$, $\pi_v$ and $q_v$ denote a prime element of $F_v$ and the norm of the prime ideal at $v$. We denote by $B_l$ and $B_l,\psi$ the $l$th Bernoulli number and $l$th generalized Bernoulli number associated with a Dirichlet character $\psi$. For a rational prime $p$, $(\frac{a}{p})$ denotes the Legendre symbol.
1. Preliminaries. Throughout the paper, $F$ is a real quadratic field and $\varphi = 1_n$. Let $\mathfrak{g}$ be the ring of integers of $F$ and put $V = F_1^n$. We put
\[ G = G^\varphi = \{ \gamma \in \text{GL}_n(F) \mid \gamma \varphi \cdot t\gamma = \varphi \} = \{ \gamma \in \text{GL}_n(F) \mid \gamma \cdot t\gamma = 1_n \} \]  
For a $\mathfrak{g}$-lattice $L$ in $V$, which is a finitely generated $\mathfrak{g}$-submodule in $V$ containing a basis of $V$, and $h \in \mathfrak{g}$, we put
\[ N(L, h) = \#\{ x \in L \mid \varphi[x] = h \}, \]
\[ \tilde{L} = \{ y \in V \mid 2\varphi(x, y) \in \mathfrak{g} \text{ for every } x \in L \}. \]
Let $G_A$ be the adelization of $G$.

For $\alpha \in G_A$, we denote by $L\alpha$ the $\mathfrak{g}$-lattice in $V$ such that $(L\alpha)_v = L_v\alpha_v$ for any nonarchimedean prime $v$ of $F$. Here $L_v$ is the $\mathfrak{g}_v$-lattice in $(F_v)_n$ which is spanned by $L$ over $\mathfrak{g}_v$. We call \{ $L\alpha \mid \alpha \in G_A$ \} (resp. \{ $L\alpha \mid \alpha \in G$ \}) the genus (resp. class) of $L$ with respect to $G$. We put
\[ C = \{ \alpha \in G_A \mid L\alpha = L \}, \quad \Gamma^\alpha = G \cap \alpha C \alpha^{-1} \quad (\alpha \in G_A). \]
The mapping $G\alpha C \mapsto \alpha^{-1}$ gives a bijection of $G \backslash G_A/C$ onto the set of classes in the genus of $L$. It is known that $G \backslash G_A/C$ is a finite set ([5, Lemma 8.7(4)]). Thus the genus of $L$ is decomposed into a disjoint union of finitely many classes. Let $\mathfrak{B}$ be a complete set of representatives for $G \backslash G_A/C$. We then put
\[ m(L) = m(G, C) = \sum_{\alpha \in \mathfrak{B}} [\Gamma^\alpha : 1]^{-1}, \]
\[ R(L, h) = \sum_{\alpha \in \mathfrak{B}} [\Gamma^\alpha : 1]^{-1} N(L\alpha^{-1}, h). \]
We call $m(G, C)$ the mass of $G$ relative to $C$.

By a $\mathfrak{g}$-maximal lattice $L$ with respect to $\varphi$, we understand a $\mathfrak{g}$-lattice $L$ in $V$ which is maximal among $\mathfrak{g}$-lattices on which the values $\varphi[x]$ are contained in $\mathfrak{g}$. The genus of a $\mathfrak{g}$-maximal lattice $L$ consists of all $\mathfrak{g}$-maximal lattices (cf. [5, Lemma 5.9]).

Let $v \in \mathfrak{a} \cup \mathfrak{h}$. For symmetric matrices $A$ and $B$ in $\text{GL}_n(F_v)$, we say that $A$ is equivalent to $B$ over $F_v$ if there exists $X \in \text{GL}_n(F_v)$ such that $XA \cdot tX = B$. We call $A$ isotropic over $F_v$ if there exists $0 \neq x \in (F_v)_n^1$ such that $xA \cdot tx = 0$. Otherwise we say that $A$ is anisotropic over $F_v$. Put
\[ \eta_r = \begin{bmatrix} 0 & 1_r \\ 1_r & 0 \end{bmatrix}. \]
Then $\eta_r$ is a symmetric matrix of $\text{GL}_{2r}(F_v)$. For every $x \in (F_v)_r^1$, we have $\eta_r[x \mid 0_r] = 0$, and so $\eta_r$ is isotropic over $F_v$. 
For $\varphi$ and $v \in a \cup h$ there exists $\alpha_v \in \text{GL}_n(F_v)$ such that

$$
(1.1) \quad \alpha_v \varphi \cdot t \alpha_v = \begin{bmatrix}
0 & 0 & 1_{r_v} \\
0 & \theta_v & 0 \\
1_{r_v} & 0 & 0
\end{bmatrix}
$$

with an anisotropic symmetric matrix $\theta_v \in \text{GL}_{\ell_v}(F_v)$ over $F_v$. We call (1.1) a **Witt decomposition** for $\varphi$ over $F_v$. Then $n = 2r_v + t_v$. It is known that $t_v \leq 4$ for $v \in h$ (cf. [5, Proposition 5.2]). In Section 3, we determine a Witt decomposition for $\varphi$ over $F_v$ ($v \in h$).

### 2. Mass formula for $\varphi = 1_n$ over real quadratic fields

Let $L$ be a $g$-maximal lattice in $V$ with respect to $\varphi$. The formula of $m(L)$ for $G$ is given in [6, Theorem 5.8]. In fact, we can apply this formula to the case of the maximal lattices with respect to an arbitrary totally definite quadratic form over an arbitrary totally real algebraic number field.

We first define a constant $\lambda_v$ in [6, Theorem 5.8]. Let $e$ be the product of all nonarchimedean primes $v$ satisfying $\tilde{L}_v \neq L_v$. For $v \mid e$, put

$$
(2.1) \quad \lambda_v = \begin{cases}
1 & \text{if } t_v = 1, \\
2^{-1}(1 + q_v)^{-1}(1 + q_v^{1-u})(1 + q_v^{-u}) & \text{if } t_v = 2, \vartheta_v = \tau_v, \text{ and } c_v \in \pi_v g_v^\times, \\
2^{-1} & \text{if } t_v = 2, \vartheta_v \neq \tau_v, \\
2^{-1}(1 + q_v)^{-1}(1 - q_v^{1-n}) & \text{if } t_v = 3, \\
2^{-1}(1 + q_v)^{-1}(1 - q_v^{1-u})(1 - q_v^{-u}) & \text{if } t_v = 4,
\end{cases}
$$

where $u = 2^{-1}n$, and if $t_v = 2$, then $\tau_v$ is the maximal order of $K_v = F_v(\sqrt{-\det(\theta_v)})$, $\vartheta_v$ is the different of $K_v$ relative to $F_v$, and $c_v$ is a constant from [6, §3.2]. Here $c_v$ can be taken as follows: Let $\theta_v$ be as in (1.1). This is equivalent to a matrix $\text{diag}(a_v, b_v)$ over $F_v$ with a suitable basis $\{e_i\}$ such that $a_v = \theta_v[e_1], b_v = \theta_v[e_2] \in F_v^\times$ and $\theta_v(e_1, e_2) = 0$. We may assume that $b_v \in g_v^\times \cup \pi_v^{-1}g_v^\times$. The mapping $y_1e_1 + y_2e_2 \mapsto b_vy_2 + y_1\sqrt{-a_vb_v}$ yields an isomorphism $(F_v)^1 \cong K_v$ such that

$$
\theta_v[y_1e_1 + y_2e_2] = b_v^{-1} \cdot N_{K_v/F_v}(b_vy_2 + y_1\sqrt{-a_vb_v}).
$$

Here $N_{K_v/F_v}(x) = xx^\vartheta$, where $\vartheta$ is the nontrivial automorphism of $K_v$ over $F_v$. Then we can take $c_v = b_v^{-1}$. Notice that $v \nmid e$ if $t_v = 0$, or $t_v = 2$, $\vartheta_v = \tau_v$ and $c_v \in g_v^\times$ (see (3.6) below).

Applying [6, Theorem 5.8] to the case where $F$ is a real quadratic field and $\varphi = 1_n$, we have the following

**Proposition 2.1.** Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer $m$, and let $n > 1$. Put $V = F_1^n$ and $\varphi = 1_n$. Let $L$ be a $g$-maximal lattice in $V$
with respect to $\varphi$, $\epsilon$ the product of all nonarchimedean primes $v$ satisfying $\tilde{L}_v \neq L_v$, and $\chi$ the Dirichlet character corresponding to $F/\mathbb{Q}$.

