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Modular parametrizations of certain elliptic curves

by

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1. Introduction. By the modularity theorem [2, 6], an elliptic curve E over $\mathbb{Q}$ admits a modular parametrization $\Phi_E : X_0(N) \to E$ for some integer N. If N is the smallest such integer, then it is equal to the conductor of E and the pullback of the Néron differential of E under Φ_E is a rational multiple of $2\pi i f_E(\tau)$, where $f_E(\tau) \in S_2(\Gamma_0(N))$ is a newform with rational Fourier coefficients. The fact that the L-function of $f_E(\tau)$ coincides with the Hasse–Weil zeta function of E (which follows from Eichler–Shimura theory) is central to the proof of Fermat's last theorem, and is related to the Birch and Swinnerton–Dyer conjecture. In addition to this, modular parametrization is used for constructing rational points on elliptic curves, and appears in the Gross–Zagier formula.

In this paper, we study some general properties of Φ_E, and as a consequence we explain and generalize the results of Kaneko and Sakai [8].

Kaneko and Sakai (inspired by the paper of Guerzhoy [7]) observed that certain elliptic curves whose associated newforms (by the modularity theorem) are given by the eta-quotients from the list of Martin and Ono [9] can be characterized by a particular differential equation involving holomorphic modular forms.

To give an example of this phenomena, let $f_{20}(\tau) = \eta(\tau)^4\eta(5\tau)^4$ be a unique newform of weight 2 on $\Gamma_0(20)$, where $\eta(\tau)$ is the Dedekind eta function $\eta(\tau) = q^{1/24}\prod_{n>0}(1-q^n)$, $q = e^{2\pi i\tau}$, and put $\Delta_{5,4}(\tau) = f_{20}(\tau/2)^2$. Then an Eisenstein series $Q_5(\tau)$ on $M_4(\Gamma_0(5))$ associated either to cusp $i\infty$ or to cusp 0 is a solution of the differential equation

$$\partial_{5,4}(Q_5)^2 = Q_5^3 - \frac{89}{13}Q_5^2\Delta_{5,4} - \frac{3500}{169}Q_5\Delta_{5,4}^2 - \frac{125000}{2197}\Delta_{5,4}^3, \tag{1.1}$$

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where $\partial_{5,4}(Q_5(\tau)) = \frac{1}{2\pi i}Q_5(\tau)' - \frac{1}{2\pi i}Q_5(\tau)\Delta_{5,4}(\tau)'/\Delta_{5,4}(\tau)$ is a Ramanujan–Serre differential operator. (Throughout the paper, we use the symbol $'$ to denote $d/d\tau$.) This differential equation defines a parametrization of the elliptic curve $E : y^2 = x^3 - \frac{89}{13}x^2 - \frac{3500}{169}x - \frac{125000}{2197}$ by the modular functions

$$x = \frac{Q_5(\tau)}{\Delta_{5,4}(\tau)}, \quad y = \frac{\partial_{5,4}(Q_5)(\tau)}{\Delta_{5,4}(\tau)^{3/2}},$$

and $f_{20}(\tau)$ is the newform associated to E. One finds that $\Delta_{5,4}(\tau) \in S_4(\Gamma_0(5))$, so curiously the modular forms $\Delta_{5,4}, Q_5$ and $\partial(Q_5)$ appearing in this parametrization are modular for $\Gamma_0(5)$, although the conductor of E is 20.

Using Eichler–Shimura theory, we generalize (1.1) to the arbitrary elliptic curve E of conductor $4N$, $E : y^2 = x^3 + ax^2 + bx + c$, where $a, b, c \in \mathbb{Q}$, which admits a modular parametrization $\Phi : X \to E$ satisfying

$$\Phi^*\left(\frac{dx}{2y}\right) = \pi i f_{4N}(\tau/2)d\tau.$$

Here X is the modular curve $\mathbb{H}/\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\Gamma_0(4N)\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, and $f_{4N}(\tau) \in S_2(\Gamma_0(4N))$ is a newform with rational Fourier coefficients associated to E. It follows from the modularity theorem that in any $\mathbb{Q}$-isomorphism class of elliptic curves there is an elliptic curve E admitting such a parametrization (note that for $u \in \mathbb{Q}^\times$ the change of variables $x = u^2X$ and $y = u^3Y$ implies $\frac{dX}{Y} = u\frac{dx}{y}$).

