ACTA ARITHMETICA
110.3 (2003)

An analog of van der Corput's A^4-process for exponential sums

by

Patrick Sargos (Nancy)

1. Introduction. Let $k \geq 2$ be a fixed integer and M be a large integer (say, once for all, $M \geq 10$). Let $f : [1, M] \to \mathbb{R}$ be a C^k function which satisfies van der Corput's hypothesis:

$$\lambda_k \leq |f^{(k)}(x)| \ll \lambda_k \quad \text{for } 1 \leq x \leq M, \tag{1}$$

where λ_k is a small positive number (say, once for all, $\lambda_k \leq 1/10$).

In order to bound the exponential sum

$$S = \sum_{m=1}^{M} e(f(m)) \tag{2}$$

with the notation $e(x) = e^{2i\pi x}$, van der Corput's method consists in applying the A^{k-2}-process, that is, the A-process iterated $k-2$ times, combined with van der Corput's inequality (that we shall denote by ξ), to get the following well-known kth derivative test for exponential sums:

$$S \ll M\lambda_k^{1/(2^k-2)} \quad \text{provided that} \quad M \gg \lambda_k^{-2^{k-2}/(2^{k-1}-1)}. \tag{3}$$

For all details concerning this bound, the reader should refer to [2, Chapter 2]. To be brief, we shall say that the proof is obtained by the $A^{k-2}\xi$-process.

The aim of this paper is to provide an alternative to the A^4-process and to give applications in the case $k = 8$ and $k = 9$. This new inequality, which we shall denote by A_4, is the content of our Theorem 1 in Section 3. The A^4 and A_4 inequalities are quite similar; the former yields a saving with exponent $1/16$ and the latter yields a worse saving, but with exponent $1/12$.

For $k \geq 6$, we can compare the kth derivative test obtained by the $A_4A^{k-6}\xi$ with the above bound (3). We have to point out that the applications of our A_4 inequality are not as straightforward as in the case of van

2000 *Mathematics Subject Classification*: Primary 11L07.

der Corput's A-process, because of the term $u(m)$ in Theorem 1. However, we easily obtain the following results.

In the case $k = 6$, the $A_4\xi$-process provides a worse bound than $A^4\xi$ does. In the case $k = 7$, $A_4A\xi$ yields a barely better result than $A^5\xi$ does. But for $k = 8$ and mainly for $k \ge 9$, $A_4A^{k-6}\xi$ is noticeably stronger than $A^{k-2}\xi$. In the case $k = 8$, we get

$$S \ll_\varepsilon M^{1+\varepsilon}\lambda_8^{1/216} \quad \text{for } M \gg \lambda_8^{-4/9}, \tag{4}$$

instead of van der Corput's bound (3): $S \ll M\lambda_8^{1/254}$ for $M \gg \lambda_8^{-64/127}$. Similarly, in the case $k = 9$, the process $A_4A^3\xi$ yields

$$S \ll_\varepsilon M^{1+\varepsilon}\lambda_9^{1/420} \quad \text{for } M \gg \lambda_9^{-48/105}, \tag{5}$$

while van der Corput's $A^7\xi$ gives: $S \ll M\lambda_9^{1/510}$ for $M \gg \lambda_9^{-128/255}$.

The scheme of our paper is as follows. In Section 2, we recall two lemmas which are essential in the proof of Theorem 1. In Section 3, we state and prove Theorem 1. The proof of Lemma 3 is somewhat similar to that of formula (35) on page 79 of [4]. A more precise version of the A_4 inequality, extended to exponential sums with a parameter, is stated in Theorem 2, although we shall not use it in this paper.

In Section 4, we study the eighth derivative test deduced from the $A_4\xi_4$ inequality (where ξ_4 denotes the fourth derivative test of [6]). This may be written as

$$S \ll_\varepsilon M^{1+\varepsilon}\lambda_8^{1/204} \quad \text{for } M \gg \lambda_8^{-8/17}, \tag{6}$$

and this bound improves on (4).

In Section 5, we study similarly $A_4\xi_5$ (where ξ_5 denotes the fifth derivative test of [7]), which yields

$$S \ll_\varepsilon M^{1+\varepsilon}\lambda_9^{7/2640} \quad \text{for } M \gg \lambda_9^{-21/55} \tag{7}$$

with $7/2640 = 1/377.1\ldots$; the condition in (7), on the relative size of M and λ, shows that this bound is valid for "very short exponential sums".

Finally, in Section 6, we show that the A_4-process may be used as a transformation of exponent pairs in the sense of [2, Chapter 3].

