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The growth rates of digits in the Oppenheim series expansions

by

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1. Introduction. Let $a_n(j)$ and $b_n(j)$, $n \ge 1$, be two sequences of positive integer-valued functions of the positive integer $j \ge 1$. The algorithm $0 < x \le 1$, $x = x_1$, and, for any $n \ge 1$ and some positive integers $d_n(x)$,

$$\frac{1}{d_n(x)} < x_n \le \frac{1}{d_n(x)-1}, \qquad x_n = \frac{1}{d_n(x)} + \frac{a_n(d_n(x))}{b_n(d_n(x))}\, x_{n+1} \tag{1}$$

leads to the series expansion

$$x = \frac{1}{d_1(x)} + \sum_{n=1}^{\infty} \frac{a_1(d_1(x))\cdots a_n(d_n(x))}{b_1(d_1(x))\cdots b_n(d_n(x))} \cdot \frac{1}{d_{n+1}(x)}, \tag{2}$$

which is called the *Oppenheim series expansion* of x. Set

$$h_n(j) = \frac{a_n(j)}{b_n(j)}\, j(j-1), \qquad j \ge 2. \tag{3}$$

If $h_n(j)$ is integer-valued ($n \ge 1$, $j \ge 2$), then (2) is termed the *restricted Oppenheim series expansion* of x. Here and in what follows, we always assume h_j is integer-valued for all $j \ge 1$.

The algorithm (1) implies

$$d_1(x) \ge 2, \qquad d_{n+1}(x) \ge h_n(d_n(x)) + 1 \quad \text{for any } n \ge 1. \tag{4}$$

On the other hand, any integer sequence $\{d_n, n \ge 1\}$ satisfying (4) is an Oppenheim admissible sequence, that is, there exists a unique $x \in (0,1]$ such that $d_n(x) = d_n$ for any $n \ge 1$. The representation (2) under (1) is unique.

The representation (2) under (1) was first studied by A. Oppenheim [7] who established its arithmetical properties, including the question of rationality of the expansion. The foundations of the metric theory were laid down by J. Galambos [2]–[4], [6]; see also the monographs of J. Galambos [5],

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F. Schweiger [8] and W. Vervaat [9]. In particular, concerning the growth of $\{d_n(x), n \geq 1\}$ J. Galambos [5, p. 93] obtained the following interesting result:

DEFINITION 1.1. Let $\beta \geq 1$. We say that the function $h_n(j)$ is *of order* β if there are constants $0 < C_1 \leq C_2$ such that

$$C_1 \leq h_n(j)/j^\beta \leq C_2 \quad \text{for all } n \text{ and } j. \tag{5}$$

THEOREM 1.2. *Let $h_n(j)$ be of order $\beta > 1$. Then, for almost all $x \in (0,1]$, the limit*

$$\lim_{n\to\infty} \beta^{-n} \log d_n(x) = G(x)$$

exists. Its value equals the finite series

$$G(x) = \beta^{-1}\Big\{ \log d_1(x) + \sum_{n=1}^{\infty} \beta^{-n} \log(d_{n+1}(x) d_n(x)^{-\beta})\Big\}.$$

From Theorem 1.2, we deduce that when $h_n(j)$ is of order $\beta > 1$, then for almost all $x \in (0,1]$,

$$\lim_{j\to\infty} \frac{\log d_{j+1}(x)}{\log h_j(d_j(x))} = 1. \tag{6}$$

Hence a natural problem is to discuss the size of the sets with different growth rates of $\{d_n(x),\, n \geq 1\}$. More precisely, for any $\alpha \geq 1$, let

$$B_\alpha = \Big\{ x \in (0,1] : \lim_{j\to\infty} \frac{\log d_{j+1}(x)}{\log h_j(d_j(x))} = \alpha \Big\}.$$

What is the size of B_α? In this paper, we calculate its Hausdorff dimension. The situation here is quite complicated and we carefully construct a Cantor set $E \subset B_\alpha$ such that the Hausdorff dimension of E approximates that of B_α. Some other exceptional sets associated with the Oppenheim series expansion were discussed in [10]–[12].

We use $|\cdot|$ to denote the diameter of a subset of $(0,1]$, $\dim_{\mathrm{H}}$ to denote the Hausdorff dimension and cl for the closure of a subset of $(0,1]$.

2. Hausdorff dimension of B_α.

In this section, we give the main result of this paper.

We start with the mass distribution principle (see [1, Proposition 2.3]) that will be used later.

LEMMA 2.1. *Let $E \subset (0,1]$ be a Borel set, and μ a measure with $\mu(E) > 0$. If for any $x \in E$,*

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then $\dim_{\mathrm{H}} E \geq s$.

The following result is proved in [10].

LEMMA 2.2. *Suppose $h_n(j) \geq j-1$ for any $n \geq 1$ and $j \geq 2$. Then for any $m \geq 3$, the set*

$$C_m = \left\{x \in (0,1] : 1 < \frac{d_j(x)}{h_{j-1}(d_{j-1}(x))} \leq m \text{ for any } j \geq 2\right\}$$

has Hausdorff dimension 1.

Now we state our main result.

THEOREM 2.3. *Let $h_n(j)$ be of order $\beta \geq 1$ and $h_n(j) \geq j-1$ for any $n \geq 1$ and $j \geq 2$. Then for any $\alpha \geq 1$,*

$$\dim_{\mathrm{H}} B_\alpha = \frac{1}{(\alpha-1)\beta+1}.$$

Proof. Let

$$D = \{x \in (0,1] : \limsup_{j\to\infty} d_j(x) < \infty\}.$$

By (4) and the assumption $h_n(j) \geq j-1$ for any $n \geq 1$ and $j \geq 2$, we have $d_{j+1}(x) \geq h_j(d_j(x)) + 1 \geq d_j(x)$. Thus for any $x \in D$, $d_{j+1}(x) = d_j(x)$ ultimately, and therefore

$$D \subset \bigcup_{k=1}^{\infty}\bigcup_{t=2}^{\infty}\{x \in (0,1] : d_j(x) = t \text{ for any } j \geq k\},$$

which implies D is countable.

If $\alpha = 1$, then for any $x \in C_m \setminus D$, where C_m is defined in Lemma 2.2, we have

$$\lim_{j\to\infty} \frac{\log d_{j+1}(x)}{\log h_j(x)} = 1.$$

Thus $C_m \setminus D \subset B_1$. By Lemma 2.2, we have $\dim_{\mathrm{H}} B_1 = 1$.

In the following, we always assume that $\alpha > 1$. We divide the proof into two parts.