(1) Suppose $n$ is odd; let $\mathfrak{f}$ be the product of all nonarchimedean primes $v$ satisfying $t_v = 3$. Then

\[(2.2)\quad m(L) = 2^{1-n}\left(\prod_{k=1}^{(n-1)/2} (4k)^{-2}B_{2k}B_{2k,\chi}\right)\left[\tilde{L} : L\right]^{(n-1)/2} \prod_{v|\mathfrak{f}} \frac{1 - q_v^{1-n}}{2(1 + q_v)}.
\]

(2) Suppose $n$ is even; put $F' = \mathbb{Q}(\sqrt{-1})$ and $F'' = \mathbb{Q}(\sqrt{-m})$, and denote by $\chi'$ (resp. $\chi''$) the Dirichlet character corresponding to $F'/\mathbb{Q}$ (resp. $F''/\mathbb{Q}$). Let $\lambda_v$ be as in (2.1).

(i) If $n \equiv 0 \pmod{4}$, then

\[(2.3)\quad m(L) = n^{-2}B_{n/2}B_{n/2,\chi}\left(\prod_{k=1}^{(n-1)/2} (4k)^{-2}B_{2k}B_{2k,\chi}\right)\times\left[\tilde{L} : L\right]^{(n-1)/2} \prod_{v|\epsilon} \lambda_v.
\]

(ii) If $n \equiv 2 \pmod{4}$, then

\[(2.4)\quad m(L) = n^{-2}B_{n/2,\chi'}B_{n/2,\chi''} \times \left(\prod_{k=1}^{(n-1)/2} (4k)^{-2}B_{2k}B_{2k,\chi}\right)\left[\tilde{L} : L\right]^{(n-1)/2} \times \prod_{v|\epsilon} \lambda_v \cdot \begin{cases} 2^{2(1-n)} & \text{if } m \equiv 1 \pmod{4}, \\ 2^{1-n} & \text{if } m \equiv 2 \pmod{4}, \\ 1 & \text{if } m \equiv 3 \pmod{4}. \end{cases}
\]

Proof. By [6, Lemma 5.6(1)], we have

$$m(G, C) = \frac{1}{2} m(G_+, C^+),$$

where

$$G_+ = \{ \gamma \in G \mid \det(\gamma) = 1 \}$$

and $C^+ = C \cap (G_+)_{\mathbb{A}}$ with the adelization $(G_+)_{\mathbb{A}}$ of $G_+$. Taking $\mathfrak{g}$ as an ideal $\epsilon$ of $F$ in [6, Theorem 5.8], we have $C^+ = D^+$, where $D^+$ is a subgroup of $(G_+)_{\mathbb{A}}$ in that theorem (cf. [6, (5.7.3), (5.7.4), and (5.7.5)]). Since $\varphi$ is anisotropic over $F$ and satisfies the condition [6, (5.7.1)], we can apply [6, Theorem 5.8] to $m(G_+, C^+)$. Hence we obtain

$$m(G_+, C^+) = m_n(\mathfrak{g})[\tilde{L} : L]^{\mu} \prod_{v|\epsilon} \lambda_v,$$
where
\[
\mathbf{m}_n(g) = 2D_F^{[\mu^2]} \prod_{k=1}^{[\mu]} \left\{ D_F^{1/2}((2k-1)!/(2\pi)^{-2k})^2 \zeta_F(2k) \right\} \times \begin{cases} 2^{-2\mu} & \text{if } n \text{ is odd,} \\ D_F^{1/2}((u-1)!/(2\pi)^{-u})^2 L(u, \psi_{K/F}) & \text{if } n \text{ is even.} \end{cases}
\]

Here \( \mu = (n-1)/2 \), and \( D_F \) is the discriminant of \( F \); if \( n \) is even, put \( u = n/2 \), \( K = F(\sqrt{(-1)^u}) \), and denote by \( \psi_{K/F} \) the Hecke character of \( F \) corresponding to \( K/F \).

Suppose \( n \) is odd. Then we have \( D_F^{[\mu^2]} = \prod_{k=1}^{[\mu]} D_F^{2k-1} \), and hence
\[
(2.5) \quad \mathbf{m}_n(g) = 2^{-(n-2)} \prod_{k=1}^{[\mu]} D_F^{2k-1/2}((2k-1)!/(2\pi)^{-2k})^2 \zeta_F(2k).
\]

Put \( \hat{\zeta}_F(s) = \pi^{-s}\Gamma(s/2)^2\zeta_F(s) \), where \( \Gamma(s) \) is the gamma function \( s \in \mathbb{C} \). By the functional equation \( \hat{\zeta}_F(s) = |D_F|^{1/2-s}\hat{\zeta}_F(1-s) \), we have
\[
\zeta_F(2k) = |D_F|^{1/2-2k}\pi^{4k} \left( \frac{2^{2k-1}}{(2k-1)!} \right)^2 \zeta_F(1-2k).
\]

Thus
\[
D_F^{2k-1/2}((2k-1)!/(2\pi)^{-2k})^2 \zeta_F(2k) = 2^{-2}\zeta_F(1-2k).
\]

Since
\[
\zeta_F(s) = \zeta_Q(s)L(s, \chi), \\
\zeta_Q(1-2k) = -B_{2k}/2k, \\
L(1-2k, \chi) = -B_{2k, \chi}/2k,
\]
we have
\[
D_F^{2k-1/2}((2k-1)!/(2\pi)^{-2k})^2 \zeta_F(2k) = (4k)^{-2}B_{2k}B_{2k, \chi}.
\]

Substituting this into (2.5) gives
\[
\mathbf{m}_n(g) = 2^{-(n-2)} \prod_{k=1}^{[\mu]} (4k)^{-2} B_{2k} B_{2k, \chi},
\]
and hence we obtain (2.2).

Suppose \( n \) is even. Since \( D_F^{[\mu^2]} = D_F^{u-1} \prod_{k=1}^{[\mu]} D_F^{2k-1} \), we have
\[
\mathbf{m}_n(g) = 2D_F^u((u-1)!/(2\pi)^{-u})^2 L(u, \psi_{K/F}) \prod_{k=1}^{[\mu]} (4k)^{-2} B_{2k} B_{2k, \chi},
\]
Then
\begin{equation}
(2.6) \quad m(L) = D_F^\mu((u - 1)! (2\pi)^{-u})^2 L(u, \psi_{K/F}) \times \left( \prod_{k=1}^{[\mu]} (4k)^{-2} B_{2k} B_{2k,\chi} \mid \tilde{L} : L \mid^\mu \prod_{v|\mathcal{c}} \lambda_v. \right)
\end{equation}

Next, we compute the value \( L(u, \psi_{K/F}) \). If \( n \equiv 0 \pmod{4} \), then \( K = F \), and so \( L(u, \psi_{K/F}) = \zeta_F(u) \). Since this value can be obtained as above, we easily deduce (2.3) from (2.6). Let \( n \equiv 2 \pmod{4} \). Then \( K \) is a totally imaginary quadratic field \( F(\sqrt{-1}) = \mathbb{Q}(\sqrt{m}, \sqrt{-1}) \) over \( F \). In this case, the functional equation is \( R(s, \psi_{K/F}) = R(1-s, \psi_{K/F}) \) by [5, Theorems A6.2, A6.3], where
\[ R(s, \psi_{K/F}) = \left| D_F N(D_{K/F}) \right|^{s/2} \left( \frac{\pi^{-s+1/2}}{\Gamma((s+1)/2)} \right)^2 L(s, \psi_{K/F}). \]

From this,
\[ L(s, \psi_{K/F}) = \left| D_F N(D_{K/F}) \right|^{(1-2s)/2} \pi^{2s-1} \left( \frac{\Gamma((2-s)/2)}{\Gamma((s+1)/2)} \right)^2 L(1-s, \psi_{K/F}). \]

Now, since \( K/\mathbb{Q}, F/\mathbb{Q}, \) and \( K/F \) are abelian extensions, we have
\[ \zeta_K(s) = \zeta_Q(s) L(s, \chi)L(s, \chi')L(s, \chi''), \]
\[ \zeta_F(s) = \zeta_Q(s) L(s, \chi), \]
\[ \zeta_K(s) = \zeta_F(s) L(s, \psi_{K/F}). \]

Hence we obtain
\begin{equation}
(2.7) \quad L(s, \psi_{K/F}) = \zeta_K(s) \zeta_F(s)^{-1} = L(s, \psi') L(s, \psi'').
\end{equation}