Application to the order of $\zeta(\sigma + it)$. Let $\sigma \in [1/2, 1]$. The order of growth of the Riemann zeta function in the critical strip is characterized by the function

$$\mu(\sigma) = \inf\{\alpha > 0 \mid \zeta(\sigma + it) \ll_{\sigma,\alpha} t^\alpha\}.$$

Our exponent pair $(1/204, 1 - 7/204 + \varepsilon)$ (for each $\varepsilon > 0$) implies at once that

$$\mu(1 - 8/204) \le 1/204$$

while our exponent pair $(1/370, 1-8/370)$, given in Section 6, implies that

$$\mu(1-9/370) \le 1/370.$$

For these deductions, the reader should refer to [3, Chapter 21.2]. These bounds are not covered by Huxley's formula (21.2.5) of [3]. To see this, we set $\gamma = 0.026958\ldots$, so that formula (21.2.5) of [3] in the case $R = 5$ may be written as

$$\mu(1-\gamma) \le 0.00373\ldots$$

Using the convexity of the function $\mu(\sigma)$ and our two bounds above, we get the slightly better bound

$$\mu(1-\gamma) \le 0.00309\ldots$$

Notations. The symbol $u \ll v$ means that u is a complex number and v a positive real number and that there exists an absolute constant $C > 0$ which depends at most on previous absolute constants, such that $|u| \le Cv$. The symbol $\ll_s$ means that the constant may also depend on the parameter s. The symbol $\ll_\varepsilon$ means furthermore that the bound holds for each fixed $\varepsilon > 0$. Finally, $u \asymp v$ means that both $u \ll v$ and $v \ll u$.

2. Two preliminary lemmas. The starting point of this paper is a kind of Weyl–van der Corput's inequality, which is Lemma 1 of [7] and which we recall now.

LEMMA 1. *Let $\Phi : \mathbb{Z} \to \mathbb{C}$ be zero outside the set $\{1, \ldots, M\}$, where $M \ge 10$ is an integer. Let N be an integer with $1 \le N \le M$. Then*

$$\Big|\sum_m \Phi(m)\Big|^2 \ll \frac{M}{N} \max_{0 \le N_1, N_2 \le N} \sum_m \Big(\Big|\sum_{|n| \le 2N_1} \Phi(m+n)\Phi(m-n)\Big| + \Big|\sum_{|n| \le N_2} \Phi(m+2n)\Phi(m-2n)\Big|\Big). \tag{8}$$

For the proof, the reader should refer to [7]. ∎

Our second lemma deals with a Diophantine system which is treated in [5] with the method of [1]. To recall it, we introduce some notations.

Let N be a positive integer. For $\mathbf{n} = (n_1, n_2, n_3) \in \{1, \ldots, N\}^3$, and for any positive integer p, we set

$$s_p(\mathbf{n}) = n_1^p + n_2^p + n_3^p. \tag{9}$$

Let c be any real number. We denote by $Y = Y(N, c)$ the number of pairs $(\mathbf{n}, \mathbf{n}') \in \{1, \ldots, N\}^6$ such that

$$s_2(\mathbf{n}) = s_2(\mathbf{n}'), \qquad c \le s_4(\mathbf{n}) - s_4(\mathbf{n}') \le c + N^3. \tag{10}$$

LEMMA 2. *With the above notations*,

$$Y \ll_\varepsilon N^{3+\varepsilon}.$$

Proof. When $c = 0$, this is Theorem 1′ of [5]. For $c \neq 0$, the proof reduces to the case $c = 0$ by Lemma 1 of [5]. ∎

3. The A_4 inequality

3.1. *Statement of the result.* As in the introduction, let $f : [1, M] \to \mathbb{R}$ be a C^k function. Here we suppose $k \geq 6$.

THEOREM 1. *Let $N \leq M$ be a positive integer. Then there exists an integer M' $(1 \leq M' \leq M)$, a real number τ with $N^3 \leq \tau \leq N^4$ and a C^{k-6} function $u : [1, M] \to \mathbb{R}$ such that*

$$|u^{(j)}(x)| \leq N^6 \sup_{1\leq x\leq M} |f^{(j+6)}(x)| \quad \textit{for } 1 \leq x \leq M \textit{ and } 0 \leq j \leq k-6 \tag{11}$$

and

$$S \ll_\varepsilon M^{1+\varepsilon}\left(\frac{1}{N} + \frac{\tau}{MN^4}\Big|\sum_{m=1}^{M'} e(\tau f^{(4)}(m) + u(m))\Big|\right)^{1/12}. \tag{12}$$

3.2. *A lemma.* The first part of the proof can be stated as a lemma that involves only Lemma 1. For this we introduce some notations.

