Proof of a conjecture of Hirschhorn and Sellers on overpartitions

by

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1. Introduction. The objective of this paper is to give a proof of a conjecture of Hirschhorn and Sellers on the number of overpartitions. We shall use the technique of dissections of theta functions.

Let us begin with some notation and terminology on q-series and partitions. We adopt the common notation

$$(1.1) \qquad (a;q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n),$$

where $|q| < 1$, and we write

$$(1.2) \qquad (a_1, \ldots, a_n; q)_\infty = (a_1; q)_\infty \cdots (a_n; q)_\infty.$$

Recall that the *Ramanujan theta function* $f(a,b)$ is defined by

$$(1.3) \qquad f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2},$$

where $|ab| < 1$. The Jacobi triple product identity can be restated as

$$(1.4) \qquad f(a,b) = (-a, -b, ab; ab)_\infty.$$

In particular, let $f(-q) = f(-q, -q^2)$. Then we have

$$(1.5) \qquad f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty.$$

For any positive integer n, we use f_n to denote $f(-q^n)$, that is,

$$(1.6) \qquad f_n = (q^n; q^n)_\infty = \prod_{k=1}^{\infty} (1 - q^{nk}).$$

[59]

The function f_n is related to the generating function of overpartitions. A *partition* of a positive integer n is a nonincreasing sequence of positive integers whose sum is n. An *overpartition* of n is a partition in which the first occurrence of a number may be overlined (see Corteel and Lovejoy [CL]). For $n \geq 1$, we let $\bar{p}(n)$ denote the number of overpartitions of n, and set $\bar{p}(0) = 1$. Corteel and Lovejoy [CL] showed that the generating function for $\bar{p}(n)$ is given by

$$(1.7) \qquad \sum_{n=0}^{\infty} \bar{p}(n)q^n = \frac{f_2}{f_1^2}.$$

Hirschhorn and Sellers [HS-1] and Fortin, Jacob and Mathieu [FJM] obtained the following Ramanujan-type generating function formulas for $\bar{p}(2n+1)$, $\bar{p}(4n+3)$ and $\bar{p}(8n+7)$:

$$(1.8) \qquad \sum_{n=0}^{\infty} \bar{p}(2n+1)q^n = 2\frac{f_2^2 f_8^2}{f_1^4 f_4},$$

$$(1.9) \qquad \sum_{n=0}^{\infty} \bar{p}(4n+3)q^n = 8\frac{f_2 f_4^6}{f_1^8},$$

$$(1.10) \qquad \sum_{n=0}^{\infty} \bar{p}(8n+7)q^n = 64\frac{f_2^{22}}{f_1^{23}}.$$

The above identities lead to congruences modulo 2, 8 and 64 for the overpartition function. Mahlburg [M] proved that $\bar{p}(n)$ is divisible by 64 for almost all n by using relations between $\bar{p}(n)$ and the number of representations of n as a sum of squares. Using the theory of modular forms, Treneer [T] showed that the coefficients of a wide class of weakly holomorphic modular forms have infinitely many congruence relations for powers of every prime p other than 2 and 3. In particular, he proved that $\bar{p}(5m^3n) \equiv 0 \pmod{5}$ for any n that is coprime to m, where m is a prime satisfying $m \equiv -1 \pmod{5}$.

The following conjecture was posed by Hirschhorn and Sellers [HS-1].

CONJECTURE 1.1. *For* $n \geq 0$,

$$(1.11) \qquad \bar{p}(40n + 35) \equiv 0 \pmod{40}.$$

To prove the conjecture, we derive a generating function for $\bar{p}(40n + 35)$ modulo 5 by using 2-dissection formulas for quotients of theta functions given by Ramanujan [B], and Hirschhorn and Sellers [HS-2]. Then we use the (p, k)-parametrization of theta functions due to Alaca, Alaca and Williams [AAW, AW, W] to show that $\bar{p}(40n+35) \equiv 0 \pmod{5}$ for $n \geq 0$. Combining this congruence and $\bar{p}(4n + 3) \equiv 0 \pmod{8}$ for $n \geq 0$, we confirm the conjecture.

