On additive properties of two special sequences

by

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1. Introduction. Let $A$ be an infinite sequence of positive integers. For each positive integer $n$, let $R_1(A, n)$, $R_2(A, n)$ and $R_3(A, n)$ denote the number of solutions of

- $x + y = n, \ x, y \in A,$
- $x + y = n, \ x < y, \ x, y \in A,$
- $x + y = n, \ x \leq y, \ x, y \in A,$

respectively. A. Sárkőzy asked whether there exist two sets $A$ and $B$ of positive integers with infinite symmetric difference, i.e.

$$|(A \cup B) \setminus (A \cap B)| = \infty,$$

and

$$R_i(A, n) = R_i(B, n), \quad n \geq n_0,$$

for $i = 1, 2, 3$. For $i = 1$, the answer is no. For $i = 2$, G. Dombi [1] proved that the set $\mathbb{N}$ of positive integers can be partitioned into two subsets $A$ and $B$ such that $R_2(A, n) = R_2(B, n)$ for all $n \in \mathbb{N}$. For $i = 3$, G. Dombi [1] conjectured that the answer is no. For other related results, the reader is referred to [2–4]. Let

$$U(A, n) = \{(x, y) \mid x + y = n, \ x, y \in A, \ x \leq y\},$$
$$U_0(A, n) = \{(x, y) \mid (x, y) \in U(A, n), \ 2 \mid x\},$$
$$U_1(A, n) = \{(x, y) \mid (x, y) \in U(A, n), \ 2 \nmid x\}.$$

Then $R_3(A, n) = |U(A, n)| = |U_0(A, n)| + |U_1(A, n)|$.

In this note, we prove the following theorem.

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THEOREM. The set $\mathbb{N}$ of positive integers can be partitioned into two subsets $A$ and $B$ such that $R_3(A,n) = R_3(B,n)$ for all $n \geq 3$.

Proof. Let $T(n)$ denote the number of zero digits in the dyadic representation of $n \geq 0$ (thus $T(0) = 1$, $T(1) = 0$, etc.). Then $T(2n+1) = T(2n) - 1$ ($n \geq 0$) and $T(2n) = T(n) + 1$ ($n \geq 1$). Let

$$A = \{ n \in \mathbb{N} \mid 2 \mid T(n-1) \}, \quad B = \{ n \in \mathbb{N} \mid 2 \mid T(n-1) \}.$$  

We use induction on $n$ to prove that $|U(A,n)| = |U(B,n)|$ for all $n \geq 3$. By calculation we have $|U(A,n)| = |U(B,n)|$ for $n = 3, 4, 5$. Now suppose that $|U(A,n)| = |U(B,n)|$ for $3 \leq n \leq k-1$ ($k \geq 6$).

CASE 1: $2 \mid k$. Define a map

$$f : U_0(A,k) \setminus \{(2, k-2)\} \to U\left(A, \frac{k}{2}\right) \setminus \left\{ \left( 1, \frac{k-2}{2} \right) \right\}$$

by

$$f(a,b) = \left( \frac{a}{2}, \frac{b}{2} \right).$$

Noting that $b \geq a \geq 4$, $2 \mid a$ and $2 \mid b$, we have

$$T\left( \frac{a}{2} - 1 \right) = T(a-2) - 1 = T(a-1),$$

$$T\left( \frac{b}{2} - 1 \right) = T(b-2) - 1 = T(b-1).$$

Hence, $f$ is well defined. It is easy to verify that $f$ is bijective. Thus

$$(1) \quad |U_0(A,k) \setminus \{(2, k-2)\}| = \left| U\left(A, \frac{k}{2}\right) \setminus \left\{ \left( 1, \frac{k-2}{2} \right) \right\} \right|.$$  

Similarly, we have

$$(2) \quad |U_0(B,k) \setminus \{(2, k-2)\}| = \left| U\left(B, \frac{k}{2}\right) \setminus \left\{ \left( 1, \frac{k-2}{2} \right) \right\} \right|.$$  