Let $\Phi : \mathbb{Z} \to \mathbb{C}$ be an arithmetic function which is zero outside the finite set $\{1, \ldots, M\}$. Let N be a positive integer, $N \leq M$. For $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{n}' = (n_1', n_2', n_3')$ both in $\{1, \ldots, N\}^3$, we set

$$\Psi_{\mathbf{n},\mathbf{n}'}(m) = \prod_{1\leq i\leq 3} \Phi(m+n_i)\Phi(m-n_i)\overline{\Phi(m+n_i')\Phi(m-n_i')}.$$

LEMMA 3. *With the above notations, let* $S = \sum_{m=1}^{M} \Phi(m) = \sum_{m\in\mathbb{Z}} \Phi(m) = \sum_m \Phi(m)$. *Then*

$$S \ll M\left\{\frac{1}{MN^4} \sum_{s_2(\mathbf{n})=s_2(\mathbf{n}')} \Big|\sum_m \Psi_{\mathbf{n},\mathbf{n}'}(m)\Big|\right\}^{1/12} + \left(\frac{M}{N}\sum_m |\Phi(m)|^2\right)^{1/2}. \tag{13}$$

In the above formula, the sum $\sum_{s_2(\mathbf{n})=s_2(\mathbf{n}')}$ runs over all $\mathbf{n}, \mathbf{n}' \in \{1, \ldots, N\}^3$ which satisfy $s_2(\mathbf{n}) = s_2(\mathbf{n}')$ in the sense of (9).

Proof of Lemma 3. We apply Lemma 1 to the sum S to get

$$S^2 \ll \frac{M}{N}\sum_m\Big|\sum_{|n_1|\leq N_1}\Phi(m+n)\Phi(m-n)\Big| + \frac{M}{N}\sum_m\Big|\sum_{|n_2|\leq N_2}\Phi(m+2n)\Phi(m-2n)\Big|$$

for some $N_1 \le N$ and some $N_2 \le N/2$. Thus

$$S^2 \ll \frac{M}{N}\sum_m |\Phi(m)|^2 + \frac{M}{N}\,S_1 + \frac{M}{N}\,S_2$$

with

$$S_1 = \sum_m \Big|\sum_{n=1}^{N_1} \Phi(m+n)\Phi(m-n)\Big|, \qquad S_2 = \sum_m \Big|\sum_{n=1}^{N_2} \Phi(m+2n)\Phi(m-2n)\Big|.$$

These two sums are quite similar, so that we may restrict our proof to the bound of S_1. By Hölder's inequality, we have

$$\begin{aligned} S_1^3 &\le M^2 \sum_m \Big|\sum_{n=1}^{N_1} \Phi(m+n)\Phi(m-n)\Big|^3 \\ &\le M^2 \sum_m \Big|\sum_{\mathbf{n}} \prod_{1\le i\le 3} \Phi(m+n_i)\Phi(m-n_i)\Big| \end{aligned}$$

where the inner sum runs over all $\mathbf{n} \in \{1,\ldots,N_1\}^3$. We introduce the parameter $a = s_2(\mathbf{n})$:

$$S_1^3 \le M^2 \sum_m \sum_{a=3}^{3N_1^2} \Big|\sum_{s_2(\mathbf{n})=a} \prod_{1\le i\le 3} \Phi(m+n_i)\Phi(m-n_i)\Big|.$$

By Cauchy's inequality, we have

$$\begin{aligned} S_1^6 \le M^5 N^2 \sum_m \sum_a \sum_{s_2(\mathbf{n})=a} \sum_{s_2(\mathbf{n}')=a} \\ \times\Big(\prod_{1\le i\le 3} \Phi(m+n_i)\Phi(m-n_i)\overline{\Phi(m+n_i')\Phi(m-n_i')}\Big), \end{aligned}$$

from which we deduce Lemma 3 at once. ■

3.3. *Proof of Theorem 1.* In Lemma 3, we have to take $\Phi(m) = \chi(m)e(f(m))$, where χ is the characteristic function of the interval $[1,M]$.

1) First, we want to detail the expression $P_{\mathbf{n},\mathbf{n}'} = |\sum_m \Psi_{\mathbf{n},\mathbf{n}'}(m)|$ for $\mathbf{n},\mathbf{n}' \in \{1,\ldots,N\}^3$ such that $s_2(\mathbf{n}) = s_2(\mathbf{n}')$. By Taylor's formula up to the sixth order, we can write

$$f(m+n) + f(m-n) = 2f(m) + f''(m)n^2 + \frac{1}{12}\,f^{(4)}(m)n^4 + v_n(m)$$

with

$$v_n(m) = \frac{1}{5!}\int_0^n (f^{(6)}(m+t) + f^{(6)}(m-t))(n-t)^5\,dt.$$

On the other hand, the function $m \mapsto \chi(m+n)\chi(m-n)$ is the characteristic function of the interval $[n+1, M-n]$, so that

$$(14)\quad P_{\mathbf{n},\mathbf{n}'} = \bigg| \sum_{m \in I_{\mathbf{n},\mathbf{n}'}} e\bigg(f''(m)(s_2(\mathbf{n}) - s_2(\mathbf{n}')) + \frac{1}{12} f^{(4)}(m)(s_4(\mathbf{n}) - s_4(\mathbf{n}')) + u_{\mathbf{n},\mathbf{n}'}(m) \bigg) \bigg|$$

with

$$u_{\mathbf{n},\mathbf{n}'}(m) = \sum_{i=1}^{3} v_{n_i}(m) - \sum_{i=1}^{3} v_{n_i'}(m),$$

where $I_{\mathbf{n},\mathbf{n}'} \subset [1, M]$ is an interval. Furthermore

$$|u^{(j)}_{\mathbf{n},\mathbf{n}'}(m)| \le \frac{1}{60} N^6 \max_{1 \le x \le M} |f^{(6+j)}(x)|.$$