It is known that \( L(1-k, \omega) = -B_{k,\omega}/k \) for a positive odd integer \( k \) with \( \omega = \chi' \) or \( \chi'' \). Combining this with (2.7), we find that \( L(1-u, \psi_{K/F}) = u^{-2} B_{u,\chi'} B_{u,\chi''}. \) Observing
\[ \frac{\Gamma((2-u)/2)}{\Gamma((u+1)/2)} = \frac{(-1)^{(u-1)/2} 2^{u-1} \pi^{1/2}}{(u-1)!}, \]
we see that
\[ D_F^\mu((u - 1)! (2\pi)^{-u})^2 L(u, \psi_{K/F}) = n^{-2} N(D_{K/F})^{-\mu} B_{u,\chi'} B_{u,\chi''}. \]

It is known that \( D_K = D_F D_{F'} D_{F''}. \) Since \( D_K = N(D_{K/F}) D_F^2 \), we have
\begin{equation}
(2.8) \quad N(D_{K/F}) = D_F^{-1} D_{F'} D_{F''} = 4 D_F^{-1} D_{F''}
\end{equation}
\[ = \begin{cases} 
2^4 & \text{if } m \equiv 1 \pmod{4}, \\
2^2 & \text{if } m \equiv 2 \pmod{4}, \\
1 & \text{if } m \equiv 3 \pmod{4}.
\end{cases} \]

Substituting these into (2.6), we obtain (2.4).
3. Computation of the mass

3.1. Local Witt decompositions for $\varphi = 1_n$. Throughout this section, $L$ is a $g$-maximal lattice in $V$ with respect to $\varphi$. To compute $\mathfrak{m}(L)$, we need to obtain $e, f, [\tilde{L} : L]$, and $\lambda_v$ of Proposition 2.1. First of all, we determine a Witt decomposition for $\varphi = 1_n$ at each nonarchimedean prime $v$ of $F$ (Lemma 3.3). For this purpose, we use a quaternion algebra for $v | 2$ and $n \equiv \pm 3, 4 \pmod{8}$. By Wedderburn’s theorem, every quaternion algebra over $F_v$ is isomorphic to the matrix algebra $(F_v)^2_2$ or a division algebra. For a quaternion algebra $B$ over $F$ and $v \in \mathfrak{a} \cup \mathfrak{h}$, we say that $B$ is ramified at $v$ if $B \otimes_F F_v$ is a division algebra, and that $B$ is unramified at $v$ if $B \otimes_F F_v$ is isomorphic to $(F_v)^2_2$. It is well known that $B$ is ramified at $v$ if and only if the reduced norm of $B \otimes_F F_v$ is anisotropic over $F_v$, and that the reduced norm of the matrix algebra $(F_v)^2_2$ is equivalent to the isotropic symmetric matrix $\eta_2$.

**Lemma 3.1.** Let $p$ be a rational prime. Then we can take an anisotropic symmetric matrix $\theta_p$ of a Witt decomposition for $\varphi$ over $\mathbb{Q}_p$ of the following form:

1. If $p \equiv 1 \pmod{4}$, then
   \[ \theta_p = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases} \]

2. If $p \equiv 3 \pmod{4}$, then
   \[ \theta_p = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{4}, \\ \pm 1 & \text{if } n \equiv \pm 1 \pmod{4}, \\ 1_2 & \text{if } n \equiv 2 \pmod{4}. \end{cases} \]

3. If $p = 2$, then
   \[ \theta_p = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{8}, \\ \pm 1 & \text{if } n \equiv \pm 1 \pmod{8}, \\ \pm 1_2 & \text{if } n \equiv \pm 2 \pmod{8}, \\ \pm 1_3 & \text{if } n \equiv \pm 3 \pmod{8}, \\ 1_4 & \text{if } n \equiv 4 \pmod{8}. \end{cases} \]

Here $\theta_p = \emptyset$ means that $t_p = 0$, i.e. $\varphi$ is equivalent to $\eta_{n/2}$ over $\mathbb{Q}_p$, and “$\theta_p = \pm 1$ if $n \equiv \pm 1 \pmod{4}$” means that $\theta_p = 1$ (resp. $\theta_p = -1$) if $n \equiv 1 \pmod{4}$ (resp. $n \equiv -1 \pmod{4}$). The other double signs should be read similarly.

This fact can be seen by [6, Examples 5.16] or [8, (1.2)]. We remark that the constant $c_p$ in [8, (1.2)] is determined by the determinant of the decomposition.
We next give a Witt decomposition for $\varphi$ over $F_v$ for a totally real number field $F$, though we are interested in real quadratic ones. To do this, we define the Hilbert symbol $(a, b)_{F_v}$ over $F_v$ by

\[(a, b)_{F_v} = \begin{cases} 1 & \text{if } \text{diag}[a, b, -1] \text{ is isotropic over } F_v, \\ -1 & \text{otherwise}, \end{cases} \tag{3.1} \]

for $a, b \in F_v^\times$. We can verify that $(a, b)_{F_v} = 1$ if and only if there exists $0 \neq [x \ y] \in (F_v)^1_2$ such that $ax^2 + by^2 = 1$.

**Lemma 3.2.** Let $F$ be a totally real number field. Let $v \in h$, and let $p$ be the rational prime which lies below $v$. Then we can take an anisotropic symmetric matrix $\theta_v$ of a Witt decomposition for $\varphi$ over $F_v$ of the following form:

(1) Suppose $v \nmid 2$.

(i) If $p \equiv 1 \pmod{4}$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases} \]

(ii) If $p \equiv 3 \pmod{4}$ and $\sqrt{-1} \not\in F_v$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{4}, \\ \pm 1 & \text{if } n \equiv \pm 1 \pmod{4}, \\ 1_2 & \text{if } n \equiv 2 \pmod{4}. \end{cases} \]

(iii) If $p \equiv 3 \pmod{4}$ and $\sqrt{-1} \in F_v$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd}. \end{cases} \]

(2) Suppose $v | 2$.

(i) If $n \equiv 0 \pmod{8}$, then $\theta_v = \emptyset$.

(ii) If $n \equiv \pm 1 \pmod{8}$, then $\theta_v = \pm 1$.

(iii) If $n \equiv \pm 2 \pmod{8}$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } \sqrt{-1} \in F_v, \\ \pm 1_2 & \text{if } \sqrt{-1} \not\in F_v. \end{cases} \]

(iv) If $n \equiv \pm 3 \pmod{8}$, then

\[\theta_v = \begin{cases} \pm 1_3 & \text{if } (-1, -1)_{F_v} = -1, \\ \mp 1 & \text{if } (-1, -1)_{F_v} = 1. \end{cases} \]

(v) If $n \equiv 4 \pmod{8}$, then

\[\theta_v = \begin{cases} 1_4 & \text{if } (-1, -1)_{F_v} = -1, \\ \emptyset & \text{if } (-1, -1)_{F_v} = 1. \end{cases} \]

Here $\theta_v = \emptyset$ means that $t_v = 0$. 
Proof. (1) Suppose $v \nmid 2$. Let $\theta_p$ be as in Lemma 3.1. If $p \equiv 1 \pmod{4}$, then we can take $\theta_v = \theta_p$, and thus we have (i).

To prove (ii) and (iii), let $p \equiv 3 \pmod{4}$. We show only the case of $n \equiv 2 \pmod{4}$, since the other cases are trivial. In this case,

\[(3.2) \quad 1_2 \text{ is isotropic over } F_v \iff \sqrt{-1} \in F_v,\]

which proves (ii) and (iii).

(2) Suppose $v \mid 2$. If $n \equiv 0$, $\pm 1$, or $\pm 2 \pmod{8}$, then we can determine a decomposition for $\varphi$ over $F_v$ in the same way as in (1), which proves (i)–(iii).

To prove (iv) and (v), we take the quaternion algebra $B_0$ of Examples 5.16 which is only ramified at 2 and infinity: $B_0 = \mathbb{Q} + \mathbb{Q}a + \mathbb{Q}b + \mathbb{Q}ab$, where $a^2 = b^2 = -1$ and $ba = -ab$. Then the reduced norm of $B_0$ is equivalent to $1_4$ over $\mathbb{Q}$. Put $B = B_0 \otimes \mathbb{Q} F$. Then it can be verified that

\[(-1, -1)_{F_v} = 1 \iff -1 \in N_{F_v(\sqrt{-1})/F_v}(F_v(\sqrt{-1})^\times) \]

\[\iff \text{the reduced norm } N_{B/F} \text{ of } B \text{ is isotropic over } F_v,\]

which leads to (v). Furthermore, set $W = \{ x \in B \mid \text{Tr}_{B/F}(x) = 0 \}$ and let $\psi$ be the restriction of $N_{B/F}$ to $W$. Then $\psi$ is equivalent to $1_3$ over $F$, and it can be verified that $N_{B/F}$ is isotropic over $F_v$ if and only if $\psi$ is isotropic over $F_v$, which proves (iv).