To such a Φ we associate a solution $Q(\tau) = x(\Phi(\tau))f_{4N}(\tau/2)^2$ of a differential equation

$$\partial_{N,4}(Q)^2 = Q^3 + aQ^2\Delta_{N,4} + bQ\Delta_{N,4}^2 + c\Delta_{N,4}^3, \tag{1.2}$$

where $\Delta_{N,4}(\tau) = f_{4N}(\tau/2)^2$, and

$$\partial_{N,4}(Q(\tau)) = \frac{1}{2\pi i}Q(\tau)' - \frac{1}{2\pi i}Q(\tau)\frac{\Delta_{N,4}(\tau)'}{\Delta_{N,4}(\tau)}.$$

We show in Corollary 12 that $f_{4N}(\tau/2)^2$ is modular for $\Gamma_0(N)$. In general the solution $Q(\tau)$ will not be holomorphic and will be modular only for $\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1}\Gamma_0(4N)\left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, but if the preimage of the point at infinity of E under Φ is contained in cusps of X and is invariant under the action of $\left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ (acting on X by Möbius transformations), then $Q(\tau)$ will be both holomorphic and modular for $\Gamma_0(N)$ (for more details see Proposition 5 and Theorem 7). Moreover, in Theorem 6 we show that there are only finitely many (up to isomorphism) elliptic curves E admitting Φ with these two properties.

We also obtain similar results generalizing the other examples from [8] that correspond to the elliptic curves over $\mathbb{Q}$ with j-invariant 0 and 1728 (see the next section).

2. Main results. Throughout the paper, let N be a positive integer and $k \in \{4, 6, 8, 12\}$. Let $E_k/\mathbb{Q}$ be an elliptic curve given by the short Weierstrass equation $y^2 = f_k(x)$, where

$$\begin{aligned} f_4(x) &= x^3 + a_2x^2 + a_4x + a_6, \\ f_6(x) &= x^3 + b_6, \\ f_8(x) &= x^3 + c_4x, \\ f_{12}(x) &= x^3 + d_6, \end{aligned}$$

and $a_2, a_4, a_6, b_6, c_4, d_6 \in \mathbb{Q}$. Moreover, we assume $j(E_4) \neq 0, 1728$. Let

$$f_{N,k}(\tau) \in S_2(\Gamma_0(k^2N/4))$$

be a newform with rational Fourier coefficients, and let $\Gamma_k := \left(\begin{smallmatrix} 2/k & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \cdot \Gamma_0(k^2N/4)\left(\begin{smallmatrix} 2/k & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define

$$\Delta_{N,k}(\tau) := f_{N,k}(2\tau/k)^{k/2} \in S_k(\Gamma_k).$$

For $f(\tau) \in M_4^{\mathrm{mer}}(\Gamma_k)$, we define the (Ramanujan–Serre) differential operator by

$$\partial_{N,k}(f(\tau)) = \frac{k}{8\pi i} f'(\tau) - \frac{1}{2\pi i} f(\tau) \frac{\Delta'_{N,k}(\tau)}{\Delta_{N,k}(\tau)} \in M_6^{\mathrm{mer}}(\Gamma_k).$$

Finally, assume that there is a meromorphic modular form $Q_k(\tau) \in M_4^{\mathrm{mer}}(\Gamma_k)$ such that the corresponding differential equation holds:

$$\begin{aligned} \partial_{N,4}(Q_4(\tau))^2 &= Q_4(\tau)^3 + a_2Q_4(\tau)^2\Delta_{N,4}(\tau) \\ &\quad + a_4Q_4(\tau)\Delta_{N,4}(\tau)^2 + a_6\Delta_{N,4}(\tau)^3, \\ \partial_{N,6}(Q_6(\tau))^2 &= Q_6(\tau)^3 + b_6\Delta_{N,6}(\tau)^2, \\ \partial_{N,8}(Q_8(\tau))^2 &= Q_8(\tau)^3 + c_4Q_8(\tau)\Delta_{N,8}(\tau), \\ \partial_{N,12}(Q_{12}(\tau))^2 &= Q_{12}(\tau)^3 + d_6\Delta_{N,12}(\tau). \end{aligned} \tag{2.1}$$

Each of these identities defines a modular parametrization $\Phi_k : X_k \to E_k$ by

$$\Phi_k(\tau) = \left(\frac{Q_k(\tau)}{\Delta_{N,k}(\tau)^{4/k}}, \frac{\partial_{N,k}(Q_k)(\tau)}{\Delta_{N,k}(\tau)^{6/k}} \right),$$

where X_k is the compactified modular curve $\mathbb{H}/\Gamma_k$.