2) We set $V = 12N^3$ and we split up the sum over $(\mathbf{n}, \mathbf{n}')$ according to the value of $\sigma := s_4(\mathbf{n}) - s_4(\mathbf{n}')$:

$$(15)\quad \sum_{s_2(\mathbf{n}) = s_2(\mathbf{n}')} P_{\mathbf{n},\mathbf{n}'} = \sum_{(\mathbf{n},\mathbf{n}') \in X} P_{\mathbf{n},\mathbf{n}'} + \sum_{l=0}^{L} \sum_{(\mathbf{n},\mathbf{n}') \in X_l} P_{\mathbf{n},\mathbf{n}'},$$

where we have set $X = \{(\mathbf{n}, \mathbf{n}') \in \{1, \ldots, N\}^6 : s_2(\mathbf{n}) = s_2(\mathbf{n}') \text{ and } |\sigma| \le V\}$ and $X_l = \{(\mathbf{n}, \mathbf{n}') \in \{1, \ldots, N\}^6 : s_2(\mathbf{n}) = s_2(\mathbf{n}') \text{ and } 2^l V < |\sigma| \le 2^{l+1} V\}$. We have $L \ll \log M$; the number of elements of X is $\operatorname{card} X \ll_\varepsilon N^{3+\varepsilon}$, according to Lemma 2, and similarly $\operatorname{card} X_l \ll_\varepsilon 2^l N^{3+\varepsilon}$.

The first term on the right hand side of (15) is trivially $\ll_\varepsilon MN^{3+\varepsilon}$. Now, among the $O(\log M)$ terms $\sum_{(\mathbf{n},\mathbf{n}') \in X_l} P_{\mathbf{n},\mathbf{n}'}$, there is at least one which dominates and which corresponds to some l that we fix. Similarly, there exists some $(\mathbf{n}, \mathbf{n}') \in X_l$ such that the second term on the right hand side of (15) is $\ll (\operatorname{card} X_l) P_{\mathbf{n},\mathbf{n}'} \log M \ll_\varepsilon M^\varepsilon |\sigma| P_{\mathbf{n},\mathbf{n}'}$. But, according to (14),

$$P_{\mathbf{n},\mathbf{n}'} = \bigg| \sum_{m \in I_{\mathbf{n},\mathbf{n}'}} e\bigg(\frac{\sigma f^{(4)}(m)}{12} + u_{\mathbf{n},\mathbf{n}'}(m) \bigg) \bigg|.$$

We insert the above equality and inequalities into (15) and then into Lemma 3. This yields a result that, with some obvious modifications, is precisely (12). The proof of Theorem 1 is complete. ■

3.4. *The A_4 inequality for exponential sums with a parameter*

THEOREM 2. *Let $H \ge 1$ and $M \ge 10$ be integers. Let $k \ge 6$ be a fixed integer. For each $h = 1, \ldots, H$, let M_h $(1 \le M_h \le M)$ be an integer and*

$f_h : [1, M] \to \mathbb{R}$ be a C^k function. Set

$$\widetilde{S} = \frac{1}{H} \sum_{h=1}^{H} \Big| \sum_{m=1}^{M_h} e(f_h(m)) \Big|.$$

Let N $(1 \le N \le M)$ be an integer. Then there exist an integer M' $(1 \le M' \le M)$ and a sextuplet $(n_1, n_2, n_3, n_1', n_2', n_3') \in \{1, \ldots, N\}^6$ such that

$$\widetilde{S} \ll_\varepsilon M^{1+\varepsilon} \bigg\{ \frac{1}{N} + \frac{\tau}{HMN^4} \Big| \sum_{h=1}^{H} \sum_{m=1}^{M'} e(\tau f_h^{(4)}(m) + u_h(m)) \Big| \bigg\}^{1/12},$$

where we have set $\tau = \frac{1}{12}(n_1^4 + n_2^4 + n_3^4 - n_1'^4 - n_2'^4 - n_3'^4)$ and

$$u_h(m) = \sum_{i=1}^{3} \int_0^{n_i} \frac{(n_i - t)^5}{5!} \, (f_h^{(6)}(m+t) + f_h^{(6)}(m-t)) \, dt$$
$$- \sum_{i=1}^{3} \int_0^{n_i'} \frac{(n_i' - t)^5}{5!} \, (f_h^{(6)}(m+t) + f_h^{(6)}(m-t)) \, dt.$$