Returning to real quadratic fields, we have

**Lemma 3.3.** Let $F = \mathbb{Q}(\sqrt{m})$ with a square free positive integer $m$. Let $v \in \mathfrak{h}$, and let $p$ be the rational prime which lies below $v$. Then we can take an anisotropic symmetric matrix $\theta_v$ of a Witt decomposition for $\varphi$ over $F_v$ of the following form:

(1) Suppose $v \nmid 2$.

(i) If $p \equiv 1 \pmod{4}$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd.} \end{cases}\]

(ii) If $p \equiv 3 \pmod{4}$ and $(\frac{D_F}{p}) \neq -1$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \equiv 0 \pmod{4}, \\ \pm 1 & \text{if } n \equiv \pm 1 \pmod{4}, \\ 1_2 & \text{if } n \equiv 2 \pmod{4}. \end{cases}\]

(iii) If $p \equiv 3 \pmod{4}$ and $(\frac{D_F}{p}) = -1$, then

\[\theta_v = \begin{cases} \emptyset & \text{if } n \text{ is even}, \\ 1 & \text{if } n \text{ is odd.} \end{cases}\]
(2) Suppose \( v \nmid 2 \).

(i) If \( n \equiv 0 \pmod{8} \), then \( \theta_v = \emptyset \).

(ii) If \( n \equiv \pm 1 \pmod{8} \), then \( \theta_v = \pm 1 \).

(iii) If \( n \equiv \pm 2 \pmod{8} \), then

\[
\theta_v = \begin{cases} 
0 & \text{if } m \equiv -1 \pmod{8}, \\
\pm 1_2 & \text{if } m \not\equiv -1 \pmod{8}.
\end{cases}
\]

(iv) If \( n \equiv \pm 3 \pmod{8} \), then

\[
\theta_v = \begin{cases} 
\pm 1_3 & \text{if } m \equiv 1 \pmod{8}, \\
\mp 1 & \text{if } m \not\equiv 1 \pmod{8}.
\end{cases}
\]

(v) If \( n \equiv 4 \pmod{8} \), then

\[
\theta_v = \begin{cases} 
1_4 & \text{if } m \equiv 1 \pmod{8}, \\
\emptyset & \text{if } m \not\equiv 1 \pmod{8}.
\end{cases}
\]

Here \( \theta_v = \emptyset \) means that \( t_v = 0 \).

Proof. First, we note that \( \sqrt{d} \in \mathbb{Q}_p \) if and only if \( p \) splits in the quadratic field \( \mathbb{Q}(\sqrt{d}) \). We see that if \( p \not\equiv 1 \pmod{4} \), that is, \( \sqrt{-1} \not\in \mathbb{Q}_p \), then \( \sqrt{-1} \in \mathbb{Q}_p \) if and only if \( \sqrt{-1} \in \mathbb{Q}_p \).

(3.3) \( \sqrt{-1} \in F_v \iff \sqrt{-m} \in \mathbb{Q}_p \).

To show this equivalence, we may assume that \( [F_v : \mathbb{Q}_p] = 2 \). Let \( \sqrt{-1} \in F_v \). As \( \sqrt{m} \in F_v \), there exist \( x, y \in \mathbb{Q}_p \) such that \( \sqrt{-m} = x + y\sqrt{m} \). Thus \( -m = x^2 + my^2 + 2xy\sqrt{m} \). If \( x = 0 \), then \( \sqrt{-1} \in \mathbb{Q}_p \). This contradicts \( \sqrt{-1} \not\in \mathbb{Q}_p \). Hence \( y = 0 \) and \( x^2 = -m \). The converse is trivial.

(1) Suppose \( v \nmid 2 \). Let \( \theta_p \) be as in Lemma 3.1. If \( p \equiv 1 \pmod{4} \), then we can take \( \theta_v = \theta_p \), and thus we have (i).

To prove (ii) and (iii), let \( p \equiv 3 \pmod{4} \). We show only the case of \( n \equiv 2 \pmod{4} \), since the other cases are trivial. In this case, \( \theta_p = 1_2 \). If \( p \) splits in \( F \), then \( \sqrt{-1} \not\in \mathbb{Q}_p = F_v \). If \( p \) does not split in \( F \), then \( p \) remains prime in \( F \), so \( \sqrt{-1} \in \mathbb{Q}_p \) if and only if \( \sqrt{-1} \in \mathbb{Q}_p \). Hence \( y = 0 \) and \( x^2 = -m \). The converse is trivial.

(2) Suppose \( v \mid 2 \). We note that \( \sqrt{-1} \not\in \mathbb{Q}_2 \). By (3.3) and (3.2), we know that \( m \equiv -1 \pmod{8} \) if and only if \( \sqrt{-1} \in F_v \). In this case, using the same method as in case (1)(iii), we can find a decomposition for \( \varphi \) over \( F_v \), since \( 1 \) is equivalent to \(-1 \) over \( F_v \).
We assume $m \not\equiv -1 \pmod{8}$ until the end of this proof. If $n \equiv 0$, ±1, or ±2 (mod 8), then we can take the decomposition for $\varphi$ over $F_v$ as in Lemma 3.2 because of (3.3), which proves (i)–(iii) for $m \not\equiv -1 \pmod{8}$.

To prove (v), though we have given a decomposition for $1_4$ by using the Hilbert symbol in Lemma 3.2, we now determine it in another way. We again take $B_0$ and $B$ as in Lemma 3.2. Since $B_0 \otimes \mathbb{Q} \mathbb{Q}_2$ is a division algebra, $1_4$ is anisotropic over $\mathbb{Q}_2$. By the local theory of quaternion algebras over algebraic number fields, a quaternion algebra over a nonarchimedean local field splits over an arbitrary quadratic field over the local field (cf. [1, VII, §2, Satz 4]). From this, we obtain

$$B \otimes_F F_v \cong (B_0 \otimes \mathbb{Q} \mathbb{Q}_2) \otimes_{\mathbb{Q}_2} F_v \cong \begin{cases} B_0 \otimes \mathbb{Q}_2 & \text{if 2 splits in } F, \\ (F_v)_{\beta} \otimes \mathbb{Q}_2 & \text{if 2 ramifies or remains prime in } F. \end{cases}$$

Therefore we have

$$\begin{cases} 1_4 \text{ is anisotropic over } F_v & \text{if } m \equiv 1 \pmod{8}, \\ 1_4 \text{ is equivalent to } \eta_2 \text{ over } F_v & \text{if } m \not\equiv 1 \pmod{8}. \end{cases}$$

Next, we prove the case $n \equiv 3 \pmod{8}$; the proof of the case $n \equiv -3 \pmod{8}$ can be obtained in the same way. Then we have

$$\begin{cases} 1_3 \text{ is anisotropic over } F_v & \text{if } m \equiv 1 \pmod{8}, \\ 1_3 \text{ is equivalent to } \text{diag}[\eta_1, -1] \text{ over } F_v & \text{if } m \not\equiv 1 \pmod{8}. \end{cases}$$

To show this, we first assume that $m \equiv 1 \pmod{8}$. Since $1_4$ is anisotropic over $F_v$, so is $1_3$. Next assume $m \not\equiv 1 \pmod{8}$. Then there exists an element $\beta_v$ in $\text{GL}_4(F_v)$ such that $\beta_v1_4 \cdot \beta_v = \eta_2$. We identify $(F_v)_{\beta}^3$ with the subspace of $(F_v)^4$ by the mapping $[x_1 x_2 x_3 x_4] \mapsto [x_1 x_4 x_2 a x_3 b t]$. Put $H = (F_v)^4_{\beta} \beta_v^{-1}$ and $W = (F_v)^3_{\beta} \beta_v^{-1}$. Now $W \cap (F_v)^3_{\beta} \beta_v^{-1} = \{0\}$ in $H$. Hence there exists $0 \neq y \in (F_v)^3_{\beta} \beta_v^{-1}$ such that $y_{\beta_v^{-1}} \in W \cap (F_v)^3_{\beta} \beta_v^{-1}$. Then $1_3[y] = 1_4[y] = \eta_2[y_{\beta_v^{-1}}] = 0$, and so $1_3$ is isotropic over $F_v$. Put $y = [y_1 y_2 y_3]$. Then $y_1^2 + y_2^2 + y_3^2 = 0$. We may assume that $y_2 \neq 0$. Then we have $N_{F_v(\sqrt{-1})/F_v}(z) = -1$ with $z = y_1 \sqrt{-1}(y_2 + y_3 \sqrt{-1})^{-1} \in F_v(\sqrt{-1})$. From this, $1_2$ is equivalent to $-1_2$ over $F_v$. Thus $1_3 = \text{diag}[1, 1, 2]$ is equivalent to $\text{diag}[\eta_1, -1]$, which is the desired decomposition.