PROPOSITION 1. *Let $\frac{dx}{2y}$ be the Néron differential on E_k. Then*

$$\Phi_k^*\left(\frac{dx}{2y}\right) = \frac{4\pi i}{k} f_{N,k}(2\tau/k)d\tau. \tag{2.2}$$

In particular, the conductor of E_k is $k^2N/4$ and $f_{N,k}(\tau)$ is the cusp form associated to E_k by the modularity theorem.

Remark 2. Note that when $k = 6, 8$ or 12, $f_{N,k}(\tau)$ is a modular form with complex multiplication by the ring of integers of $\mathbb{Q}(\sqrt{-3})$, $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$, respectively.

Conversely, given a modular parametrization $\Phi_k : X_k \to E_k$ satisfying (2.2), we construct a differential equation (2.1) and its solution $Q_k(\tau)$ as follows.

Let x and y be two functions on E_k satisfying the Weierstrass equation $y^2 = f_k(x)$. The functions $x(\tau) := x \circ \Phi_k(\tau)$ and $y(\tau) := y \circ \Phi_k(\tau)$ satisfy $y(\tau)^2 = f_k(x(\tau))$. Moreover (2.2) implies that

$$\left(\frac{k}{8\pi i}x'(\tau)\right)^2 = f_{N,k}(2\tau/k)^2 y(\tau)^2 = \Delta_{N,k}(\tau)^{4/k} f_k(x(\tau)). \tag{2.3}$$

Define $Q_k(\tau) := x(\tau)\Delta_{N,k}(\tau)^{4/k}$.

Proposition 3. *The following formula holds:*

$$\partial_{N,k}(Q_k(\tau))^2 = \Delta_{N,k}(\tau)^{12/k} f_k(x(\tau)).$$

In particular, $Q_k(\tau)$ is a solution of (2.1).

Now we investigate conditions under which $Q_k(\tau)$ is holomorphic. The following lemma easily follows from the formula above.

Lemma 4. *Assume that $\tau_0 \in X_k$ is a pole of $x(\tau)$. Then*

$$\operatorname{ord}_{\tau_0}(Q_k(\tau)) = \begin{cases} 0 & \text{if } \tau_0 \text{ is a cusp}, \\ -2 & \text{if } \tau_0 \in \mathbb{H}. \end{cases}$$

As a consequence, we have the following characterization of the holomorphicity of $Q_k(\tau)$ in terms of the modular parametrization Φ_k. Denote by $\mathcal{C}$ the set of cusps of X_k, and by $\mathcal{O}$ the point at infinity of E_k.

Proposition 5. *$Q_k(\tau)$ is holomorphic if and only if $\Phi_k^{-1}(\mathcal{O}) \subset \mathcal{C}$.*

In Section 3.2 we show that the degree of Φ_k (as a function of the conductor) grows faster than the total ramification index at cusps, hence the following theorem holds.

Theorem 6. *There are finitely many elliptic curves $E/\mathbb{Q}$ (up to a $\mathbb{Q}$-isomorphism) that admit a modular parametrization $\Phi : X_k \to E$ with the property that $\Phi^{-1}(\mathcal{O}) \subset \mathcal{C}$.*

In particular, there are finitely many elliptic curves E_k (up to a $\mathbb{Q}$-isomorphism) for which $Q_k(\tau)$ (which satisfies (2.1)*) is holomorphic.*

Define $A := \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$ and $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. It is easy to see that $\Gamma_0(N)$ is generated by Γ_k and A and T (Lemma 9 below), hence $Q_k(\tau)$ is modular for $\Gamma_0(N)$ if and only if it is invariant under the action of the slash operators $|A$ and $|T$. The following theorem describes the modularity in terms of the parametrization Φ_k.

