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On non-intersecting arithmetic progressions

by

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I. Introduction. Suppose that $a_1 \pmod{q_1}, a_2 \pmod{q_2}, \ldots, a_k \pmod{q_k}$ is a collection of arithmetic progressions, where $2 \le q_1 < \ldots < q_k \le x$, with the property that

$$\{a_i \pmod{q_i}\} \cap \{a_j \pmod{q_j}\} = \emptyset \quad \text{if } i \ne j.$$

We say that such a collection of arithmetic progressions is *disjoint* or *non-intersecting*. Let $f(x)$ be the maximum value for k, maximized over all choices of progressions $a_i \pmod{q_i}$. Define

$$\begin{aligned} L(c,x) &:= \exp(c\sqrt{\log x \log\log x}), \\ \psi(x,y) &:= \#\{n \le y : p \text{ prime},\ p \mid n \Rightarrow p \le y\}, \\ \psi^*(x,y) &:= \#\{n \le y : p \text{ prime},\ p^a \mid n \Rightarrow p^a \le y\}. \end{aligned}$$

In [3], Erdős and Szemerédi prove that

$$\frac{x}{\exp((\log x)^{1/2+\varepsilon})} < f(x) < \frac{x}{(\log x)^c}$$

for some constant $c > 0$. (This result is also mentioned in [2].) Their lower bound can be refined by using more exact estimates for $\psi(x, L(c,x))$ than was used in their paper. Specifically, as direct consequence of [1, Lemma 3.1], we have the following estimate:

LEMMA 1. *For any constant $c > 0$,*

$$\psi(x, L(c,x)) = \frac{x}{L(1/(2c) + o(1), x)}. \tag{1}$$

We also have the same estimate for $\psi^*(x, L(c,x))$, since

$$\begin{aligned} \psi(x, L(c,x)) &> \psi^*(x, L(c,x)) \\ &> \psi(x, L(c,x)) - \sum_{n^2 > L(c,x)} \psi(x/n^2, L(c,x)) \end{aligned} \tag{2}$$

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$$= \psi(x, L(c,x)) - O\biggl(\frac{x}{L(c/2+1/(2c)+o(1),x)}\biggr).$$

Now, let p be the largest prime number less than or equal to $L(1/\sqrt{2},x)$. Let $q_1,\ldots,q_t$ be the collection of all integers $\le x$ which are divisible by p, and whose prime power factors are all $< p$. From (1) and (2), we deduce that

$$t = \frac{x}{pL(1/\sqrt{2}+o(1),x)} = \frac{x}{L(\sqrt{2}+o(1),x)}.$$

For each $q_i = pl_r^{h_r} l_{r-1}^{h_{r-1}} \ldots l_1^{h_1}$, where $p > l_r^{h_r} > l_{r-1}^{h_{r-1}} > \ldots > l_1^{h_1}$ are the powers of the distinct primes dividing q_i, we choose the residue class $a_i \pmod{q_i}$ using the Chinese Remainder Theorem as follows:

$$a_i \equiv l_r^{h_r} \pmod{p}; \qquad a_i \equiv l_{j-1}^{h_{j-1}} \pmod{l_j^{h_j}} \qquad \text{for } 2 \le j \le r;$$

and finally,

$$a_i \equiv 0 \pmod{l_1^{h_1}}.$$

This is exactly the construction which appears in [3] (except that their progressions were all square-free), and it is easy to see that our progressions $a_i \pmod{q_i}$ are disjoint. Thus, we have

$$f(x) > \frac{x}{L(\sqrt{2}+o(1),x)}.$$

In this paper we will prove the following results:

Theorem 1. *If* $a_1 \pmod{q_1},\ldots,a_k \pmod{q_k}$ *are a collection of disjoint arithmetic progressions, where the* q_i*'s are square-free and* $2 \le q_1 < \ldots < q_k \le x$, *then*

$$k < \frac{x}{L(1/2-o(1),x)}.$$

Corollary to Theorem 1.

$$f(x) < \frac{x}{L(1/6-o(1),x)}.$$

Thus, we will have shown that

$$\frac{x}{L(\sqrt{2}+o(1),x)} < f(x) < \frac{x}{L(1/6-o(1),x)}.$$

To see how the Corollary follows from Theorem 1, let $b_1 \pmod{r_1},\ldots,b_{f(x)} \pmod{r_{f(x)}}$ be a maximal collection of disjoint arithmetic progressions with $2 \le r_1 < \ldots < r_{f(x)} \le x$. Suppose, for proof by contradiction, that for some $\varepsilon < 1/6$,

$$f(x) > \frac{x}{L(1/6-\varepsilon,x)}. \tag{3}$$

Write each $r_i = \alpha_i\beta_i$, where β_i is square-free, $\gcd(\alpha_i,\beta_i) = 1$, and every prime dividing α_i divides to a power ≥ 2. (Note: we may have α_i or $\beta_i = 1$.)