3.2. The relative discriminant $D_{K/F}$. Put $K = F(\sqrt{-1})$. We give the relative discriminant $D_{K/F}$ and the relative different $\mathfrak{d}_{K/F}$ of $K/F$, which is needed in the proof of Lemma 3.4:

$$D_{K/F} = \begin{cases} 4g & \text{if } m \equiv 1 \pmod{4}, \\ 2g & \text{if } m \equiv 2 \pmod{4}, \\ g & \text{if } m \equiv 3 \pmod{4}, \end{cases}$$
where \( P \) is the ring of integers of \( K \), and the product runs through primes \( v \) in \( F \) dividing 2. To show these, we first determine \( D_{K/F} \). We see that
\[
 v \mid D_{K/F} \quad \text{only if} \quad v \mid 2 \quad \text{by (2.8). If} \quad m \equiv 5 \pmod{8}, \quad \text{then} \quad D_{K/F} = 2g, \quad \text{since} \quad 2 \quad \text{remains prime in} \quad F \quad \text{and (2.8) holds. If} \quad m \equiv 2 \pmod{4}, \quad \text{then} \quad D_{K/F} = 2g, \quad \text{since} \quad 2 \quad \text{ramifies in} \quad F \quad \text{and we have (2.8). If} \quad m \equiv 3 \pmod{4}, \quad \text{then} \quad D_{K/F} = g \quad \text{by (2.8). Let} \quad m \equiv 1 \pmod{8}. \quad \text{Then} \quad 2 \quad \text{splits in} \quad F. \quad \text{For every} \quad \sigma \in \text{Gal}(K/Q), \quad \text{we have} \quad \delta_{K/F}^\sigma = \delta_{K/Q}^\sigma/(\delta_F/Q)^\sigma = \delta_{K/Q}/\delta_F/Q \tau = \delta_{K/F}, \quad \text{since} \quad K/Q \quad \text{is an abelian extension. Thus} \quad D_{K/F}^\sigma = D_{K/F}, \quad \text{which implies} \quad D_{K/F} = 2g \quad \text{by (2.8).}
\]

Next we prove (3.5). The case \( m \equiv 3 \pmod{4} \) is clear. If \( m \neq 3 \pmod{4} \), then \( v \) is ramified in \( K \) for any \( v \mid 2 \) from (3.4) and Dedekind’s discriminant theorem, that is, there exists a prime \( w \) in \( K \) such that \( v \tau = w^2. \) If \( m \equiv 1 \pmod{8} \), then \( D_{K/F} = N_{K/F}(\delta_{K/F}) = 2g = \prod v v^2 \) by (3.4), and hence we have \( \delta_{K/F} = \prod_{v|2} \prod_{w|v} w^2 = \prod v \tau^2. \) The other cases can be shown in the same way.

3.3. The index \([\widetilde{L}_v : L_v]\). The index \([\widetilde{L}_v : L_v]\) for \( v \in h \) is given by \([6, (3.2.1)]\):
\[
 [\widetilde{L}_v : L_v] = \begin{cases} 
 1 & \text{if} \quad t_v = 0, \\
 [g_v : 2g_v] & \text{if} \quad t_v = 1 \text{ and } v \mid 2, \\
 1 & \text{if} \quad t_v = 1 \text{ and } v \nmid 2, \\
 q_v^2 & \text{if} \quad t_v = 2, \quad \delta_v = r_v, \quad \text{and} \quad c_v \in \pi_v g_v^\times, \\
 1 & \text{if} \quad t_v = 2, \quad \delta_v = r_v, \quad \text{and} \quad c_v \in g_v^\times, \\
 [r_v : \delta_v] & \text{if} \quad t_v = 2 \text{ and } \delta_v \neq r_v, \\
 [g_v : 2g_v]q_v^2 & \text{if} \quad t_v = 3, \\
 q_v^2 & \text{if} \quad t_v = 4.
\end{cases}
\]

Here \( r_v, \delta_v, \) and \( c_v \) are as in (2.1), and \( \widetilde{L}_v = \{ y \in (F_v)^n | 2 \varphi(x, y) \in g_v \) for every \( x \in L_v \}. \) Using this result, we have the following

**Lemma 3.4.** Let \( L \) be a \( g \)-maximal lattice in \( V \) with respect to \( \varphi \). Let \( v \in h \).

1. Suppose \( v \nmid 2 \). Then \( [\widetilde{L}_v : L_v] = 1 \).
2. Suppose \( v \mid 2 \).
   
   (i) If \( n \equiv 0 \pmod{8} \), then \( [\widetilde{L}_v : L_v] = 1 \).
   
   (ii) If \( n \equiv \pm 1 \pmod{8} \), then
   \[
   [\widetilde{L}_v : L_v] = \begin{cases} 
 2 & \text{if} \quad m \equiv 1 \pmod{8}, \\
 2^2 & \text{otherwise}.
\end{cases}
\]
(iii) If \( n \equiv \pm 2 \pmod{8} \), then
\[
\tilde{L}_v : L_v = \begin{cases} 
1 & \text{if } m \equiv 3 \pmod{4}, \\
2^4 & \text{if } m \equiv -3 \pmod{8}, \\
2^2 & \text{otherwise}.
\end{cases}
\]

(iv) If \( n \equiv \pm 3 \pmod{8} \), then
\[
\tilde{L}_v : L_v = \begin{cases} 
2^3 & \text{if } m \equiv 1 \pmod{8}, \\
2^2 & \text{otherwise}.
\end{cases}
\]

(v) If \( n \equiv 4 \pmod{8} \), then
\[
\tilde{L}_v : L_v = \begin{cases} 
2^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise}.
\end{cases}
\]

Proof. (1) There exists a \( g \)-maximal lattice \( M \) in \( V \) with respect to \( \varphi \) including \( g_n \). Then \( g_n \subset M \subset \tilde{M} \subset g_n \). Since \( \tilde{g}_n = 2^{-1}g_n \), we have \( \tilde{(g_v)_n} = (g_v)_n \) for every \( v \nmid 2 \), so that \( \tilde{M}_v = M_v \). Since \( M \) is in the genus of \( L \), there exists \( \alpha \in G_A \) such that \( L\alpha = M \). Thus we have \( L_v\alpha_v = M_v \). Hence
\[
[\tilde{L}_v : L_v] = [\tilde{M}_v : M_v] = 1 \text{ if } v \nmid 2.
\]

(2) Suppose \( v \mid 2 \). (i) If \( n \equiv 0 \pmod{8} \), then \( 1_n \) is equivalent to \( \eta_{n/2} \) over \( F_v \), that is, \( t_v = 0 \) for every \( v \in h \) by Lemma 3.3. Thus the assertion (i) can be easily obtained by (3.6).

(ii) If \( n \equiv \pm 1 \pmod{8} \), then \( t_v = 1 \) for every \( v \in h \) by Lemma 3.3. Thus, by (3.6), we have
\[
[\tilde{L}_v : L_v] = [g_v : 2g_v] = \begin{cases} 
[g : v] & \text{if } \text{2 splits in } F, \\
[g : 2g] & \text{if } \text{2 remains prime in } F, \\
[g : v]^2 & \text{if } \text{2 ramifies in } F
\end{cases}
= \begin{cases} 
2 & \text{if } m \equiv 1 \pmod{8}, \\
2^2 & \text{otherwise}.
\end{cases}
\]

Thus we have (ii).