PART I: *Upper bound*. Let $\varepsilon < \min\{\alpha - 1/\beta, \alpha - 1\}$. For any $x \in B_\alpha \setminus D$, from the definition of B_α, there exists j_0 such that for any $j \geq j_0$,

$$h_j^{\alpha-\varepsilon}(d_j(x)) < d_{j+1}(x) < h_j^{\alpha+\varepsilon}(d_j(x)). \tag{7}$$

Thus

$$B_\alpha \setminus D \subset \bigcup_{j_0=1}^{\infty} B_\alpha(\varepsilon, j_0),$$

where

$$B_\alpha(\varepsilon, j_0) = \{x \in (0,1] : h_j^{\alpha-\varepsilon}(d_j(x)) < d_{j+1}(x) < h_j^{\alpha+\varepsilon}(d_j(x)) \text{ for } j \geq j_0\}.$$

Fix $j_0 \geq 1$; we now estimate $\dim_{\mathrm{H}} B_\alpha(\varepsilon, j_0)$ from above.

Since $h_j(d)$ is of order β, for any

$$0<\eta<\min\left(1,\frac{\beta(\alpha-1-\varepsilon)}{\alpha+1-\varepsilon},\beta-\frac{1}{\alpha-\varepsilon}\right)$$

there exists $d_0>3^{1/2\beta\varepsilon}$ such that for any $d\ge d_0$,

$$d^{\beta-\eta}<h_j(d)<d^{\beta+\eta}. \tag{8}$$

For any $x\in B_\alpha(\varepsilon,j_0)\setminus D$, since $d_j(x)\to\infty$ as $j\to\infty$, there exists j_1 such that $d_j(x)\ge d_0$ for any $j\ge j_1$. Thus we have

$$B_\alpha(\varepsilon,j_0)\setminus D\subset\bigcup_{j_1=1}^{\infty}B_\alpha(\varepsilon,j_0,\eta,j_1),$$

where

$$B_\alpha(\varepsilon,j_0,\eta,j_1)=\{x\in(0,1]:h_j^{\alpha-\varepsilon}(d_j(x))<d_{j+1}(x)<h_j^{\alpha+\varepsilon}(d_j(x))\text{ for }\\ j\ge j_0\text{ and }d_j(x)\ge d_0\text{ for }j\ge j_1\}.$$

For any $j_1\ge 1$, let $j_2=\max\{j_1,j_0\}$. Then $B_\alpha(\varepsilon,j_0,\eta,j_1)$ is contained in

$$\begin{aligned}
&\{x\in(0,1]:h_j^{\alpha-\varepsilon}(d_j(x))<d_{j+1}(x)<h_j^{\alpha+\varepsilon}(d_j(x))\\
&\qquad\qquad\text{and }d_j(x)\ge d_0\text{ for any }j\ge j_2\}\\
&=\bigcup_{d_1,\dots,d_{j_2-1}}\{x\in(0,1]:d_1(x)=d_1,\dots,d_{j_2-1}(x)=d_{j_2-1},\ d_{j_2}(x)\ge d_0,\\
&\qquad\qquad\text{and }h_j^{\alpha-\varepsilon}(d_j(x))<d_{j+1}(x)<h_j^{\alpha+\varepsilon}(d_j(x))\text{ for any }j\ge j_2\}\\
&\subset\bigcup_{d_1,\dots,d_{j_2-1},d_{j_2}\ge d_0}\{x\in(0,1]:d_1(x)=d_1,\dots,d_{j_2}(x)=d_{j_2},\\
&\qquad\qquad d_j^{(\beta-\eta)(\alpha-\varepsilon)}(x)<d_{j+1}(x)<d_j^{(\beta+\eta)(\alpha+\varepsilon)}(x)\text{ for any }j\ge j_2\},
\end{aligned}$$

where the union is over all $d_1,\dots,d_{j_2-1},d_{j_2}$ such that $d_1\ge 2$, $d_{j_2}\ge d_0$ and $d_{j+1}\ge h_j(d_j)+1$ for any $1\le j\le j_2-1$.

For any $\bar d=(d_1,\dots,d_{j_2-1},d_{j_2})$ satisfying the above conditions, let

$$\begin{aligned}
\varGamma(\varepsilon,j_0,\eta,j_2,\bar d)&=\{x\in(0,1]:d_1(x)=d_1,\dots,d_{j_2}(x)=d_{j_2},\\
&\quad[d_j^{(\beta-\eta)(\alpha-\varepsilon)}(x)]<d_{j+1}(x)\le[d_j^{(\beta+\eta)(\alpha+\varepsilon)}(x)]+1\text{ for any }j\ge j_2\}.
\end{aligned}$$

By the σ-stability of Hausdorff dimension (notice that D is countable), in order to get an upper bound of $\dim_{\mathrm H}B_\alpha(\varepsilon,j_0)$, it suffices to give an upper bound of $\dim_{\mathrm H}\varGamma(\varepsilon,j_0,\eta,j_2,\bar d)$ for any $j_2\ge j_0$ and any $\bar d=(d_1,\dots,d_{j_2-1},d_{j_2})$ as above.

Now we introduce a kind of symbolic space defined as follows. For any $k \geq j_2 + 1$, let

$$D_k = \{\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{N}^k : \sigma_1 = d_1, \ldots, \sigma_{j_2} = d_{j_2},\ [\sigma_j^{(\beta-\eta)(\alpha-\varepsilon)}] < \sigma_{j+1} \leq [\sigma_j^{(\beta+\eta)(\alpha+\varepsilon)}] + 1 \text{ for } j_2 \leq j < k\},$$

and define

$$D^* = \bigcup_{k=j_2+1}^{\infty} D_k.$$

For any $k \geq j_2 + 1$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, let J_σ and I_σ denote the following closed subintervals of $(0,1]$:

$$J_\sigma = \bigcup_{[\sigma_k^{(\beta-\eta)(\alpha-\varepsilon)}] < d \leq [\sigma_k^{(\beta+\eta)(\alpha+\varepsilon)}]+1} \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, \ldots, d_k(x) = \sigma_k,\ d_{k+1}(x) = d\},$$

$$I_\sigma = \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, \ldots, d_k(x) = \sigma_k\}.$$

By the restriction on η, we know that $J_\sigma \neq \emptyset$, since, for any $j \geq j_2$,

$$\sigma_{j+1} > \sigma_j^{(\beta-\eta)(\alpha-\varepsilon)} > \sigma_j > \cdots > \sigma_{j_2} \geq d_0.$$

It follows that

$$\sigma_{j+1} > \sigma_j^{(\beta-\eta)(\alpha-\varepsilon)} > h_j^{\frac{(\beta-\eta)(\alpha-\varepsilon)}{\beta+\eta}}(\sigma_j) \geq h_j(\sigma_j).$$

Moreover, we know that $\sigma_k \geq d_0$ for any $k \geq j_2 + 1$, which yields

$$\sigma_k^{2(\alpha\eta+\beta\varepsilon)} \geq d_0^{2(\alpha\eta+\beta\varepsilon)} \geq 3^{\frac{2(\alpha\eta+\beta\varepsilon)}{2\beta\varepsilon}} \geq 3. \tag{9}$$

Each J_σ is called an *interval of kth order*. Finally, define

$$E = \bigcap_{k=j_2+1}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma.$$