Now, at least half of α_i's must be $\le L(1/3,x)$, for if not we would deduce from our assumption (3) that

$$\begin{aligned}\frac{x}{2L(1/6-\varepsilon,x)} &< f(x)/2 < \#\{r_i : \alpha_i > L(1/3,x)\}\\ &< x \sum_{n^2 > L(1/3,x)} \frac{1}{n^2} \prod_{p\ \text{prime}} \left(1+\frac{1}{p^2}+\frac{1}{p^3}+\dots\right)\\ &\ll \frac{x}{L(1/6,x)},\end{aligned}$$

which is impossible for x large enough in terms of ε. Thus, there must exist an $\alpha < L(1/3,x)$ for which at least $f(x)/(2L(1/3,x))$ of the r_i's have $\alpha_i = \alpha$. Let $R(\alpha) \subseteq \{r_1,\dots,r_{f(x)}\}$ be such a collection of r_i's, where

$$|R(\alpha)| > \frac{f(x)}{2L(1/3,x)} > \frac{x}{2L(1/2-\varepsilon,x)};$$

this last inequality follows from our assumption (3). Now there must exist a residue class $b \pmod{\alpha}$ for which at least $|R(\alpha)|/\alpha$ of the progressions $b_i \pmod{r_i}$ satisfy

$$r_i \in R(\alpha) \quad \text{and} \quad b_i \equiv b \pmod{\alpha}. \tag{4}$$

Thus, the arithmetic progressions $b_i \pmod{r_i/\alpha}$, where r_i satisfies (4), form a collection of $\ge |R(\alpha)|/\alpha \gg x/(\alpha L(1/2-\varepsilon,x))$ disjoint progressions, with distinct square-free moduli $\le x/\alpha$. This contradicts Theorem 1 for x sufficiently large in terms of ε. We must conclude, therefore, that the bound in (3) is false for all $\varepsilon < 1/6$ and $x > x_0(\varepsilon)$, and so the Corollary to Theorem 1 follows.

II. Proof of Theorem 1. Before we prove Theorem 1, we will need the following lemma:

LEMMA 2. *There are at most $x/L(c/2+o(1),x)$ positive integers $n \le x$ such that $\omega(n) > c\sqrt{\log x/\log\log x}$ (recall: $\omega(n) = \sum_{p\mid n,\, p\ \text{prime}} 1$), where c is some positive constant.*

Proof. We observe that

$$\begin{aligned}\#\{n \le x : \omega(n) > c\sqrt{\log x/\log\log x}\} &< x \sum_{j > c\sqrt{\frac{\log x}{\log\log x}}} \frac{(\sum_{p^a \le x,\, p\ \text{prime}} 1/p^a)^j}{j!}\\ &= \frac{x}{(c\sqrt{\log x/\log\log x})^{\{c+o(1)\}\sqrt{\log x/\log\log x}}}\\ &= \frac{x}{L(c/2+o(1),x)}.\end{aligned}$$

We now resume the proof of Theorem 1. Consider the collection of all the q_i's with the properties:

(A) $\omega(q_i) < \sqrt{\log x/\log\log x}$.

(B) There exists a prime $p > L(1,x)$ such that $p \mid q_i$.

Let $\{r_1,\ldots,r_{k'}\}$ be the collection of all q_i's satisfying (A) and (B), and let $\{b(r_1),\ldots,b(r_{k'})\}$ be their corresponding residue classes.

To prove our theorem, we start with the set $S_0 = \{r_1,\ldots,r_{k'}\}$, and construct a sequence of subsets $S_0 \supseteq S_1 \supseteq S_2 \supseteq \ldots$, and a sequence of primes $p_1, p_2, \ldots$ (and let $p_0 = 1$), such that for each $i \ge 1$, the following three properties hold:

1. Each member of S_i is divisible by the primes $p_1,\ldots,p_i$.

2. There exists an integer A_i such that for each $r_j \in S_i$, we have $b(r_j) \equiv A_i \pmod{p_1\ldots p_i}$.

3. $|S_i| > |S_{i-1}|/(p_i\sqrt{\log x/\log\log x})$.