2. The generating function. In this section, we derive a generating function of $\bar{p}(40n+35)$ modulo 5. We first recall several 2-dissection formulas for quotients of theta functions due to Ramanujan [B] and Hirschhorn and Sellers [HS-2].

The following relations are consequences of dissection formulas of Ramanujan collected in Entry 25 in Berndt's book [B, p. 40]. Recall that $f_n = (q^n; q^n)_\infty$, as given by (1.6).

LEMMA 2.1. *We have*

$$(2.1) \qquad f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q \frac{f_2 f_{16}^2}{f_8},$$

$$(2.2) \qquad \frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q \frac{f_4^2 f_{16}^2}{f_2^5 f_8},$$

$$(2.3) \qquad f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q \frac{f_2^2 f_8^4}{f_4^2},$$

$$(2.4) \qquad \frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q \frac{f_4^2 f_8^4}{f_2^{10}}.$$

Hirschhorn and Sellers [HS-2] established the following 2-dissection formula.

LEMMA 2.2. *We have*

$$(2.5) \qquad \frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q \frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}.$$

By Lemmas 2.1 and 2.2, we are led to a generating function of $\bar{p}(40n+35)$ modulo 5.

THEOREM 2.3. *We have*

$$(2.6) \qquad \sum_{n=0}^{\infty} \bar{p}(40n + 35)q^n$$

$$\equiv 2\frac{f_2^{122}}{f_1^{63} f_4^{40}} + 3\frac{f_1 f_2^{26}}{f_4^8} + 4q\frac{f_2^{98}}{f_1^{55} f_4^{24}} + 3qf_1^9 f_2^2 f_4^8 + 4q^2 \frac{f_2^{74}}{f_1^{47} f_4^8}$$

$$+ 4q^3 \frac{f_2^{50} f_4^8}{f_1^{39}} + 4q^4 \frac{f_2^{26} f_4^{24}}{f_1^{31}} + 2q^5 \frac{f_2^2 f_4^{40}}{f_1^{23}} \pmod{5}.$$

Proof. Recall that the theta functions $\varphi(q)$ and $\psi(q)$ are defined by

$$(2.7) \qquad \varphi(q) = f(q, q) = \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$(2.8) \qquad \psi(q) = f(q, q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2}.$$

By the Jacobi triple product identity, we find

$$(2.9) \qquad \varphi(q) = \frac{f_2^5}{f_1^2 f_4^2},$$

$$(2.10) \qquad \psi(q) = \frac{f_2^2}{f_1}.$$

Replacing q by $-q$ in (2.9) and (2.10), and using the fact that

$$(2.11) \qquad f(q) = (-q; -q)_\infty = \frac{f_2^3}{f_1 f_4},$$

we deduce that

$$(2.12) \qquad \varphi(-q) = \frac{f_1^2}{f_2},$$

$$(2.13) \qquad \psi(-q) = \frac{f_1 f_4}{f_2}.$$

In view of (1.3), (1.4) and (2.7), we see that

$$(2.14) \quad \varphi(q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \sum_{n=-\infty}^{\infty} q^{25n^2} + 2q \sum_{n=-\infty}^{\infty} q^{25n^2 + 10n} + 2q^4 \sum_{n=-\infty}^{\infty} q^{25n^2 + 20n}$$

$$= \varphi(q^{25}) + 2q D(q^5) + 2q^4 E(q^5),$$

where

$$(2.15) \qquad D(q) = \sum_{n=-\infty}^{\infty} q^{5n^2 + 2n} = (-q^3, -q^7, q^{10}; q^{10})_\infty,$$

$$(2.16) \qquad E(q) = \sum_{n=-\infty}^{\infty} q^{5n^2 + 4n} = (-q, -q^9, q^{10}; q^{10})_\infty.$$