It is obvious that

$$\Gamma(\varepsilon, j_0, \eta, j_2, \bar{d}) = E.$$

From the proof of Theorem 6.1 in [5], we have, for any $k \geq j_2 + 1$ and $\sigma \in D_k$,

$$|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k}, \tag{10}$$

$$\begin{aligned} |J_\sigma| &= \sum_{[\sigma_k^{(\beta-\eta)(\alpha-\varepsilon)}] < d \leq [\sigma_k^{(\beta+\eta)(\alpha+\varepsilon)}]+1} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{(d-1)d} \\ &= \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left(\frac{1}{[\sigma_k^{(\beta-\eta)(\alpha-\varepsilon)}]} - \frac{1}{[\sigma_k^{(\beta+\eta)(\alpha+\varepsilon)}] + 1} \right). \end{aligned} \tag{11}$$

For any

$$(12)\qquad s > \frac{(\alpha+\varepsilon)(\beta+\eta)}{(\alpha-\varepsilon)(\beta-\eta)[2+(\alpha-\varepsilon)(\beta-\eta)-(\beta+\eta)]-(\alpha+\varepsilon)(\beta+\eta)},$$

by (9) and (11), we have

$$\begin{aligned}
\mathbf{H}^s(E) &\le \liminf_{k\to\infty} \sum_{\sigma\in D_k} |J_\sigma|^s \\
&= \liminf_{k\to\infty} \sum_{\sigma\in D_k} \left(\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\left(\frac{1}{[\sigma_k^{(\beta-\eta)(\alpha-\varepsilon)}]}-\frac{1}{[\sigma_k^{(\beta+\eta)(\alpha+\varepsilon)}]+1}\right)\right)^s \\
&= \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{[\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)}]<\sigma_k\le[\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}]+1} \left(\frac{h_k(\sigma_k)}{\sigma_k(\sigma_k-1)}\right)^s \\
&\quad\times\left(\frac{1}{[\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}]}-\frac{1}{[\sigma_k^{(\alpha+\varepsilon)(\beta+\eta)}]+1}\right)^s \\
&\quad\times\left(\frac{1}{[\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)}]}-\frac{1}{[\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}]+1}\right)^{-s} \\
&\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{[\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)}]<\sigma_k\le[\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}]+1} \left(\frac{\sigma_k^{\beta+\eta}}{\sigma_k^2}\cdot\frac{\sigma_k}{\sigma_k-1}\right)^s \\
&\quad\times\left(\frac{1}{[\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}]}\cdot\frac{\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}+1}{\sigma_{k-1}^{2(\alpha\eta+\beta\varepsilon)}-1}\right)^s \\
&\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{[\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)}]<\sigma_k\le[\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}]+1} \left(\frac{\sigma_k^{\beta+\eta}}{\sigma_k^2}\cdot\frac{\sigma_k}{\sigma_k-1}\right)^s \\
&\quad\times\left(\frac{1}{\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}-1}\,\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}\right)^s.
\end{aligned}$$

For any $k \ge j_2+1$, let

$$a_k := \frac{\sigma_k}{\sigma_k-1}\cdot\frac{\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}}{\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}-1}.$$

Since $\log(1+x) < x$ for any $x>0$, and $(\alpha-\varepsilon)(\beta-\eta)>1$, we have

$$\begin{aligned}
(13)\qquad \log a_k &< \frac{1}{\sigma_k-1}+\frac{1}{\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}-1} < \frac{4}{\sigma_k} < \frac{4}{d_0^{((\alpha-\varepsilon)(\beta-\eta))^{k-j_2}}} \\
&< \frac{4}{d_0^{((\alpha-\varepsilon)(\beta-\eta)-1)(k-j_2)}} =: \frac{4}{r^{k-j_2}} \qquad (r>1).
\end{aligned}$$

Thus

$$\mathbf{H}^s(E) \le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s \sum_{[\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)}]<\sigma_k\le[\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}]+1} (e^{4/r^{k-j_2}})^s$$
$$\times \left(\frac{\sigma_k^{\beta+\eta}}{\sigma_k^2}\cdot\frac{1}{\sigma_k^{(\alpha-\varepsilon)(\beta-\eta)}}\,\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}\right)^s$$
$$\le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s e^{4s/r^{k-j_2}} \sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}$$
$$\times \left(\frac{1}{\sigma_{k-1}^{(\alpha-\varepsilon)(\beta-\eta)(2+(\alpha-\varepsilon)(\beta-\eta)-(\beta+\eta))}}\,\sigma_{k-1}^{(\alpha+\varepsilon)(\beta+\eta)}\right)^s.$$

By (12) we have

$$\mathbf{H}^s(E) \le \liminf_{k\to\infty} \sum_{\sigma\in D_{k-1}} |J_\sigma|^s e^{4s/r^{k-j_2}}$$
$$\le \cdots \le \lim_{k\to\infty} \prod_{i=j_2+1}^{k} e^{4s/r^{i-j_2}} \sum_{\sigma\in D_{j_2+1}} |J_\sigma|^s < \infty.$$

Thus

$$\dim_{\mathrm{H}} E \le \frac{(\alpha+\varepsilon)(\beta+\eta)}{(\alpha-\varepsilon)(\beta-\eta)[2+(\alpha-\varepsilon)(\beta-\eta)-(\beta+\eta)]-(\alpha+\varepsilon)(\beta+\eta)}.$$

By the σ-stability of Hausdorff dimension, we have, for any $j_1\ge 1$,

$$\dim_{\mathrm{H}} B_\alpha(\varepsilon,j_0,\eta,j_1) \le \sup_{d_1,\dots,d_{j_2-1},d_{j_2}\ge d_0} \dim_{\mathrm{H}} \varGamma(\varepsilon,j_0,\eta,j_2,\bar{d})$$
$$\le \frac{(\alpha+\varepsilon)(\beta+\eta)}{(\alpha-\varepsilon)(\beta-\eta)[2+(\alpha-\varepsilon)(\beta-\eta)-(\beta+\eta)]-(\alpha+\varepsilon)(\beta+\eta)}.$$

This implies

$$\dim_{\mathrm{H}} B_\alpha(\varepsilon,j_0) \le \frac{(\alpha+\varepsilon)(\beta+\eta)}{(\alpha-\varepsilon)(\beta-\eta)[2+(\alpha-\varepsilon)(\beta-\eta)-(\beta+\eta)]-(\alpha+\varepsilon)(\beta+\eta)}.$$

Since η is arbitrary, we get

$$\dim_{\mathrm{H}} B_\alpha(\varepsilon,j_0) \le \frac{(\alpha+\varepsilon)\beta}{(\alpha-\varepsilon)\beta[2+(\alpha-\varepsilon)\beta-\beta]-(\alpha+\varepsilon)\beta}.$$

The σ-stability of Hausdorff dimension yields

$$\dim_{\mathrm{H}} B_\alpha \le \frac{(\alpha+\varepsilon)\beta}{(\alpha-\varepsilon)\beta[2+(\alpha-\varepsilon)\beta-\beta]-(\alpha+\varepsilon)\beta}.$$

Since ε is arbitrary, we have

$$\dim_{\mathrm{H}} B_\alpha \le \frac{1}{(\alpha-1)\beta+1}.$$

This completes the proof of Part I.