THEOREM 7. *If $\Phi_k^{-1}(\mathcal{O})$ is invariant under A and T, then $Q_k(\tau)$ is modular for $\Gamma_0(N)$.*

3. Proofs

3.1. Proof of Propositions 1 and 3

Proof of Proposition 1. We have

$$\begin{aligned}\Phi_k^*\left(\frac{dx}{2y}\right) &= \frac{1}{2}\,\frac{d}{d\tau}\left(\frac{Q_k(\tau)}{\Delta_{N,k}(\tau)^{4/k}}\right)\frac{\Delta_{N,k}(\tau)^{6/k}}{\partial_{N,k}(Q_k)(\tau)}\,d\tau \\ &= \frac{1}{2}\,\frac{\frac{d}{d\tau}Q_k(\tau)f_{N,k}(2\tau/k)^2 - \frac{d}{d\tau}f_{N,k}(2\tau/k)^2Q_k(\tau)}{f_{N,k}(2\tau/k)^4} \\ &\quad\times\frac{f_{N,k}(2\tau/k)^3}{\frac{k}{8\pi i}\frac{d}{d\tau}Q_k(\tau) - Q_k(\tau)\frac{\frac{d}{d\tau}f_{N,k}(2\tau/k)^{k/2}}{2\pi i f_{N,k}(2\tau/k)^{k/2}}}\,d\tau \\ &= \frac{4\pi i}{k}f_{N,k}(2\tau/k)d\tau. \ \blacksquare\end{aligned}$$

Proof of Proposition 3. By definition,

$$\begin{aligned}\partial_{N,k}(Q_k(\tau)) &= \frac{k}{8\pi i}\big(x(\tau)\Delta_{N,k}(\tau)^{4/k}\big)' - \frac{1}{2\pi i}x(\tau)\Delta_{N,k}(\tau)^{4/k}\frac{\Delta'_{N,k}(\tau)}{\Delta_{N,k}(\tau)} \\ &= \frac{k}{8\pi i}x'(\tau)\Delta_{N,k}(\tau)^{4/k}.\end{aligned}$$

Hence the claim follows from (2.3). ∎

3.2. Proof of Theorem 6. Let $e_x \in \mathbb{Z}$ be the ramification index of Φ_k at $x \in X_k$, and let $\deg(\Phi_k)$ be the degree of Φ_k. It follows from the Hurwitz formula that $\sum_{x\in X_k}(e_x - 1) = 2g - 2$, where g is the genus of X_k (equal to the genus of $\Gamma_0(k^2N/4)$). Therefore $\Phi_k^{-1}(\mathcal{O}) \subset \mathcal{C}$ implies

$$\deg(\Phi_k) \le \sum_{x\in\mathcal{C}} e_x \le 2g - 2 + \#\mathcal{C}. \tag{3.1}$$

In [11], Watkins proved a lower bound for the degree of a modular parametrization Φ of an elliptic curve over $\mathbb{Q}$ of conductor M:

$$\deg(\Phi) \ge \frac{M^{7/6}}{\log M}\,\frac{1/10300}{\sqrt{0.02 + \log\log M}}.$$

On the other hand, an upper bound (see [4]) for the genus g of $X_0(M)$ is

$$g < M\frac{e^\gamma}{2\pi^2}(\log\log M + 2/\log\log M) \quad \text{for } M > 2,$$

where $\gamma = 0.5772\ldots$ is Euler's constant.

If we use a trivial bound $\#\mathcal{C} \le M$, an easy calculation shows that (3.1) cannot hold for curves E_k of conductor greater than 10^{50}. Thus, we have proved Theorem 6.

REMARK 8. If we assume that the ramification index at cusps is bounded by 24 (as suggested in the paper of Brunault [3]), and if we use Abramovich's [1] lower bound for the modular degree, $\deg(\Phi) \ge 7M/1600$, we conclude that (3.1) cannot hold for elliptic curves of conductor greater than 2^{19}.

3.3. Proof of Theorem 7. In this section we investigate conditions on the modular parametrization Φ_k under which $\Delta_{N,k}(\tau)$ and $Q_k(\tau)$, initially modular for Γ_k, are modular for $\Gamma_0(N)$.