(iii) If \( m \equiv -1 \pmod{8} \), then \( t_v = 0 \) by Lemma 3.3, and so \( [\tilde{L}_v : L_v] = 1 \). If \( m \neq -1 \pmod{8} \), then \( t_v = 2 \) by Lemma 3.3. In this case, by Dedekind’s discriminant theorem and (3.4), \( F_v(\sqrt{-1})/F_v \) is a ramified (resp. an unramified) quadratic extension if \( m \equiv 1, 2 \pmod{4} \) (resp. \( m \equiv 3 \pmod{8} \)). Let \( m \equiv 3 \pmod{8} \). Then \( t_v = 2 \) and \( d_v = r_v \). Since we can take \( \theta_v = \pm 1 \) in Lemma 3.3, we have \( c_v = b_v^{-1} = \pm 1 \in g_v^x \) as shown below (2.1). Thus
\[
[\tilde{L}_v : L_v] = 1 \text{ by } (3.6). \]
If \( m \equiv 1, 2 \pmod{4} \), then \( d_v = \pi_{K_v}^2 r_v \) by (3.5). Here \( \pi_{K_v} \) is a prime element of \( K_v = F_v(\sqrt{-1}) \). Thus by (3.6), we have
\[
[\tilde{L}_v : L_v] = [r_v : d_v] = [r_v : \pi_{K_v}^2 r_v]. \]
Since \( v \) ramifies in \( K \), we have
\( r_v / \pi_{K_v}^2 r_v \cong g_v / \pi_v g_v \), and hence \([r_v : \pi_{K_v}^2 r_v] = q_v^2 \). Therefore

\[
[\tilde{L}_v : L_v] = q_v^2 = \begin{cases} 
2^2 & \text{if } m \equiv 1, \pm 2 \pmod{8}, \\
2^4 & \text{if } m \equiv -3 \pmod{8}.
\end{cases}
\]

(iv) Let \( n \equiv \pm 3 \pmod{8} \). Combining Lemma 3.3 with (3.6), we have

\[
[\tilde{L}_v : L_v] = \begin{cases} 
[\tilde{g}_v : 2\tilde{g}_v] q_v^2 & \text{if } m \equiv 1 \pmod{8}, \\
[\tilde{g}_v : 2\tilde{g}_v] & \text{if } m \not\equiv 1 \pmod{8},
\end{cases}
\]

and hence we get (iv).

(v) Let \( n \equiv 4 \pmod{8} \). Then

\[
[\tilde{L}_v : L_v] = \begin{cases} 
q_v^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{if } m \not\equiv 1 \pmod{8}
\end{cases}
\]

by Lemma 3.3 and (3.6), which proves (v). 

We note that \([\tilde{L}_v : L_v] \neq 1\) if and only if \( v | D_{K/F} \) when \( n \equiv \pm 2 \pmod{8} \).

It is well known that \([\tilde{L} : L] = \prod_{v \in \mathfrak{h}}[\tilde{L}_v : L_v] \). Combining this with Lemma 3.4, we have the following

**Lemma 3.5.** Let \( L \) be a \( g \)-maximal lattice in \( V \) with respect to \( \varphi \).

1. If \( n \equiv 0 \pmod{8} \), then \([\tilde{L} : L] = 1 \).
2. If \( n \equiv \pm 1 \pmod{8} \), then \([\tilde{L} : L] = 2^2 \).
3. If \( n \equiv \pm 2 \pmod{8} \), then

\[
[\tilde{L} : L] = \begin{cases} 
1 & \text{if } m \equiv 3 \pmod{4}, \\
2^4 & \text{if } m \equiv -3 \pmod{8}, \\
2^2 & \text{otherwise}.
\end{cases}
\]

4. If \( n \equiv \pm 3 \pmod{8} \), then

\[
[\tilde{L} : L] = \begin{cases} 
2^6 & \text{if } m \equiv 1 \pmod{8}, \\
2^2 & \text{otherwise}.
\end{cases}
\]

5. If \( n \equiv 4 \pmod{8} \), then

\[
[\tilde{L} : L] = \begin{cases} 
2^4 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise}.
\end{cases}
\]

**3.4. Formula for computation.** Summing up Lemmas 3.3–3.5 and Proposition 2.1, we obtain the following

**Theorem 3.6.** Let \( F = \mathbb{Q}(\sqrt{m}) \) with a square free positive integer \( m \), and let \( n > 1 \). Let \( L \) be a \( g \)-maximal lattice in \( V \) with respect to \( \varphi \). Let \( \chi, \chi', \) and \( \chi'' \) be the Dirichlet characters corresponding to \( F/\mathbb{Q}, \mathbb{Q}(\sqrt{-1})/\mathbb{Q}, \) and \( \mathbb{Q}(\sqrt{-m})/\mathbb{Q}, \) respectively.
(1) If \( n \equiv 0 \pmod{8} \), then
\[
m(L) = n^{-2} B_{n/2} B_{n/2, \chi} \left( \prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k, \chi} \right).
\]

(2) If \( n \equiv \pm 1 \pmod{8} \), then
\[
m(L) = \prod_{k=1}^{(n-1)/2} (4k)^{-2} B_{2k} B_{2k, \chi}.
\]

(3) If \( n \equiv \pm 2 \pmod{8} \), then
\[
m(L) = n^{-2} B_{n/2, \chi} B_{n/2, \chi''} \left( \prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k, \chi} \right)
\times \begin{cases} 
2^{-2} & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{if } m \equiv 3 \pmod{4}, \\
2^{-1} & \text{otherwise}.
\end{cases}
\]

(4) If \( n \equiv \pm 3 \pmod{8} \), then
\[
m(L) = \left( \prod_{k=1}^{(n-1)/2} (4k)^{-2} B_{2k} B_{2k, \chi} \right)
\times \begin{cases} 
2^{-2} \cdot 3^{-2} (2^{n-1} - 1)^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise}.
\end{cases}
\]

(5) If \( n \equiv 4 \pmod{8} \), then
\[
m(L) = n^{-2} B_{n/2} B_{n/2, \chi} \left( \prod_{k=1}^{[(n-1)/2]} (4k)^{-2} B_{2k} B_{2k, \chi} \right)
\times \begin{cases} 
2^{-2} \cdot 3^{-2} (2^{n/2-1} - 1)^2 (2^{n/2} - 1)^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise}.
\end{cases}
\]

Proof. (1) If \( n \equiv 0 \pmod{8} \), then substituting \( e = g \) and \( [\tilde{L} : L] = 1 \) into (2.3), we obtain (1).

(2) If \( n \equiv \pm 1 \pmod{8} \), then \( t_v = 1 \) for every \( v \in h \) and \( f = g \) by Lemma 3.3. By Lemma 3.4, we have \( e = \prod_{v|2} v \). Hence by Lemma 3.5 and (2.1),
\[
[\tilde{L} : L]^{(n-1)/2} \prod_{v|e} \lambda_v = 2^{n-1}.
\]
Substituting this into (2.2), we obtain (2).

(3) Suppose \( n \equiv \pm 2 \pmod{8} \). If \( m \equiv 1, 2 \pmod{4} \) and \( v|2 \), then \( t_v = 2 \) by Lemma 3.3. By (2.1) and Lemma 3.4, we have \( \lambda_v = 2^{-1} \) and \( e = \prod_{v|2} v \).
If $m \equiv 3 \pmod{4}$, then $e = g$. Thus by Lemma 3.5,

$$
[\tilde{L} : L]^{(n-1)/2} \prod_{v \mid \xi} \lambda_v = \begin{cases} 
2^{2(n-2)} & \text{if } m \equiv 1 \pmod{8}, \\
2^{2n-3} & \text{if } m \not\equiv -3 \pmod{8}, \\
2^{n-2} & \text{if } m \equiv 2 \pmod{4}, \\
1 & \text{if } m \equiv 3 \pmod{4}.
\end{cases}
$$

Combining this with (2.4), we obtain (3).

(4) Let $m \equiv 1 \pmod{8}$. By Lemma 3.3, we have $t_v = 3$ if $v \mid 2$. Thus, by Lemma 3.4, $e = \prod_{v \mid 2} v = f$. If $m \not\equiv 1 \pmod{8}$, then $f = g$ and $e = \prod_{v \mid 2} v$.

Hence by Lemma 3.5,

$$
2^{-n}[\tilde{L} : L]^{(n-1)/2} \prod_{v \mid \xi} \lambda_v = \begin{cases} 
2^{-2} \cdot 3^{-2} (2^{n-1} - 1)^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise}.
\end{cases}
$$

(5) Suppose $n \equiv 4 \pmod{8}$. If $m \equiv 1 \pmod{8}$, then $t_v = 4$ for $v \mid 2$ and $e = \prod_{v \mid 2} v$ by Lemmas 3.3 and 3.4. If $m \not\equiv 1 \pmod{8}$, then $e = g$. By Lemma 3.5,

$$
[\tilde{L} : L]^{(n-1)/2} \prod_{v \mid \xi} \lambda_v = \begin{cases} 
2^{-2} \cdot 3^{-2} (2^{n/2-1} - 1)^2 (2^{n/2} - 1)^2 & \text{if } m \equiv 1 \pmod{8}, \\
1 & \text{otherwise},
\end{cases}
$$

which proves (5). $\blacksquare$

4. Numerical example. In this section, we take $F = \mathbb{Q}(\sqrt{5})$ and $\varphi = 1_4$, and as an application of Theorem 3.6 we show that the genus of a $g$-maximal lattice in $V = F_4^1$ with respect to $\varphi$ consists of one class; moreover, we give a formula for $N(L, h)$ by using this fact and the formula due to Shimura in [7, Theorem 1.5].