Part II: *Lower bound.* Since $h_j(d)$ is of order β, there exists $c > 2$ such that for any $j \ge 1$ and $d \ge 2$,

$$c^{-1}d^\beta \le h_j(d) \le cd^\beta. \tag{14}$$

Since $\alpha > 1$, there exist $d_0 > 4$ and K_0 such that for any $d \ge d_0$, $j \ge K_0$, we have

$$c^{-\alpha}d^{\beta\alpha} \ge cd^\beta + 1, \qquad (d-1)^\alpha - 1 \ge (d-1)^{(j+1)/j}. \tag{15}$$

This implies, for any $d \ge d_0$,

$$h_j^\alpha(d) \ge (c^{-1}d^\beta)^\alpha \ge cd^\beta + 1 \ge h_j(d) + 1. \tag{16}$$

Let $j_0 \ge \max(\beta/3, K_0)$, $j_0 \in \mathbb{N}$. Choose an integer sequence $d_1, \dots, d_{j_0}$ satisfying $d_1 \ge 2$, $d_{j+1} \ge h_j(d_j) + 1$, $1 \le j \le j_0 - 1$, and

$$d_{j_0} > \max\{d_0, c^3 2^{\beta+2} + 1\}, \qquad (d_{j_0} - 1)^\alpha[(d_{j_0} - 1)^{2/j_0} - 1] > 2. \tag{17}$$

Define

$$\begin{aligned} B_\alpha^{(1)}(j_0) = \{x \in (0,1] : d_1(x) = d_1, \dots, d_{j_0}(x) = d_{j_0},& \\ h_j^\alpha(d_j(x)) < d_{j+1}(x) \le h_j^{\alpha+2/j}(d_j(x)) \text{ for } j \ge j_0\}.& \end{aligned}$$

Then $B_\alpha^{(1)}(j_0) \ne \emptyset$. In fact, by (17) and $h_j(d) \ge d - 1$ for any j and d, we have

$$\begin{aligned} h_{j_0}^{\alpha+2/j_0}(d_{j_0}) - h_{j_0}^\alpha(d_{j_0}) &= h_{j_0}^\alpha(d_{j_0})(h_{j_0}^{2/j_0}(d_{j_0}) - 1) \\ &\ge (d_{j_0} - 1)^\alpha[(d_{j_0} - 1)^{2/j_0} - 1] > 2. \end{aligned}$$

Then there exists $d_{j_0+1} \in \mathbb{N}$ satisfying $h_{j_0}^\alpha(d_{j_0}) < d_{j_0+1} \le h_{j_0}^{\alpha+2/j_0}(d_{j_0})$, and, by (16),

$$d_{j_0+1} \ge h_{j_0}^\alpha(d_{j_0}) \ge h_{j_0}(d_{j_0}) + 1.$$

Suppose by induction there exist $d_{j_0+1}, d_{j_0+2}, \dots, d_j \in \mathbb{N}$ satisfying

$$h_{k-1}^\alpha(d_{k-1}) < d_k \le h_{k-1}^{\alpha+2/(k-1)}(d_{k-1}), \qquad j_0 + 1 \le k \le j,$$

and

$$d_k \ge h_{k-1}(d_{k-1}) + 1, \qquad j_0 + 1 \le k \le j.$$

By (15) and (17), we have

$$\begin{aligned} h_j^{\alpha+2/j}(d_j) - h_j^\alpha(d_j) &= h_j^\alpha(d_j)(h_j^{2/j}(d_j) - 1) \\ &\ge (d_j - 1)^\alpha((d_j - 1)^{2/j} - 1) \\ &\ge (h_{j-1}^\alpha(d_{j-1}) - 1)^\alpha[(h_{j-1}^\alpha(d_{j-1}) - 1)^{2/j} - 1] \\ &\ge ((d_{j-1} - 1)^\alpha - 1)^\alpha[((d_{j-1} - 1)^\alpha - 1)^{2/j} - 1] \\ &\ge (d_{j-1} - 1)^{\frac{j}{j-1}\alpha}[(d_{j-1} - 1)^{\frac{2}{j}\cdot\frac{j}{j-1}} - 1] \end{aligned}$$

$$\geq (d_{j-1}-1)^{\alpha}[(d_{j-1}-1)^{2/(j-1)}-1]$$
$$\geq \cdots \geq (d_{j_0}-1)^{\alpha}[(d_{j_0}-1)^{2/j_0}-1] > 2.$$

Thus there exists $d_{j+1} \in \mathbb{N}$ satisfying $h_j^{\alpha}(d_j) < d_{j+1} \leq h_j^{\alpha+2/j}(d_j)$, and, by (16),

$$d_{j+1} > h_j^{\alpha}(d_j) \geq h_j(d_j)+1.$$

Therefore $B_{\alpha}^{(1)}(j_0) \neq \emptyset$. From (16), it is clear that for any $x \in B_{\alpha}^{(1)}(j_0)$, $d_j(x) \to \infty$ as $j \to \infty$ and $B_{\alpha}^{(1)}(j_0) \subset B_{\alpha}$.

Fix $x_0 \in B_{\alpha}^{(1)}(j_0)$ and choose any t satisfying

$$t > \frac{\alpha}{\alpha\beta-1} + \frac{2\alpha}{\beta(\alpha\beta-1)}. \tag{18}$$

Since $d_j(x_0) \to \infty$ as $j \to \infty$, there exists $j_1 \geq j_0$ such that for any $j \geq j_1$,

$$d_j(x_0) \geq \max(9^{3/2\alpha}, c)^t. \tag{19}$$

Now

$$t > \frac{\alpha}{\alpha\beta-1} + \frac{2\alpha}{\beta(\alpha\beta-1)}$$

implies that there exists $j_2 \geq j_1$ such that for any $j \geq j_2$,

$$\left(t-\frac{\alpha}{\alpha\beta-1}\right)(\alpha\beta-1)(j-j_1) > \frac{2(\alpha j+1)}{\beta} - \frac{\alpha}{\alpha\beta-1}. \tag{20}$$

Define

$$\begin{aligned} B_{\alpha}^{(2)}(j_1) = \{x \in (0,1] : d_1(x) = d_1(x_0), \ldots, d_{j_1}(x) = d_{j_1}(x_0),\\ h_j^{\alpha}(d_j(x)) < d_{j+1}(x) \leq h_j^{\alpha+2/j}(d_j(x)) \text{ for any } j \geq j_1\},\\ B_{\alpha}^{(3)}(j_2) = \{x \in (0,1] : d_1(x) = d_1(x_0), \ldots, d_{j_2}(x) = d_{j_2}(x_0),\\ h_j^{\alpha}(d_j(x)) < d_{j+1}(x) \leq h_j^{\alpha+2/j}(d_j(x)) \text{ for any } j \geq j_2\}. \end{aligned}$$