It is easily checked that

$$(2.17) \qquad D(q) E(q) = \frac{f_2^2 f_5 f_{20}}{f_1 f_4}.$$

By the binomial theorem, we see that for any positive integer k

$$(2.18) \qquad (1 - q^k)^5 \equiv (1 - q^{5k}) \pmod{5},$$

which implies that

$$(2.19) \qquad \varphi^5(q) \equiv \varphi(q^5) \pmod 5.$$

Based on (1.7), (2.9) and (2.19), we have

$$(2.20) \quad \sum_{n=0}^{\infty} \bar{p}(n)(-q)^n$$

$$= \frac{1}{\varphi(q)} = \frac{\varphi^4(q)}{\varphi^5(q)}$$

$$\equiv \frac{\varphi^4(q)}{\varphi(q^5)} \equiv \frac{(\varphi(q^{25}) + 2qD(q^5) + 2q^4E(q^5))^4}{\varphi(q^5)}$$

$$\equiv \frac{1}{\varphi(q^5)} \big(\varphi^4(q^{25}) + 3q\varphi^3(q^{25})D(q^5) + 4q^2\varphi^2(q^{25})D^2(q^5)$$

$$+ 2q^3\varphi(q^{25})D^3(q^5) + 3q^4\varphi^3(q^{25})E(q^5) + q^4D^4(q^5)$$

$$+ 3q^5\varphi^2(q^{25})D(q^5)E(q^5) + q^6\varphi(q^{25})D^2(q^5)E(q^5)$$

$$+ 4q^7D^3(q^5)E(q^5) + 4q^8\varphi^2(q^{25})E^2(q^5) + q^9\varphi(q^{25})D(q^5)E^2(q^5)$$

$$+ q^{10}D^2(q^5)E^2(q^5) + 2q^{12}\varphi(q^{25})E^3(q^5)$$

$$+ 4q^{13}D(q^5)E^3(q^5) + q^{16}E^4(q^5)^4 \big) \pmod 5,$$

which yields

$$(2.21) \quad \sum_{n=0}^{\infty} \bar{p}(5n)(-q)^n$$

$$\equiv \frac{\varphi^4(q^5) + 3q\varphi^2(q^5)D(q)E(q) + q^2D^2(q)E^2(q)}{\varphi(q)} \pmod 5.$$

Plugging (2.9) and (2.17) into (2.21), we get

$$(2.22) \quad \sum_{n=0}^{\infty} \bar{p}(5n)(-q)^n \equiv \frac{f_1^2 f_4^2 f_{20}^{20}}{f_2^5 f_5^8 f_{20}^8} + 3q\frac{f_1 f_4 f_{10}^{10}}{f_2^3 f_5^3 f_{20}^3} + q^2\frac{f_5^2 f_{20}^2}{f_2} \pmod 5.$$

Replacing q by $-q$ in (2.22) and invoking (2.11), we arrive at

$$(2.23) \quad \sum_{n=0}^{\infty} \bar{p}(5n)q^n \equiv \frac{f_2 f_5^8}{f_1^2 f_{10}^4} - 3q\frac{f_5^3 f_{10}}{f_1} + q^2\frac{f_{10}^6}{f_2 f_5^2} \pmod 5.$$

According to the 2-dissection formulas (2.1), (2.2), (2.3), (2.5) and congru-

ence (2.23), we obtain

$$(2.24) \quad \sum_{n=0}^{\infty} \bar{p}(5n)q^n$$

$$\equiv \frac{f_2}{f_{10}^4}\left(\frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}\right)\left(\frac{f_{20}^{10}}{f_{10}^2 f_{40}^4} - 4q^5\frac{f_{10}^2 f_{40}^4}{f_{20}^2}\right)^2$$

$$- 3qf_{10}\left(\frac{f_{10}f_{40}^5}{f_{20}^2 f_{80}^2} - 2q^5\frac{f_{10}f_{80}^2}{f_{40}}\right)\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10}f_{40}}{f_2^3 f_8 f_{20}}\right)$$

$$+ q^2\frac{f_{10}^6}{f_2}\left(\frac{f_{40}^5}{f_{10}^5 f_{80}^2} + 2q^5\frac{f_{20}^2 f_{80}^5}{f_{10}^5 f_{40}}\right)$$