Let $g = \mathbb{Z} + \mathbb{Z} \omega$ with $\omega = (1 + \sqrt{5})/2$. We note that $\omega$ is a fundamental unit of $F$. We consider a $g$-maximal lattice $L$ with respect to $\varphi$. Set

$$
L = \sum_{i=1}^{4} g \alpha_i = g_4^1 \alpha, \quad \alpha = \begin{bmatrix} \alpha_1 \\
\vdots \\
\alpha_4 \end{bmatrix} \in \text{GL}_4(F).
$$

A $g$-lattice $L$ in $V$ is $g$-maximal with respect to $\varphi$ if and only if $\varphi[x] \in g$ for every $x \in L$ and $[\tilde{L}_v : L_v] = 1$ for every $v \in h$. These conditions are equivalent to

$$(4.1) \quad \begin{cases} 
\varphi[\alpha_i] \in g & (1 \leq i \leq 4), \\
2\varphi(\alpha_i, \alpha_j) \in g & (i \neq j), \\
\det(\alpha) \in 2^{-2}g^\times.
\end{cases}$$
The last condition can be seen by using elementary divisors. Put
\[
\alpha = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
1/2 & (1 + \omega)/2 & \omega/2 & 0 \\
1/2 & 1/2 & 1/2 & 1/2
\end{bmatrix};
\]
then \( \alpha \) satisfies (4.1). Hence \( L = g_{4}^{1} \alpha \) is a \( g \)-maximal lattice with respect to \( \varphi \). We remark that \( L \) contains \( g_{4}^{1} \).

We shall calculate the order of \( \Gamma = \{ \gamma \in G \mid L \gamma = L \} \) for this \( L \). We take the subgroup
\[
\Gamma_{1} = \{ \gamma \in \Gamma \mid e_{1} \gamma = e_{1} \},
\]
where \( \{ e_{i} \}_{i=1}^{4} \) is the standard basis of \( V \). We will calculate \([ \Gamma : \Gamma_{1} ]\) and \([ \Gamma_{1} : 1 ]\). For \( \gamma = t^{1}[\gamma_{1} \cdots \gamma_{4}] \in F_{4}, \gamma \in \Gamma_{1} \) if and only if
\[
\begin{align*}
\gamma_{i} &\in n(L, 1) \quad (1 \leq i \leq 4), \quad \gamma_{i} \cdot t_{i} \gamma_{j} = 0 \quad (i \neq j), \\
2^{-1}(\gamma_{1} + (1 + \omega)\gamma_{2} + \omega \gamma_{3}) &\in n(L, 1 + \omega), \\
2^{-1}(\gamma_{1} + \gamma_{2} + \gamma_{3} + \gamma_{4}) &\in n(L, 1).
\end{align*}
\]
Here \( n(M, h) = \{ x \in M \mid \varphi[x] = h \} \) for a \( g \)-lattice \( M \) in \( V \) and \( h \in F \). This can be verified as in [2].

We need \( n(L, 1) \) to use condition (4.3). As \( g_{4}^{1} \subset L \), we see that \( L \subset \widetilde{L} \subset g_{4}^{1} = (2^{-1}g_{4})^{1} \). So we only have to determine \( n((2^{-1}g_{4})^{1}, 1) \), and find the elements in the set satisfying the following condition: For \( y = [x_{1}/2 \cdots x_{4}/2] \in (2^{-1}g_{4})^{1} \) and \( x_{i} = a_{i} + b_{i}\omega \), the condition \( y \in L \) is equivalent to the conditions
\[
\begin{align*}
\begin{cases}
a_{2} + b_{3} + a_{4} + b_{4} \equiv b_{2} + a_{3} + b_{3} + a_{4} \equiv 0 \pmod{2}, \\
a_{1} + a_{3} + b_{3} + b_{4} \equiv b_{1} + a_{3} + a_{4} + b_{4} \equiv 0 \pmod{2}.
\end{cases}
\end{align*}
\]
The set \( n((2^{-1}g_{4})^{1}, 1) \) can be determined in the following way: Let \( x/2 \) be an element of \( (2^{-1}g_{4})^{1} \), and solve the equation \( \varphi[x] = 4 \) instead of \( \varphi[x/2] = 1 \). Furthermore, \( \varphi[x] \) is of the form
\[
\varphi[x] = \begin{bmatrix} 1_{4} & 0 \\
0 & 1_{4} \end{bmatrix} \begin{bmatrix} [\widehat{x}] \\
[\widehat{x}] \end{bmatrix} + \begin{bmatrix} 0 & 1_{4} \\
1_{4} & 1_{4} \end{bmatrix} \begin{bmatrix} [\widehat{x}] \end{bmatrix} \cdot \omega.
\]
Here we set \( \widehat{x} = [a_{1} \cdots a_{4} b_{1} \cdots b_{4}] \in \mathbb{Z}_{8}^{4} \) for \( x = [x_{1} \cdots x_{4}] \in g_{4}^{1} \) and \( x_{i} = a_{i} + b_{i}\omega \). Calculating \( \widehat{x} \) in \( \mathbb{Z}_{8}^{4} \) satisfying the conditions
\[
\begin{align*}
\begin{cases}
\begin{bmatrix} 1_{4} & 0 \\
0 & 1_{4} \end{bmatrix} \begin{bmatrix} [\widehat{x}] \end{bmatrix} = 4, \\
\begin{bmatrix} 0 & 1_{4} \\
1_{4} & 1_{4} \end{bmatrix} \begin{bmatrix} [\widehat{x}] \end{bmatrix} = 0,
\end{cases}
\end{align*}
\]
we get \( n((2^{-1}g)_{1}, 1) \). Employing condition (4.4) for this set, we find
\[
\begin{align*}
n(L, 1) &= n((2^{-1}g)_{1}, 1) \cap L \\
&= \{ \pm e_{i} \mid 1 \leq i \leq 4 \} \cup \{(\delta_{1}/2, \ldots, \delta_{4}/2)\} \\
&\quad \cup\{0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2\} \cup \{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \\
&\quad \cup\{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \cup \{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \\
&\quad \cup\{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \cup \{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \\
&\quad \cup\{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \cup \{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \\
&\quad \cup\{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\} \cup \{(0, \delta_{2}/2, \delta_{3}/2, \delta_{4}/2)\}.
\end{align*}
\]

Here we allow \( \delta_{i} \) to take the values \( \pm 1 \) at random in each set, and \( \omega^{e} = 1 - \omega = -\omega^{-1} \) with the nontrivial automorphism \( \omega \) of \( F \) over \( \mathbb{Q} \).

We compute \( [\Gamma : \Gamma_{1}] \). For \( \gamma, \delta \in \Gamma \), we denote by \( \gamma_{1} \) the first row vector of \( \gamma \in \Gamma \). Since \( \Gamma_{1}\gamma = \Gamma_{1}\delta \) if and only if \( \gamma_{1} = \delta_{1} \), we see that \( [\Gamma : \Gamma_{1}] \leq N(L, 1) \); moreover, for any \( \gamma_{1} \in n(L, 1) \) there exists an element of \( \Gamma \) which has \( \gamma_{1} \) as the first row vector. This can be verified by using \( n(L, 1) \) and taking an element of \( \Gamma \) satisfying (4.3). Thus
\[
[\Gamma : \Gamma_{1}] = N(L, 1) = 120.
\]

To compute \([\Gamma_{1} : 1]\), we use the condition
\[
2^{-1}(\omega^{-1}\gamma_{1} + \omega^{-1}(1 + \omega)\gamma_{2} + \gamma_{3}) \in n(L, 1)
\]
instead of
\[
2^{-1}(\gamma_{1} + (1 + \omega)\gamma_{2} + \omega\gamma_{3}) \in n(L, 1 + \omega)
\]
in (4.3), because \( n(L, 1 + \omega) = \omega \cdot n(L, 1) \). From this, \( \Gamma_{1} \) consists of all elements \( \gamma = t^{t}e_{1} t^{t}g_{2} t^{t}\gamma_{3} t^{t}\gamma_{4} \in F_{4}^{t} \) such that
\[
\begin{align}
\gamma_{i} \in n(L, 1) & \quad (2 \leq i \leq 4), \quad \gamma_{i}, t^{t}\gamma_{j} = 0 \quad (i \neq j), \\
2^{-1}(\omega^{-1}\gamma_{1} + \omega^{-1}(1 + \omega)\gamma_{2} + \gamma_{3}) & \in n(L, 1), \\
2^{-1}(\gamma_{1} + \gamma_{2} + \gamma_{3} + \gamma_{4}) & \in n(L, 1).
\end{align}
\]

Since \( n(L, 1) \) has been determined, we can choose the pair \( (e_{1}, \gamma_{2}, \gamma_{3}, \gamma_{4}) \) satisfying (4.5). After some calculations, we obtain \([\Gamma_{1} : 1] = 120\). Thus \([\Gamma : 1] = 120 \cdot 120 = 2^{6} \cdot 3^{2} \cdot 5^{2}\).