For any $x \in B_{\alpha}^{(3)}(j_2)$ and $j \geq j_2$, by (19), we have

$$\begin{aligned} d_j(x) &> h_{j-1}^{\alpha}(d_{j-1}(x)) \geq (c^{-1}d_{j-1}^{\beta}(x))^{\alpha} = (c^{-1})^{\alpha} d_{j-1}^{\alpha\beta}(x)\\ &> (c^{-1})^{\alpha} h_{j-2}^{\alpha^2\beta}(d_{j-2}(x)) \geq (c^{-1})^{\alpha}(c^{-1}d_{j-2}^{\beta}(x))^{\alpha^2\beta}\\ &= (c^{-1})^{\alpha}(c^{-1})^{\alpha^2\beta} d_{j-2}^{(\alpha\beta)^2}(x) \geq \cdots\\ &\geq (c^{-1})^{\alpha}\cdots(c^{-1})^{(\alpha\beta)^{j-j_1-1}\alpha} d_{j_1}^{(\alpha\beta)^{j-j_1}}(x)\\ &\geq (c^{-1})^{\alpha\frac{(\alpha\beta)^{j-j_1}-1}{\alpha\beta-1}} \cdot c^{t(\alpha\beta)^{j-j_1}}\\ &\geq c^{(t-\frac{\alpha}{\alpha\beta-1})(\alpha\beta-1)(j-j_1)+\frac{\alpha}{\alpha\beta-1}} > c^{\frac{2(\alpha j+1)}{\beta}}. \end{aligned} \tag{21}$$

On the other hand, if $c > 9^{3/2\alpha}$, from (21), we have

$$d_j(x) \geq 9^{\frac{3}{2\alpha}\cdot\frac{2(\alpha j+1)}{\beta}} \geq 9^{3j/\beta},$$

while if $c \leq 9^{3/2\alpha}$, then $c^{-1} \geq 9^{-3/2\alpha}$, and from (19), in the same way as in the proof of (21), we have

$$\begin{aligned} d_j(x) &> (c^{-1})^{\alpha} \cdots (c^{-1})^{(\alpha\beta)^{j-j_1-1}\alpha} d_{j_1}^{(\alpha\beta)^{j-j_1}}(x) \\ &\geq (9^{-3/2\alpha})^{\alpha} \cdots (9^{-3/2\alpha})^{(\alpha\beta)^{j-j_1-1}\alpha} \cdot (9^{3/2\alpha})^{t(\alpha\beta)^{j-j_1}} \geq 9^{3j/\beta}. \end{aligned}$$

So for any $x \in B_{\alpha}^{(3)}(j_2)$ and $j \geq j_2$, we have

$$d_j(x) > 9^{3j/\beta}. \tag{22}$$

Let

$$\begin{aligned} B_{\alpha}^{(4)}(j_2) = \{x \in (0,1] : {}& d_1(x) = d_1(x_0), \ldots, d_{j_2}(x) = d_{j_2}(x_0), \\ & (c d_j^{\beta}(x))^{\alpha} < d_{j+1}(x) \leq (c^{-1} d_j^{\beta}(x))^{\alpha+2/j} \text{ for any } j \geq j_2\}, \\ B_{\alpha}^{(5)}(j_2) = \{x \in (0,1] : {}& d_1(x) = d_1(x_0), \ldots, d_{j_2}(x) = d_{j_2}(x_0), \\ & d_j^{(\beta+\frac{\beta}{2(\alpha j+1)})\alpha}(x) < d_{j+1}(x) \leq d_j^{(\beta-\frac{\beta}{2(\alpha j+1)})(\alpha+\frac{2}{j})}(x) \text{ for } j \geq j_2\}. \end{aligned}$$

By (21), we have

$$B_{\alpha}^{(5)}(j_2) \subset B_{\alpha}^{(4)}(j_2) \subset B_{\alpha}^{(3)}(j_2) \subset B_{\alpha}^{(2)}(j_1) \subset B_{\alpha}^{(1)}(j_0) \subset B_{\alpha}. \tag{23}$$

For any $j \geq j_2$, write

$$s_j = \alpha\left(\beta + \frac{\beta}{2(\alpha j+1)}\right), \qquad t_j = \left(\beta - \frac{\beta}{2(\alpha j+1)}\right)\left(\alpha + \frac{2}{j}\right). \tag{24}$$

Then for any $j \geq j_2$,

$$t_j = s_j + \frac{\beta}{j}. \tag{25}$$

For any $j \geq j_2 + 1$, define

$$L_j = \left(1 + \frac{1}{j^2}\right) d_{j_2}^{\prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}}(x_0), \qquad M_j = d_{j_2}^{\prod_{i=j_2}^{j-2} \frac{s_i+2t_i}{3} t_{j-1}}(x_0),$$

where $M_{j_2+1} = d_{j_2}^{t_{j_2}}(x_0)$. From (24), we have, for any $j \geq j_2 + 1$,

$$\left(t_{j-1} - \frac{\beta}{3(j-1)}\right)\left(s_j + \frac{2\beta}{3j}\right) > t_{j-1} s_j,$$

thus

$$d_{j_2}^{\prod_{i=j_2}^{j} \frac{s_i+2t_i}{3}}(x_0) > d_{j_2}^{\prod_{i=j_2}^{j-2} \frac{s_i+2t_i}{3} t_{j-1} s_j}(x_0),$$

that is,

$$L_{j+1} > M_j^{s_j}. \tag{26}$$

At the same time, it is evident that, for any $j \geq j_2 + 1$,

(27) $$M_{j+1} < L_j^{t_j}.$$

Let

$$B_\alpha^{(6)}(j_2) = \{x \in (0,1] : d_1(x) = d_1(x_0), \ldots, d_{j_2}(x) = d_{j_2}(x_0), \\ [L_{j+1}] < d_{j+1}(x) \leq [M_{j+1}] \text{ for } j \geq j_2 + 1\}.$$

From (26), (27) and (23), we have

(28) $$B_\alpha^{(6)}(j_2) \subset B_\alpha^{(5)}(j_2) \subset B_\alpha^{(4)}(j_2) \subset B_\alpha^{(3)}(j_2) \subset B_\alpha^{(2)}(j_1) \subset B_\alpha^{(1)}(j_0) \subset B_\alpha.$$

In the following, we find a lower bound of Hausdorff dimension of $B_\alpha^{(6)}(j_2)$ by using the mass distribution principle (Lemma 2.1).