Proof. We write

$$\sum_{m=1}^{M_h} e(f_h(m)) = \int_{-1/2}^{1/2} \Big(\sum_{m=1}^{M} e(f_h(m) - \theta m) \Big) \Big(\sum_{j=1}^{M_h} e(\theta j) \Big) \, d\theta,$$

so that

$$\widetilde{S} \ll \frac{1}{H} \int_{-1/2}^{1/2} \Big(\sum_{h=1}^{H} \Big| \sum_{m=1}^{M} e(f_h(m) - \theta m) \Big| \min(M, |\theta^{-1}|) \Big) \, d\theta$$
$$\ll \frac{1}{H} \max_{\theta \in \mathbb{R}} \Big(\sum_{h=1}^{H} \Big| \sum_{m=1}^{M} e(f_h(m) - \theta m) \Big| \Big) \log M.$$

The rest of the proof can be obtained by adjusting the proof of Theorem 1. ∎

4. An eighth derivative test for exponential sums

4.1. *Statement of the result.* As in the introduction, let $f : [1, M] \to \mathbb{R}$ be a C^8 function whose eighth derivative satisfies

$$\lambda_8 \le f^{(8)}(x) \ll \lambda_8. \tag{16}$$

Theorem 3. *With the above hypothesis*,

$$\sum_{m=1}^{M} e(f(m)) \ll_\varepsilon M^\varepsilon (M \lambda_8^{1/204} + \lambda_8^{-95/204}). \tag{17}$$

We notice that this formulation is equivalent to (6), the proof of this assertion being similar to step 0 in §4 of [6].

4.2. *A fourth derivative test.* The fourth derivative test for exponential sums of [6] does not apply directly here because of the term $u(m)$ in Theorem 1. However, some slight modifications yield the following statement.

LEMMA 4. *Let $g, u : [1, M] \to \mathbb{R}$ be respectively C^4 and C^2 with*

$$\lambda_4 \le g^{(4)}(x) \ll \lambda_4 \quad \text{and} \quad u''(x) \ll \lambda_4^{9/13} \qquad \text{for } 1 \le x \le M.$$

Then

$$\sum_{m=1}^{M} e(g(m) + u(m)) \ll_\varepsilon M^\varepsilon (M\lambda_4^{1/13} + \lambda_4^{-7/13}).$$

Proof. In the proof of Theorem 1 of [6], set $f(m) = g(m) + u(m)$. The first change occurs in step 1 of that proof, where we have to ensure that, for $h \le H_1$, the hypotheses of Lemma 3 of [6] are satisfied. But, for this point, the above rough condition $u''(x) \ll \lambda_4^{9/13}$ is widely sufficient.

The main change occurs in formula (4.16) of [6], where we have to remove the term $z = \Delta_h u(m+n+q) - \Delta_{h+r} u(m)$. As we are in a triple exponential sum with variables h, q, n, a summation by parts according to Lemma 2 of [6] should involve higher derivatives for u.

Thus, we proceed as follows. We write

$$\begin{aligned} z &= \Delta_h u(m+n+q) - \Delta_h u(m) + \Delta_h u(m) - \Delta_{h+r} u(m) \\ &= \Delta_{n+q}\Delta_h u(m) - \Delta_r u(m+h) = z_1 - z_2(h), \end{aligned}$$

say. Now, in the triple exponential sum of formula (4.12) of [6] we remove the term $e(-z_2(h))$ by a summation on the variable h (this only requires $u''(x) \ll \lambda_4^{3/13}$), while for the term $e(z_1)$, we write roughly $e(z_1) = 1 + O(|z_1|)$. The term $O(|z_1|)$ yields the desired saving when $u''(x) \ll \lambda_4^{9/13}$. ■

4.3. *Proof of Theorem 3.* Let $S = \sum_{m=1}^{M} e(f(m))$. According to Theorem 1, we have

$$S \ll_\varepsilon M^{1+\varepsilon} \left\{ \frac{1}{N} + \frac{\tau}{MN^4} \Big| \sum_{m=1}^{M'} e(\tau f^{(4)}(m) + u(m)) \Big| \right\}^{1/12}$$

with $N^3 \le \tau \le N^4$ and where u is a C^2 function such that $|u''(x)| \ll N^6 \lambda_8$ for $1 \le x \le M'$. We want to bound the sum

$$S(\tau) := \frac{1}{N} + \frac{\tau}{MN^4} \Big| \sum_{m=1}^{M'} e(\tau f^{(4)}(m) + u(m)) \Big|$$

with the use of Lemma 3. This will be possible if $u''(x) \ll (\tau\lambda_8)^{9/13}$ and in particular if $N^6 \lambda_8 \ll (\tau\lambda_8)^{9/13}$. Thus, the restrictions on N are