Define a map $g : U_1(A,k) \setminus \{(1, k-1)\} \to U\left(B, \frac{k+2}{2}\right) \setminus \left\{ \left( 1, \frac{k}{2} \right) \right\}$ by $f(a,b) = \left( \frac{a+1}{2}, \frac{b+1}{2} \right)$. Noting that $b \geq a \geq 2$, $2 \mid a$ and $2 \mid b$, we have

$$T\left( \frac{a+1}{2} - 1 \right) = T\left( \frac{a-1}{2} \right) = T(a-1) - 1,$$

$$T\left( \frac{b+1}{2} - 1 \right) = T\left( \frac{b-1}{2} \right) = T(b-1) - 1.$$  

Hence, $g$ is well defined. It is easy to verify that $g$ is bijective. Thus

$$(3) \quad |U_1(A,k) \setminus \{(1, k-1)\}| = \left| U\left(B, \frac{k+2}{2}\right) \setminus \left\{ \left( 1, \frac{k}{2} \right) \right\} \right|.$$
Similarly, we have

\[ |U_1(B, k) \setminus \{(1, k - 1)\}| = \left| U \left( A, \frac{k + 2}{2} \right) \setminus \left\{ \left( 1, \frac{k}{2} \right) \right\} \right|. \]

Noting that \( 1 \not\in A \) and \( 2 \not\in B \), by (1)–(4), we have

\[ |U(A, k) \setminus \{(2, k - 2)\}| = \left| U \left( A, \frac{k}{2} \right) \right| + \left| U \left( B, \frac{k + 2}{2} \right) \setminus \left\{ \left( 1, \frac{k}{2} \right) \right\} \right|, \]

\[ |U(B, k) \setminus \{(1, k - 1)\}| = \left| U \left( B, \frac{k}{2} \right) \setminus \left\{ \left( 1, \frac{k - 2}{2} \right) \right\} \right| + \left| U \left( A, \frac{k + 2}{2} \right) \right|. \]

Noting that \( T(k - 2) = T \left( \frac{k}{2} - 1 \right) + 1 \) and \( T(k - 3) = T(k - 4) - 1 = T \left( \frac{k - 4}{2} \right) = T \left( \frac{k - 2}{2} - 1 \right) \), we have the following possibilities:

(i) If \( 2 \mid T \left( \frac{k}{2} - 1 \right) \) and \( 2 \mid T \left( \frac{k - 2}{2} - 1 \right) \), then

\[ \left( 1, \frac{k}{2} \right) \in U \left( B, \frac{k + 2}{2} \right), \quad \left( 1, \frac{k - 2}{2} \right) \in U \left( B, \frac{k}{2} \right), \]

\[ (2, k - 2) \not\in U(A, k), \quad (1, k - 1) \not\in U(B, k). \]

In this case, by (5) and (6), we have

\[ |U(A, k)| = \left| U \left( A, \frac{k}{2} \right) \right| + \left| U \left( B, \frac{k + 2}{2} \right) \right| - 1, \]

\[ |U(B, k)| = \left| U \left( B, \frac{k}{2} \right) \right| + \left| U \left( A, \frac{k + 2}{2} \right) \right| - 1. \]

(ii) If \( 2 \nmid T \left( \frac{k}{2} - 1 \right) \) and \( 2 \mid T \left( \frac{k - 2}{2} - 1 \right) \), then

\[ \left( 1, \frac{k}{2} \right) \in U \left( B, \frac{k + 2}{2} \right), \quad \left( 1, \frac{k - 2}{2} \right) \not\in U \left( B, \frac{k}{2} \right), \]

\[ (2, k - 2) \in U(A, k), \quad (1, k - 1) \not\in U(B, k). \]

In this case, by (5) and (6), we have

\[ |U(A, k)| = \left| U \left( A, \frac{k}{2} \right) \right| + \left| U \left( B, \frac{k + 2}{2} \right) \right|, \]

\[ |U(B, k)| = \left| U \left( B, \frac{k}{2} \right) \right| + \left| U \left( A, \frac{k + 2}{2} \right) \right|. \]