$$\equiv \frac{f_8^5 f_{20}^{20}}{f_2^4 f_{10}^8 f_{16}^2 f_{40}^8} + 2q\frac{f_4^2 f_{16}^2 f_{20}^{20}}{f_2^4 f_8 f_{10}^8 f_{40}^8} - 3q\frac{f_8 f_{10}^2 f_{40}^4}{f_2^2 f_{80}^2} - 3q^2\frac{f_4^3 f_{10}^3 f_{40}^6}{f_2^3 f_8 f_{20}^3 f_{80}^2}$$

$$+ q^2\frac{f_{10}f_{40}^5}{f_2 f_{80}^2} - 3q^5\frac{f_8^5 f_{20}^8}{f_2^4 f_{10}^4 f_{16}^2} + q^6\frac{f_8 f_{10}^2 f_{20}^2 f_{80}^2}{f_2^2 f_{40}^2} - q^6\frac{f_4^2 f_{16}^2 f_{20}^8}{f_2^4 f_8 f_{10}^4}$$

$$+ q^7\frac{f_4^3 f_{10}^3 f_{80}^2}{f_2^3 f_8 f_{20}} + 2q^7\frac{f_{10}f_{20}^2 f_{80}^2}{f_2 f_{40}} + q^{10}\frac{f_8^5 f_{40}^8}{f_2^4 f_{16}^2 f_{20}^4}$$

$$+ 2q^{11}\frac{f_4^2 f_{16}^2 f_{40}^8}{f_2^4 f_8 f_{20}^4} \pmod 5.$$

Extracting the terms of odd powers of q on both sides of (2.24), we have

$$(2.25) \quad \sum_{n=0}^{\infty} \bar{p}(10n+5)q^{2n+1}$$

$$\equiv 2q\frac{f_4^2 f_{16}^2 f_{20}^{20}}{f_2^4 f_8 f_{10}^8 f_{40}^8} - 3q\frac{f_8 f_{10}^2 f_{40}^4}{f_2^2 f_{80}^2} - 3q^5\frac{f_8^5 f_{20}^8}{f_2^4 f_{10}^4 f_{16}^2} + q^7\frac{f_4^3 f_{10}^3 f_{80}^2}{f_2^3 f_8 f_{20}}$$

$$+ 2q^7\frac{f_{10}f_{20}^2 f_{80}^2}{f_2 f_{40}} + 2q^{11}\frac{f_4^2 f_{16}^2 f_{40}^8}{f_2^4 f_8 f_{20}^4} \pmod 5.$$

Dividing both sides of (2.25) by q and replacing q^2 by q, we get

$$(2.26) \quad \sum_{n=0}^{\infty} \bar{p}(10n+5)q^n$$

$$\equiv 2\frac{f_2^2 f_8^2 f_{10}^{20}}{f_1^4 f_4 f_5^8 f_{20}^8} - 3\frac{f_4 f_5^2 f_{20}^4}{f_1^2 f_{40}^2} - 3q^2\frac{f_4^5 f_{10}^8}{f_1^4 f_5^4 f_8^2}$$

$$+ q^3\frac{f_2^3 f_5^3 f_{40}^2}{f_1^3 f_4 f_{10}} + 2q^3\frac{f_5 f_{10}^2 f_{40}^2}{f_1 f_{20}} + 2q^5\frac{f_2^2 f_8^2 f_{20}^8}{f_1^4 f_4 f_{10}^4} \pmod 5.$$

Employing the 2-dissection formulas (2.4), (2.5) and congruence (2.26), we

deduce that

$$(2.27) \quad \sum_{n=0}^{\infty} \bar{p}(10n+5)q^n$$

$$\equiv 2\frac{f_2^2 f_8^2 f_{10}^{20}}{f_4 f_{20}^8}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14} f_{40}^4} + 4q^5\frac{f_{20}^2 f_{40}^4}{f_{10}^{10}}\right)^2$$

$$- 3\frac{f_4 f_{20}^4}{f_{40}^2}\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)^2$$