We continue constructing these subsets until we reach a subset S_t which has the additional property:

4. There exists a prime $p \ne p_1,\ldots,p_t$, $p \ge L(1,x)$, such that at least $|S_t|/\sqrt{\log x/\log\log x}$ of the elements of S_t are divisible by p.

Let us suppose for the time being that we can construct these sets $S_1,\ldots,S_t$. Applying property 3 iteratively, together with property 4, we find that the number of elements of S_t which are divisible by p (which are already divisible by $p_1\ldots p_t$ by property 1) is at least

$$\frac{|S_0|}{p_1\ldots p_t(\sqrt{\log x/\log\log x})^{t+1}} \ge \frac{|S_0|}{p_1\ldots p_t L(1/2+o(1),x)}.$$

(*Note*: By property (A) above we have $t \le \sqrt{\log x/\log\log x}$ since every element of S_0 has at most $\sqrt{\log x/\log\log x}$ prime factors.) From this, together with the fact that $p > L(1,x)$, we have

$$\begin{aligned}\frac{x}{p_1\ldots p_k L(1,x)} &\ge \#\{n \le x : pp_1\ldots p_t \mid n\} > \#\{q \in S_t : p \mid q\}\\ &\ge \frac{|S_0|}{p_1\ldots p_t L(1/2+o(1),x)}.\end{aligned}$$

It follows that

$$|S_0| < \frac{x}{L(1/2-o(1),x)}.$$

From this, together with Lemmas 1 and 2 and the fact that the elements of

S_0 satisfy (A) and (B) above, we have

$$\frac{x}{L(1/2-o(1),x)} > |S_0| > k - \#\{n \le x : \omega(n) \ge \sqrt{\log x/\log\log x}\} - \psi(x, L(1,x)) > k - \frac{x}{L(1/2-o(1),x)},$$

and so

$$k < \frac{x}{L(1/2-o(1),x)},$$

which proves our theorem.

To construct our sets S_i, we apply the following iterative procedure: suppose we have constructed the sets $S_1, \ldots, S_i$ which satisfy 1 through 3 above. To construct S_{i+1}, first pick any element $r \in S_i$. Now let $e_1, \ldots, e_j$ be all those primes dividing $r/(p_1 \ldots p_i)$ (note: $j < \sqrt{\log x/\log\log x}$). Each element $s \in S_i$, $s \ne r$, is divisible by at least one of these primes, since otherwise $\gcd(r,s) = p_1 \ldots p_i$ and so we would have $b(r) \equiv A_i \equiv b(s) \pmod{\gcd(r,s)}$, which would mean that $\{b(r) \pmod{r}\} \cap \{b(s) \pmod{s}\} \ne \emptyset$.

Now, there must be at least $|S_i|/j > |S_i|/\sqrt{\log x/\log\log x}$ of the elements of S_i which are divisible by one of these primes e_h. Let $C_i \subseteq S_i$ be the collection of all elements S_i divisible by this prime e_h. There exists at least one residue class $B \pmod{e_h}$ for which more than $|C_i|/e_h > |S_i|/(e_h\sqrt{\log x/\log\log x})$ of the elements $r \in C_i$ satisfy $b(r) \equiv B \pmod{e_h}$. Now let S_{i+1} be the collection of all such $r \in C_i$, set $p_{i+1} = e_h$, and let $A_{i+1} \equiv A_i \pmod{p_1 \ldots p_i}$ and $A_{i+1} \equiv B \pmod{p_{i+1}}$ by the Chinese Remainder Theorem. Then properties 1, 2, and 3 above follow immediately for this set S_{i+1}.

If there exists a prime $p > L(1,x)$ which divides more than

$$\frac{|S_{i+1}|}{\sqrt{\log x/\log\log x}}$$

of the elements of S_{i+1}, then we set $t = i+1$ and we are finished. If not, we continue constructing these sets S_j. We are guaranteed to eventually hit upon such a prime p since all our r_j's are divisible by at least one prime $p > L(1,x)$ by property (B).

References

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