For $S = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$, and a (meromorphic) modular form $f(\tau)$ of weight l, we define the usual slash operator as $f(\tau)|_l S := f(S\tau)(c\tau + d)^{-l}$, where $S\tau = \frac{a\tau+b}{c\tau+d}$. Define $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ and $A := \left(\begin{smallmatrix} 1 & 0 \\ N & 1 \end{smallmatrix}\right)$.

LEMMA 9. *The group $\Gamma_0(kN/2)$ is generated by Γ_k and T, while $\Gamma_0(N)$ is generated by $\Gamma_0(kN/2)$ and A.*

Proof. To prove the first statement, let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(kN/2)$. Then $\gcd(a, k/2) = 1$, and there is $r \in \mathbb{Z}$ such that $ar \equiv -b \pmod{k/2}$. Then $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) T^r \in \Gamma_k = \Gamma_0(kN/2) \cap \Gamma^0(k/2)$, and the claim follows.

The second statement is proved analogously. ■

Thus, to prove that $\Delta_{N,k}(\tau)$ and $Q_k(\tau)$ are modular for $\Gamma_0(N)$ it suffices to show their invariance under the slash operators $|T$ and $|A$.

LEMMA 10. *The matrices A and T normalize Γ_k.*

Proof. Let $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_k = \Gamma_0(kN/2) \cap \Gamma^0(k/2)$. Then $kN/2 | c$ and $k/2 | c$, and $ad \equiv 1 \pmod{k/2}$. In particular, since $k/2 \in \{2, 3, 4, 6\}$, it follows that $a \equiv d \pmod{k/2}$.

Since

$$A^{-1} \begin{pmatrix} ab \\ cd \end{pmatrix} A = \begin{pmatrix} a + bN & b \\ -aN - bN^2 + c + dN & -bN + d \end{pmatrix},$$

$$T^{-1} \begin{pmatrix} ab \\ cd \end{pmatrix} T = \begin{pmatrix} a - c & a + b - c - d \\ c & c + d \end{pmatrix},$$

the claim follows. ■

For a prime p, define the Hecke operator T_p as a double coset operator $\Gamma_k \left(\begin{smallmatrix} 1 & 0 \\ 0 & p \end{smallmatrix}\right) \Gamma_k$ acting on the space of cusp forms on Γ_k. The slash operators $|A$ and $|T$ correspond to $\Gamma_k A \Gamma_k$ and $\Gamma_k T \Gamma_k$ (see [6, Chapter 5]).

Define the Fricke involution $|_2 B$ on $S_2(\Gamma_k)$ by the matrix $B := \left(\begin{smallmatrix} 0 & -k/2 \\ kN/2 & 0 \end{smallmatrix}\right)$. Note that $|_2 B$ is the conjugate of the usual Fricke involution on $\Gamma_0(k^2N/4)$.

In particular, B normalizes Γ_k, and $|_2 B$ commutes with all the Hecke operators T_p with $p \nmid k^2N/4$. Hence, $f_{N,k}(2\tau/k)|_2 B = \lambda_{k,N} f_{N,k}(2\tau/k)$ for some $\lambda_{k,N} = \pm 1$.

Lemma 11. *The following are true:*

(a) $f_{N,k}(2\tau/k)|_2 T = e^{4\pi i/k} f_{N,k}(2\tau/k)$,
(b) $f_{N,k}(2\tau/k)|_2 A = e^{-4\pi i/k} f_{N,k}(2\tau/k)$.

In particular, $|_2 A$ *and* $|_2 B$ *have order* $k/2$ *when acting on* $f_{N,k}(2\tau/k)$.

Proof. A key observation is that the Fourier coefficients of $f_{N,k}(\tau)$ are supported at integers that are 1 (mod $k/2$). This implies

$$f_{N,k}(2\tau/k)|_2 T = e^{4\pi i/k} f_{N,k}(2\tau/k).$$

When $k = 4$ (and $k = 12$) this is a consequence of the general fact that $a_f(2) = 0$ whenever $f(\tau) = \sum a_f(n)q^n$ is a newform of level divisible by 4 (see [10, p. 29]). In the other three cases, $f_{N,k}(\tau)$ is a modular form with complex multiplication by the ring of integers of $\mathbb{Q}(\sqrt{-3})$ or $\mathbb{Q}(\sqrt{-1})$, hence its Fourier coefficients $a_{f_{N,k}}(p)$ are zero when p is an inert prime (i.e. $p \equiv 2$ (mod 3) or $p \equiv 3$ (mod 4), respectively). Multiplicativity of the Fourier coefficients then implies the observation.