$$- 3q^2\frac{f_4^5 f_{10}^8}{f_8^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)\left(\frac{f_{20}^{14}}{f_{10}^{14} f_{40}^4} + 4q^5\frac{f_{20}^2 f_{40}^4}{f_{10}^{10}}\right)$$

$$+ q^3\frac{f_2^3 f_{40}^2}{f_4 f_{10}}\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)^3$$

$$+ 2q^3\frac{f_{10}^2 f_{40}^2}{f_{20}}\left(\frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}\right)$$

$$+ 2q^5\frac{f_2^2 f_8^2 f_{20}^8}{f_4 f_{10}^4}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)$$

$$\equiv -3\frac{f_4 f_8^2 f_{20}^8}{f_2^4 f_{40}^4} + 2\frac{f_4^{13} f_{20}^{20}}{f_2^{12} f_8^2 f_{10}^8 f_{40}^8} + 3q\frac{f_4 f_8^6 f_{20}^{20}}{f_2^8 f_{10}^8 f_{40}^8} - q\frac{f_4^4 f_{10} f_{20}^5}{f_2^5 f_{40}^2}$$

$$- 3q^2\frac{f_4^{19} f_{20}^{14}}{f_2^{14} f_8^6 f_{10}^6 f_{40}^4} - 3q^2\frac{f_4^7 f_{10}^2 f_{20}^2}{f_2^6 f_8^2} - 2q^3\frac{f_4^7 f_8^2 f_{20}^{14}}{f_2^{10} f_{10}^6 f_{40}^4}$$

$$+ 2q^3\frac{f_8 f_{10}^2 f_{20} f_{40}}{f_2^2} + q^3\frac{f_8^3 f_{20}^6}{f_2^3 f_4 f_{10} f_{40}} + 2q^4\frac{f_4^3 f_{10}^3 f_{40}^3}{f_2^3 f_8 f_{20}^2}$$

$$+ 3q^4\frac{f_4^2 f_8 f_{20}^3 f_{40}}{f_2^4} + 3q^5\frac{f_4^5 f_{10} f_{40}^3}{f_2^5 f_8} + 3q^5\frac{f_4^{13} f_8^8}{f_2^{12} f_8^2 f_{10}^4}$$

$$+ 2q^6\frac{f_4 f_8^6 f_{20}^8}{f_2^8 f_{10}^4} + q^6\frac{f_8^8 f_{10}^2 f_{40}^5}{f_2^6 f_8^3 f_{20}^3} - 2q^7\frac{f_4^{19} f_{20}^2 f_{40}^4}{f_2^{14} f_8^6 f_{10}^2}$$

$$- 3q^8\frac{f_4^7 f_8^2 f_{20}^2 f_{40}^4}{f_2^{10} f_{10}^2} + 2q^{10}\frac{f_4^{13} f_8^8}{f_2^{12} f_8^2 f_{20}^4} + 3q^{11}\frac{f_4 f_8^6 f_{40}^8}{f_2^8 f_{20}^4} \pmod{5}.$$

Extracting the terms of odd powers of q on both sides of (2.27), then dividing by q and replacing q^2 by q, we find that

$$(2.28) \quad \sum_{n=0}^{\infty} \bar{p}(20n+15)q^n$$

$$\equiv 3\frac{f_2 f_4^6 f_{10}^{20}}{f_1^8 f_5^8 f_{20}^8} - \frac{f_2^4 f_5 f_{10}^5}{f_1^5 f_{20}^2} - 2q\frac{f_2^7 f_4^2 f_{10}^{14}}{f_1^{10} f_5^6 f_{20}^4} + 2q\frac{f_4 f_5^2 f_{10} f_{20}}{f_1^2}$$

$$+ q\frac{f_4^3 f_{10}^6}{f_1^3 f_2 f_5 f_{20}} + 3q^2 \frac{f_2^5 f_5 f_{20}^3}{f_1^5 f_4} + 3q^2 \frac{f_2^{13} f_{10}^8}{f_1^{12} f_4^2 f_5^4}$$

$$- 2q^3 \frac{f_2^{19} f_{10}^2 f_{20}^4}{f_1^{14} f_4^6 f_5^2} + 3q^5 \frac{f_2 f_4^6 f_{20}^8}{f_1^8 f_{10}^4} \pmod 5.$$