On the other hand, we see that \( m(L) = 2^{-6} \cdot 3^{-2} \cdot 5^{-2} \) by Theorem 3.6. Therefore the genus of \( L \) consists of one class.
Next, to introduce the formula in [7, Theorem 1.5(II)], we define $\gamma_v(s)$ by the following formulas (cf. [7, §1.6]):

$$
\gamma_v(s) = \begin{cases}
1 - \frac{q_v^{(\nu+1)(1-s)}}{1 - q_v^{1-s}} & \text{if } t_v = 0, \\
1 - \frac{(-q_v^{1-s})^{\nu+1}}{1 + q_v^{-1}} & \text{if } t_v = 2, \mathfrak{d}_v = \tau_v, \text{ and } c_v \in g_v^\times, \\
1 - \frac{(-q_v^{1-s})^{\nu+2} + q_v^{2-s}(1 - (-q_v^{1-s})^\nu)}{(1 + q_v^{-s})(1 + q_v^{1-s})} & \text{if } t_v = 2, \mathfrak{d}_v = \tau_v, \text{ and } c_v \in \pi_v g_v^\times, \\
1 + \psi_v(c_v h)(|h|_v^{-1} N(\mathfrak{d}_v))^{1-s} & \text{if } t_v = 2 \text{ and } \mathfrak{d}_v \neq \tau_v, \\
1 - \frac{q_v^{(\nu+2)(1-s)} - q_v^{2-s}(1 - q_v^{\nu(1-s)})}{(1 - q_v^{-s})(1 - q_v^{1-s})} & \text{if } t_v = 4,
\end{cases}
$$

where $\nu \in \mathbb{Z}$ is determined by $|h|_v^{-1} = q_v^\nu$ with the normalized valuation $|v|$ at $v$ of $F_v$.

**Theorem 4.1 (Shimura).** Let $F$ be a totally real algebraic number field, and put $n = 2u$ and $V = F_n^1$. Let $\varphi$ be a totally positive symmetric matrix of $\text{GL}_n(F)$, $L$ a $g$-maximal lattice in $V$ with respect to $\varphi$, and $h$ a totally positive element of $g$. Put $K = F(\sqrt{(-1)^{n/2} \det(\varphi)})$, and denote by $\mathfrak{d}$ the different of $K/F$ and by $\psi_{K/F}$ the Hecke character of $F$ corresponding to the extension $K/F$. Then

$$
(4.6) \quad \frac{R(L, h)}{m(L)} = \frac{N_{F/Q}(h)^{u-1}}{D_F^{(n-1)/2}N(\mathfrak{d})^{1/2}N(f)\{(u - 1)! (2\pi)^{-u}\}^2 L(u, \psi_{K/F})} \times \prod_{v \mid h \mathfrak{e}} \gamma_v(u).
$$

Here $\mathfrak{e}$ is the product of all primes $v$ satisfying $\mathfrak{L}_v \neq L_v$, $f$ is the product of prime factors of $\mathfrak{e}$ unramified in $K$, and $\gamma_v(s)$ is given as above.

We specialize this formula to the case where $F = \mathbb{Q}(\sqrt{5})$, $V = F_n^1$, and $\varphi = 1_4$. We easily see that $K = F$. By Lemma 3.4, we have $\mathfrak{e} = g$. Since $t_v = 0$ for every $v \in h$ by Lemma 3.3, $\gamma_v(2)$ is given by

$$
\gamma_v(2) = \frac{1 - q_v^{(e_v+1)}}{1 - q_v^{-1}}
$$

for a totally positive element $h$ of $g$, where the principal ideal of $h$ is...
(h)_F = \prod_v v^{e_v}. Now, the genus of a g-maximal lattice with respect to \varphi consists of one class. Thus N(L, h) = R(L, h)m(L)^{-1}. Hence by Theorem 4.1, we obtain

\begin{equation}
N(L, h) = \frac{N_{F/Q}(h)}{5^{3/2}(2\pi)^{-4}\zeta_F(2)} \prod_{v|h} \frac{1 - q_v^{-(e_v+1)}}{1 - q_v^{-1}}
= 120 \prod_{v|h} \left( \sum_{i=0}^{e_v} q_v^i \right).
\end{equation}

Thus N(L, h)/120 coincides with the Fourier coefficient of the Eisenstein series \(E_2\) in the space \(M_{(2,2)}(g, \text{id})\) of all Hilbert modular forms for \(GL(2)\) over \(F = \mathbb{Q}(\sqrt{5})\) of weight \(2, 2\), level \(g\), and with the identity character. It is known that the dimension of this space is 1 (see [3, §3.4], for example), that is, \(M_{(2,2)}(g, \text{id}) = \mathbb{C} \cdot E_2\) in this case.

5. Numerical table of \(m(L)\). Let \(F = \mathbb{Q}(\sqrt{m})\) with a square free positive integer \(m\), and let \(\varphi = 1_n\). Using Theorem 3.6, we give a table of \(m(L)\) for several quadratic fields \(F\).

We note that the \(l\)th generalized Bernoulli number \(B_{l, \psi}\) associated with a Dirichlet character \(\psi\) can be computed by the following recursion formula (cf. [4]):

\[ B_{l, \psi} = \frac{1}{f} \sum_{a=1}^{f} \psi(a)a^l - \sum_{k=1}^{l-1} \frac{f^{l-k}}{k} \binom{l}{k-1} B_{k, \psi} \quad (0 < l \in \mathbb{Z}), \]

\[ B_{0, \psi} = 0. \]

Here \(f\) is the conductor of \(\psi\) and we put \(\binom{l}{r} = l!/(l-r)!r!\).

\[
\begin{array}{|c|cccccc|}
\hline
m & 2 & 3 & 4 & 5 & 6 \\
\hline
2 & 2^7 & 2^{14} & 3^2 & 2^{9.3}.5 & 2^{11} & 2^{11} \cdot 3.5 \\
3 & 2^{13} & 2^{14} & 3^2 & 2^{23} & 2^{23} & 2^{23} \cdot 3.5 \\
5 & 2^7 & 2^{13} & 3^2 & 2^{2^3 \cdot 5^2}.2^2 & 2^{113} & 2^{113} \cdot 2^{2^3 \cdot 5^2} \\
6 & 2^7 & 2^7 & 3^2 & 2^{3.7^2} & 2^{7.2^3} & 2^{7.2^3} \cdot 7^2 \cdot 5 \\
7 & 2^7 & 2^7 & 2^{2^3 \cdot 5^2}.2^2 & 2^{113} & 2^{113} \cdot 2^{2^3 \cdot 5^2} \\
10 & 2^7 & 2^7 & 3^2 & 2^{2^3 \cdot 5^2}.2^2 & 2^{7.2^3} & 2^{7.2^3} \cdot 5 \\
11 & 2^7 & 2^7 & 3^2 & 2^{2^3 \cdot 5^2}.2^2 & 2^{7.2^3} & 2^{7.2^3} \cdot 5 \\
13 & 2^7 & 2^7 & 3^2 & 2^{2^3 \cdot 5^2}.2^2 & 2^{29} & 2^{29} \cdot 2^{2^3 \cdot 5^2} \\
14 & 2^7 & 2^7 & 3^2 & 2^{2^3 \cdot 5^2}.2^2 & 2^{503} & 2^{11.2503} \\
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\end{array}
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Shimura’s mass formula

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Department of Mathematical Sciences
Ritsumeikan University
Kusatsu, Shiga 525-8577, Japan
E-mail: rp056956@se.ritsumei.ac.jp

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