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# On a problem of Erdős regarding binomial coefficients

by

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**1. Introduction and main result.** Arithmetical properties of binomial coefficients have been studied by many authors (cf. [1], [3], [4], [5]). Of particular interest is the sequence of middle binomial coefficients $d_n = \binom{2n}{n}$. Moser [7] proved that no $d_n$ is a product of two others. That is, the equation

$$\binom{2n}{n} = \binom{2a}{a}\binom{2b}{b}$$

has no solutions with $a, b \geq 1$. Erdős [2] proved that $\binom{2a}{a} \nmid \binom{2n}{n}$ for each $a \in (n/2, n)$. This enabled him to show that

$$\binom{2n}{n} = \prod_{i=1}^{r} \binom{2a_i}{a_i}, \qquad a_i \geq 1,$$

has no solutions for any $r \geq 2$. In the same paper he raised the following

QUESTION 1 ([2]). *Do there exist distinct finite sets $A, B \subseteq \mathbb{N}$ with*

$$\prod_{a\in A} \binom{2a}{a} = \prod_{b\in B} \binom{2b}{b}?$$

Our main result is

THEOREM 2. *For each positive rational number $d$ there exist infinitely many pairs $(A, B)$ of disjoint finite subsets of $\mathbb{N}$ with*

$$\prod_{a\in A} \binom{2a}{a} = d \prod_{b\in B} \binom{2b}{b}.$$

In particular, taking $d = 1$ we provide a positive answer for Question 1.

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**2. Proof of Theorem 2.** In this section, all subsets of $\mathbb{N}$ are assumed to be finite (unless explicitly specified otherwise). Given a pair $(A, B)$ of (finite) subsets of $\mathbb{N}$, define

$$F(A,B) = \frac{\prod_{a\in A}\binom{2a}{a}}{\prod_{b\in B}\binom{2b}{b}}.$$

The main component of our proof is

PROPOSITION 3. *Let*

$$\mathcal{G} = \{d \in \mathbb{Q} : \exists A, B \subseteq \mathbb{N},\ F(A,B) = d\}.$$

*Then $\mathcal{G}$ is closed under multiplication and division by 2 (i.e., $\{2^l d_0 : l \in \mathbb{Z}\} \subseteq \mathcal{G}$ for each $d_0 \in \mathcal{G}$). Moreover, for each $d_0 \in \mathcal{G}$ there are infinitely many pairs $(A, B)$ of disjoint subsets of $\mathbb{N}$ with $F(A,B) = d_0$.*

Since $1 = F(\emptyset, \emptyset) \in \mathcal{G}$, this already solves Question 1.

LEMMA 4. *For every $M \geq 0$ there exist disjoint sets $A, B \subseteq \mathbb{N}$, with $|A| = |B| = 3$ and $\min(A \cup B) > M$, such that $F(A,B) = 4$.*

*Proof.* Let $n, m, r$ be positive integers and assume that $n$, $m$, $r$, $n-1$, $m-1$, $r-1$ are distinct. Take

$$A = \{n, m, r-1\}, \qquad B = \{n-1, m-1, r\}.$$

Observing that $\binom{2t}{t} = 4\big(1 - \frac{1}{2t}\big)\binom{2(t-1)}{t-1}$ for each $t > 0$, we obtain

$$F(A,B) = \frac{4(1-1/2n)(1-1/2m)}{1-1/2r}.$$

Thus, $F(A,B) = 4$ if and only if $(1-1/2n)(1-1/2m) = 1-1/2r$, that is, $r(2m+2n-1) = 2mn$.

Let $k$ be an odd integer and put

$$n = \frac{k(k-1)^2}{4}, \qquad m = \frac{k^2+1}{2}, \qquad r = \frac{(k-1)^2}{2}.$$

Taking a large enough $k$, we see that $r, r-1, n, n-1, m, m-1$ are distinct integers larger than $M$. Note that $2m+2n-1 = k(k^2+1)/2$. Thus we get $r(2m+2n-1) = 2mn$ and so $F(A,B) = 4$. ∎

*Proof of Proposition 3.* We begin by proving that for each $l \in \mathbb{Z}$, $M \in \mathbb{N}$ there are infinitely many pairs $((A_n, B_n))_{n=1}^{\infty}$ of disjoint subsets of $\mathbb{N}$ with $F(A_n, B_n) = 2^l$ and $(A_n \cup B_n) \cap [0, M] \subseteq \{1\}$.