By (2.18), we see that

(2.29) $f_5 \equiv f_1^5 \pmod 5.$

Substituting (2.29) into (2.28) gives

(2.30) $\displaystyle\sum_{n=0}^{\infty} \bar{p}(20n+15)q^n$

$$\equiv 3\frac{f_2^{101}}{f_1^{48} f_4^{34}} - \frac{f_2^{29}}{f_4^{10}} - 2q\frac{f_2^{77}}{f_1^{40} f_4^{18}} + 2q f_1^8 f_2^5 f_4^6 + q\frac{f_2^{29}}{f_1^8 f_4^2}$$

$$+ 3q^2 f_2^5 f_4^{14} + 3q^2 \frac{f_2^{53}}{f_1^{32} f_4^2} - 2q^3 \frac{f_2^{29} f_4^{14}}{f_1^{24}}$$

$$+ 3q^5 \frac{f_4^{46}}{f_1^8 f_2^{19}} \pmod 5.$$

Combining the 2-dissection formulas (2.3), (2.4) and congruence (2.30), we see that

(2.31) $\displaystyle\sum_{n=0}^{\infty} \bar{p}(20n+15)q^n$

$$\equiv 3\frac{f_2^{101}}{f_4^{34}}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^{12} + 2q f_2^5 f_4^6 \left(\frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}\right)^2$$

$$- \frac{f_2^{29}}{f_4^{10}} - 2q\frac{f_2^{77}}{f_4^{18}}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^{10}$$

$$+ q\frac{f_2^{29}}{f_4^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2 + 3q^2 f_2^5 f_4^{14}$$

$$+ 3q^2 \frac{f_2^{53}}{f_4^2}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^8$$

$$- 2q^3 f_2^{29} f_4^{14}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^6$$

$$+ 3q^5 \frac{f_4^{46}}{f_2^{19}}\left(\frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}\right)^2$$

$$\equiv 3\frac{f_4^{134}}{f_2^{67}f_8^{48}} - \frac{f_2^{29}}{f_4^{10}} + 2q\frac{f_4^{122}}{f_2^{63}f_8^{40}} + 3q\frac{f_2f_4^{26}}{f_8^8} + q^2\frac{f_4^{110}}{f_2^{59}f_8^{32}}$$

$$+ 4q^3\frac{f_4^{98}}{f_2^{55}f_8^{24}} + 3q^3 f_2^9 f_4^2 f_8^8 + q^4\frac{f_4^{86}}{f_2^{51}f_8^{16}} + 4q^5\frac{f_4^{74}}{f_2^{47}f_8^8}$$

$$+ 4q^7\frac{f_4^{50}f_8^8}{f_2^{39}} + q^8\frac{f_4^{38}f_8^{16}}{f_2^{35}} + 4q^9\frac{f_4^{26}f_8^{24}}{f_2^{31}} + q^{10}\frac{f_4^{14}f_8^{32}}{f_2^{27}}$$

$$+ 2q^{11}\frac{f_4^2 f_8^{40}}{f_2^{23}} + 3q^{12}\frac{f_8^{48}}{f_2^{19}f_4^{10}} \pmod 5.$$

Extracting the terms of odd powers of q on both sides of (2.31), then dividing by q and replacing q^2 by q, we obtain (2.6). ∎

3. Proof of Conjecture 1.1. In this section, we use the (p, k)-para-metrization of theta functions given by Alaca, Alaca and Williams [AAW, AW, W] to represent the generating function of $\bar{p}(40n + 35)$ modulo 5 as a linear combination of functions in p and k, where p and k are defined in terms of the theta function $\varphi(q)$ as follows:

$$(3.1) \qquad p := p(q) = \frac{\varphi^2(q) - \varphi^2(q^3)}{2\varphi^2(q^3)}, \qquad k := k(q) = \frac{\varphi^3(q^3)}{\varphi(q)}$$

(see Alaca, Alaca and Williams [AAW]). Williams [W] proved that

$$(3.2) \qquad p = 2\frac{f_2^3 f_3^3 f_{12}^6}{f_1 f_4^2 f_9^6}.$$

We have the following congruence.