On the other hand $A = BT^{-1}B^{-1}$, therefore

$$\begin{aligned} f_{N,k}(2\tau/k)|_2 A &= (f_{N,k}(2\tau/k)|_2 B)|_2 T^{-1}|_2 B^{-1} \\ &= (\lambda_{k,N} f_{N,k}(2\tau/k)|_2 T^{-1})|_2 B^{-1} \\ &= \lambda_{k,N}\lambda_{k,N}^{-1} e^{-4\pi i/k} f_{N,k}(2\tau/k). \ \blacksquare \end{aligned}$$

Corollary 12. *We have:*

(a) $\Delta_{N,k}(\tau) \in S_k(\Gamma_0(N))$,
(b) $\Delta_{N,8}(\tau)^{1/2}|_4 A = -\Delta_{N,8}(\tau)^{1/2}$ *and* $\Delta_{N,8}(\tau)^{1/2}|_4 T = -\Delta_{N,8}(\tau)^{1/2}$,
(c) $\Delta_{N,12}(\tau)^{1/2}|_6 A = -\Delta_{N,12}(\tau)^{1/2}$ *and* $\Delta_{N,12}(\tau)^{1/2}|_6 T = -\Delta_{N,12}(\tau)^{1/2}$.

We now recall some basic facts about Jacobians of modular curves. For more details see Chapter 6 of [6]. Denote by $\mathrm{Jac}(X_k)$ the Jacobian of X_k. We will view it either as $S_2(\Gamma_k)^\wedge/H_1(X_k,\mathbb{Z})$ (where $\gamma \in H_1(X_k,\mathbb{Z})$ acts on $f(\tau) \in S_2(\Gamma_k)$ by $f(\tau) \mapsto \int_\gamma f(\tau)\,d\tau$), or as the Picard group $\mathrm{Pic}^0(X_k)$ of X_k, which is the quotient $\mathrm{Div}^0(X_k)/\mathrm{Div}^l(X_k)$ of the degree zero divisors of X_k modulo principal divisors. If x_0 is a base point in X_k, then X_k embeds into its Picard group under the Abel–Jacobi map

$$X_k \to \mathrm{Pic}^0(X_k), \qquad x \mapsto (x) - (x_0),$$

where $(x) - (x_0)$ denotes the equivalence class of divisors $(x) - (x_0) + \mathrm{Div}^l(X_k)$.

It is known that the parametrization $\Phi_k : X_k \to E_k$ can be factored as

$$X_k \hookrightarrow \mathrm{Jac}(X_k) \xrightarrow{\psi_k} \tilde{E}_k \xrightarrow{\phi_k} E_k. \tag{3.2}$$

Here $X_k \hookrightarrow \mathrm{Jac}(X_k)$ is the Abel–Jacobi map (for some base point x_0 in X_k), ϕ_k is a rational isogeny and $\tilde{E}_k$ (together with ψ_k) is the strong Weil curve associated to the newform $f_{N,k}(2\tau/k)$ via the Eichler–Shimura construction as follows.

Let V_k be the $\mathbb{C}$-span of $f_{N,k}(2\tau/k) \in S_2(\Gamma_k)$, and define $\Lambda_k := H_1(X_k)|V_k$. Restriction to V_k gives a homomorphism

$$\psi_k : \mathrm{Jac}(X_k) \to V_k^\wedge/\Lambda_k \cong \tilde{E}_k.$$

Here $V_k^\wedge/\Lambda_k$ is a one-dimensional complex torus isomorphic to the rational elliptic curve $\tilde{E}_k$ with the Weierstrass equation

$$\tilde{E}_k : y^2 = x^3 - \frac{g_2(\Lambda_k)}{4}x - \frac{g_3(\Lambda_k)}{4}.$$

Let S be either A or T. Since by Lemma 10, S normalizes Γ_k, we can define the action of S on $\mathrm{Jac}(X_k)$ in two equivalent ways: for $\phi \in S_2(\Gamma_k)^\wedge/H_1(X_k,\mathbb{Z})$ and $f(\tau) \in S_2(\Gamma_k)$ let $S(\phi)(f(\tau)) := \phi(f(\tau)|_2 S)$, or for $P = (x) - (x_0) \in \mathrm{Pic}^0(X_k)$ let $S(P) = (Sx) - (Sx_0)$. Now Lemma 11 implies that the action of S on $\mathrm{Jac}(X_k)$ descends to an automorphism of $\tilde{E}_k$ of order $k/2$.