$$N \ll \tau^{9/78} \lambda_8^{-4/78}, \qquad N \ll M.$$

We deduce that

$$S(\tau) \ll \frac{1}{N} + \frac{\tau^{14/13}}{N^4}\lambda_8^{1/13} + \frac{\tau^{6/13}}{MN^4\lambda_8^{7/18}},$$

provided that the above restrictions on N hold. For fixed τ, we could optimize the parameter N as in Srinivasan's Lemma (cf. [2, Lemma 2.4]), but this would involve intricate calculations which are not necessary for the simplified bound (6). We only introduce the inequality $N^3 \le \tau \le N^4$, which yields

$$S(\tau) \ll \frac{1}{N} + N^{4/13}\lambda_8^{1/13} + \frac{1}{MN^{28/13}\lambda_8^{7/13}},$$

provided that $N \ll \lambda_8^{-4/51}$ and $N \le M$.

To prove Theorem 3, we may assume that

$$(*) \qquad M \asymp \lambda_8^{-8/17}.$$

Indeed, we set $M_0 = [\lambda_8^{-8/17}]$. If $M \ge M_0$, we divide the sum S_M into $O(M\lambda_8^{-8/17})$ shorter sums and the problem reduces to $(*)$. We may also choose $N \asymp \lambda_8^{-1/17}$, so that the three terms on the right hand side of the above inequality are $\asymp \lambda_8^{1/17}$. This yields the desired result (6). ∎

5. A ninth derivative test for exponential sums

5.1. *A fifth derivative test*

LEMMA 5. *Let $g, u : [1, M] \to \mathbb{R}$ be respectively C^5 and C^3 with*

$$\lambda_5 \le g^{(5)}(x) \ll \lambda_5 \quad \text{and} \quad u'''(x) \ll \lambda_5^{3/8}/M \qquad \text{for } 1 \le x \le M.$$

Then

$$\sum_{m=1}^{M} e(g(m) + u(m)) \ll_\varepsilon M^\varepsilon(M\lambda_5^{7/192} + \lambda_5^{-77/192}).$$

Proof. The only change to be done in the proof of [7, Theorem 1] occurs in [7, (3.3)]. Set $f(x) = g(x) + u(x)$. Then, by Taylor's formula,

$$\begin{aligned} f(m+n) + f(m-n) &= 2g(m) + g''(m)n^2 + \frac{1}{12}g^{(4)}(m)n^4 \\ &\quad + \frac{1}{4!}\int_0^n (n-t)^4(g^{(5)}(m+t) - g^{(5)}(m-t))\,dt \\ &\quad + 2u(m) + u''(m)n^2 \\ &\quad + \frac{1}{2}\int_0^n (n-t)^2(u'''(m+t) - u'''(m-t))\,dt \\ &= (u''(1) + g''(m))n^2 + \frac{1}{12}g^{(4)}(m)n^4 + v_m(n) \end{aligned}$$

with

$$v_m(n) = 2g(m) + 2u(m) + (u''(m) - u''(1))n^2 + \frac{1}{4!}\int_0^n (n-t)^4 (g^{(5)}(m+t) - g^{(5)}(m-t))\,dt + \frac{1}{2}\int_0^n (n-t)^2 (u'''(m+t) - u'''(m-t))\,dt.$$

In the situation of Theorem 1 of [7], the hypothesis on the size of u'' is precisely what is needed to ensure that $v'_m(n) \ll N^{-1}$. In the exponential sum $|\sum_{n\sim N} e(f(m+n) + f(m-n))|$, the term $e(v_m(n))$ can be removed by partial summation, while the term $u''(1)n^2$ in the principal part of the phase does not cause any problem. The proof of Lemma 5 is complete. ∎

5.2. *Application to a ninth derivative test*

THEOREM 4. *Let* $f : [1, M] \to \mathbb{R}$ *be a* C^9 *function such that*

$$\lambda_9 \le f^{(9)}(x) \ll \lambda_9 \quad \textit{for } 1 \le x \le M,$$

for some positive small number λ_9. *Then*

$$\sum_{m=1}^{M} e(f(m)) \ll_\varepsilon M^\varepsilon (M\lambda_9^{7/2640} + \lambda_9^{-1001/2640}). \tag{18}$$