THEOREM 3.1. *For* $n \geq 0$,

$$(3.3) \qquad \bar{p}(40n + 35) \equiv 0 \pmod 5.$$

Proof. The following representations of $q^{1/24}f_1$, $q^{1/12}f_2$ and $q^{1/6}f_4$ in terms of p and k are due to Alaca and Williams [AW]:

$$(3.4) \quad q^{1/24}f_1 = 2^{-1/6}p^{1/24}(1-p)^{1/2}(1+p)^{1/6}(1+2p)^{1/8}(2+p)^{1/8}k^{1/2},$$

$$(3.5) \quad q^{1/12}f_2 = 2^{-1/3}p^{1/12}(1-p)^{1/4}(1+p)^{1/12}(1+2p)^{1/4}(2+p)^{1/4}k^{1/2},$$

$$(3.6) \quad q^{1/6}f_4 = 2^{-2/3}p^{1/6}(1-p)^{1/8}(1+p)^{1/24}(1+2p)^{1/8}(2+p)^{1/2}k^{1/2}.$$

Substituting (3.4), (3.5) and (3.6) into (2.6), we find that

$$(3.7) \quad 2^{19}\sum_{n=0}^{\infty}\bar{p}(40n + 35)q^n$$

$$\equiv \frac{\sqrt{2}\,p^{7/8}(1+2p)^{21/8}(2+p)^{21/8}}{16q^{7/8}(1-p)^6(1+p)^2\sqrt{k}}F(p, k) \pmod 5,$$

where $F(p, k)$ is given by

$$(3.8) \quad F(p, k) = 5k^{10}(524288 + 6029312p + 88735744p^2 + 840761344p^3$$
$$+ 5072977920p^4 + 22470361088p^5 + 75791417344p^6$$
$$+ 196034666496p^7 + 392385622016p^8 + 610286094336p^9$$
$$+ 731633712128p^{10} + 663209854464p^{11} + 441020946176p^{12}$$
$$+ 204189055872p^{13} + 59086163776p^{14} + 8129694944p^{15}$$
$$+ 138932400p^{16} + 2477318p^{19} - 16585772p^{18}$$
$$+ 33708184p^{17} + 19661p^{20}).$$

By (3.4) and (3.5), we have

$$(3.9) \qquad \frac{f_2^{22}}{f_1^{23}} = \frac{\sqrt{2}\, p^{7/8}(1+2p)^{21/8}(2+p)^{21/8}}{16q^{7/8}(1-p)^6(1+p)^2\sqrt{k}}.$$

Hence (3.7) can be rewritten as

$$(3.10) \qquad 2^{19} \sum_{n=0}^{\infty} \bar{p}(40n + 35)q^n \equiv \frac{f_2^{22}}{f_1^{23}} F(p, k) \pmod 5,$$

where $F(p, k)$ is defined by (3.8). Clearly, f_2^{22}/f_1^{23} is a formal power series in q with integer coefficients. By (3.1) and (3.2), we see that p and k are also formal power series in q with integer coefficients. It can be seen that the coefficients of $F(p, k)$ are divisible by 5. So we reach the assertion. ∎

To complete the proof of Conjecture 1.1, we recall that Hirschhorn and Sellers [HS-1], and Fortin, Jacob and Mathieu [FJM] independently, derived the congruence

$$(3.11) \qquad\qquad \bar{p}(4n + 3) \equiv 0 \pmod 8$$

for $n \geq 0$. This yields

$$(3.12) \qquad\qquad \bar{p}(40n + 35) \equiv 0 \pmod 8$$

for $n \geq 0$. Combining (3.12) and $\bar{p}(40n + 35) \equiv 0 \pmod 5$ for $n \geq 0$, we come to the conclusion that $\bar{p}(40n + 35) \equiv 0 \pmod{40}$ for $n \geq 0$. ∎

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