First we introduce a kind of symbolic space defined in a similar way to the proof of Part I. For any $k \geq j_2$, let

$$D_k = \{\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{N}^k : \sigma_j = d_j(x_0) \text{ for } 1 \leq j \leq j_2, \\ \text{and } [L_{j+1}] < \sigma_{j+1} \leq [M_{j+1}] \text{ for } j_2 \leq j < k\},$$

and define

$$D^* = \bigcup_{k=j_2}^{\infty} D_k.$$

For any $k \geq j_2$ and $\sigma = (\sigma_1, \ldots, \sigma_k) \in D_k$, let J_σ and I_σ denote the following closed subintervals of $(0,1]$:

$$J_\sigma = \bigcup_{[L_{k+1}] < d \leq [M_{k+1}]} \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, \ldots, d_k(x) = \sigma_k, d_{k+1}(x) = d\},$$

$$I_\sigma = \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, \ldots, d_k(x) = \sigma_k\};$$

each J_σ is called an *interval of nth order*. Let

$$E = \bigcap_{k=j_2}^{\infty} \bigcup_{\sigma \in D_k} J_\sigma.$$

It is obvious that

$$E = B_\alpha^{(6)}(j_2).$$

From (10) and (11), we have

(29) $$|I_\sigma| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdot \frac{a_2(\sigma_2)}{b_2(\sigma_2)} \cdots \frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})} \cdot \frac{1}{(\sigma_k - 1)\sigma_k},$$

(30) $$|J_\sigma| = \sum_{[L_{k+1}] < d \leq [M_{k+1}]} \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \cdot \frac{1}{(d-1)d} \\ = \frac{a_1(\sigma_1)}{b_1(\sigma_1)} \cdots \frac{a_k(\sigma_k)}{b_k(\sigma_k)} \left(\frac{1}{[L_{k+1}]} - \frac{1}{[M_{k+1}]}\right).$$

Let μ be a probability mass distribution supported on E such that for any $k \geq j_2$ and $\sigma \in D_k$,

$$\mu(J_\sigma) = \frac{1}{\sharp D_k}, \tag{31}$$

where $\sharp$ denotes cardinality. We shall use μ to give a lower bound of the Hausdorff dimension of E.

For any $k \geq j_2$, write

$$A_k = \sum_{j=j_2+1}^{k} \prod_{i=j_2}^{j-1} \frac{s_i + 2t_i}{3}, \quad B_k = \sum_{j=j_2+1}^{k+1} \prod_{i=j_2}^{j-2} \frac{s_i + 2t_i}{3}\, t_{j-1}, \quad C_k = \prod_{j=j_2}^{k} \frac{s_i + 2t_i}{3}.$$

Then

$$\lim_{k\to\infty} \frac{A_k}{B_k} = \lim_{k\to\infty} \frac{A_{k+1} - A_k}{B_{k+1} - B_k} = \frac{1}{\alpha\beta}, \tag{32}$$

$$\lim_{k\to\infty} \frac{C_k}{B_k} = \lim_{k\to\infty} \frac{C_{k+1} - C_k}{B_{k+1} - B_k} = \frac{\alpha\beta - 1}{\alpha\beta}. \tag{33}$$

Now we estimate $\sharp D_k$. Notice that $x^a + 1 > (x+1)^a$ for any $0 < a < 1$ and $x > 0$. By (22), for any $j \geq j_2 \geq j_0 \geq \beta/3$ we have

$$\left(\frac{1}{2}\, L_j\right)^{\beta/3j} > \frac{1}{2}\left((L_j + 1)^{\beta/3j} - 1\right) > \frac{1}{2}\left(9^{\frac{3j}{\beta}\cdot\frac{\beta}{3j}} - 1\right) = 4.$$

Then for any $j \geq j_2 + 1$,

$$\begin{aligned}
[M_j] - [L_j] &> d_{j_2}^{\prod_{i=j_2}^{j-2} \frac{s_i+2t_i}{3} t_{j-1}} - 1 - \left(1 + \frac{1}{j^2}\right) d_{j_2}^{\prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}} \\
&\geq d_{j_2}^{\prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}} \left(d_{j_2}^{\prod_{i=j_2}^{j-2} \frac{s_i+2t_i}{3} \frac{t_{j-1}-s_{j-1}}{3}} - 3\right) \\
&\geq d_{j_2}^{\prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}} \left(\left(\frac{1}{2}\, L_{j-1}\right)^{\frac{\beta}{3(j-1)}} - 3\right) \\
&\geq d_{j_2}^{\prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}} = d_{j_2}^{C_{j-1}}.
\end{aligned} \tag{34}$$

Thus for any $k \geq j_2 + 1$,

$$\sharp D_k = \prod_{j=j_2+1}^{k} ([M_j] - [L_j]) \geq d_{j_2}^{\sum_{j=j_2+1}^{k} \prod_{i=j_2}^{j-1} \frac{s_i+2t_i}{3}} = d_{j_2}^{A_k}. \tag{35}$$

For any s satisfying

$$0 < s < \frac{1}{\alpha\beta\Big(2 - \Big(\alpha + \dfrac{2}{j_2}\Big)^{-1}\Big) - (\alpha\beta - 1)},$$

there exists $1/2 > \eta_0 > 0$ such that for any $0 < \eta < \eta_0$,

$$s < \frac{1-\eta}{\alpha\beta\left(2-\left(\alpha+\frac{2}{j_2}\right)^{-1}\right)-(\alpha\beta-1-\eta)}. \tag{36}$$

For any fixed $0 < \eta < \eta_0$, by (32) and (33), there exists $k_0(\eta)$ such that for any $k \geq k_0(\eta)$,

$$\frac{A_k}{B_k} > \frac{1-\eta}{\alpha\beta}, \qquad \frac{C_k}{B_k} > \frac{\alpha\beta-1-\eta}{\alpha\beta}. \tag{37}$$

For any $x \in E$, we prove that

$$\liminf_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} \geq s.$$

For such x, there exists $\sigma = (\sigma_1, \sigma_2, \ldots)$ such that $\sigma_i = d_i(x_0)$ for $1 \leq i \leq j_2$, and for any $k \geq j_2$, $(\sigma|k) := (\sigma_1, \ldots, \sigma_k) \in D_k$ and $d_j(x) = \sigma_j$ for any $j \geq 1$. Thus $x \in J_{\sigma_1\cdots\sigma_k}$ for any $k \geq j_2$.

From the proof of Theorem 6.1 in [5], we know that, for any $k \geq j_2$, the right endpoint of the interval $J_{\sigma_1\cdots\sigma_k}$, i.e., $\max\{y \in (0,1] : y \in J_{\sigma_1\cdots\sigma_k}\}$, is

$$\frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{1}{\sigma_2} + \cdots + \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\cdot\frac{1}{[L_{k+1}]}, \tag{38}$$

and the left endpoint, i.e., $\min\{y \in (0,1] : y \in J_{\sigma_1\cdots\sigma_k}\}$, is

$$\frac{1}{\sigma_1} + \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdot\frac{1}{\sigma_2} + \cdots + \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\cdot\frac{1}{[M_{k+1}]}. \tag{39}$$

(i) If $\sigma_k - 1 > [L_k]$, from the definition of $h_j(d)$, (38) and (39), we know that the gap between $J_{\sigma_1\cdots\sigma_k}$ and $J_{\sigma_1\cdots\sigma_k-1}$, denoted by $g_k^r(x)$, is

$$\begin{aligned}
&\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\cdot\frac{a_k(\sigma_k-1)}{b_k(\sigma_k-1)}\cdot\frac{1}{[M_{k+1}]}\\
&\qquad\qquad + \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\left(\frac{1}{h_k(\sigma_k)} - \frac{1}{[L_{k+1}]}\right)\\
&\qquad \geq \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\cdot\frac{a_k(\sigma_k-1)}{b_k(\sigma_k-1)}\cdot\frac{1}{[M_{k+1}]} =: G_k^r(x).
\end{aligned}$$

(ii) Suppose $\sigma_k = [L_k] + 1$. If $\sigma_j = [L_j] + 1$ for any $j_2 + 1 \leq j \leq k$, let $g_k^r(x) = g_{k-1}^r(x) = \cdots = g_{j_2+1}^r(x) = \infty$ and $G_k^r(x) = G_{k-1}^r(x) = \cdots = G_{j_2+1}^r(x) = \infty$. If there exists $j_2 + 1 \leq j \leq k$ such that $\sigma_j > [L_j] + 1$, let $\widetilde{j} = \max\{j : j_2 + 1 \leq j \leq k,\ \sigma_j > [L_j] + 1\}$. Define $g_k^r(x) = g_{k-1}^r(x) = \cdots = g_{\widetilde{j}}^r(x)$ and $G_k^r(x) = G_{k-1}^r(x) = \cdots = G_{\widetilde{j}}^r(x)$.