Recall that x and y are functions on E_k satisfying the Weierstrass equation $y^2 = f_k(x)$, and that $x(\tau) = x \circ \Phi_k(\tau)$ and $y(\tau) = y \circ \Phi_k(\tau)$ are modular functions on X_k.

Proposition 13. *Let S be either A or T. If $\Phi_k^{-1}(\mathcal{O})$ is invariant under A and T, then:*

(a) $x(\tau)|S = \begin{cases} x(\tau) & \text{if } k = 4, \\ -x(\tau) & \text{if } k = 8. \end{cases}$

(b) $y(\tau)|S = \begin{cases} y(\tau) & \text{if } k = 6, \\ -y(\tau) & \text{if } k = 12. \end{cases}$

Proof. For $P \in E_k$, we define $S(P) := \phi_k(S(\tilde{P}))$ for any $\tilde{P} \in \phi_k^{-1}(P)$. This is well defined since the S-invariance of $\Phi_k^{-1}(\mathcal{O})$ implies the S-invariance of $\mathrm{Ker}(\phi_k)$. We have $\phi_k(S(P)) = S(\phi_k(P))$, hence S is an automorphism of E_k.

Let x_0 be a base point of the Abel–Jacobi map in (3.2). Then x_0 is in $\Phi_k^{-1}(\mathcal{O})$, hence $\phi_k \circ \psi_k$ maps $(Sx_0) - (x_0)$ to $\mathcal{O}$ in E_k. In particular, for $x \in X_k$ we have

$$\Phi_k(Sx) = \phi_k \circ \psi_k((Sx) - (x_0)) = \phi_k \circ \psi_k((Sx) - (Sx_0)) = S(\Phi_k(x)). \tag{3.3}$$

Assume first that $k = 4$. Then $j(E_4) \neq 0, 1728$, and the automorphism group of E_4 is of order 2 generated by $(x, y) \mapsto (x, -y)$. In particular $x(S(P)) = x(P)$ for every $P \in E_4$.

If $k = 8$, then S is an automorphism of order $k/2 = 4$ of $\tilde{E}_k$, therefore $j(\tilde{E}_k) = 1728$ and $g_3(\Lambda_8) = 0$. Moreover ϕ_k is an isomorphism (defined over $\mathbb{Q}$), which implies that S is an isomorphism of order 4 of E_8 as well. The automorphism group is generated by $(x, y) \mapsto (-x, iy)$, hence $x(S(P)) = -x(P)$ for every $P \in E_8$.

If $k = 6$ or 12, then $j(\tilde{E}_k) = 0$, $g_2(\Lambda_k) = 0$ and ϕ_k is an isomorphism (defined over $\mathbb{Q}$). Therefore, S has order 3 on E_k if $k = 6$, and order 6 if $k = 12$. The automorphism group is generated by $(x, y) \mapsto (e^{2\pi i/3}x, -y)$, and in particular $y(S(P)) = y(P)$ if $k = 6$, and $y(S(P)) = -y(P)$ if $k = 12$, for every $P \in E_k$.

Now (3.3) implies

$$x(\tau)|S = x(S\tau) = x(\Phi_k(S\tau)) = x(S(\Phi_k(\tau))),$$
$$y(\tau)|S = y(S\tau) = y(\Phi_k(S\tau)) = y(S(\Phi_k(\tau))),$$

and the claim follows from the previous paragraph. ∎

We need the following technical lemma. Recall $Q_k(\tau) := x(\tau)\Delta_{N,k}(\tau)^{4/k}$.