Since $F(B,A) = F(A,B)^{-1}$, we may assume without loss of generality that $l \geq 0$. Write $l = 2t+s$ with $s \in \{0,1\}$. Assume first that $s = 0$. Lemma 4 enables us to construct an infinite sequence of pairs $((X_i, Y_i))_{i=1}^{\infty}$, with $X_i, Y_i \subseteq \mathbb{N}$, $F(X_i, Y_i) = 4$, $\min(X_i \cup Y_i) > M$, such that $X_1, Y_1, X_2, Y_2, \ldots$

are pairwise disjoint. If $t > 0$ then put

$$A_n = \bigcup_{i=n}^{n+t-1} X_i, \quad B_n = \bigcup_{i=n}^{n+t-1} Y_i, \qquad n = 1, 2, \ldots.$$

Otherwise $t = 0$ and put

$$A_n = X_n \cup Y_{n+1}, \quad B_n = X_{n+1} \cup Y_n, \qquad n = 1, 2, \ldots.$$

We conclude that $F(A_n, B_n) = 4^t = 2^l$, $A_n \cap B_n = \emptyset$ and $\min(A_n \cup B_n) > M$. The proof for the case $s = 1$ is obtained by replacing $A_n$ with $A_n \cup \{1\}$.

Now let $d_0 \in \mathcal{G}$, and write $d_0 = F(A, B)$ with disjoint $A, B \subseteq \mathbb{N}$. Assume first that $1 \notin A \cup B$. Taking $M > \max(A, B)$, we see that $A_n \cup B_n$, $A \cup B$ are disjoint, and thus $F(A \cup A_n, B \cup B_n) = 2^l d_0$ for each $n$. This completes the proof for this case.

If $1 \in A \cup B$, then the proof is obtained by repeating the same arguments on the triple $(A', B', d_0')$ where $A' = A \setminus \{1\}$, $B' = B \setminus \{1\}$ and $d_0' = F(A', B')$. (Observe that $d_0' \in \{2d_0, d_0/2\}$.) ∎

Lemma 5. *For each* $c \in \{1, 3, \ldots, 15\}$, $t \in \{1, 3\}$ *there exist* $A, B \subseteq \{1, \ldots, 7\}$ *such that* $F(A, B) = 2^l c/t$ *for some* $l \in \mathbb{Z}$.

*Proof.* Table 1 provides for each $c \in \{1, 3, \ldots, 15\}$ a pair $(A, B)$ with $F(A, B) = 2^l c$ and a pair $(A', B')$ with $F(A', B') = 2^{l'} c/3$ for some $l, l' \in \mathbb{Z}$. ∎

**Table 1.** A solution for $F(A, B) = 2^l c/t$ when $c \in \{1, 3, \ldots, 15\}$, $t \in \{1, 3\}$

| $c$ | $(A, B)$ | $(A', B')$ |
|---|---|---|
| 1 | $(\emptyset, \emptyset)$ | $(\emptyset, \{2\})$ |
| 3 | $(\{2\}, \emptyset)$ | $(\emptyset, \emptyset)$ |
| 5 | $(\{3\}, \emptyset)$ | $(\{3\}, \{2\})$ |
| 7 | $(\{4\}, \{3\})$ | $(\{4\}, \{3, 2\})$ |
| 9 | $(\{3, 5\}, \{4\})$ | $(\{2\}, \emptyset)$ |
| 11 | $(\{2, 6\}, \{5\})$ | $(\{6\}, \{5\})$ |
| 13 | $(\{4, 7\}, \{3, 6\})$ | $(\{4, 7\}, \{2, 3, 6\})$ |
| 15 | $(\{2, 3\}, \emptyset)$ | $(\{3\}, \emptyset)$ |

Given a positive integer $n$, let $[n]_2$ denote the binary representation of $n$. Thus, $[n]_2 = \varepsilon_t \ldots \varepsilon_0$ is a binary word, with $n = \sum_{k=0}^{t} \varepsilon_k 2^k$ and $\varepsilon_t = 1$. Let $\nu(n)$ denote the 2-adic valuation of $n$ (that is, $2^{\nu(n)}$ is the exact power of 2 dividing $n$).