(iii) If $\sigma_k+1 \le [M_k]$, the gap between $J_{\sigma_1\cdots\sigma_k}$ and $J_{\sigma_1\cdots\sigma_k+1}$, denoted by $g_k^l(x)$, satisfies

$$g_k^l(x) \ge \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\cdot\frac{1}{[M_{k+1}]} =: G_k^l(x).$$

(iv) If $\sigma_k = [M_k]$, let $g_k^l(x)$ denote the distance between the left endpoint of $J_{\sigma_1\cdots\sigma_{k-1}}$ and the left endpoint of $J_{\sigma_1\cdots\sigma_k}$. Then

$$g_k^l(x) = \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\cdot\frac{1}{[M_{k+1}]} =: G_k^l(x).$$

Define

$$G_k(x) := \min\{G_k^l(x), G_k^r(x)\}.$$

Let $k > \max(1/3(\alpha-1), j_2)$. By (24), we have $\alpha\beta/3k < (\alpha-1)t_k$, which implies $t_k\beta < (t_k - \beta/3k)\alpha\beta$. Thus $t_k\beta < ((s_k+2t_k)/3)t_{k+1}$, so

$$M_{k+1}^{\beta} < M_{k+2}. \tag{40}$$

On the other hand, by (17), we have

$$\sigma_{k+1} - 1 \ge d_{j_2}(x_0) - 1 = d_{j_2} - 1 > c^3\cdot 2^{\beta+2}. \tag{41}$$

Combining (40) and (41), we have

$$c^3\cdot 2^{\beta+2}\,\frac{\sigma_{k+1}^{\beta}}{\sigma_{k+1}(\sigma_{k+1}-1)}\,M_{k+1} < M_{k+2}.$$

By the definition of $h_j(d)$, and since $h_j(d)$ is of order β, we have

$$\frac{h_k(\sigma_k)}{\sigma_k(\sigma_k-1)}\cdot\frac{h_{k+1}(\sigma_{k+1}-1)}{(\sigma_{k+1}-1)(\sigma_{k+1}-2)}\cdot\frac{1}{[M_{k+2}]} < \frac{h_k(\sigma_k-1)}{(\sigma_k-1)(\sigma_k-2)}\cdot\frac{1}{[M_{k+1}]},$$

i.e.

$$\frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{k+1}(\sigma_{k+1}-1)}{b_{k+1}(\sigma_{k+1}-1)}\cdot\frac{1}{[M_{k+2}]} < \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_{k-1}(\sigma_{k-1})}{b_{k-1}(\sigma_{k-1})}\cdot\frac{a_k(\sigma_k-1)}{b_k(\sigma_k-1)}\cdot\frac{1}{[M_{k+1}]},$$

that is, $G_{k+1}^r(x) < G_k^r(x)$. In the same way, we have $G_{k+1}^l(x) < G_k^l(x)$. So $k > \max(1/3(\alpha-1), j_2)$ implies $G_{k+1}(x) < G_k(x)$.

Let $K_1 = \max(1/3(\alpha-1), j_2)$. For any $0 < r < \min_{j_2<j\le K_1}\{G_j(x)\}$, there exists $k \ge K_1$ such that $G_{k+1}(x) \le r < G_k(x)$. Thus $B(x,r)$ can intersect only one kth-order interval, which is $J_{\sigma_1\cdots\sigma_k}$. Now we find an upper bound of the number of $(k+1)$th-order intervals, the $(k+1)$th-order subintervals of $J_{\sigma_1\cdots\sigma_k}$, which intersect $B(x,r)$. Since $J_\sigma \subset I_\sigma$, we only need to consider the number of $\{I(\sigma_1,\ldots,\sigma_k,j)\}_{[L_{k+1}]<j\le[M_{k+1}]}$ which intersect $B(x,r)$. By (29), the definition of $h_j(d)$ and the fact that $E = B_\alpha^{(6)}(j_2) \subset B_\alpha^{(1)}(j_0)$,

we have

$$|I_{\sigma_1\cdots\sigma_k j}| = \frac{a_1(\sigma_1)}{b_1(\sigma_1)}\cdots\frac{a_k(\sigma_k)}{b_k(\sigma_k)}\cdot\frac{1}{j(j-1)}$$
$$= \prod_{l=1}^{j_2-1}\frac{a_l(\sigma_l)}{b_l(\sigma_l)}\cdot\frac{1}{\sigma_{j_2}(\sigma_{j_2}-1)}\cdot\frac{h_{j_2}(\sigma_{j_2})}{\sigma_{j_2+1}(\sigma_{j_2+1}-1)}\cdots\frac{h_{k-1}(\sigma_{k-1})}{\sigma_k(\sigma_k-1)}\cdot\frac{h_k(\sigma_k)}{j(j-1)}$$
$$\geq c(j_2)\,\frac{\sigma_{j_2+1}^{(\alpha+2/j_2)^{-1}}}{\sigma_{j_2+1}^2}\cdots\frac{\sigma_k^{(\alpha+2/(k-1))^{-1}}}{\sigma_k^2}\cdot\frac{j^{(\alpha+2/k)^{-1}}}{j^2}$$
$$\geq c(j_2)\,\frac{1}{(\sigma_{j_2+1}\cdots\sigma_k\cdot j)^{2-(\alpha+2/j_2)^{-1}}}$$
$$\geq c(j_2)\,\frac{1}{(M_{j_2+1}\cdots M_{k+1})^{2-(\alpha+2/j_2)^{-1}}} = c(j_2)d_{j_2}^{-(2-(\alpha+2/j_2)^{-1})B_k},$$

where

$$c(j_2) = \prod_{l=1}^{j_2-1}\frac{a_l(\sigma_l)}{b_l(\sigma_l)}\cdot\frac{1}{\sigma_{j_2}(\sigma_{j_2}-1)} = \prod_{l=1}^{j_2-1}\frac{a_l(d_l(x_0))}{b_l(d_l(x_0))}\cdot\frac{1}{d_{j_2}(x_0)(d_{j_2}(x_0)-1)},$$