Lemma 14. *If $\partial_{N,k}(Q_k(\tau)) \in M_6^{\mathrm{mer}}(\Gamma_0(N))$, then $Q_k(\tau) \in M_4^{\mathrm{mer}}(\Gamma_0(N))$.*

Proof. As in the proof of Proposition 3, we have

$$\partial_{N,k}(Q_k(\tau)) = \frac{k}{8\pi i}x'(\tau)\Delta_{N,k}(\tau)^{4/k} = \frac{k}{8\pi i}\,\frac{x'(\tau)}{x(\tau)}Q_k(\tau).$$

Let S be either A or T. Then $(x(S\tau))' = x'(\tau)|_2 S$, and the invariance of $x'(\tau)/x(\tau)$ under S (hence under $\Gamma_0(N)$) follows from the fact that $x(\tau)$ is an eigenfunction for S, which follows from the proof of Proposition 13. ∎

Since $Q_k(\tau) := x(\tau)\Delta_{N,k}(\tau)^{4/k}$, Theorem 7 for $k = 4$ and 8 now follows from Corollary 12(a), (b) and Proposition 13(a), while the $k = 6$ and 12 case follows from $\partial_{N,k}(Q_k)(\tau) = y(\tau)\Delta_{N,k}(\tau)^{6/k}$ together with Corollary 12(a), (c), Proposition 13(b) and Lemma 14.

4. Example. Let

$$f_{19,4}(\tau) = \sum_{n=1}^{\infty} a(n)q^n = q + 2q^3 - q^5 - 3q^7 + q^9 + \cdots$$

be a unique newform in $S_2(\Gamma_0(76))$, and denote $\Delta_{19,4}(\tau) = f_{19,4}(\tau/2)^2 \in S_4(\Gamma_0(19))$.

Set $\Gamma = \left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)^{-1} \Gamma_0(76) \left(\begin{smallmatrix} 1/2 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. For $\tau \in \bar{\mathbb{H}}$ we define

$$\Psi(\tau) := \pi i \int_{i\infty}^{\tau} f(z/2)\, dz.$$

For $\gamma \in \Gamma$ and $\tau \in \bar{\mathbb{H}}$, define $\omega(\gamma) := \Psi(\gamma\tau) - \Psi(\tau)$. One easily checks that $\frac{d}{d\tau}\omega(\gamma) = 0$, hence $\omega(\gamma)$ does not depend on τ. Denote by Λ the image of Γ under ω. By Eichler–Shimura theory, Λ is a lattice, and $\Psi(\tau)$ induces a parametrization $X := \mathbb{H}/\Gamma \to \mathbb{C}/\Lambda$. The complex torus $\mathbb{C}/\Lambda$ is isomorphic to $E : y^2 = x^3 - \frac{g_2(\Lambda)}{4}x - \frac{g_3(\Lambda)}{4}$ by the map given by the Weierstrass $\wp$-function and its derivative, $z \mapsto (\wp(z,\Lambda), \wp'(z,\Lambda)/2)$, thus by composing these two maps we obtain a modular parametrization $\Phi : X \to E$.

One finds that Λ has generators

$$\omega_1 = 1.1104197465122\ldots,$$
$$\omega_2 = 0.5552098732561\ldots + 2.1752061725591\ldots \times i.$$

Moreover, $g_2(\Lambda) = 256/3$ and $g_3(\Lambda) = 4112/27$, hence Proposition 3 implies that

$$\begin{aligned} Q(\tau) &= \Delta_{19,4}(\tau)\wp(\Psi(\tau),\Lambda) \\ &= 1 + \tfrac{1}{3}\big(8q + 8q^2 + 64q^3 + 232q^4 + 336q^5 + 256q^6 + 512q^7 + \cdots\big) \end{aligned}$$

satisfies the differential equation

$$\partial_{19,4}(Q)^2 = Q^3 - \tfrac{64}{3} Q \Delta_{19,4}^2 - \tfrac{1028}{27}\Delta_{19,4}^3. \tag{4.1}$$

One finds that

$$\mathrm{GCD}(\{p + 1 - a(p) : p \text{ prime}, p \equiv 1 \pmod{76}\}) = 1,$$

hence it follows from the special case of the Drinfeld–Manin theorem (see [5, Theorem 2.20]) that $\Psi(\tau)$ maps cusps of X to the lattice Λ, or equivalently that Φ maps cusps of X to the point at infinity of E. The modular curve X has six cusps, and one can check (for example by using Magma) that the degree of Φ is six, therefore the conditions of Proposition 5 and Theorem 7 are satisfied, and we conclude that $Q(\tau) \in M_4(\Gamma_0(19))$.

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