Proof. The theorem is stated in a different way than in the introduction, but in fact, (18) is equivalent to (7). We are thus going to prove (7). Analogously to our proof of Theorem 3, we may assume that $M \asymp \lambda_9^{-21/55}$. We introduce the parameter $N \asymp \lambda_9^{-7/220}$. Now, we apply Theorem 1:

$$S = \sum_{m=1}^{M} e(f(m)) \ll_\varepsilon M^{1+\varepsilon} \left\{ \frac{1}{N} + \frac{\tau}{MN^4} \left| \sum_{m=1}^{M'} e(\tau f^{(4)}(m) + u(m)) \right| \right\}^{1/12},$$

where M' is a positive integer at most equal to M and where u is a C^3 function such that $u''' \ll N^6\lambda_9$. To apply Lemma 5, we have to ensure that $N^6\lambda_9 \ll (\tau\lambda_9)^{3/8} M^{-1}$. As $\tau \gg N^3$ and according to the choice of M and N, this condition is widely satisfied. We may now apply Lemma 5:

$$\begin{aligned} \frac{1}{N} + \frac{\tau}{MN^4} \left| \sum_{m=1}^{M'} e(\tau f^{(4)}(m) + u(m)) \right| &\ll_\varepsilon \frac{1}{N} + \frac{\tau M^\varepsilon}{MN^4} (M(\tau\lambda_9)^{7/192} + (\tau\lambda_9)^{-77/192}) \\ &\ll_\varepsilon M^\varepsilon \left(\frac{1}{N} + \frac{\tau}{N^4} (\tau\lambda_9)^{7/192} + \frac{\tau}{N^4 M (\tau\lambda_9)^{77/192}} \right). \end{aligned}$$

We see that, in the above bound, τ appears only with positive exponents, so that we may replace τ by its maximum value, which is N^4. But for this value of τ and for the above choice of M and N, the three terms in the latter bound above are of the same size. The final saving has thus size $N^{-1/12}$ or $\lambda_9^{7/2640}$. We have proved (7). ∎

6. A transformation of exponent pairs. In this section, we show that Theorem 1 yields a transformation of exponent pairs. For the definition of exponent pairs, the reader should refer to [2, Chapter 3].

THEOREM 5. *Let* $(\mu,\nu)\in[0,1/2]\times[1/2,1]$. *Suppose that, for any small enough positive* ε, *the pair* $(\mu,\nu+\varepsilon)$ *is an exponent pair. Then, for any small enough positive* ε, *the pair*

$$(\mu',\nu'+\varepsilon)=\left(\frac{\mu}{12+48\mu},\frac{11+44\mu+\nu}{12+48\mu}+\varepsilon\right)$$

is also an exponent pair.

Proof. STEP 1. We first recall what is to be proved. Let $s<1$ be a fixed real number, and ε be a fixed small enough positive real number. We have to prove that there exist a positive real number $\eta=\eta(s,\varepsilon,\mu,\nu)\le 1/2$ small enough, and a positive integer $l=l(s,\varepsilon,\mu,\nu)$ large enough such that, for any integer $M\ge 10$ and any positive real number T, for any "semi-monomial" function $f:[M,2M]\to\mathbb{R}$ which can be written as $f(x)=\phi(x)+v(x)$, with

$$\phi(x)=Tx^s/M^s\quad\text{if } s\neq 0\qquad\text{and}\qquad \phi(x)=T\log x\quad\text{if } s=0 \tag{19}$$

and where $v:[M,2M]\to\mathbb{R}$ is a C^l function such that

$$|v^{(j)}(x)|\le\eta|\phi^{(j)}(x)|\qquad\text{for } M\le x\le 2M \text{ and } 1\le j\le l \tag{20}$$

and for any interval $I\subset[M,2M]$, we have

$$\Big|\sum_{m\in I\cap\mathbb{N}}e(f(m))\Big|\le CM^{2\varepsilon}P(\mu,\nu)+M/T \tag{21}$$

where we have set $P(\mu,\nu)=T^{\mu'}M^{\nu'-\mu'}$ and where $C=C(s,\varepsilon,\mu,\nu,\eta,l)$ is a constant which does not depend on M and T.

STEP 2. We now treat the main part of the proof, where we suppose

$$T\ge M^{(1+\mu+3\nu+\varepsilon)/(1+\mu)}. \tag{22}$$

We bound the sum $S=\sum_{m\in I\cap\mathbb{N}}e(f(m))$ by means of Theorem 1:

$$S\ll M^{1+\varepsilon}\left(\frac{1}{N}+\frac{\tau}{MN^4}\Big|\sum_{m\in I'\cap\mathbb{N}}e(\tau f^{(4)}(m)+u(m))\Big|\right)^{1/12} \tag{23}$$

where τ and u are as in Theorem 1 and where $I'\subset I$ is an interval.

Our aim is to apply the exponent pair $(\mu, \nu+\varepsilon)$ to the inner sum in (23). For this, we need the function $g(x) = \tau f^{(4)}(x) + u(x)$ to satisfy some "semi-monomiality" properties. But if we suppose for example $s \neq 0$, we have $g(x) = \pm\phi_1(x) + v_1(x)$ with

$$\phi_1(x) = \frac{T_1}{M^{s-4}}\, x^{s-4}, \qquad T_1 = |s(s-1)(s-2)(s-3)|\,\frac{\tau T}{M^4},$$
$$v_1(x) = \tau v^{(4)}(x) + u(x).$$

By definition of the exponent pair $(\mu, \nu+\varepsilon)$, there exist $\eta_1 > 0$ and an integer l_1 such that the condition

$$|v_1(x)| \le \eta_1 |\phi_1^{(j)}(x)| \quad \text{for } M \le x \le 2M \text{ and } 1 \le j \le l_1 \tag{24}$$

is sufficient to guarantee that the bound

$$S_1 := \sum_{m \in I' \cap \mathbb{N}} e(g(m)) \ll T_1^{\mu} M^{\nu-\mu+\varepsilon} + M/T_1 \tag{25}$$

holds uniformly in M and T, recalling that μ, ν, ε and $s-4$ are fixed.