which does not depend on x. So the number of $(k+1)$th-order intervals which intersect $B(x,r)$ is not more than

$$4r\big(c(j_2)d_{j_2}^{-(2-(\alpha+2/j_2)^{-1})B_k}\big)^{-1}. \tag{42}$$

By (31), (34)–(37), (42), we have

$$\mu(B(x,r)) \leq \frac{1}{\sharp D_{k+1}}\min\left\{\frac{4r}{c(j_2)d_{j_2}^{-(2-(\alpha+2/j_2)^{-1})B_k}}, [M_{k+1}]-[L_{k+1}]\right\}$$
$$\leq \frac{1}{\sharp D_{k+1}}\left(\frac{4r}{c(j_2)d_{j_2}^{-(2-(\alpha+2/j_2)^{-1})B_k}}\right)^s([M_{k+1}]-[L_{k+1}])^{1-s}$$
$$\leq \frac{1}{\sharp D_k}\left(\frac{4r}{c(j_2)}\,d_{j_2}^{(2-(\alpha+2/j_2)^{-1})B_k}\cdot\frac{1}{[M_{k+1}]-[L_{k+1}]}\right)^s$$
$$\leq d_{j_2}^{-A_k}\left(\frac{4r}{c(j_2)}\right)^s d_{j_2}^{(2-(\alpha+2/j_2)^{-1})B_k s}d_{j_2}^{-C_k s}$$
$$\leq \left(\frac{4r}{c(j_2)}\right)^s d_{j_2}^{B_k(-\frac{1-\eta}{\alpha\beta}+s(2-(\alpha+2/j_2)^{-1})-s\frac{\alpha\beta-1-\eta}{\alpha\beta})}$$
$$\leq \left(\frac{4r}{c(j_2)}\right)^s = \left(\frac{4}{c(j_2)}\right)^s r^s.$$

By Lemma 2.1, we have

$$\dim_{\mathrm{H}} B^{(6)}(j_2) = \dim_{\mathrm{H}} E \geq s.$$

Therefore,

$$\dim_{\mathrm{H}} B^{(1)}(j_0) \geq \dim_{\mathrm{H}} B^{(6)}(j_2) \geq \frac{1}{\alpha\beta(2-(\alpha+2/j_2)^{-1})-(\alpha\beta-1)} \geq \frac{1}{\alpha\beta(2-(\alpha+2/j_0)^{-1})-(\alpha\beta-1)}.$$

Thus

$$\dim_{\mathrm{H}} B_\alpha \geq \frac{1}{\alpha\beta(2-(\alpha+2/j_0)^{-1})-(\alpha\beta-1)}$$

for any $j_0 > \beta/3$, $j_0 \in \mathbb{N}$, which implies

$$\dim_{\mathrm{H}} B_\alpha \geq \frac{1}{(\alpha-1)\beta+1}.$$

This completes the proof of Theorem 2.3. ∎

We now list some special cases which satisfy the assumptions of Theorem 2.3.

Example 1 (Engel expansion). Let $a_n(d_n) = 1$, $b_n(d_n) = d_n$ ($n = 1, 2, \ldots$). Then (2), together with the algorithm (1), yields the *Engel expansion* of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\cdots d_n(x)} + \cdots. \tag{43}$$

In this case, $h_n(j) = j - 1$ is of order 1. By Theorem 2.3, we have

Corollary 2.4. *For the Engel expansion*,

$$\dim_{\mathrm{H}} \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{\log d_{n+1}(x)}{\log d_n(x)} = \alpha \right\} = \frac{1}{\alpha} \quad \textit{for any } \alpha \geq 1.$$

Example 2 (Sylvester expansion). Choose $a_n(d_n) = 1$, $b_n(d_n) = 1$ ($n = 1, 2, \ldots$). We get the *Sylvester expansion* of x,

$$x = \frac{1}{d_1(x)} + \frac{1}{d_2(x)} + \cdots + \frac{1}{d_n(x)} + \cdots. \tag{44}$$

Here $h_n(j) = j(j-1)$ is of order 2. By Theorem 2.3, we have

Corollary 2.5. *For the Sylvester expansion*,

$$\dim_{\mathrm{H}} \left\{ x \in (0,1] : \lim_{n\to\infty} \frac{\log d_{n+1}(x)}{\log d_n(x)} = \alpha \right\} = \frac{1}{\alpha-1} \quad \textit{for any } \alpha \geq 2.$$

Example 3 (Cantor product). Take $a_n(d_n) = d_n + 1$ $b_n(d_n) = d_n$ ($n = 1, 2, \ldots$). The expansion (2) yields the *Cantor product*

$$1 + x = \left(1 + \frac{1}{d_1(x)}\right)\left(1 + \frac{1}{d_2(x)}\right) \cdots \left(1 + \frac{1}{d_n(x)}\right) \cdots. \tag{45}$$

Here $h_n(j) = j^2 - 1$ is of order 2. By Theorem 2.3, we have

Corollary 2.6. *For the Cantor product*,

$$\dim_{\mathrm{H}}\left\{x\in(0,1]:\lim_{n\to\infty}\frac{\log d_{n+1}(x)}{\log d_n(x)}=\alpha\right\}=\frac{1}{\alpha-1}\quad \textit{for any } \alpha\geq 2.$$

Example 4 (Modified Engel expansion). Let $a_n(d_n)=1, b_n(d_n)=d_n-1$ $(n=1,2,\ldots)$. We get the *modified Engel expansion* of x,

$$x=\frac{1}{d_1(x)}+\cdots+\frac{1}{(d_1(x)-1)(d_2(x)-1)\cdots(d_{n-1}(x)-1)d_n(x)}+\cdots. \tag{46}$$

Thus $h_n(j)=j$ is of order 1. By Theorem 2.3, we have

Corollary 2.7. *For the modified Engel expansion*,

$$\dim_{\mathrm{H}}\left\{x\in(0,1]:\lim_{n\to\infty}\frac{\log d_{n+1}(x)}{\log d_n(x)}=\alpha\right\}=\frac{1}{\alpha}\quad \textit{for any } \alpha\geq 1.$$

Example 5 (Daróczy–Kátai–Birthday expansion). Choose $a_n(d_n)=d_n$, $b_n(d_n)=1$ $(n=1,2,\ldots)$. The resulting series expansion of x takes the form

$$x=\frac{1}{d_1(x)}+\frac{d_1(x)}{d_2(x)}+\cdots+\frac{d_1(x)d_2(x)\cdots d_{n-1}(x)}{d_n(x)}+\cdots. \tag{47}$$

This *Daróczy–Kátai–Birthday expansion* was introduced for the first time in Galambos [6]. Here $h_n(j)=j^2(j-1)$ is of order 3. By Theorem 2.3, we have

Corollary 2.8. *For the Daróczy–Kátai–Birthday expansion*,

$$\dim_{\mathrm{H}}\left\{x\in(0,1]:\lim_{n\to\infty}\frac{\log d_{n+1}(x)}{\log d_n(x)}=\alpha\right\}=\frac{1}{\alpha-2}\quad \textit{for any } \alpha\geq 3.$$

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