*Proof of Theorem 2.* Write $d = x/y$ with $x, y \in \mathbb{N}$. A theorem of Kummer [6] implies that for most numbers $k$ (i.e., for a set of density 1) we have $y \mid \binom{2k}{k}$. Thus, we may take a $k_0 \geq 8$ such that $\binom{2k_0}{k_0} x/y \in \mathbb{N}$. (In fact, any $k_0 \geq 8$ with $k_0 \equiv -1 \pmod{y}$ is such.) A simple calculation shows that for any integer $n > 0$, the base 2 representations of $n$ and $3n$ cannot begin with the same three letters. In particular, we may take a $K = t\binom{2k_0}{k_0}x/y$ with $t \in \{1,3\}$ so that $[k_0]_2$ is not a prefix of $[K]_2$. The main part of the proof will be a construction of sets $A_0, B_0$ such that $\min(A_0 \cup B_0) \geq 8$, $k_0 \notin B_0$ and $F(A_0, B_0) = 2^l K/c$ for some $l \in \mathbb{Z}$ and $c \in \{1, 3, 5, \ldots, 15\}$. Lemma 5 provides sets $A', B' \subseteq \{1, \ldots, 7\}$ such that $F(A', B') = 2^{l'} c/t$ for some $l' \in \mathbb{Z}$. Thus we will get

$$F(A_0 \cup A', B_0 \cup B' \cup \{k_0\}) = 2^{l+l'} \frac{K}{t\binom{2k_0}{k_0}} = 2^{l+l'} \frac{x}{y} \in \mathcal{G},$$

and the theorem will then follow by Proposition 3.

Construct by induction a sequence of odd positive integers $(K_n)_{n=1}^{\infty}$ given by

$$K_1 = \frac{K}{2^{\nu(K)}}, \qquad K_{n+1} = \frac{K_n + 1}{2^{\nu(K_n+1)}}, \quad n = 1, 2, \ldots.$$

If $K_1 \leq 15$ then the pair $(A_0, B_0) = (\emptyset, \emptyset)$ satisfies the required properties (take $c = K_1$, $l = -\nu(K)$). Thus, we may assume that $K_1 > 15$. Note that $K_{n+1} < K_n$, unless $K_n = 1$ (in which case $K_{n+1} = 1$ as well). Let $m$ denote the maximal index with $K_m > 15$. Put

$$a_n = \frac{K_n + 1}{2}, \quad b_n = \frac{K_n - 1}{2}, \qquad n = 1, \ldots, m,$$

and

$$A_0 = \{a_1, \ldots, a_m\}, \qquad B_0 = \{b_1, \ldots, b_m\}, \qquad c = K_{m+1}.$$

Thus $c \leq 15$. Since $K_m > 15$ we obtain $\min(A_0 \cup B_0) = b_m \geq 8$.

Note that $a_n = b_n + 1$ and thus

$$\frac{\binom{2a_n}{a_n}}{\binom{2b_n}{b_n}} = \frac{2(2a_n - 1)}{a_n} = 2^{2-\nu(K_n+1)} \frac{K_n}{K_{n+1}}, \qquad n = 1, \ldots, m.$$

Since $a_1, b_1, a_2, b_2, \ldots, a_m, b_m$ are distinct, we conclude that $F(A_0, B_0) = 2^l K_1/K_{m+1} = 2^{l'} K/c$ for some $l, l' \in \mathbb{Z}$. It can be easily observed that $[b_n]_2$ is a prefix of $[K]_2$ for each $n \leq m$. Thus, our assumptions ensure that $k_0 \notin B_0$. This completes the proof. ∎

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