We choose $\eta = \eta_1/2$ and $l = l_1 + 6$. Using formulas (11) and (20), we get, for $M \le x \le 2M$ and $1 \le j \le l-6$,

$$|v_1(x)| \le \eta |\phi_1^{(j)}(x)| + O\left(\frac{TN^6}{M^{j+6}}\right)$$
$$\le |\phi_1^{(j)}(x)| \left(\eta + O\left(\frac{N^6}{\tau M^2}\right)\right) \le |\phi_1^{(j)}(x)| \left(\eta + O\left(\frac{N^3}{M^2}\right)\right).$$

Suppose now that there exists $\delta = \delta(\mu, \nu, s, \varepsilon) > 0$ such that

$$N \ll M^{2/3-\delta}. \tag{26}$$

Then, for M large enough, condition (24) is satisfied and we may insert (25) into (23) to get

$$S \ll M^{1+\varepsilon}\left(\frac{1}{N} + N^{4\mu} T^{\mu} M^{\nu-1-5\mu+\varepsilon} + \frac{M^4}{TN^4}\right)^{1/12}.$$

The first two terms in the bracket are of the same size if we choose

$$N \asymp \frac{M^{(1+5\mu-\nu-\varepsilon)/(1+4\mu)}}{T^{\mu/(1+4\mu)}}. \tag{27}$$

With this choice of N, we may take $\delta = \varepsilon(1+\mu)/(1+4\mu)$ in (26). On the other hand, we shall have the bound

$$S \ll \frac{M^{1+\varepsilon}}{N^{1/12}} \tag{28}$$

whenever we can assume that $M^4(TM^3)^{-1} \ll 1$; but this last condition is precisely (22), so that (27) and (28) imply (21) in this case.

STEP 3. We now suppose that $T < M^{(1+\mu+3\nu+3\varepsilon)/(1+\mu)}$. For these relative sizes of T and M, the bound (21) is not relevant and may be deduced from some known result; here we use the classical exponent pair $(1/30, 26/30) = A^3B(0,1)$ which, in the range considered, is widely sufficient to recover (21), providing that ε is small enough. The details are left to the reader. ∎

EXAMPLES. Theorems 3 and 4 give rise to exponent pairs. Indeed, both theorems rely on kth derivative tests (with $k = 4, 5$) which correspond in fact to exponent pairs. Thus, $(1/204, 1-7/204+\varepsilon)$ and $(1/378, 1-8/378)$ are exponent pairs for each $\varepsilon > 0$. Furthermore, the latter can be improved if we use Theorem 3 of [7] instead of Theorem 1 of [7], so that $(1/370, 1-8/370)$ is an exponent pair.

As a last example, we apply A_4A to the exponent pair $(17/456, 388/456 + \varepsilon)$ of Theorem 3 of [7] and we get the exponent pair $(1/716, 1-9/716)$.

References

[1] E. Bombieri and H. Iwaniec, *Some mean value theorems for exponential sums*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 13 (1986), 473–486.

[2] S. W. Graham and G. Kolesnik, *Van der Corput's Method of Exponential Sums*, London Math. Soc. Lecture Note Ser. 126, Cambridge Univ. Press, 1991.

[3] M. N. Huxley, *Area, Lattice Points and Exponential Sums*, Clarendon Press, Oxford, 1996.

[4] H. L. Montgomery, *Ten Lectures on the Interface Between Analytic Number Theory and Harmonic Analysis*, CBMS Reg. Conf. Ser. Math. 84, Amer. Math. Soc., 1994.

[5] O. Robert et P. Sargos, *Un théorème de moyenne pour les sommes d'exponentielles. Application à l'inégalité de Weyl*, Publ. Inst. Math. (Beograd) 67 (81) (2000), 14–30.

[6] —, —, *A fourth derivative test for exponential sums*, Compositio Math., to appear.

[7] P. Sargos, *Un critère de la dérivée cinquième pour les sommes d'exponentielles*, Bull. London Math. Soc. 32 (2000), 398–402.

Institut Elie Cartan
Université Henri Poincaré–Nancy 1
BP 239
54506 Vandœuvre-lès-Nancy Cedex, France
E-mail: Sargos@iecn.u-nancy.fr

Received on 14.11.2001
and in revised form on 3.2.2003 (4148)