(iii) If \( 2 \mid T \left( \frac{k}{2} - 1 \right) \) and \( 2 \nmid T \left( \frac{k - 2}{2} - 1 \right) \), then

\[ \left( 1, \frac{k}{2} \right) \not\in U \left( B, \frac{k + 2}{2} \right), \quad \left( 1, \frac{k - 2}{2} \right) \in U \left( B, \frac{k}{2} \right), \]

\[ (2, k - 2) \not\in U(A, k), \quad (1, k - 1) \in U(B, k). \]
In this case, by (5) and (6), we have

\[ |U(A, k)| = \left| U \left( A, \frac{k}{2} \right) \right| + \left| U \left( B, \frac{k + 2}{2} \right) \right|, \]
\[ |U(B, k)| = \left| U \left( B, \frac{k}{2} \right) \right| + \left| U \left( A, \frac{k + 2}{2} \right) \right|. \]

(iv) If \( 2 \mid T \left( \frac{k}{2} - 1 \right) \) and \( 2 \mid T \left( \frac{k-2}{2} - 1 \right) \), then

\[ \left( 1, \frac{k}{2} \right) \notin U \left( B, \frac{k + 2}{2} \right), \quad \left( 1, \frac{k - 2}{2} \right) \notin U \left( B, \frac{k}{2} \right), \]
\[ (2, k - 2) \in U(A, k), \quad (1, k - 1) \in U(B, k). \]

In this case, by (5) and (6), we have

\[ |U(A, k)| = \left| U \left( A, \frac{k}{2} \right) \right| + \left| U \left( B, \frac{k + 2}{2} \right) \right| + 1, \]
\[ |U(B, k)| = \left| U \left( B, \frac{k}{2} \right) \right| + \left| U \left( A, \frac{k + 2}{2} \right) \right| + 1. \]

Since \( 3 \leq k/2 < k \), \( 3 \leq (k + 2)/2 < k \), by the induction hypothesis, we have

\[ \left| U \left( A, \frac{k}{2} \right) \right| = \left| U \left( B, \frac{k}{2} \right) \right|, \quad \left| U \left( A, \frac{k + 2}{2} \right) \right| = \left| U \left( B, \frac{k + 2}{2} \right) \right|. \]

By (i)–(iv), we have

\[ |U(A, k)| = |U(B, k)|. \]

**Case 2:** \( 2 \nmid k \). Define a map \( h : U_0(A, k) \rightarrow U_1(B, k) \) by

\[ h(a, b) = (a - 1, b + 1). \]

Since \( 2 \mid a \), we have \( 2 \nmid b, b + 1 \geq a - 1 \geq 1 \) and \( 2 \nmid a - 1 \). By \( T(a - 2) = T(a - 1) + 1 \) and \( T(b) = T(b - 1) - 1 \), we know that \( h \) is well defined. It is clear that \( h \) is injective. Now we show that \( h \) is surjective. Assume that \( (u, v) \in U_1(B, k) \). Let \( a' = u + 1 \) and \( b' = v - 1 \). Then \( 2 \mid u + 1, 2 \mid v - 2, T(a' - 1) = T(u - 1) = T(v - 1) - 1 \) and \( T(b' - 1) = T(v - 2) = T(v - 1) + 1 \). To prove that \( (a', b') \in U_0(A, k) \), it is sufficient to prove that \( a' \leq b' \). If \( a' > b' \), then, since \( u \leq v, 2 \nmid u \) and \( 2 \mid v \), we have \( a' - 1 = b' \). But

\[ T(a' - 1) = T(u - 1) - 1 \equiv T(v - 1) - 1 = T(b') - 1 \pmod{2}, \]

a contradiction. So \( a' \leq b' \) and then \( (a', b') \in U_0(A, k) \). Hence \( h \) is bijective. Thus

\[ |U_0(A, k)| = |U_1(B, k)|. \]

Similarly, \( |U_0(B, k)| = |U_1(A, k)| \). Therefore \( |U(A, k)| = |U(B, k)| \). This completes the proof.
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References


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